Stochastic nonlinear Schrödinger equations: no blow-up in the non-conservative case

Viorel Barbu[1], **Michael Röckner**[2], **Deng Zhang**[3]

Abstract. This paper is devoted to the study of noise effects on blow-up solutions to stochastic nonlinear Schrödinger equations. It is a continuation of our recent work [2], where the (local) well-posedness is established in H^1, also in the non-conservative critical case. Here we prove that in the non-conservative focusing mass-(super)critical case, by adding a large multiplicative Gaussian noise, with high probability one can prevent the blow-up on any given bounded time interval $[0, T]$, $0 < T < \infty$. Moreover, in the case of spatially independent noise, the explosion even can be prevented with high probability on the whole time interval $[0, \infty)$. The noise effects obtained here are completely different from those in the conservative case studied in [5].

Keywords: (stochastic) nonlinear Schrödinger equation, Wiener process, noise effect, blow-up.

2000 Mathematics Subject Classification: 60H15, 35Q55, 60H30

[1]Octav Mayer Institute of Mathematics (Romanian Academy) and Al.I. Cuza University and, 700506, Iaşi, Romania. This work was supported by the DFG through CRC 701.

[2]Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany. This research was supported by the DFG through CRC 701.

[3]Department of Mathematics, Shanghai Jiao Tong University, 200240 Shanghai, China. Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany.

1 Introduction and main results.

We consider the stochastic nonlinear Schrödinger equation with linear multiplicative noise,

$$\begin{aligned}
&idX(t,\xi) = \Delta X(t,\xi)dt + \lambda |X(t,\xi)|^{\alpha-1}X(t,\xi)dt \\
&\qquad\qquad - i\mu(\xi)X(t,\xi)dt + iX(t,\xi)dW(t,\xi),\ t\in(0,T),\ \xi\in\mathbb{R}^d, \quad (1.1)\\
&X(0) = x \in H^1.
\end{aligned}$$

Here, the exponents of particular interest lie in the focusing mass-(super)critical range, namely,

$$\lambda = 1, \quad \alpha \in [1+\frac{4}{d}, 1+\frac{4}{(d-2)^+}). \tag{1.2}$$

W is the colored Wiener process

$$W(t,\xi) = \sum_{j=1}^{N} \mu_j e_j(\xi)\beta_j(t),\ t\geq 0,\ \xi\in\mathbb{R}^d, \tag{1.3}$$

where $N<\infty$, $\mu_j\in\mathbb{C}$, e_j are real-valued functions, and $\beta_j(t)$ are independent real Brownian motions on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with natural filtation $(\mathcal{F}_t)_{t\geq 0}$, $1\leq j\leq N$. Moreover, as required by the physical context (see [3] and [4]), μ is of the form

$$\mu = \sum_{j=1}^{N} |\mu_j|^2 e_j^2. \tag{1.4}$$

Hence $|X(t)|_2^2$ is a martingale, which allows to define the so-called "physical probability law". In particular, in the conservative case (i.e. $Re\mu_j = 0$, $1\leq j\leq N$), the last two terms in (1.1) coincide with the Stratonovitch integration. We also refer to [1] for discussions on the physical background.

Definition 1.1 *A solution X to (1.1) on $[0,\tau]$, where τ is an $(\mathscr{F}_t)$-stopping time, is an H^1-valued continuous $(\mathscr{F}_t)$-adapted process, such that $|X|^{\alpha-1}X\in L^1(0,\tau;H^{-1})$, $\mathbb{P}-a.s$, and it satisfies $\mathbb{P}-a.s$*

$$\begin{aligned}
X(t) =& x - \int_0^t (i\Delta X(s) + \mu X(s) + \lambda i|X(s)|^{\alpha-1}X(s))ds \\
&+ \int_0^t X(s)dW(s), \quad t\in[0,\tau], \qquad (1.5)
\end{aligned}$$

as an equation in H^{-1}.

The well-posedness of (1.1) is studied in our recent paper [2], based on the rescaling transformation used in [1] and the Strichartz estimates established in [17] for perturbations of the Laplacian. We also refer to the standard monographs [9] and [16] for the deterministic case (i.e. $\mu_j = 0$, $1 \leq j \leq N$) and to [5] and [8] for the stochastic conservative case (i.e. $Re\mu_j = 0$, $1 \leq j \leq N$).

The main interest of this article is to study the noise effects on blow-up in the focusing mass-(super)critical case. Our motivations mainly come from two aspects. On the one hand, the blow-up phenomenon in the deterministic case is extensively studied in the literature, and it is well known that there exist blow-up solutions in the focusing mass-(super)critical case (1.2), especially for initial data with negative Hamiltonian (cf. e.g. [9], [16]). On the other hand, when there is noise in the system, it is of great interest to investigate the noise effects on the formation of singularities. For example, in the conservative case, it is proved in [6] in the supercritical case that noise can accelerate blow-up with positive probability. But in the critical case numerical results suggest that noise has the effect to delay explosion (cf. [7], [10] and [11])

Here, we focus on the noise effects on blow-up, but in the non-conservative case, i.e.,

$$\exists j_0 : 1 \leq j_0 \leq N, \ \textit{such that } Re\mu_{j_0} \neq 0 \tag{1.6}$$

(Without loss of generality, we assume that $Re\mu_1 \neq 0$). Surprisingly, the noise effects here are completely different from those in the conservative cases. We will prove that, in the non-conservative case by adding a large noise, with high probability one can prevent blow-up on any given bounded time interval $[0, T]$, $0 < T < \infty$. Moreover, when the noise is spatially independent, the explosion even can be prevented with high probability on the whole time interval $[0, \infty)$.

To state our resutls precisely, we assume for the spatial functions in the noise that

(H) $e_j = f_j + c_j$, $1 \leq j \leq N$, where c_j are real constants and f_j are real-valued functions, such that $f_j \in C_b^\infty$ and

$$\lim_{|\xi| \to \infty} \zeta(\xi) \sum_{1 \leq |\gamma| \leq 3} |\partial^\gamma f_j(\xi)| = 0,$$

where γ is a multi-index and

$$\zeta = \begin{cases} 1+|\xi|^2, & \text{if } d \neq 2; \\ (1+|\xi|^2)(\ln(1+|\xi|^2))^2, & \text{if } d = 2. \end{cases}$$

(In Section 3 we will take c_1 large enough such that $c_1 > |f_1|_\infty$. Hence, without loss of generality, we assume that f_1 is positive.)

The main result is then as follows:

Theorem 1.2 *Consider* (1.1) *in the non-conservative case* (1.6). *Let λ and α satisfy* (1.2). *Assume (H) with f_j, $1 \leq j \leq N$, and c_k, $2 \leq k \leq N$ being fixed. Then for any $x \in H^1$ and $0 < T < \infty$,*

$$\mathbb{P}(X(t) \textit{ does not blow up on } [0,T]) \to 1, \quad \textit{as } c_1 \to \infty.$$

(where we recall that by (1.6) *we have $Re\mu_1 \neq 0$.)*

Furthermore, if f_j, $1 \leq j \leq N$, are also constants, then for any $x \in H^1$,

$$\mathbb{P}(X(t) \textit{ does not blow up on } [0,\infty)) \to 1, \quad \textit{as } c_1 \to \infty.$$

Remark 1.3 *Theorem 1.2 can be viewed as a complement to [6]. It was proved there that in the conservative supercritical case, i.e., $Re\mu_j = 0$, $1 \leq j \leq N$, $\alpha \in (1+\frac{4}{d}, \infty)$ if $d = 1,2$ and $\alpha \in (\frac{7}{3}, 5)$ if $d = 3$, the non-degenerate multiplicative noise can accelerate blow-up with positive probability (see Theorem* 5.1 *in [6]). In contrast to [6], Theorem 1.2 reveals that in the non-conservative supercritical and also critical cases specified in* (1.2) *with $d \geq 1$, the large multiplicative noise has the effect to stabilize the system.*

Similar phenomena happen for the deterministic damped nonlinear Schrödinger equation,

$$i\partial_t u + \Delta u + |u|^{\alpha-1}u + iau = 0, \quad a > 0. \tag{1.7}$$

Note that, this equation is analogous to (2.2) below in the special case where the noise $W(t)$ is spatially independent and $\mu_k \in \mathbb{R}$, $1 \leq k \leq N$, i.e.

$$i\partial_t y - \Delta y - e^{(\alpha-1)ReW(t)}|y|^{\alpha-1}y + i\widehat{\mu}y = 0, \quad \widehat{\mu} > 0.$$

This similarity indeed indicates the dissipative effects produced by the multiplicative noise in the non-conservative case.

The global well-posedness of (1.7) is proved in [18, Theorem 1] (see also [19, p.98]), provided a is large enough, and the proof is based on the decay estimate of $e^{it\Delta}$ (see [18, Lemma 4]).

However, since the decay estimates do not necessarily hold for the general Schrödinger-type operator $A(t)$ in (2.3), we employ here quite different arguments based on the contraction mapping arguments as in [1, 2], involving a second transformation (see (2.8) below) and the Strichartz estimates established in [17]. The advantage of this proof is that it is also applicable to the case of spatially dependent noise.

This article is structured as follows. In Section 2 we apply two transformations to reduce the original stochastic equation (1.1) to a random equation (2.9) below, which reveals the dissipative effect produced by the noise in the non-conservative case. Then the non-explosion results in Theorem 1.2 are established in Section 3. Furthermore, we also show that these results do not generally hold with probability 1. Finally, the Appendix contains Itô-formulas for the Hamiltonian, variance and momentum that are used in the proof.

2 Preliminaries.

Following [1] and [2], we apply the rescaling transformation

$$X = e^W y \tag{2.1}$$

to (1.1) and obtain the random equation

$$\begin{aligned} &\frac{\partial y}{\partial t}(t,\xi) = A(t)y(t,\xi) - ie^{(\alpha-1)ReW(t,\xi)}|y(t,\xi)|^{\alpha-1}y(t,\xi), \qquad (2.2)\\ &y(0) = x, \end{aligned}$$

where

$$A(t) = -i(\Delta + b(t)\cdot\nabla + c(t)), \tag{2.3}$$

$$b(t) = 2\nabla W(t), \tag{2.4}$$

$$c(t) = \sum_{j=1}^{d}(\partial_j W(t))^2 + \Delta W(t) - i\widehat{\mu}, \tag{2.5}$$

and

$$\widehat{\mu} := \sum_{j=1}^{N} (|\mu_j|^2 + \mu_j^2) e_j^2. \tag{2.6}$$

We stress that on a heuristic level (2.2) follows easily by Itô's product rule. The rigorous proof is more involved. We refer to [1, Lemma 6.1] for the L^2-case and [20, Theorem 2.1.3] for the H^1-case.

Note that the real part of the damped term $\widehat{\mu}$ is positive in the non-conservative case, namely,

$$Re\widehat{\mu} = \sum_{j=1}^{N} (Re\mu_j)^2 e_j^2 \geq (Re\mu_1)^2 c_1^2 > 0, \tag{2.7}$$

but it vanishes in the conservative case, which indicates the different noise effects between the two cases.

To explore this damped term, we apply to (2.2) a second transformation

$$z(t,\xi) = e^{\widehat{\mu}t} y(t,\xi), \tag{2.8}$$

and derive that

$$\begin{aligned} &\frac{\partial z(t)}{\partial t} = \widehat{A}(t) z(t) - i e^{-(\alpha-1)(Re\widehat{\mu}t - ReW(t))} |z(t)|^{\alpha-1} z(t), \\ &z(0) = x \in H^1, \end{aligned} \tag{2.9}$$

where

$$\widehat{A}(t) = -i(\Delta + \widehat{b}(t) \cdot \nabla + \widehat{c}(t)) \tag{2.10}$$

with

$$\widehat{b}(t) = -2t\nabla\widehat{\mu} + 2\nabla W(t), \tag{2.11}$$

and

$$\begin{aligned} \widehat{c}(t) =& t^2 \sum_{j=1}^{N} (\partial_j \widehat{\mu})^2 - t\Delta\widehat{\mu} - 2t\nabla W(t) \cdot \nabla\widehat{\mu} \\ &+ \left[\sum_{j=1}^{N} (\partial_j W(t))^2 + \Delta W(t) \right]. \end{aligned} \tag{2.12}$$

The key fact here is that, an exponential decay term $e^{-(\alpha-1)Re\widehat{\mu}t}$ appears in (2.9), which weakens the nonlinearity and thus can be expected to prevent blow-up, provided that μ is sufficiently large (or the noise is sufficiently large in some other appropriate sense). For this purpose, let us rewrite equation (2.9) in the mild form

$$z(t) = V(t,0)x + \int_0^t (-i)V(t,s)\left[h(s)|z(s)|^{\alpha-1}z(s)\right] ds, \tag{2.13}$$

where

$$h(s) := e^{-(\alpha-1)(Re\widehat{\mu}s - ReW(s))} \tag{2.14}$$

and $V(t,s)$ is the evolution operator generated by the homogenous part of (2.2), namely, $V(t,s)x = z(t)$, $s \leq t \leq T$, solves

$$\begin{aligned} \frac{dz(t)}{dt} &= \widehat{A}(t)z(t), \quad a.e\ t \in (s,T), \\ z(s) &= x \in H^1. \end{aligned} \tag{2.15}$$

(The existence and uniqueness of the evolution operator $V(t,s)$ follow mainly from [12, 13]. For more details, we refer to [1, 2].)

Remark 2.1 *The solutions to* (2.9) *are understood analogously to Definition 1.1, and Assumption* (H) *is sufficient to establish the local existence and uniqueness of solutions for* (2.9)*, hence also for* (1.1)*, by the transformations* (2.1) *and* (2.8)*. Indeed, the proofs follow by similar arguments as in [2, Proposition* 2.5*] (see also [1, Lemma* 4.2*]), and one can remove the additional decay assumption* $\lim_{|\xi|\to 0} \zeta(\xi)|e_j(\xi)| = 0$ *in [2], due to the fact that* $\widehat{b}, \widehat{c}$ *in* (2.9) *only involve the gradient of* $\widehat{\mu}$ *and* $W(t)$*. This fact allows us later to take* c_1 *very large to prevent blow-up.*

As in [2, Lemma 2.7] one can check from [17] and Assumption (H) that Strichartz estimates hold for $V(t,s)$,

Lemma 2.2 *Assume* (H)*. Then for any* $T > 0$, $u_0 \in H^1$ *and* $f \in L^{q_2'}(0,T; W^{1,p_2'})$*, the solution of*

$$u(t) = V(t,0)u_0 + \int_0^t V(t,s)f(s)ds, 0 \leq t \leq T, \tag{2.16}$$

satisfies the estimates

$$\|u\|_{L^{q_1}(0,T;L^{p_1})} \le C_T(|u_0|_2 + \|f\|_{L^{q_2'}(0,T;L^{p_2'})}), \tag{2.17}$$

and

$$\|u\|_{L^{q_1}(0,T;W^{1,p_1})} \le C_T(|u_0|_{H^1} + \|f\|_{L^{q_2'}(0,T;W^{1,p_2'})}), \tag{2.18}$$

where (p_1, q_1) *and* (p_2, q_2) *are Strichartz pairs, i.e.,*

$$(p_i, q_i) \in [2, \infty] \times [2, \infty] : \frac{2}{q_i} = \frac{d}{2} - \frac{d}{p_i},\ if\ d \neq 2,$$

or

$$(p_i, q_i) \in [2, \infty) \times (2, \infty] : \frac{2}{q_i} = \frac{d}{2} - \frac{d}{p_i},\ if\ d = 2,$$

Furthermore, the process C_t, $t \ge 0$, *can be taken to be* $(\mathscr{F}_t)$*-progressively measurable, increasing and continuous.*

3 Proof of the main results

Proof of Theorem 1.2. (i). For convenience, let us first consider the easier case of spatially independent noise to illustrate the main idea.

By the transformations (2.1) and (2.8), it is equivalent to prove the assertion for the random equation (2.9). Note that in this case $\widehat{b} = \widehat{c} = 0$, hence $V(t, s) = e^{-i(t-s)\Delta}$ and the Strichartz coefficient $C_t \equiv C$ is independent of t.

Choose the Strichartz pair $(p, q) = (\alpha + 1, \frac{4(\alpha+1)}{d(\alpha-1)})$. Set

$$\mathcal{Z}_M^\tau = \{u \in C(0, \tau; L^2) \cap L^q(0, \tau; L^p) : \|u\|_{L^\infty(0,\tau;H^1)} + \|u\|_{L^q(0,\tau;W^{1,p})} \le M\}, \tag{3.1}$$

and define the integral operator G on $\mathcal{Z}_M^\tau$ by

$$G(u)(t) = V(t, 0)x + \int_0^t (-i)V(t, s) \left[h(s)|u(s)|^{\alpha-1}u(s)\right] ds,\ u \in \mathcal{Z}_M^\tau. \tag{3.2}$$

We claim that, for $u \in \mathcal{Z}_M^\tau$,

$$\|G(u)\|_{L^\infty(0,\tau;H^1)} + \|G(u)\|_{L^q(0,\tau;W^{1,p})} \le 2C|x|_{H^1} + 2CD_1(\tau)M^\alpha, \tag{3.3}$$

where

$$D_1(t) = \alpha D^{\alpha-1} \|h\|_{L^v(0,t)} \tag{3.4}$$

with D the Sobolev coefficient such that $\|u\|_{L^p} \le D|u|_{H^1}$, $v > 1$ and $\frac{1}{v} = 1 - \frac{2}{q} > 0$.

Indeed, by Lemma 2.2,

$$\begin{aligned} &\|G(u)\|_{L^\infty(0,\tau;H^1)} + \|G(u)\|_{L^q(0,\tau;W^{1,p})} \\ \le& 2C|x|_{H^1} + 2C\|h|u|^{\alpha-1}u\|_{L^{q'}(0,\tau;W^{1,p'})}. \end{aligned} \tag{3.5}$$

Moreover, Hölder's inequality and Sobolev's imbedding theorem yield

$$\begin{aligned} \|h|u|^{\alpha-1}u\|_{L^{q'}(0,\tau;L^{p'})} \le& |h|_{L^v(0,\tau)} \||u|^{\alpha-1}u\|_{L^q(0,\tau;L^{p'})} \\ \le& D^{\alpha-1}|h|_{L^v(0,\tau)} \|u\|^{\alpha-1}_{L^\infty(0,\tau;H^1)} \|u\|_{L^q(0,\tau;L^p)}, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \|h\nabla(|u|^{\alpha-1}u)\|_{L^{q'}(0,\tau;L^{p'})} \le& \alpha\|h|u|^{\alpha-1}|\nabla u|\|_{L^{q'}(0,\tau;L^{p'})} \\ \le& \alpha D^{\alpha-1}|h|_{L^v(0,\tau)} \|u\|^{\alpha-1}_{L^\infty(0,\tau;H^1)} \|\nabla u\|_{L^q(0,\tau;L^p)}. \end{aligned} \tag{3.7}$$

Hence, plugging (3.6) and (3.7) into (3.5) implies (3.3), as claimed.

Similarly to (3.3), for $u_1, u_2 \in \mathcal{Z}^\tau_M$,

$$\begin{aligned} &\|G(u_1) - G(u_2)\|_{L^\infty(0,\tau;L^2)} + \|G(u_1) - G(u_2)\|_{L^q(0,\tau;L^p)} \\ \le& 4CD_1(\tau)M^{\alpha-1}\|u_1 - u_2\|_{L^q(0,\tau;L^p)}. \end{aligned} \tag{3.8}$$

Now, let $M = 3C|x|_{H^1}$, choose the $(\mathscr{F}_t)$-stopping time $\tau = \tau(c_1)$,

$$\tau := \inf\left\{t > 0 : 2 \cdot 3^\alpha |x|^{\alpha-1}_{H^1} C^\alpha D_1(t) > 1\right\}. \tag{3.9}$$

Then, as in the proof of Proposition 2.5 in [2], we obtain a local solution z of (2.9) on $[0, \tau]$.

Next we show that $\mathbb{P}(\tau = \infty) \to 1$, as $c_1 \to \infty$. As the definition of τ involves the term $D_1(t)$, we shall use (3.4) to estimate $\|h\|_{L^v(0,\infty)}$.

Set $\phi_k = \mu_k e_k$, $1 \leq k \leq N$. By the scaling property of Brownian motion, i.e. $\mathbb{P} \circ [Re\phi_k \ \beta_k(\cdot)]^{-1} = \mathbb{P} \circ [\beta_k((Re\phi_k)^2 \cdot)]^{-1}$, for any $c \geq 0$,

$$
\begin{aligned}
&\mathbb{P}(\|h\|_{L^v(0,\infty)}^v \geq c) \\
=&\mathbb{P}\left(\int_0^\infty \prod_{k=1}^N e^{-(\alpha-1)v[(Re\phi_k)^2 s - Re\phi_k \beta_k(s)]} ds \geq c\right) \\
=&\mathbb{P}\left(\int_0^\infty \prod_{k=1}^N e^{-(\alpha-1)v[(Re\phi_k)^2 s - \beta_k((Re\phi_k)^2 s)]} ds \geq c\right). \qquad (3.10)
\end{aligned}
$$

Note that, by the law of the iterated logarithm of Brownian motion,

$$
C_1^* := \int_0^\infty e^{-(\alpha-1)v[s-\beta_1(s)]} ds < \infty, \ a.s, \qquad (3.11)
$$

and

$$
C := 1 \vee \max_{2\leq k \leq N} \sup_{s \geq 0} e^{-(\alpha-1)v[(Re\phi_k)^2 s - \beta_k((Re\phi_k)^2 s)]} < \infty, \ a.s. \qquad (3.12)
$$

Then $\mathbb{P}$-a.s.,

$$
\begin{aligned}
&\int_0^\infty \prod_{k=1}^N e^{-(\alpha-1)v[(Re\phi_k)^2 s - \beta_k((Re\phi_k)^2 s)]} ds \\
\leq& C^N \int_0^\infty e^{-(\alpha-1)v[(Re\phi_1)^2 s - \beta_1((Re\phi_1)^2 s)]} ds \\
\leq& \frac{1}{(Re\phi_1)^2} C^N C_1^*. \qquad (3.13)
\end{aligned}
$$

Hence, plugging (3.13) into (3.10), since $C^N C_1^* < \infty$ $a.s.$ and $(Re\phi_1)^2 \to \infty$ as $c_1 \to \infty$, we deduce that for any fixed $c \geq 0$,

$$
\mathbb{P}(\|h\|_{L^v(0,\infty)}^v \geq c) \leq \mathbb{P}\left(C^N \widetilde{C}_1^* \geq c(Re\phi_1)^2\right) \to 0, \quad as \ c_1 \to \infty. \qquad (3.14)
$$

Consequently, choose $c = \left[4 \cdot 3^\alpha \alpha |x|_{H^1}^{\alpha-1} C^\alpha D^{\alpha-1}\right]^{-v} > 0$. By the definition of τ in (3.9) and (3.14), we then derive that

$$
\begin{aligned}
&\mathbb{P}(\tau=\infty) \\
=&\mathbb{P}\left(2\cdot 3^\alpha |x|_{H^1}^{\alpha-1} C^\alpha D_1(t) < 1,\ \forall t\in[0,\infty)\right) \\
\geq&\mathbb{P}\left(2\cdot 3^\alpha \alpha |x|_{H^1}^{\alpha-1} C^\alpha D^{\alpha-1}\|h\|_{L^v(0,\infty)} \leq \frac{1}{2}\right) \\
\geq&1-\mathbb{P}\left(\|h\|_{L^v(0,\infty)}^v \geq c\right) \\
\rightarrow&1, \quad as\ c_1\rightarrow\infty,
\end{aligned}
$$

which completes the proof for spatially independent noise.

(ii). Now, we consider the general case when the noise $W(t)$ is space-dependent. Again it is equivalent to prove the assertion for the random equation (2.9).

Let $\mathcal{Z}_M^\tau$, G be as in (3.1) and (3.2) respectively. Similarly to (3.3), for $u\in\mathcal{Z}_M^\tau$,

$$
\begin{aligned}
&\|G(u)\|_{L^\infty(0,\tau;H^1)}+\|G(u)\|_{L^q(0,\tau;W^{1,p})} \\
\leq&2C_\tau|x|_{H^1}+2C_\tau D_2(\tau)M^\alpha,
\end{aligned} \tag{3.15}
$$

where C_t is the Strichartz coefficient, and

$$
D_2(t)=\alpha D^{\alpha-1}\|h\|_{L^v(0,t;W^{1,\infty})}. \tag{3.16}
$$

with $v>1$ and $\frac{1}{v}=1-\frac{2}{q}>0$.

Moreover, for $u_1,u_2\in\mathcal{Z}_M^\tau$,

$$
\begin{aligned}
&\|G(u_1)-G(u_2)\|_{L^\infty(0,\tau;L^2)}+\|G(u_1)-G(u_2)\|_{L^q(0,\tau;L^p)} \\
\leq&4C_\tau D_2(\tau)M^{\alpha-1}\|u_1-u_2\|_{L^q(0,\tau;L^p)}.
\end{aligned} \tag{3.17}
$$

Set $M=3C_\tau|x|_{H^1}$, choose the $(\mathscr{F}_t)$-stopping time $\tau=\tau(c_1)$,

$$
\tau:=\inf\{t\in[0,T], 2\cdot 3^\alpha|x|_{H^1}^{\alpha-1}C_t^\alpha D_2(t)>1\}\wedge T. \tag{3.18}
$$

It follows from (3.15) and (3.17) that $G(\mathcal{Z}_M^\tau)\subset\mathcal{Z}_M^\tau$ and G is a contraction on $C([0,\tau];L^2)\cap L^q(0,\tau;L^p)$. Therefore, using the same arguments as in [2], we obtain a local solution z on $[0,\tau]$.

To show that $\mathbb{P}(\tau = T) \to 1$, as $c_1 \to \infty$, using (3.18) and (3.16), we shall estimate $\|h\|_{L^v(0,t;W^{1,\infty})}$ below. For simplicity, set $|f|_\infty := |f|_{L^\infty}$ for any $f \in L^\infty(\mathbb{R}^d)$ and $\phi_k := \mu_k e_k$, $1 \le k \le N$.

As regards the norm $\|h\|_{L^v(0,t;L^\infty)}$, by (2.14) and (2.7),

$$|h(t)|_{L^\infty} \le e^{-(\alpha-1)\sum_{k=1}^{N}[\frac{(Re\mu_1)^2 c_1^2}{N}t-|Re\phi_k|_\infty|\beta_k(t)|]}. \tag{3.19}$$

Analogously to (3.12),

$$\widetilde{C} := 1 \vee \max_{2\le k\le N}\sup_{t\ge 0} e^{-(\alpha-1)v[\frac{(Re\mu_1)^2c_1^2}{N}t-|\beta_k(|Re\phi_k|_\infty^2 t)|]} < \infty, \quad a.s. \tag{3.20}$$

Moreover, choosing c_1 large enough such that $c_1 > |f_1|_\infty$, we have

$$\begin{aligned}
&\int_0^T e^{-(\alpha-1)v[\frac{(Re\mu_1)^2c_1^2}{N}t-|\beta_1(|Re\phi_1|_\infty^2 t)|]}dt \\
=&\frac{1}{|Re\phi_1|_\infty^2}\int_0^{|Re\phi_1|_\infty^2 T} e^{-(\alpha-1)v[\frac{(Re\mu_1)^2c_1^2}{N|Re\phi_1|_\infty^2}t-|\beta_1(t)|]}dt \\
\le&\frac{1}{|Re\phi_1|_\infty^2}\widetilde{C}_1^*,
\end{aligned} \tag{3.21}$$

where $\widetilde{C}_1^* := \int_0^\infty e^{-(\alpha-1)v[\frac{1}{4N}t-|\beta_1(t)|]}dt < \infty$ $\mathbb{P}$-a.s.

Thus, as in (3.14), it follows from (3.19)-(3.21) and the scaling property of β_k, $1 \le k \le N$, that for any $c > 0$ fixed,

$$\begin{aligned}
&\mathbb{P}\left(C_T^{\alpha v}\|h\|_{L^v(0,T;L^\infty)}^v \ge c\right) \\
\le&\mathbb{P}\left(C_T^{\alpha v}\widetilde{C}^N\ \widetilde{C}_1^* \ge |Re\phi_1|_\infty^2 c\right) \\
\to&0, \quad as\ c_1 \to \infty, \quad \mathbb{P}-a.s.,
\end{aligned} \tag{3.22}$$

where C_T is the Strichartz coefficient.

Similar arguments can also be applied to the norm $\|\nabla h\|_{L^v(0,t;L^\infty)}$. Indeed, from (2.14) and (2.7),

$$\nabla h(t) = h(t)\left[-(\alpha-1)\sum_{k=1}^{N}\left(2Re\phi_k(Re\mu_k\nabla f_k)t - Re\mu_k\nabla f_k\beta_k(t)\right)\right],$$

which implies

$$|\nabla h(t)|_\infty \leq (\alpha-1)|h(t)|_\infty \sum_{k=1}^{N} \left(2|Re\phi_k|_\infty |Re\mu_k \nabla f_k|_\infty t + |Re\mu_k \nabla f_k|_\infty |\beta_k(t)|\right).$$

Hence, for any $c>0$ fixed,

$$\begin{aligned}
&\mathbb{P}(C_T^{\alpha v}\|\nabla h\|^v_{L^v(0,T;L^\infty)} \geq c)\\
\leq&\mathbb{P}\Big(C_T^{\alpha v}\int_0^T (\alpha-1)^v |h(t)|^v_{L^\infty}\\
&\qquad\left[\sum_{k=1}^N 2|Re\phi_k|_\infty |Re\mu_k\nabla f_k|_\infty t + |Re\mu_k \nabla f_k|_\infty |\beta_k(t)|\right]^v dt \geq c\Big)\\
\leq&\mathbb{P}\Bigg(C_T^{\alpha v}\int_0^T (\alpha-1)^v \left[\prod_{k=1}^N e^{-(\alpha-1)v[\frac{(Re\mu_1)^2c_1^2}{N}t - |\beta_k(|Re\phi_k|^2_\infty t)|]}\right]\\
&\qquad\left[\sum_{k=1}^N 2|Re\phi_k|_\infty |Re\mu_k\nabla f_k|_\infty t + |Re\mu_k \nabla f_k|_\infty |\beta_k(t)|\right]^v dt \geq c\Bigg)\\
\leq&\mathbb{P}\Bigg(C_T^{\alpha v}\widetilde{C}^N \frac{1}{|Re\phi_1|^2_\infty}\int_0^\infty e^{-(\alpha-1)v[\frac{1}{4N}t-|\beta_1(t)|]}\\
&\quad\left[\sum_{k=1}^N \frac{2|Re\phi_k|_\infty|Re\mu_k\nabla f_k|_\infty}{|Re\phi_1|^2_\infty}t + |Re\mu_k\nabla f_k|_\infty \left|\beta_k(\frac{t}{|Re\phi_1|^2_\infty})\right|\right]^v dt \geq \frac{c}{(\alpha-1)^v}\Bigg).
\end{aligned}$$

Choosing c_1 large enough, such that $\sum_{k=1}^N \frac{2|Re\phi_k|_\infty|Re\mu_k\nabla f_k|_\infty}{|Re\phi_1|^2_\infty} < 1$ and $\frac{|Re\mu_k\nabla f_k|_\infty}{|Re\phi_1|_\infty} < 1$, we have as $c_1\to\infty$,

$$\begin{aligned}
&\mathbb{P}(C_T^{\alpha v}\|\nabla h\|^v_{L^v(0,T;L^\infty)} \geq c)\\
\leq&\mathbb{P}(C_T^{\alpha v}\widetilde{C}^N\widetilde{C}_1' \geq \frac{c}{(\alpha-1)^v}|Re\phi_1|^2_\infty)\\
\to&0. \qquad (3.23)
\end{aligned}$$

where C_T is the Stichartz coefficient and $\widetilde{C}_1' := \int_0^\infty e^{-(\alpha-1)v[\frac{1}{4N}t-|\beta_1(t)|]}\left[t+\sum_{k=1}^N \beta_k(t)\right]^v dt < \infty$ $\mathbb{P}$-a.s.

Now we come back to the definition of τ in (3.18). Choosing

$$c = [4 \cdot 3^{\alpha} \alpha D^{\alpha-1} |x|_{H^1}^{\alpha-1}]^{-v} > 0,$$

we deduce from (3.22) and (3.23) that

$$\begin{aligned}
&\mathbb{P}(\tau = T) \\
\geq &\mathbb{P}(2 \cdot 3^{\alpha} |x|_{H^1}^{\alpha-1} C_t^{\alpha} D_1(t) < 1, \forall t \in [0, T]) \\
\geq &\mathbb{P}(2 \cdot 3^{\alpha} \alpha D^{\alpha-1} |x|_{H^1}^{\alpha-1} C_T^{\alpha} \|h\|_{L^v(0,T,W^{1,\infty})} < \frac{1}{2}) \\
\geq &1 - \mathbb{P}(C_T^{\alpha v} \|h\|_{L^v(0,T,W^{1,\infty})}^v \geq c) \\
\geq &1 - \mathbb{P}(C_T^{\alpha v} \|h\|_{L^v(0,T,L^{\infty})}^v \geq \frac{1}{2} c) - \mathbb{P}(C_T^{\alpha v} \|\nabla h\|_{L^v(0,T,L^{\infty})}^v \geq \frac{1}{2} c) \\
\to &1, \quad as \ c_1 \to \infty.
\end{aligned}$$

Therefore, we complete the proof of Theorem 1.2. □

One may further ask whether the non-explosion results in Theorem 1.2 hold with probability 1. This is, unfortunately, not generally true. In fact, define the Hamiltonian

$$H(z) = \frac{1}{2} |\nabla z|_2^2 - \frac{1}{\alpha+1} |z|_{\alpha+1}^{\alpha+1}, \quad z \in H^1,$$

and set $\sum = \{u \in H^1, \int |\xi|^2 |u(\xi)|^2 d\xi < \infty.\}$. We have the following result

Proposition 3.1 *Consider* (1.1) *in the non-conservative case* (1.6)*. Let* λ *and* α *satisfy* (1.2)*. Assume* (H) *with* f_j*,* $1 \leq j \leq N$*, and* c_k*,* $2 \leq k \leq N$ *being fixed. Furthermore, assume* $\mu_k \in \mathbb{R}$*,* $1 \leq k \leq N$*. Let* $x \in \sum$ *with* $H(x) < 0$*,*

Then there exists $\epsilon_0 > 0$*, such that for* $0 < \epsilon < \epsilon_0$ *and* $0 \leq \sum\limits_{1 \leq k \leq N} |\nabla f_k|_{L^\infty} < \epsilon$*, the solution to (1.1) blows up in finite time with positive probability.*

In particular, in the case that f_j*,* $1 \leq j \leq N$*, are fixed constants, the solution to (1.1) blows up in finite time with positive probability.*

The proof follows from the standard virial analysis (see e.g [14]). For any $u \in \sum$, define the variance

$$V(u) = \int |\xi|^2 |u(\xi)|^2 d\xi, \tag{3.24}$$

and the momentum

$$G(u) = Im \int \xi u(\xi) \cdot \overline{\nabla u(\xi)} d\xi. \tag{3.25}$$

Proof of Proposition 3.1. We prove the assertion by contradiction. Assume that the solution $X(t)$ to (1.1) exists globally in H^1 $\mathbb{P} - a.s.$

By Lemmas 4.1, 4.2 and 4.3 in the Appendix,

$$\begin{aligned} V(X(t)) =& V(x) + 4G(x)t + 8H(x)t^2 \\ &+ 4\sum_{k=1}^N \int_0^t (t-s)^2 |\nabla \phi_k X(s)|_2^2 ds \\ &- 4(\alpha-1)\sum_{k=1}^N \int_0^t (t-s)^2 \int \phi_k^2 |X(s)|^{\alpha+1} d\xi ds \\ &+ \frac{16}{\alpha+1}\left[1 - \frac{d(\alpha-1)}{4}\right] \int_0^t (t-s)|X(s)|_{\alpha+1}^{\alpha+1} ds \\ &+ M_t, \end{aligned} \tag{3.26}$$

where $\phi_k = \mu_k e_k$, $1 \le k \le N$, and

$$\begin{aligned} M_t :=& 8\sum_{k=1}^N \int_0^t (t-s)^2 \left[Re \left\langle \nabla(\phi_k X(s)), \nabla X(s) \right\rangle_2 - \int \phi_k |X(s)|^{\alpha+1} d\xi \right] d\beta_k(s) \\ &- 8\sum_{k=1}^N \int_0^t (t-s) Im \int \xi \cdot \nabla X(s) \overline{X(s)} \phi_k d\xi d\beta_k(s) \\ &+ 2\sum_{k=1}^N \int_0^t \int |\xi|^2 |X(s)|^2 \phi_k d\xi d\beta_k(s). \end{aligned}$$

Fix $t > 0$ and define for $r \in [0, \infty)$,

$$\begin{aligned} \widetilde{M}(t,r) :=& 8\sum_{k=1}^N \int_0^r (t-s)^2 \left[Re \left\langle \nabla(\phi_k X(s)), \nabla X(s) \right\rangle_2 - \int \phi_k |X(s)|^{\alpha+1} d\xi \right] d\beta_k(s) \\ &- 8\sum_{k=1}^N \int_0^r (t-s) Im \int \xi \cdot \nabla X(s) \overline{X(s)} \phi_k d\xi d\beta_k(s) \\ &+ 2\sum_{k=1}^N \int_0^r \int |\xi|^2 |X(s)|^2 \phi_k d\xi d\beta_k(s). \end{aligned} \tag{3.27}$$

Set $\sigma_m := \inf\{s \in [0,t], |\nabla X_m(s)|_2^2 > m\} \wedge t$. Then $\sigma_m \to t$, as $m \to \infty$.

Direct computations show that $\widetilde{M}(t, \cdot \wedge \sigma_m)$ is a square integrable martingale, in particular,

$$\mathbb{E}[\widetilde{M}(t, t \wedge \sigma_m)] = 0. \tag{3.28}$$

Indeed, e.g. in regard to the second term in the right hand side of (3.27) , we note that

$$\begin{aligned}
&\mathbb{E}\int_0^{r\wedge\sigma_m} \sum_{k=1}^N \left|(t-s)Im \int \xi \cdot \nabla X(s)\overline{X(s)}\phi_k d\xi\right|^2 ds \\
\leq & C\mathbb{E}\int_0^{r\wedge\sigma_m} (t-s)^2 V(X(s))|\nabla X(s)|_2^2 ds \\
\leq & mC\mathbb{E} \sup_{s\in[0,\sigma_m]} V(X(s)) \int_0^r (t-s)^2 ds,
\end{aligned} \tag{3.29}$$

where $C = \sum\limits_{k=1}^N |\phi_k|_{L^\infty}^2 < \infty$. Then, as in the proof of (4.10) below, we deduce that the right hand side in (3.29) is finite. The other terms can be estimated even more easily.

Now, take the expectation in (3.26). Since the fifth and sixth terms in the right hand side of (3.26) are non-positive for α satisfying (1.2), it follows that

$$\begin{aligned}
\mathbb{E}V(X(\sigma_m \wedge t)) \leq & V(x) + 4G(x)(\sigma_m \wedge t) + 8H(x)(\sigma_m \wedge t)^2 \\
& + 4\mathbb{E}\int_0^{\sigma_m\wedge t} (\sigma_m \wedge t - s)^2 \sum_{k=1}^N |\nabla\phi_k X(s)|_2^2 ds, \quad t < \infty.
\end{aligned}$$

Then, taking $m \to \infty$, by Fatou's lemma, and since $\nabla\phi_k = \mu_k \nabla f_k$ and $\mathbb{E}|X(t)|_2^2 = |x|_2^2$, we obtain

$$\mathbb{E}V(X(t)) \leq V(x) + 4G(x)t + 8H(x)t^2 + at^3 \tag{3.30}$$

with

$$a = \frac{4}{3}\sum_{k=1}^N |\mu_k||\nabla f_k|_{L^\infty}^2 |x|_2^2.$$

Let $f(t)$ denote the right hand side of (3.30), i.e.,

$$f(t) := V(x) + 4G(x)t + 8H(x)t^2 + at^3.$$

We claim that, if $\sum_{k=1}^{N} |\nabla f_k|_{L^\infty}$ is small enough, then there exists $T > 0$ such that $f(T) < 0$. But, taking into account $\mathbb{E}V(X(t)) \geq 0$ and (3.30), we get a contradiction.

It remains to prove the claim. Since

$$f'(t) = 3at^2 + 16H(x)t + 4G(x),$$

for $\sum_{k=1}^{N} |\nabla f_k|_{L^\infty}$ small enough, the discriminant is positive and the largest root of $f(t)$ is

$$t_* := \frac{2G(x)}{-4H(x) - \sqrt{16(H(x))^2 - 3aG(x)}} > 0. \tag{3.31}$$

Note that, proving the claim is equivalent to showing that $f(t_*) < 0$. Since $f'(t^*) = 0$, simple computations show that

$$f(t_*) = \frac{8}{3}H(x)t_*^2 + \frac{8}{3}G(x)t_* + V(x).$$

Since the largest roof of

$$g(t) := \frac{8}{3}H(x)t^2 + \frac{8}{3}G(x)t + V(x)$$

is

$$\widetilde{t_*} := \frac{-G(x) - \sqrt{(G(x))^2 - \frac{3}{2}H(x)V(x)}}{2H(x)},$$

which is independent of a. But by (3.31), $t_* \to \infty$ as $a \to 0$, yielding that $\widetilde{t_*} < t_*$ for a small enough, thereby implying $f(t_*) < 0$ and completing the proof. $\square$

4 Appendix.

This appendix contains the Itô-formulas for the Hamiltonian, variance and momentum. As mentioned in Remark 2.1, one can obtain a local solution X

to (1.1) on $[0,\tau_n]$, $n\in\mathbb{N}$, where τ_n are $(\mathscr{F}_t)$-stopping times, and X satisfies $\mathbb{P}$-a.s. for any Strichartz pair (ρ,γ),

$$X|_{[0,t]}\in C([0,t];H^1)\cap L^\gamma(0,t;W^{1,\rho}),\quad t<\tau^*(x) \tag{4.1}$$

with $\tau^*(x)=\lim\limits_{n\to\infty}\tau_n$.

Let us start with the Itô-formula for the Hamiltonian $H(X(t))$ proved in [2, Theorem 3.1].

Theorem 4.1 *Let α satisfy* (1.2). *Set $\phi_j:=\mu_j e_j$, $j=1,...,N$. Then $\mathbb{P}$-a.s*

$$\begin{aligned}&H(X(t))\\ =&H(x)+\int_0^t Re\,\langle-\nabla(\mu X(s)),\nabla X(s)\rangle_2\,ds+\frac{1}{2}\sum_{j=1}^N\int_0^t|\nabla(X(s)\phi_j)|_2^2ds\\ &-\frac{1}{2}\lambda(\alpha-1)\sum_{j=1}^N\int_0^t\int(Re\phi_j)^2|X(s)|^{\alpha+1}d\xi ds\\ &+\sum_{j=1}^N\int_0^t Re\,\langle\nabla(\phi_jX(s)),\nabla X(s)\rangle_2\,d\beta_j(s)\\ &-\lambda\sum_{j=1}^N\int_0^t\int Re\phi_j|X(s)|^{\alpha+1}d\xi d\beta_j(s),\quad 0\le t<\tau^*(x).\end{aligned}$$

The following lemma is concerned with the Itô-formula for the variance.

Lemma 4.2 *Let $\sum$ be as in Proposition 3.1 and $x\in\sum$. Then $\mathbb{P}$-a.s. for $t<\tau^*(x)$,*

$$V(X(t))=V(x)+4\int_0^t G(X(s))ds+M_1(t), \tag{4.2}$$

where G is as in (3.25) *and*

$$M_1(t):=2\sum_{k=1}^N\int_0^t\int|\xi|^2|X(s)|^2Re\phi_k d\xi d\beta_k(s)$$

with $\phi_k:=\mu_k e_k$, $1\le k\le N$.

Proof. The proof is similar to that in [2, Lemma 5.1] (see also [15]), hence we just give a sketch of it below.

Set $\varphi^\epsilon := \varphi * \phi_\epsilon$ for any locally integrable function φ mollified by ϕ_ϵ, where $\phi_\epsilon = \epsilon^{-d}\phi(\frac{x}{\epsilon})$ and $\phi \in C_c^\infty(\mathbb{R}^d)$ is a real-valued nonnegative function with unit integral. Set $V_\eta(u) = \int e^{-\eta|\xi|^2}|\xi|^2|u(\xi)|^2 d\xi$ and $V(u) = \int |\xi|^2|u(\xi)|^2 d\xi$ for $u \in \sum$.

By (1.1) it follows that $\mathbb{P}$-a.s. for every $\xi \in \mathbb{R}^d$, $t < \tau^*(x)$,

$$
\begin{aligned}
(X(t))^\epsilon(\xi) = x^\epsilon(\xi) + \int_0^t \left[-i\Delta(X(s))^\epsilon(\xi) - (\mu X(s))^\epsilon(\xi) - i(g(X(s)))^\epsilon(\xi)\right] ds \\
+ \sum_{k=1}^N \int_0^t (X(s)\phi_j)^\epsilon(\xi) d\beta_j(s),
\end{aligned}
\tag{4.3}
$$

where $g(X(s)) := |X(s)|^{\alpha-1}X(s)$. For simplicity, we set $X^\epsilon(t) := (X(t))^\epsilon(\xi)$ and correspondingly for the other arguments.

Applying the product rule yields $\mathbb{P}$-a.s.

$$
\begin{aligned}
|X^\epsilon(t)|^2 =& |x^\epsilon|^2 - 2Re\int_0^t \overline{X}^\epsilon(s) i\Delta X^\epsilon(s) ds - 2Re\int_0^t \overline{X}^\epsilon(s)(\mu X(s))^\epsilon ds \\
&- 2Re\int_0^t \overline{X}^\epsilon(s) i[g(X(s))]^\epsilon ds + \sum_{k=1}^N \int_0^t |(X(s)\phi_k)^\epsilon|^2 ds \\
&+ 2\sum_{k=1}^N Re\int_0^t \overline{X}^\epsilon(s)(X(s)\phi_k)^\epsilon d\beta_k(s), \quad t < \tau^*(x).
\end{aligned}
$$

Then, integration over $\mathbb{R}^d$ with $e^{-\eta|\xi|^2}|\xi|^2$, interchanging integrals and integrating by parts, we have $\mathbb{P}$-a.s. for $t < \tau^*(x)$,

$$\begin{aligned} V_\eta(X^\epsilon(t)) =& V_\eta(x^\epsilon) + 4Im \int_0^t \int e^{-\eta|\xi|^2}(1-\eta|\xi|^2)X^\epsilon(s)\xi\cdot\nabla\overline{X^\epsilon}(s)d\xi ds \\ & - 2Re \int_0^t \int e^{-\eta|\xi|^2}|\xi|^2\overline{X}^\epsilon(s)(\mu X(s))^\epsilon d\xi ds \\ & - 2Re \int_0^t \int e^{-\eta|\xi|^2}|\xi|^2\overline{X}^\epsilon(s)i[g(X(s))]^\epsilon d\xi ds \\ & + \sum_{k=1}^N \int_0^t \int e^{-\eta|\xi|^2}|\xi|^2|(X(s)\phi_k)^\epsilon|^2 d\xi ds \\ & + 2\sum_{k=1}^N Re \int_0^t \int e^{-\eta|\xi|^2}|\xi|^2\overline{X}^\epsilon(s)(X(s)\phi_k)^\epsilon d\xi d\beta_k(s). \end{aligned} \tag{4.4}$$

As $\sup\limits_{\xi\in\mathbb{R}^d} e^{-\eta|\xi|^2}[|(1-\eta|\xi|^2)\xi| + |\xi|^2] < \infty$, one can take the limit $\epsilon \to 0$ in (4.4), which leads to

$$\begin{aligned} V_\eta(X(t)) =& V_\eta(x) + 4Im \int_0^t \int e^{-\eta|\xi|^2}(1-\eta|\xi|^2)X(s)\xi\cdot\nabla\overline{X}(s)d\xi ds \\ & + 2\sum_{k=1}^N \int_0^t \int e^{-\eta|\xi|^2}|\xi|^2|X(s)|^2 Re\phi_k d\xi d\beta_k(s), \quad t < \tau^*(x). \end{aligned} \tag{4.5}$$

To pass to the limit $\eta \to 0$, we shall prove that

$$\sup_{s\in[0,\tau_n]} V(X(s)) \le \widetilde{C}(n) < \infty, \quad \mathbb{P}-a.s. \tag{4.6}$$

Then by (4.5), (4.6), $\sup\limits_{\eta>0}\sup\limits_{\xi\in\mathbb{R}^d} |e^{-\eta|\xi|^2}(1-\eta|\xi|^2)| = 1$ and Lebesque's dominated theorem, we obtain (4.2) for $t \le \tau_n$, $n \in \mathbb{N}$. Consequently, since $\tau_n \to \tau^*(x)$, as $n \to \infty$, we conclude (4.2) for $t < \tau^*(x)$.

It remains to prove (4.6). For every $n \in \mathbb{N}$, set

$$\sigma_{n,m} := \inf\{s \in [0,\tau_n] : |\nabla X(s)|_2^2 > m\} \wedge \tau_n.$$

Burkholder-Davis-Gundy's inequality implies that

$$\begin{aligned}
\mathbb{E} \sup_{s\in[0,t\wedge\sigma_{n,m}]} V_\eta(X(s)) \leq & 4\mathbb{E}\int_0^{t\wedge\sigma_{n,m}} \int e^{-\eta|\xi|^2}|1-\eta|\xi|^2||\overline{X}(s)\xi\cdot\nabla X(s)|d\xi ds \\
& + c\mathbb{E}\sqrt{\int_0^{t\wedge\sigma_{n,m}} \sum_{k=1}^N \left(\int e^{-\eta|\xi|^2}|\xi|^2|X(s)|^2 Re\phi_k d\xi\right)^2 ds} \\
= & J_1 + J_2, \qquad (4.7)
\end{aligned}$$

where c is independent of n, m and η.

Since $\sup_{\eta>0}\sup_{\xi\in\mathbb{R}^d} |e^{-\eta|\xi|^2}(1-\eta|\xi|^2)| = 1$ and $\mathbb{E}\sup_{s\in[0,\sigma_{n,m}]}|\nabla X(s)|_2^2 \leq m < \infty$,

$$\begin{aligned}
J_1 \leq & 4\mathbb{E}\int_0^{t\wedge\sigma_{n,m}} \sqrt{V(X(s))}|\nabla X(s)|_2 ds \\
\leq & 4\int_0^t \mathbb{E}\sup_{r\in[0,s\wedge\sigma_{n,m}]} V(X(r))ds + 4mT. \qquad (4.8)
\end{aligned}$$

Moreover,

$$\begin{aligned}
J_2 \leq & C\mathbb{E}\sqrt{\int_0^{t\wedge\sigma_{n,m}} [V_\eta(X(s))]^2 ds} \\
\leq & \epsilon C\mathbb{E}\sup_{s\in[0,t\wedge\sigma_{n,m}]} V_\eta(X(s)) + CC_\epsilon\int_0^t \mathbb{E}\sup_{r\in[0,s\wedge\sigma_{n,m}]} V_\eta(X(r))ds, \qquad (4.9)
\end{aligned}$$

where C depends on $|\phi_k|_{L^\infty}$, $1 \leq k \leq N$, and is independent of n, m and η.

Hence, plugging (4.8) and (4.9) into (4.7), taking ϵ small enough, and noting that $V_\eta(X) \leq V(X)$, we derive that

$$\mathbb{E}\sup_{s\in[0,t\wedge\sigma_{n,m}]} V_\eta(X(s)) \leq C_1\int_0^t \mathbb{E}\sup_{r\in[0,s\wedge\sigma_{n,m}]} V(X(r))ds + C_2(m,T),$$

with C_1 and $C_2(m,T)$ independent of η. Then letting $\eta \to 0$ and using Fatou's lemma, we have

$$\mathbb{E}\sup_{s\in[0,t\wedge\sigma_{n,m}]} V(X(s)) \leq C_1\int_0^t \mathbb{E}\sup_{r\in[0,s\wedge\sigma_{n,m}]} V(X(r))ds + C_2(m,T), \quad t\in[0,T],$$

which implies by Gronwall's inequality that

$$\mathbb{E} \sup_{t \in [0, \sigma_{n,m}]} V(X(t)) \le C(m, T) < \infty, \tag{4.10}$$

hence $\sup_{t \in [0, \sigma_{n,m}]} V(X(t)) \le \widetilde{C}(m, T) < \infty$, $\mathbb{P}$-a.s. But, since $\sup_{t \in [0, \tau_n]} |\nabla X(t)|_2^2 < \infty$, $\mathbb{P}$-a.s, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\exists m(\omega) < \infty$ such that $\sigma_{n,m(\omega)}(\omega) = \tau_n(\omega)$. Then $\mathbb{P}\left(\bigcup_{m \in \mathbb{N}} \{\sigma_{n,m} = \tau_n\} \right) = 1$. This implies (4.6) and completes the proof of Lemma 4.2. □

We conclude this section with the Itô-formula for the momentum.

Lemma 4.3 *Let $x \in \sum$. Then $\mathbb{P}$-a.s for $t < \tau^*(x)$,*

$$\begin{aligned} G(X(t)) =& G(x) + 4 \int_0^t P(X(s)) ds \\ & - \sum_{k=1}^N \int_0^t Im \int \xi \cdot \nabla \phi_k |X(s)|^2 \overline{\phi_k} d\xi ds + M_2(t), \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} P(X) :=& \frac{1}{2} |\nabla X|_2^2 - \frac{d(\alpha-1)}{4(\alpha+1)} |X|_{L^{\alpha+1}}^{\alpha+1} \\ =& H(X) + \frac{1}{\alpha+1} [1 - \frac{d(\alpha-1)}{4}] |X|_{L^{\alpha+1}}^{\alpha+1}, \end{aligned}$$

$\phi_k = \mu_k e_k$, $1 \le k \le N$, and

$$\begin{aligned} M_2(t) :=& d \sum_{k=1}^N \int_0^t \int |X(s)|^2 Im \phi_k d\xi d\beta_k(s) \\ & - 2 \sum_{k=1}^N \int_0^t Im \int \xi \cdot \nabla X(s) \overline{X(s)} \overline{\phi_k} d\xi d\beta_k(s). \end{aligned}$$

Here, d is the dimension of the space.

Proof. The proof is similar to that in Lemma 4.2 but involves more complicated computations. For simplicity of exposition, we omit the proof here and refer to [20, Lemma 3.3.2] for details. □

Acknowledge. Deng Zhang would like to thank DFG for the support through the IRTG 1132 at Bielefeld University.

References

[1] V. Barbu, M. Röckner, D. Zhang, The stochastic nonlinear Schrödinger equations with multiplicative noise: the rescaling approach, *J. Nonlinear Sci.* DOI 10.1007/s00332-014-9193-x.

[2] V. Barbu, M. Röckner, D. Zhang, The stochastic nonlinear Schrödinger equations, arXiv:1404.5039.

[3] A. Barchielli, M. Gregoratti, Quantum Trajectories and Measurements in Continuous Case. The Diffusive Case, Lecture Notes Physics, 782, Springer Verlag, Berlin, 2009.

[4] A. Barchielli, C. Pelegrini, F. Petruccio, Stochastic Schrödinger equations with coloured noise, *A Letter Journal Exploring the Frontiers of Physics*, EPL, 91 (2010).

[5] A. de Bouard, A. Debussche, The stochastic nonlinear Schrödinger equation in H^1, *Stoch. Anal. Appl.*, 21 (2003), 97-126.

[6] A. de Bouard, A. Debussche, Blow up for the stochastic nonlinear Schrödinger equation with multiplicative noise, *The Annals of Probability*, 33 (2005), 1078-1110.

[7] A. de Bouard, A. Debussche, L. Di Menza, Theoretical and numerical aspects of stochastic nonlinear Schrödinger equations, *Journées "Équations aux Dérivées Partielles" Plestin-les-Grèves*, 5-8 (2002), III.1-III.13.

[8] A. de Bouard, R. Fukuizumi, Representation formula for stochastic Schrödinger evolution equations and applications, *Nonlinearity* 25 (2012), no. 11, 2993-3022.

[9] T. Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics, 10, New York University Courant Institute of Mathematical Sciences, New York, 2003.

[10] A. Debussche, and L. Di Menza, Numerical simulation of focusing stochastic nonlinear Schrödinger equations, *Phys. D*, 162 (2002), 131-154.

[11] A. Debussche, and L. Di Menza, Numerical resolution of stochastic focusing NLS equations, *Appl. Math. Letters*, 15 (2002), no.6, 661-669.

[12] S. Doi, On the Cauchy problem for Schrödinger type equation and the regularity of solutions, *J. Math. Kyoto Univ.*, 1994, 34 (2), 319-328.

[13] S. Doi, Remarks on the Cauchy problem for Schrödinger-type equations, *Comm. PDE*, 21 (1996), 163-178.

[14] R. T. Glassey, On the blowing-up of solutions to the Cauchy problem for the nonlinear Schrödinger equation. *J. Math. Phys.*, 18, (1977), 1794-1797.

[15] N.V.Krylov, Itô's formula for the L_p-norm of a stochastic W_p^1 -valued process, *Probab. Theory Relat. Fields*, 147 (2010), 583-605.

[16] F. Linares, G. Ponce, Introduction to Nonlinear Dispersive Equations, Springer, 2009.

[17] J. Marzuola, J. Metcalfe, D. Tataru, Strichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations, *J. Funct. Anal.*, 255 (6) (2008), 1479-1553.

[18] M. Ohta, G. Todorova, Remarks on global existence and blowup for damped nonlinear Schrödinger equations. *Discrete Contin. Dyn. Syst.*, 23(2009), 1313- 1325.

[19] C. Sulem, P.-L. Sulem, The nonlinear Schrödinger equation. Self-focusing and wave collapse. Applied Mathematical Sciences, 139. Springer, New York, (1999).

[20] D. Zhang, Stochastic nonlinear Schrodinger equation, PhD thesis, Universität Bilefeld, http://igk.math.uni-bielefeld.de/gallery, 2014.