

Existence and uniqueness of solutions to stochastic functional differential equations in infinite dimensions ^{*}

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Abstract

In this paper, we present a general framework for solving stochastic functional differential equations in infinite dimensions in the sense of martingale solutions, which can be applied to a large class of SPDE with finite delays, e.g. d -dimensional stochastic fractional Navier-Stokes equations with delays, d -dimensional stochastic reaction-diffusion equations with delays, d -dimensional stochastic porous media equations with delays. Moreover, under local monotonicity conditions for the nonlinear term we obtain the existence and uniqueness of strong solutions to SPDE with delays.

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1 Introduction

Recently stochastic partial differential equations (SPDE) with delays have been paid a lot of attention (see e.g [BBT14], [BDT12], [CR04] [TLT02], [J10] and the references therein). There is a large amount of literature on the mathematical theory and on applications of stochastic functional (or delay) differential equations (see e.g. [M84], [RS08], [HMS09] and the reference therein). When one wants to model some evolution phenomena arising in

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physics, biology, engineering, etc., some hereditary characteristics such as after-effect, time-lag or time-delay can appear in the variables. Typical examples arise in the mathematical modelling of materials with thermal memory, biochemical reactions, population models, etc. (see, for instance, [R96], [W96] and references cited therein). So, one is naturally lead to use functional differential equations which take into account the history of the system. However, in most cases, randomness affects the model, so that the system should be modelled by a stochastic functional equation.

There are a lot of studies about existence and uniqueness of probabilistically strong solutions for various classes of nonlinear SPDEs with time delays in a variational framework (see e.g. [CGR02], [CLT00], [J10]): Let

$$V \subset H_0 \cong H_0^* \subset V^*$$

be a Gelfand triple and

$$(1.1) \quad \begin{aligned} dx(t) &= [A_1(t, x(t)) + A_2(t, x_t)]dt + B(t, x_t)dW(t), t \in [0, \infty), \\ x(s) &= \psi(s), s \in [-h, 0], \end{aligned}$$

where W is a cylindrical Wiener process on a separable Hilbert space U , $x_t(s) := x(t + s)$ and $A_1 : \mathbb{R}^+ \times V \rightarrow V^*$ is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(V)$ -measurable, $A_2 : \mathbb{R}^+ \times C([-h, 0]; H_0) \rightarrow V^*$ is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(C([-h, 0]; H_0))$ -measurable, $B : \mathbb{R}^+ \times C([-h, 0]; H_0) \rightarrow L_2(U; H_0)$ is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(C([-h, 0]; H_0))$ -measurable. In the above references the authors suppose that A_1 , i.e. the nonlinear term without delay, satisfies monotonicity conditions, whereas A_2 , i.e. the nonlinear term with delay, satisfies Lipschitz conditions with respect to a suitable norm. If we consider more general equations with nonlinear terms not satisfying monotonicity conditions such as Navier-Stokes equations and more interesting delay terms like $A_2(t, x) = \int_{-h}^0 f(x(t+r))dr \cdot \nabla x(t - r_1(t)) + \nabla b(x(t - r_2(t)))$, we cannot apply the above result. Here $r_1, r_2 : \mathbb{R} \rightarrow [0, h]$ and f is a bounded Lipschitz continuous function on \mathbb{R} , b is a linear growth and continuous function on \mathbb{R} .

To obtain the existence and uniqueness of solutions for the equations containing more general nonlinear terms and more interesting delays, we tried to use the monotonicity trick and assumed that the nonlinear terms satisfy local monotonicity conditions as Wei Liu and the first named author did in [LR10]. However, when we apply this method, the local monotonicity condition should be: There exist locally bounded measurable functions $\rho : V \rightarrow [0, +\infty)$ and $\rho_1 : C([-h, 0]; H_0) \rightarrow [0, +\infty)$ such that

$$(1.2) \quad \begin{aligned} & {}_{V^*}\langle A_1(t, \xi(t)) - A_1(t, \eta(t)) + A_2(t, \xi_t) - A_2(t, \eta_t), \xi(t) - \eta(t) \rangle_V + \|B(t, \xi_t) - B(t, \eta_t)\|_2^2 \\ & \leq [\rho(\eta(t)) + \rho_1(\eta_t)] \|\xi(t) - \eta(t)\|_{H_0}^2, \end{aligned}$$

for $\xi, \eta \in C([-h, \infty), H_0) \cap L_{\text{loc}}^p([-h, \infty); V)$ with $p \geq 1$. Here the middle norm is the H_0 -norm, not the norm for the paths like $C([-h, 0], H_0)$ -norm, which is not natural and cannot even cover the Lipschitz case mentioned above.

Instead, in this paper we take a different approach. First, we provide a general framework to prove existence of solutions (see Theorems 2.1 and 2.2) under very weak assumptions

(see (H1), (H2), (H3)) in the sense of D. W. Stroock and S.R.S. Varadhan's martingale problem (see [SV79]), which can be applied to a large class of SPDEs with delays, such as stochastic fractional Navier-Stokes equations with delay in any dimensions (see Section 5.1), stochastic reaction-diffusion equations with delay (see Section 5.2) and stochastic porous media equations with delay in any dimension (see Section 5.3). We also emphasize that for our existence results, we only assume continuity, coercivity and growth conditions written in terms of integrals over time, which enables us to cover a large number of equations with interesting delays (see Remark 2.2 (ii)). Second, under local monotonicity conditions for the nonlinear terms we obtain pathwise uniqueness for SPDEs with delays, which implies existence and uniqueness of (probabilistically) strong solution by the Yamada-Watanabe Theorem. Here we also emphasize that our local monotonicity condition (H4) can cover the examples we mentioned above. It is clearly weaker than the Lipschitz condition and (1.2), since instead of the middle norm H_0 we use a norm for the paths and the local term in the local monotonicity condition may depend on the paths of two solutions.

This paper is organized as follows: In Section 2, we introduce the general framework and state the main results on existence of martingale solutions. In Section 3, we give the proofs of the main results from Section 2. In Section 4 we concentrate on the uniqueness of probabilistically strong solutions. In Section 5, we apply the main results from Sections 2 and 4 to stochastic fractional Navier-Stokes equations with delays in any dimensions (see Section 5.1), to stochastic reaction-diffusion equations with delays (see Section 5.2) and to stochastic porous media equations with delays in any dimensions (see Section 5.3).

2 Existence of Martingale solutions

Let $(\mathbb{H}, \langle \cdot, \cdot \rangle_{\mathbb{H}})$ be a separable Hilbert space and $\mathbb{X}, V, \mathbb{Y}$ be three separable and reflexive Banach spaces with norms $\|\cdot\|_{\mathbb{X}}, \|\cdot\|_V$ and $\|\cdot\|_{\mathbb{Y}}$ such that

$$\mathbb{X} \subset V \subset \mathbb{Y} \subset \mathbb{H}$$

continuously and densely. By Kuratowski's theorem we have that $\mathbb{X} \in \mathcal{B}(V)$, $V \in \mathcal{B}(\mathbb{Y})$, $\mathbb{Y} \in \mathcal{B}(\mathbb{H})$ and $\mathcal{B}(\mathbb{Y}) = \mathcal{B}(\mathbb{H}) \cap \mathbb{Y}$, $\mathcal{B}(\mathbb{X}) = \mathcal{B}(V) \cap \mathbb{X}$, $\mathcal{B}(V) = \mathcal{B}(\mathbb{Y}) \cap V$.

If we identify the dual of \mathbb{H} with itself, we get

$$\mathbb{X} \subset \mathbb{H} \simeq \mathbb{H}^* \subset \mathbb{X}^*$$

continuously and densely.

If ${}_{\mathbb{X}^*}\langle \cdot, \cdot \rangle_{\mathbb{X}}$ denotes the dualization between \mathbb{X} and its dual space \mathbb{X}^* , then it follows that

$${}_{\mathbb{X}^*}\langle u, v \rangle_{\mathbb{X}} = \langle u, v \rangle_{\mathbb{H}}, \quad v \in \mathbb{X}, u \in \mathbb{H}^*.$$

Moreover, we assume that \mathbb{X} is a Hilbert space and $\mathbb{X} \subset \mathbb{H}$ compactly.

Let

$$\{e_1, e_2, \dots\} \subset \mathbb{X}$$

be an orthonormal basis of \mathbb{H} and let $\mathbb{H}_n := \text{span}\{e_1, \dots, e_n\}$ and $\mathcal{E} := \text{span}\{e_1, e_2, \dots\}$ is dense in \mathbb{X}^* . Let $\Pi_n : \mathbb{X}^* \rightarrow \mathbb{H}_n$ be defined by

$$\Pi_n y := \sum_{i=1}^n \langle y, e_i \rangle_{\mathbb{X}} e_i, \quad y \in \mathbb{X}^*.$$

Since $\mathbb{X} \subset \mathbb{H}$ is compact and hence so is $\mathbb{H}(\simeq \mathbb{H}^*) \subset \mathbb{X}^*$, we can choose $\{e_n : n \in \mathbb{N}\} \subset \mathbb{X}$ in such a way that

$$\|\Pi_n y\|_{\mathbb{X}^*} \leq \|y\|_{\mathbb{X}^*}, \quad \forall n \in \mathbb{N}, y \in \mathbb{X}^*,$$

(cf the proof of [AR89, Proposition 3.5]).

Remark 2.1. \mathbb{X}^* will be the space where the SPDE will initially hold. \mathbb{H} will eventually be the state space for the solutions of the equation. \mathbb{Y} and V are spaces to identify integrability properties of the solutions. In applications we choose $\mathbb{Y} = \mathbb{H}$ (see examples in Sections 5.1 and 5.2) or $\mathbb{Y} = V$ (see the example in Section 5.3).

Let $\{W_t\}_{t \geq 0}$ be a cylindrical Wiener process on a separable Hilbert space U w.r.t a complete filtered probability space $(\Omega, \mathcal{F}^0, (\mathcal{F}_t^0), \mathbb{P})$ and $(L_2(U; \mathbb{H}), \|\cdot\|_{L_2(U; \mathbb{H})})$ denotes the space of all Hilbert-Schmidt operators from U to \mathbb{H} . Let

$$\mathfrak{C} := C([-h, 0]; \mathbb{X}^*) \cap L^\infty(-h, 0; \mathbb{H}),$$

with norm

$$\|\psi\|_{\mathfrak{C}} := \sup_{-h \leq s \leq 0} \|\psi(s)\|_{\mathbb{H}}, \quad \psi \in \mathfrak{C},$$

and

$$\mathfrak{C}_\infty := C([-h, \infty); \mathbb{X}^*) \cap L_{\text{loc}}^\infty([-h, \infty); \mathbb{H}),$$

equipped with the corresponding metric,

$$L_{\mathbb{Y}}^p := L^p([-h, 0]; \mathbb{Y}), \quad L_V^p := L^p([-h, 0]; V) \quad \text{and} \quad L_{\mathbb{H}}^p := L^p([-h, 0]; \mathbb{H}), \quad p > 1.$$

Since $\mathbb{H} \subset \mathbb{X}^*$ is compact, it is easy to see that

$$\mathfrak{C} = C_{bw}([-h, 0]; \mathbb{H}),$$

where the latter denotes the set of all norm-bounded weakly continuous functions from $[-h, 0]$ to \mathbb{H} . Then obviously \mathfrak{C} with the above norm is Polish.

Given a path $y \in C([-h, +\infty); \mathbb{X}^*)$ and $t \geq 0$, we associate with it a path $y_t \in C([-h, 0]; \mathbb{X}^*)$ by setting $y_t(s) = y(t + s)$, $s \in [-h, 0]$.

We consider the following stochastic evolution equation

$$(2.1) \quad \begin{aligned} dx(t) &= [A_1(t, x(t)) + A_2(t, x_t)]dt + B(t, x_t)dW(t), \quad t \in [0, \infty), \\ x(s) &= \psi(s), \quad s \in [-h, 0], \end{aligned}$$

where $A_1 : \mathbb{R}^+ \times V \rightarrow \mathbb{X}^*$ is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(V)$ -measurable, $A_2 : \mathbb{R}^+ \times (L_V^p \cap \mathfrak{C}) \rightarrow \mathbb{X}^*$ is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(L_V^p \cap \mathfrak{C})$ -measurable, $B : \mathbb{R}^+ \times (L_V^p \cap \mathfrak{C}) \rightarrow L_2(U; \mathbb{H})$ is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(L_V^p \cap \mathfrak{C})$ -measurable for some $p \geq 2$. Now we fix p and suppose that for every $u \in \mathbb{S}_1 := L_{\text{loc}}^p([-h, \infty); V) \cap \mathfrak{C}_\infty$,

$$A_1(\cdot, u(\cdot)) \in L_{\text{loc}}^1([0, \infty); \mathbb{X}^*), A_2(\cdot, u) \in L_{\text{loc}}^1([0, \infty); \mathbb{X}^*).$$

Let us now state the precise conditions on the coefficients of equation (2.1):

First we introduce the following function class $\mathfrak{U}^q, q \geq 1$: A lower semi-continuous function $\mathcal{N} : \mathbb{Y} \rightarrow [0, \infty]$ belongs to \mathfrak{U}^q if $\mathcal{N}(y) = 0$ implies $y = 0$, and

$$\mathcal{N}(cy) \leq c^q \mathcal{N}(y), \quad \forall c \geq 0, y \in \mathbb{Y},$$

and

$$\{y \in \mathbb{Y} : \mathcal{N}(y) \leq 1\} \text{ is compact in } \mathbb{Y}.$$

Then we can extend \mathcal{N} to a $\mathcal{B}(\mathbb{X}^*)/\mathcal{B}([0, \infty])$ -measurable function on \mathbb{X}^* by setting $\mathcal{N}(x) = \infty, x \in \mathbb{X}^* \setminus \mathbb{Y}$.

Suppose that there exists $\mathcal{N}_1 \in \mathfrak{U}^p$ for $p \geq 2$ as above such that

$$(2.2) \quad \|x\|_V^p \lesssim \mathcal{N}_1(x), \quad \forall x \in \mathbb{Y},$$

where $\|x\|_V := +\infty$ if $x \in \mathbb{Y} \setminus V$, and the following holds: $\|\Pi_n A_1(\cdot, \cdot)\|_{\mathbb{X}^*}$ is locally bounded on $\mathbb{R}^+ \times \mathbb{H}_n$, $\|\Pi_n A_2(\cdot, \cdot)\|_{\mathbb{X}^*}, \|\Pi_n B(\cdot, \cdot)\|_{L_2(U, \mathbb{H})}$ are bounded on balls in $\mathbb{R}^+ \times C([-h, 0]; \mathbb{H}_n)$ and for every $t > 0$, $\Pi_n A_1(t, \cdot)$ is continuous on \mathbb{H}_n , $\Pi_n A_2(t, \cdot), \Pi_n B(t, \cdot)$ are continuous on $C([-h, 0]; \mathbb{H}_n)$ and $\mathcal{N}_1(x) \leq C_n \|x\|_{\mathbb{H}_n}^p$ for $x \in \mathbb{H}_n$.

(H1) (Weakened demicontinuity) If $\{u^n, u, n \in \mathbb{N}\} \subset \mathfrak{C}_\infty$ with u^n converging to u in $\mathbb{S}_0 := L_{\text{loc}}^p([-h, \infty); \mathbb{Y}) \cap C([-h, \infty); \mathbb{X}^*)$ and weakly in $L_{\text{loc}}^p([-h, \infty); V)$, then for every $R > 0$ and cut-off function $\chi_R \in C_0^\infty(\mathbb{R})$ satisfying

$$\chi_R(x) = \begin{cases} 1 & \text{for } |x| \leq R, \\ 0 & \text{for } |x| > 2R, \end{cases}$$

we have for all $v \in \mathbb{X}, t \geq 0$,

$$(2.3) \quad \lim_{n \rightarrow \infty} \int_0^t \mathbb{X}^* \langle A_1(s, u^n(s)), v \rangle_{\mathbb{X}} \chi_R(\|u^n(s)\|_{\mathbb{H}}) ds = \int_0^t \mathbb{X}^* \langle A_1(s, u(s)), v \rangle_{\mathbb{X}} \chi_R(\|u(s)\|_{\mathbb{H}}) ds,$$

$$(2.4) \quad \lim_{n \rightarrow \infty} \int_0^t \mathbb{X}^* \langle A_2(s, u_s^n), v \rangle_{\mathbb{X}} ds = \int_0^t \mathbb{X}^* \langle A_2(s, u_s), v \rangle_{\mathbb{X}} ds,$$

and

$$\lim_{n \rightarrow \infty} \int_0^t \|B^*(s, u_s^n)(v) - B^*(s, u_s)(v)\|_U ds = 0,$$

where B^* denotes the adjoint operator of B .

(H2) (Coercivity) There exist locally bounded measurable functions $\lambda_1, c_1 : \mathbb{R} \mapsto \mathbb{R}^+$ such that for $u \in C([-h, \infty); \mathbb{X})$, $t \geq 0$ with $\int_0^t \mathcal{N}_1(u(s))ds < \infty$

$$\begin{aligned} & \int_0^t \mathbb{X}^* \langle A_1(s, u(s)) + A_2(s, u_s), u(s) \rangle_{\mathbb{X}} ds \\ & \leq -\frac{1}{2} \int_0^t \mathcal{N}_1(u(s))ds + c_1(t) \int_{-h}^0 \mathcal{N}_1(u(s))ds + \lambda_1(t) \int_0^t (1 + \|u_s\|_{\mathcal{E}}^2)ds. \end{aligned}$$

(H3) (Growth condition) There exist locally bounded measurable functions $\lambda_2, \lambda_3, \lambda_4 : \mathbb{R} \mapsto \mathbb{R}^+$ and $\gamma' \geq \gamma > 1$ such that for all $u \in \mathbb{S}_1$, $t \geq 0$

$$\begin{aligned} \int_0^t \|A_1(s, u(s))\|_{\mathbb{X}^*}^{\gamma} ds & \leq \lambda_2(t) \left[\int_0^t \mathcal{N}_1(u(s))ds \right]^{\gamma'} + \lambda_3(t) \left(1 + \sup_{s \in [0, t]} \|u(s)\|_{\mathbb{H}}^{\gamma'} \right), \\ \int_0^t \|A_2(s, u_s)\|_{\mathbb{X}^*}^{\gamma} ds & \leq \lambda_2(t) \left[\int_{-h}^t \mathcal{N}_1(u(s))ds \right]^{\gamma'} + \lambda_3(t) \left(1 + \sup_{s \in [-h, t]} \|u(s)\|_{\mathbb{H}}^{\gamma'} \right), \\ \|B(t, u_t)\|_{L_2(U; \mathbb{H})}^2 & \leq \lambda_4(t) (1 + \|u_t\|_{\mathcal{E}}^2). \end{aligned}$$

Remark 2.2. (i) If y^n strongly converges to y in \mathbb{Y} , for every $v \in \mathbb{X}$, $s \in \mathbb{R}^+$,

$$\lim_{n \rightarrow \infty} \mathbb{X}^* \langle A_1(s, y^n), v \rangle_{\mathbb{X}} = \mathbb{X}^* \langle A_1(s, y), v \rangle_{\mathbb{X}},$$

and

$$\mathcal{N}_1(\cdot) \asymp \|\cdot\|_V^p,$$

(2.3) in (H1) holds by (H3) and the dominated convergence theorem. In this case (2.4) is equivalent to the following demicontinuity: if u^n converge to u in \mathbb{S}_0 and weakly in $L_{loc}^p([-h, \infty); V)$, $A_2(\cdot, u^n)$ converge to $A_2(\cdot, u)$ weakly in $L_{loc}^1([0, \infty); \mathbb{X}^*)$.

(ii) Since we consider stochastic differential equations with delay, the coercivity conditions and growth conditions we assume are written in terms of integrals over time, more precisely in the dualization between $L_{loc}^1([-h, \infty); \mathbb{X}^*)$ and $L_{loc}^\infty([-h, \infty); \mathbb{X})$, which is of course weaker than the general coercivity conditions and growth conditions without time integrals. For (H2) and (H3) it is very natural to use $\int_{-h}^0 \mathcal{N}_1(u(s))ds$ to control the growth of the nonlinear term, which also enables us to cover more interesting nonlinear terms with delay (see Section 5).

Let $\Omega_0 := C([-h, \infty); \mathbb{X}^*)$ with the metric

$$\rho(x, y) := \sum_{m=1}^{\infty} \frac{1}{2^m} \left(\sup_{s \in [-h, m]} \|x(s) - y(s)\|_{\mathbb{X}^*} \wedge 1 \right).$$

Define the canonical process $x : \Omega_0 \rightarrow \mathbb{X}^*$ as $x(t, \omega) := \omega(t)$. For $t \geq 0$ define the σ -algebra by

$$\mathcal{F}_t := \sigma\{x(r) : r \leq t\}.$$

and $-h \leq t \leq 0$, $\mathcal{F}_t := \mathcal{F}_0$, $\mathcal{F} := \vee_{t \geq 0} \mathcal{F}_t$.

Definition 2.1. (Martingale Solution) Given an initial value $\psi \in \mathfrak{C}$, a probability measure $P \in \mathcal{P}(\Omega_0)$ (=all probability measure on Ω_0), is called a martingale solution of Eq. (1.1) if
(M1) $P(x(t) = \psi(t), -h \leq t \leq 0) = 1$ and for every $n \in \mathbb{N}$

$$P \left\{ x \in \Omega_0 : \int_0^n \|A_1(s, x(s)) + A_2(s, x_s)\|_{\mathbb{X}^*} ds + \int_0^n \|B(s, x_s)\|_{L_2(U; \mathbb{H})}^2 ds < +\infty \right\} = 1;$$

(M2) for every $l \in \mathcal{E}$ the process

$$M_l(t, x) := \mathbb{X}^* \langle x(t), l \rangle_{\mathbb{X}} - \int_0^t \mathbb{X}^* \langle A_1(s, x(s)) + A_2(s, x_s), l \rangle_{\mathbb{X}} ds, t \geq 0,$$

is a continuous square integrable (\mathcal{F}_t) -martingale with respect to P , whose quadratic variation process is given by

$$\langle M_l \rangle(t, x) := \int_0^t \|B^*(s, x_s)(l)\|_U^2 ds, t \geq 0.$$

Remark 2.3. By Kuratowski's theorem $L_V^p \in \mathcal{B}(\mathfrak{C})$ and $\mathcal{B}(L_V^p \cap \mathfrak{C}) = \mathcal{B}(\mathfrak{C}) \cap L_V^p$. Furthermore, $A_2 : \mathbb{R}^+ \times (L_V^p \cap \mathfrak{C}) \rightarrow \mathbb{X}^*$ is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(L_V^p \cap \mathfrak{C})$ -measurable and the map $x \rightarrow x_s$ is \mathcal{F}_s -measurable, which imply that for $s \geq 0$, $A_2(s, x_s)$ is (\mathcal{F}_s) -adapted.

Now we can state the first main result.

Theorem 2.1. *Suppose (H1) – (H3) hold for some $p \geq 2$. Then for every $\psi \in \mathfrak{C}$ satisfying*

$$(2.5) \quad \mathcal{N}(\psi) := \sup_n \int_{-h}^0 \mathcal{N}_1(\Pi_n \psi(t)) dt + \int_{-h}^0 \mathcal{N}_1(\psi(t)) dt < \infty,$$

(2.1) has a martingale solution P such that for every $q \in \mathbb{N}$, $T > 0$

$$\mathbb{E}^P \left(\sup_{t \in [-h, T]} \|x(t)\|_{\mathbb{H}}^{2q} + \left[\int_0^T \mathcal{N}_1(x(t)) dt \right]^q \right) < \infty.$$

Remark 2.4. (i) Condition (2.5) ensures the crucial energy estimate in Lemma 3.1 below by the coercivity condition (H2). If $\mathcal{N}_1(\Pi_n \psi) \leq \mathcal{N}_1(\psi)$ for $\psi \in V$, which holds in the examples in Sections 5.1, 5.2, then condition (2.5) is satisfied if $\int_{-h}^0 \mathcal{N}_1(\psi(t)) dt < \infty$. In general, condition (2.5) holds if ψ is smooth enough.

(ii) Condition (2.5) can be dropped if $c_1 = 0$ in (H2) and (2.4) in (H1) holds if u^n converge to u in $L_{\text{loc}}^p([-h, \infty); \mathbb{H}) \cap C([-h, \infty); \mathbb{X}^*)$.

Definition 2.2. We say that there exists a weak solution of equation (2.1) if given every $\psi \in \mathfrak{C} \cap L_V^p$, there exists a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [-h, \infty)}, P)$, a cylindrical Wiener process W on the space U and a progressively measurable process $X : [-h, \infty) \times \Omega \rightarrow \mathbb{H}$, such that for P -a.e. $\omega \in \Omega$,

$$X(\cdot, \omega) \in \mathbb{S}_1,$$

and such that P -a.s.

$$X(t) = \psi(t), \quad -h \leq t \leq 0,$$

and

$$\mathbb{X}^* \langle X(t), l \rangle_{\mathbb{X}} - \int_0^t \mathbb{X}^* \langle A_1(s, X(s)) + A_2(s, X_s), l \rangle_{\mathbb{X}} ds = \langle X(0), l \rangle_{\mathbb{H}} + \langle \int_0^t B(s, X_s) dW(s), l \rangle_{\mathbb{H}},$$

for $t \in [0, \infty)$ and all $l \in \mathbb{X}$.

Moreover, by [On05, Theorem 2] and Theorem 2.1 we obtain the following results.

Theorem 2.2. *Suppose (H1) – (H3) hold for some $p \geq 2$. Then for every $\psi \in \mathfrak{C}$ with $\mathcal{N}(\psi) < \infty$, (2.1) has a weak solution $X : [-h, \infty) \times \Omega \rightarrow \mathbb{H}$ such that for every $q \in \mathbb{N}$, $T > 0$*

$$\mathbb{E}^P \left(\sup_{t \in [-h, T]} \|X(t)\|_{\mathbb{H}}^{2q} + \left[\int_{-h}^T \mathcal{N}_1(X(t)) dt \right]^q \right) < \infty.$$

3 Proof of Theorem 2.1

3.1 Finite dimensional case

Now we prove Theorem 2.1 for $U = \mathbb{H} = \mathbb{X} = \mathbb{R}^d$. In this case

$$\Omega_0 = C([-h, \infty); \mathbb{R}^d),$$

$$\mathfrak{C} := C([-h, 0]; \mathbb{R}^d),$$

$$A(t, u) := A_1(t, u(0)) + A_2(t, u), \quad \forall u \in \mathfrak{C}.$$

Define for $n \in \mathbb{N}$, $x \in C([-h, \infty); \mathbb{R}^d)$

$$A^n(t, x) := 1_{\{t \leq n\}} \chi_n \left(\sup_{-h \leq s \leq t} \|x(s)\|_{\mathbb{R}^d} \right) A(t, x_t), \quad B^n(t, x) := 1_{\{t \leq n\}} \chi_n \left(\sup_{-h \leq s \leq t} \|x(s)\|_{\mathbb{R}^d} \right) B(t, x_t),$$

where $0 \leq \chi_n \in C(\mathbb{R}^+; \mathbb{R})$ is a decreasing cutoff function with

$$\chi_n(r) = \begin{cases} 1 & \text{for } r \leq n, \\ 0 & \text{for } r > 2n, \end{cases}$$

Then for $n \in \mathbb{N}$, $A^n(\cdot, x(\cdot))$ and $B^n(\cdot, x(\cdot))$ are bounded and progressively measurable and for each $t > 0$ $A^n(t, x(\cdot))$, $B^n(t, x(\cdot))$ are continuous on Ω_0 and \mathcal{F}_t -measurable. By a similar argument as in the proof of [SV79, Theorem 6.1.6] we obtain that there exists a probability measure $P_n \in \mathcal{P}(\Omega_0)$ such that $P_n(x(t) = \psi(t), -h \leq t \leq 0) = 1$ and

$$M_n(t, x) := x(t) - x(0) - \int_0^t A^n(s, x) ds, \quad x \in \Omega_0,$$

is a continuous square integrable (\mathcal{F}_t) -martingale with square variation process

$$\langle M_n \rangle(t, x) = \int_0^t (B^n)^*(s, x) B^n(s, x) ds, t \geq 0.$$

Although in [SV79, Theorem 6.1.6] it requires that the initial value takes values in \mathbb{R}^d , we can still use a similar argument as in the proof to obtain the above results even if the initial value belongs to \mathfrak{C} .

By Itô's formula and (H2), (H3) we have

$$\begin{aligned} (3.1) \quad & \|x(t)\|_{\mathbb{H}}^2 \\ &= \|x(0)\|_{\mathbb{H}}^2 + \int_0^t \left(2\langle A^n(s, x), x(s) \rangle + \|B^n(s, x)\|_{L_2(\mathbb{R}^d; \mathbb{R}^d)}^2 \right) ds + 2 \int_0^t x(s) dM_n(s) \\ &= \|x(0)\|_{\mathbb{H}}^2 + \int_0^t \left(\mathbf{1}_{\{s \leq n\}} \chi_n \left(\sup_{-h \leq r \leq s} \|x(r)\|_{\mathbb{R}^d} \right) [2\langle A(s, x_s), x(s) \rangle + \|B^n(s, x_s)\|_{L_2(\mathbb{R}^d; \mathbb{R}^d)}^2] \right) ds \\ &\quad + 2 \int_0^t x(s) dM_n(s) \\ &= \|x(0)\|_{\mathbb{H}}^2 + \chi_n \left(\sup_{-h \leq r \leq 0} \|x(r)\|_{\mathbb{R}^d} \right) \int_0^c \left([2\langle A(s, x_s), x(s) \rangle + \|B(s, x_s)\|_{L_2(\mathbb{R}^d; \mathbb{R}^d)}^2] \right) ds \\ &\quad + 2 \int_0^t x(s) dM_n(s) \\ &\leq \|\psi(0)\|_{\mathbb{H}}^2 + C \int_0^t (\|x_s\|_{\mathfrak{C}}^2 + 1) ds + 2 \int_0^t x(s) dM_n(s) + C\mathcal{N}(\psi), \end{aligned}$$

where $0 \leq c \leq t \wedge n$, C is a generic constant (independent of n) and we used mean value theorem for integrations in the third equality. By the BDG(=Burkholder-Davis-Gundy) inequality, (H3) and Young's inequality we have

$$\begin{aligned} & \mathbb{E}^{P_n} \left(\sup_{r \in [0, t]} \|x(r)\|_{\mathbb{H}}^2 \right) \\ & \leq \|\psi(0)\|_{\mathbb{H}}^2 + C \mathbb{E}^{P_n} \int_0^t (\|x_s\|_{\mathfrak{C}}^2 + 1) ds + 2 \mathbb{E}^{P_n} \left(\sup_{r \in [0, t]} \left| \int_0^r x(s) dM_n(s) \right| \right) \\ & \quad + C\mathcal{N}(\psi) \\ & \leq \|\psi(0)\|_{\mathbb{H}}^2 + C \mathbb{E}^{P_n} \int_0^t (\|x_s\|_{\mathfrak{C}}^2 + 1) ds + C\mathcal{N}(\psi) \\ & \quad + C \mathbb{E}^{P_n} \left(\sup_{r \in [0, t]} \|x(r)\|_{\mathbb{H}}^2 \int_0^t \|B(s, x_s)\|_{L_2(\mathbb{R}^d; \mathbb{R}^d)}^2 ds \right)^{1/2} \\ & \leq \|\psi(0)\|_{\mathbb{H}}^2 + C \mathbb{E}^{P_n} \int_0^t (\|x_s\|_{\mathfrak{C}}^2 + 1) ds + C\mathcal{N}(\psi) + \frac{1}{2} \mathbb{E}^{P_n} \sup_{r \in [0, t]} \|x(r)\|_{\mathbb{H}}^2. \end{aligned}$$

Hence by Gronwall's inequality we obtain for every $T > 0$

$$\mathbb{E}^{P_n} \left(\sup_{r \in [-h, T]} \|x(r)\|_{\mathbb{H}}^2 \right) \leq C_T \left(1 + \sup_{t \in [-h, 0]} \|\psi(t)\|_{\mathbb{H}}^2 + \mathcal{N}(\psi) \right).$$

Similar as in the proof of Lemma 3.1 below we obtain that for $q \in \mathbb{N}$ there exists $C_T > 0$ such that for all $n \in \mathbb{N}$ we have

$$(3.2) \quad \mathbb{E}^{P_n} \sup_{t \in [-h, T]} \|x(t)\|_{\mathbb{H}}^{2q} \leq C_T (\|\psi\|_{\mathfrak{E}}^{2q} + 1 + \mathcal{N}(\psi)^q),$$

which combining with our assumption for \mathcal{N}_1 in finite dimensional case also implies that

$$\mathbb{E}^{P_n} \int_{-h}^T \mathcal{N}_1(x(t)) dt \leq C_T (\|\psi\|_{\mathfrak{E}}^{2p} + 1 + \mathcal{N}(\psi)^p),$$

Moreover, as in infinite dimensional case (cf. Section 3.2 below) we have for some $\beta \in (0, 1)$

$$\sup_n \mathbb{E}^{P_n} \left(\sup_{s \neq t \in [0, T]} \frac{|x(t) - x(s)|}{|t - s|^\beta} \right) \leq C,$$

which implies that for every $T > 0$ $(P_n)_{n \in \mathbb{N}}$ is tight on $C([-h, +\infty); \mathbb{R}^d)$. Selecting a subsequence if necessary, we may assume that P_n weakly converges to some probability measure P on $C([-h, +\infty); \mathbb{R}^d)$. By Skorohod's representation theorem, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and $C([-h, +\infty); \mathbb{R}^d)$ -valued random variables \tilde{x}^n and \tilde{x} such that

- (i) \tilde{x}^n has the law P_n for each $n \in \mathbb{N}$;
- (ii) $\tilde{x}^n \rightarrow \tilde{x}$ in $C([-h, +\infty); \mathbb{R}^d)$, \tilde{P} -a.e., and \tilde{x} has the law P .

By similar arguments as in the infinite dimensional case, we have obtained the existence of martingale solutions in the finite dimensional case.

3.2 Infinite dimensional case

The first step of the proof is mainly based on the Galerkin approximation.

Obviously, $\Pi_n|_{\mathbb{H}}$ is just the orthogonal projection onto \mathbb{H}_n in \mathbb{H} and we have

$$\mathbb{X}^* \langle \Pi_n A_1(t, y), v \rangle_{\mathbb{X}} = \langle \Pi_n A_1(t, y), v \rangle_{\mathbb{H}} = \mathbb{X}^* \langle A_1(t, y), v \rangle_{\mathbb{X}}, y \in \mathbb{Y}, v \in \mathbb{H}_n,$$

and

$$\mathbb{X}^* \langle \Pi_n A_2(t, u), v \rangle_{\mathbb{X}} = \langle \Pi_n A_2(t, u), v \rangle_{\mathbb{H}} = \mathbb{X}^* \langle A_2(t, u), v \rangle_{\mathbb{X}}, u \in \mathfrak{E} \cap L_{\mathbb{Y}}^p, v \in \mathbb{H}_n.$$

Let $\{g_1, g_2, \dots\}$ be an orthonormal basis of U and

$$W_t^{(n)} := \sum_{i=1}^n \langle W_t, g_i \rangle_U g_i = \tilde{\Pi}_n W_t,$$

where $\tilde{\Pi}_n$ is the orthogonal projection onto $\text{span}\{g_1, \dots, g_n\}$ in U .

Now for each finite $n \in \mathbb{N}$ we consider the following stochastic equation on \mathbb{H}_n

$$(3.3) \quad \begin{aligned} dx^{(n)}(t) &= [\Pi_n A_1(t, x^{(n)}(t)) + \Pi_n A_2(t, x_t^{(n)})] dt + \Pi_n B(t, x_t^{(n)}) dW^{(n)}(t), \\ x^{(n)}(t) &= \Pi_n \psi(t), t \in [-h, 0]. \end{aligned}$$

Set

$$\Omega^{(n)} := C([-h, \infty); \mathbb{H}_n)$$

and

$$\mathcal{F}_t^{(n)} := \mathcal{B}(C([-h, t]; \mathbb{H}_n)), \mathcal{F}^{(n)} := \vee_{t \geq 0} \mathcal{F}_t^{(n)}.$$

By the solvability of SDDE in the finite dimensional case (see Section 3.1) we know that (3.3) has a martingale solution, i.e. there exists a probability measure $P_n \in \mathcal{P}(\Omega^{(n)})$ such that (M1) and (M2) hold.

In order to construct the solution of (2.1), we need some a priori estimates for $x^{(n)}$.

Lemma 3.1. *Under the assumptions in Theorem 2.1, for every $q \in \mathbb{N}, T > 0$ there exists $C_T > 0$ such that for all $n \in \mathbb{N}$ we have*

$$(3.4) \quad \mathbb{E}^{P_n} \sup_{t \in [-h, T]} \|x^{(n)}(t)\|_{\mathbb{H}}^{2q} + \mathbb{E}^{P_n} \left(\int_{-h}^T \mathcal{N}_1(x^{(n)}(t)) dt \right)^q \leq C_T (\|\psi\|_{\mathfrak{C}}^{2q} + 1 + \mathcal{N}(\psi)^q).$$

Proof. First by (M2) the following equality holds in \mathbb{H}_n

$$x^{(n)}(t) = \Pi_n x(0) + \int_0^t [\Pi_n A_1(r, x^{(n)}(r)) + \Pi_n A_2(r, x_r^{(n)})] dr + M_n(t, x^{(n)}), t \in [0, T],$$

where $M_n(t, x^{(n)}), t \in [0, T]$, is a continuous \mathbb{H}_n -valued $(\mathcal{F}_t^{(n)})$ -martingale with respect to P_n , whose covariation operator process in \mathbb{H}_n is given by

$$\langle M_n \rangle(t, x^{(n)}) = \int_0^t \Pi_n B(s, x_s^{(n)}) \tilde{\Pi}_n \tilde{\Pi}_n^* B^*(s, x_s^{(n)}) \Pi_n^* ds, t \geq 0.$$

By Itô's formula and (H2) we have

$$(3.5) \quad \begin{aligned} \|x^{(n)}(t)\|_{\mathbb{H}}^2 &= \|x^{(n)}(0)\|_{\mathbb{H}}^2 + \int_0^t [2\langle A_1(s, x^{(n)}(s)) + A_2(s, x_s^{(n)}), x^{(n)}(s) \rangle_{\mathbb{H}_n} \\ &\quad + \|\Pi_n B(s, x_s^{(n)}) \tilde{\Pi}_n\|_{L_2(U, \mathbb{H})}^2] ds + M_n^{(1)}(t, x^{(n)}) \\ &\leq \|\psi(0)\|_{\mathbb{H}}^2 + \int_0^t (-\mathcal{N}_1(x^{(n)}(s)) + (2\lambda_1(t) + \lambda_4(t))(1 + \|x_s^{(n)}\|_{\mathfrak{C}}^2)) ds + 2c_1(t)\mathcal{N}(\psi) \\ &\quad + M_n^{(1)}(t, x^{(n)}), \quad t \in [0, T], \end{aligned}$$

where $M_n^{(1)}(t, x^{(n)})$ is a continuous real-valued $(\mathcal{F}_t^{(n)})$ -martingale with respect to P_n , whose quadratic variation process is given by

$$\langle M_n^{(1)} \rangle(t, x^{(n)}) := 4 \int_0^t \|(\Pi_n B(s, x_s^{(n)}) \tilde{\Pi}_n)^* x^{(n)}(s)\|_U^2 ds, t \geq 0.$$

By (3.5) we obtain that for $t \in [0, T]$

$$\|x^{(n)}(t)\|_{\mathbb{H}}^2 + \int_0^t \mathcal{N}_1(x^{(n)}(s)) ds \leq \|\psi(0)\|_{\mathbb{H}}^2 + (2\lambda_1(t) + \lambda_4(t)) \int_0^t (1 + \|x_s^{(n)}\|_{\mathfrak{C}}^2) ds + 2c_1(t)\mathcal{N}(\psi) + M_n^{(1)}(t, x^{(n)}),$$

which implies that for every $q \in \mathbb{N}$

(3.6)

$$\|x^{(n)}(t)\|_{\mathbb{H}}^{2q} + \left[\int_0^t \mathcal{N}_1(x^{(n)}(s)) ds \right]^q \leq C \|\psi(0)\|_{\mathbb{H}}^{2q} + C \int_0^t (1 + \|x_s^{(n)}\|_{\mathfrak{E}}^{2q}) ds + C \mathcal{N}(\psi)^q + C |M_n^{(1)}(t, x^{(n)})|^q,$$

where C depends on q and t and independent of n . For every given $n \in \mathbb{N}$ we define the stopping time

$$\tau_R^{(n)} = \inf\{t \in [0, T] : \|x^{(n)}(t)\|_{\mathbb{H}} > R\} \wedge T, \quad R > 0.$$

Here $\inf \emptyset := \infty$. It is obvious that

$$\lim_{R \rightarrow \infty} \tau_R^{(n)} = T, \quad n \in \mathbb{N}.$$

By the B-D-G inequality we have

(3.7)

$$\begin{aligned} & \mathbb{E}^{P_n} \sup_{r \in [0, t \wedge \tau_R^{(n)}]} |M_n^{(1)}(r, x^{(n)})|^q \\ & \leq C(q) \mathbb{E}^{P_n} \left(\int_0^{t \wedge \tau_R^{(n)}} \|x^{(n)}(s)\|_{\mathbb{H}}^2 \|B(s, x_s^{(n)})\|_{L_2(\mathbb{U}, \mathbb{H})}^2 ds \right)^{q/2} \\ & \leq C(q) \mathbb{E}^{P_n} \left(\int_0^{t \wedge \tau_R^{(n)}} \|x^{(n)}(s)\|_{\mathbb{H}}^2 (\|x_s^{(n)}\|_{\mathfrak{E}}^2 + 1) ds \right)^{q/2} \\ & \leq C(q) \mathbb{E}^{P_n} \left[\varepsilon \sup_{s \in [0, t \wedge \tau_R^{(n)}]} \|x^{(n)}(s)\|_{\mathbb{H}}^{2q} + C_\varepsilon \left(\int_0^{t \wedge \tau_R^{(n)}} (\|x_s^{(n)}\|_{\mathfrak{E}}^2 + 1) ds \right)^q \right] \\ & \leq \varepsilon \mathbb{E}^{P_n} \sup_{s \in [0, t \wedge \tau_R^{(n)}]} \|x^{(n)}(s)\|_{\mathbb{H}}^{2q} + t^{q-1} C_\varepsilon \mathbb{E}^{P_n} \int_0^{t \wedge \tau_R^{(n)}} \left(\sup_{r \in [0, s]} \|x^{(n)}(r)\|_{\mathbb{H}}^{2q} + 1 \right) ds + C_T \|\psi\|_{\mathfrak{E}}^{2q}, \quad t \in [0, T], \end{aligned}$$

where $\varepsilon > 0$ is a small constant and C_ε comes from Young's inequality and may change from line to line.

Then by (3.6), (3.7) and Gronwall's lemma we have

$$\mathbb{E}^{P_n} \sup_{t \in [0, \tau_R^{(n)}]} \|x^{(n)}(t)\|_{\mathbb{H}}^{2q} + \mathbb{E}^{P_n} \left[\int_0^{\tau_R^{(n)}} \mathcal{N}_1(x^{(n)}(s)) ds \right]^q \leq C (\|\psi\|_{\mathfrak{E}}^{2q} + 1 + \mathcal{N}(\psi)^q), \quad n \geq 1,$$

where C is a constant independent of n .

For $R \rightarrow \infty$, (3.4) now follows from the monotone convergence theorem. \square

Proof of Theorem 2.1. Since $\Omega^{(n)} = C([-h, \infty); \mathbb{H}_n)$ is a closed subset of Ω . We extend P_n to a probability measure \hat{P}_n on (Ω, \mathcal{F}) by setting

$$\hat{P}_n(A) := P_n(A \cap \Omega_n), \quad A \in \mathcal{F}.$$

We now show that $(\hat{P}_n)_{n \in \mathbb{N}}$ is tight on

$$\mathbb{S} := \mathbb{S}_0 \cap L_{\text{loc}}^{p,w}([-h, \infty); V),$$

where $\mathbb{S}_0 := C([-h, \infty), \mathbb{X}^*) \cap L_{\text{loc}}^p([-h, \infty); \mathbb{Y})$ and $L_{\text{loc}}^{p,w}([-h, \infty); V)$ denotes the space $L_{\text{loc}}^p([-h, \infty); V)$ equipped with the weak topology.

By [GRZ09, Lemma 4.3], the reflexivity of space $L^p(-h, T; V)$ for every T , the fact that $\mathcal{N}(\psi) < \infty$, and by Lemma 3.1 and (2.2) we only need to prove that for some $\beta > 0$ and every $T > 0$

$$\sup_{n \in \mathbb{N}} \mathbb{E}^{\hat{P}_n} \left(\sup_{s, t \in [0, T], s \neq t} \frac{\|x(t) - x(s)\|_{\mathbb{X}^*}}{|t - s|^\beta} \right) = \sup_{n \in \mathbb{N}} \mathbb{E}^{P_n} \left(\sup_{s, t \in [0, T], s \neq t} \frac{\|x^{(n)}(t) - x^{(n)}(s)\|_{\mathbb{X}^*}}{|t - s|^\beta} \right) < \infty.$$

By (H3) and Lemma 3.1 we have

$$\begin{aligned} & \mathbb{E}^{P_n} \left[\sup_{s, t \in [0, T], s \neq t} \left(\left\| \int_s^t \Pi_n A_1(r, x^{(n)}(r)) + \Pi_n A_2(r, x_r^{(n)}) dr \right\|_{\mathbb{X}^*}^\gamma / |t - s|^{\gamma-1} \right) \right] \\ (3.8) \quad & \leq C_\gamma \mathbb{E}^{P_n} \left[\int_0^T \left\| \Pi_n A_1(r, x^{(n)}(r)) \right\|_{\mathbb{X}^*}^\gamma + \left\| \Pi_n A_2(r, x_r^{(n)}) \right\|_{\mathbb{X}^*}^\gamma dr \right] \\ & \leq C_\gamma \mathbb{E}^{P_n} \left[\int_0^T \left\| A_1(r, x^{(n)}(r)) \right\|_{\mathbb{X}^*}^\gamma + \left\| A_2(r, x_r^{(n)}) \right\|_{\mathbb{X}^*}^\gamma dr \right] \\ & \leq C_\gamma \mathbb{E}^{P_n} \left[\int_0^T \mathcal{N}_1(x^{(n)}(r)) dr \right]^{\gamma'} + C_\gamma \mathbb{E}^{P_n} \left(\sup_{r \in [-h, T]} \|x^{(n)}(r)\|_{\mathbb{H}}^{\gamma'} + 1 \right) + C_\gamma \mathcal{N}(\psi)^{\gamma'} \leq C_{T, \gamma'}, \end{aligned}$$

and for every $0 \leq s < t \leq T$ and $q \in \mathbb{N}$

$$\begin{aligned} \mathbb{E}^{P_n} \|M_n(t, x^{(n)}) - M_n(s, x^{(n)})\|_{\mathbb{H}}^{2q} & \leq C_q \mathbb{E}^{P_n} \left(\int_s^t \|B(r, x_r)\|_{L_2(U; \mathbb{H})}^2 dr \right)^q \\ & \leq C_q |t - s|^{q-1} \mathbb{E}^{P_n} \left(\int_s^t \|B(r, x_r)\|_{L_2(U; \mathbb{H})}^{2q} dr \right) \\ & \leq C_q |t - s|^q \mathbb{E}^{P_n} \left(\sup_{s \in [-h, T]} \|x(s)\|_{\mathbb{H}}^{2q} + 1 \right) \\ & \leq C_q |t - s|^q (\|\psi\|_{\mathfrak{E}}^{2q} + \mathcal{N}(\psi)^q + 1). \end{aligned}$$

By Kolmogorov's criterion, for every $\alpha \in (0, \frac{q-1}{2q})$ we get

$$\mathbb{E}^{P_n} \left(\sup_{s, t \in [0, T], s \neq t} \frac{\|M_n(t, x^{(n)}) - M_n(s, x^{(n)})\|_{\mathbb{H}}^{2q}}{|t - s|^{q\alpha}} \right) \leq C_q (\|\psi\|_{\mathfrak{E}}^{2q} + \mathcal{N}(\psi)^q + 1).$$

Now we obtain that for some $\beta > 0$

$$\sup_{n \in \mathbb{N}} \mathbb{E}^{P_n} \left(\sup_{s, t \in [0, T], s \neq t} \frac{\|x^{(n)}(t) - x^{(n)}(s)\|_{\mathbb{X}^*}}{|t - s|^\beta} \right) < \infty,$$

which as we mentioned above implies that $(\hat{P}_n)_{n \in \mathbb{N}}$ is tight on \mathbb{S} .

By Skorohod's representation theorem in [J98], there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and \mathbb{S} -valued random variables $\tilde{x}^{(n)}$ and \tilde{x} such that

- (i) $\tilde{x}^{(n)}$ has the law \hat{P}_n for each $n \in \mathbb{N}$;
- (ii) $\tilde{x}^{(n)} \rightarrow \tilde{x}$ in \mathbb{S} , \tilde{P} -a.e., and \tilde{x} has the law P . (Here selecting a subsequence if necessary.)

Then we have

$$\begin{aligned} P(x(t) = \psi(t), -h \leq t \leq 0) &= \tilde{P}(\tilde{x}(t) = \psi(t), -h \leq t \leq 0) \\ &= \lim_{n \rightarrow \infty} \tilde{P}(\tilde{x}^{(n)}(t) = \Pi_n \psi(t), -h \leq t \leq 0) = \lim_{n \rightarrow \infty} \hat{P}_n(x(t) = \Pi_n \psi(t), -h \leq t \leq 0) = 1. \end{aligned}$$

For every $q \in \mathbb{N}$ set

$$\xi_q(t, x) := \sup_{r \in [-h, t]} \|x(r)\|_{\mathbb{H}}^{2q} + \left(\int_0^t \mathcal{N}_1(x(r)) dr \right)^q.$$

By Fatou's lemma, Lemma 3.1 and the lower semi-continuity of $\xi_q(t, x)$ with respect to x on \mathbb{S} , we have

$$\begin{aligned} \mathbb{E}^P(\xi_q(t, x)) &= \mathbb{E}^{\tilde{P}}[\xi_q(t, \tilde{x})] \leq \liminf_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}}[\xi_q(t, \tilde{x}^{(n)})] \\ &= \liminf_{n \rightarrow \infty} \mathbb{E}^{\hat{P}_n}[\xi_q(t, x)] \\ &\leq C[\|\psi\|_{\mathfrak{C}}^{2q} + 1 + \mathcal{N}(\psi)^q]. \end{aligned}$$

Now we verify (M2) for P .

Fixing $l \in \mathcal{E}$, we want to show $M_l(t, x), t \geq 0$, in (M2) is a continuous (\mathcal{F}_t) -martingale with respect to P , whose square variation process is given by

$$\langle M_l \rangle(t, x) = \int_0^t \|B^*(s, x_s)(l)\|_{\mathcal{U}}^2 ds, t \geq 0.$$

By Lemma 3.1 we have

$$(3.9) \quad \lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}} |_{\mathbb{X}^*} \langle \tilde{x}^{(n)}(t) - \tilde{x}(t), l \rangle_{\mathbb{X}} = 0.$$

Now for the nonlinear part, set for $R > 0$

$$G_R^1(t, x) := \int_0^t \mathbb{X}^* \langle A_1(s, x(s)), l \rangle_{\mathbb{X}} \cdot \chi_R(\|x(s)\|_{\mathbb{H}}) ds,$$

where $\chi_R \in C_0^\infty(\mathbb{R})$ is a cutoff function with $\chi_R(r) = 1$ if $|r| \leq R$ and $\chi_R(r) = 0$ if $|r| > 2R$. By (H1) we obtain \tilde{P} -a.s.

$$(3.10) \quad \int_0^t \mathbb{X}^* \langle A_2(s, \tilde{x}_s^{(n)}), l \rangle_{\mathbb{X}} ds \rightarrow \int_0^t \mathbb{X}^* \langle A_2(s, \tilde{x}_s), l \rangle_{\mathbb{X}} ds, \quad n \rightarrow \infty,$$

and

$$G_R^1(t, \tilde{x}^{(n)}) \rightarrow G_R^1(t, \tilde{x}), \quad n \rightarrow \infty.$$

Moreover, by (H3) and Lemma 3.1 we have

$$\begin{aligned} & \sup_n \mathbb{E}^{\tilde{P}} \left(\int_0^t \|A_1(s, \tilde{x}^{(n)}(s))\|_{\mathbb{X}^*}^\gamma ds \right) \\ &= \sup_n \mathbb{E}^{P_n} \left(\int_0^t \|A_1(s, x^{(n)}(s))\|_{\mathbb{X}^*}^\gamma ds \right) < \infty, \end{aligned}$$

and

$$\begin{aligned} & \sup_n \mathbb{E}^{\tilde{P}} \left(\int_0^t \|A_2(s, \tilde{x}_s^{(n)})\|_{\mathbb{X}^*}^\gamma ds \right) \\ &= \sup_n \mathbb{E}^{P_n} \left(\int_0^t \|A_2(s, x_s^{(n)})\|_{\mathbb{X}^*}^\gamma ds \right) < \infty, \end{aligned}$$

which implies that

$$\lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}} |G_R^1(t, \tilde{x}^{(n)}) - G_R^1(t, \tilde{x})| = 0,$$

and

$$(3.11) \quad \lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}} \left| \int_0^t [\mathbb{X}^* \langle A_2(s, \tilde{x}_s^{(n)}), l \rangle_{\mathbb{X}} - \mathbb{X}^* \langle A_2(s, \tilde{x}_s), l \rangle_{\mathbb{X}}] ds \right| = 0.$$

Set

$$G(t, x) := \int_0^t \mathbb{X}^* \langle A_1(s, x), l \rangle_{\mathbb{X}} ds,$$

by (H3) and Lemma 3.1 we have

$$\begin{aligned} & \lim_{R \rightarrow \infty} \sup_n \mathbb{E}^{\tilde{P}} |G_R^1(t, \tilde{x}^{(n)}) - G(t, \tilde{x}^{(n)})| \\ & \leq \|l\|_{\mathbb{X}} \lim_{R \rightarrow \infty} \sup_n [(\mathbb{E}^{\tilde{P}} \int_0^t \|A_1(s, \tilde{x}^{(n)}(s))\|_{\mathbb{X}^*}^\gamma ds)^{1/\gamma} (\int_0^t \tilde{P}(\|\tilde{x}^{(n)}(s)\|_{\mathbb{H}} \geq R) ds)^{(\gamma-1)/\gamma}] \\ & \leq \|l\|_{\mathbb{X}} \lim_{R \rightarrow \infty} \sup_n (\mathbb{E}^{\tilde{P}} \int_0^t \|A_1(s, \tilde{x}^{(n)}(s))\|_{\mathbb{X}^*}^\gamma ds)^{1/\gamma} (\mathbb{E}^{\tilde{P}} \int_0^t \|\tilde{x}^{(n)}(s)\|_{\mathbb{H}}^\gamma ds)^{(\gamma-1)/\gamma} / R^{\gamma-1} \\ & = \|l\|_{\mathbb{X}} \lim_{R \rightarrow \infty} \sup_n (\mathbb{E}^{P_n} \int_0^t \|A_1(s, x^{(n)}(s))\|_{\mathbb{X}^*}^\gamma ds)^{1/\gamma} (\mathbb{E}^{P_n} \int_0^t \|x^{(n)}(s)\|_{\mathbb{H}}^\gamma ds)^{(\gamma-1)/\gamma} / R^{\gamma-1} = 0, \end{aligned}$$

and similarly

$$\lim_{R \rightarrow \infty} \mathbb{E}^{\tilde{P}} |G_R^1(t, \tilde{x}) - G(t, \tilde{x})| = 0,$$

which imply that

$$(3.12) \quad \lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}} |G(t, \tilde{x}^{(n)}) - G(t, \tilde{x})| = 0.$$

On the other hand, by Lemma 3.1 and (H3) we have

$$\begin{aligned} (3.13) \quad & \lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}} \left| \int_0^t \mathbb{X}^* \langle \Pi_n A_1(s, \tilde{x}^{(n)}(s)) + \Pi_n A_2(s, \tilde{x}_s^{(n)}), l \rangle_{\mathbb{X}} - \mathbb{X}^* \langle A_1(s, \tilde{x}^{(n)}(s)) + A_2(s, \tilde{x}_s^{(n)}), l \rangle_{\mathbb{X}} ds \right| \\ & \leq \lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}} \left| \int_0^t \mathbb{X}^* \langle A_1(s, \tilde{x}^{(n)}(s)) + A_2(s, \tilde{x}_s^{(n)}), (\Pi_n - I)l \rangle_{\mathbb{X}} ds \right| = 0. \end{aligned}$$

Combining (3.9), (3.11), (3.12) and (3.13) we obtain that for $t > 0$

$$(3.14) \quad \lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}} |\langle M_n(t, \tilde{x}^{(n)}), l \rangle - M_l(t, \tilde{x})| = 0.$$

Let $t > s \geq 0$ and g be every bounded and real valued \mathcal{F}_s -measurable continuous function on \mathbb{S} . Using (3.14) we have

$$\begin{aligned} \mathbb{E}^P((M_l(t, x) - M_l(s, x))g(x)) &= \mathbb{E}^{\tilde{P}}((M_l(t, \tilde{x}) - M_l(s, \tilde{x}))g(\tilde{x})) \\ &= \lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}}((\langle M_n(t, \tilde{x}^{(n)}), l \rangle - \langle M_n(s, \tilde{x}^{(n)}), l \rangle)g(\tilde{x}^{(n)})) \\ &= \lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}^n}((\langle M_n(t, x), l \rangle - \langle M_n(s, x), l \rangle)g(x)) \\ &= 0. \end{aligned}$$

On the other hand by the B-D-G inequality (H3) and Lemma 3.1 we have

$$\begin{aligned} \sup_n \mathbb{E}^{\tilde{P}} |\langle M_n(t, \tilde{x}^{(n)}), l \rangle|^{2q} &\leq C \sup_n \mathbb{E}^{\tilde{P}} \left(\int_0^t \|B^*(s, \tilde{x}_s^{(n)})(l)\|_U^2 ds \right)^q \\ &\leq C \sup_n \mathbb{E}^{\tilde{P}} \left(\int_0^t \|B^*(s, \tilde{x}_s^{(n)})(l)\|_U^{2q} ds \right) < +\infty. \end{aligned}$$

Then by (3.14) we obtain that for $t \geq 0$

$$\lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}} |\langle M_n(t, \tilde{x}^{(n)}), l \rangle - M_l(t, \tilde{x})|^2 = 0,$$

and by (H1)

$$\lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}} \left(\int_0^t \|(\Pi_n B \tilde{\Pi}_n)^*(s, \tilde{x}_s^{(n)})(l) - B^*(s, \tilde{x}_s)(l)\|_U^2 ds \right) = 0,$$

which imply that

$$\mathbb{E}^P \left(M_l^2(t, x) - \int_0^t \|B^*(r, x_r)(l)\|_U^2 dr \middle| \mathcal{F}_s \right) = M_l^2(s, x) - \int_0^s \|B^*(r, x_r)(l)\|_U^2 dr.$$

Now the result follows. □

4 Uniqueness of the solutions

Now we consider the pathwise uniqueness of the solutions. We introduce another Hilbert space H_0 , which is the space where we obtain uniqueness of the solutions.

Let

$$V \subset H_0 \cong H_0^* \subset V^*$$

be a Gelfand triple, *i.e.* $(H_0, \langle \cdot, \cdot \rangle_{H_0})$ is a separable Hilbert space, identified with its dual space by the Riesz isomorphism, V is as in Section 2 and it is continuously and densely

embedded into H_0 . If ${}_{V^*}\langle \cdot, \cdot \rangle_V$ denotes the dualization between V and its dual space V^* , it follows that

$${}_{V^*}\langle u, v \rangle_V = \langle u, v \rangle_{H_0}, \quad u \in H_0, v \in V.$$

Since $\mathbb{X} \subset V$ continuously we have that

$$V^* \subset \mathbb{X}^*,$$

and there exists a bounded linear operator $\Lambda : \mathbb{X} \rightarrow V$ such that

$$(4.1) \quad {}_{V^*}\langle u, \Lambda v \rangle_V = {}_{\mathbb{X}^*}\langle u, v \rangle_{\mathbb{X}}, \quad u \in V^*, v \in \mathbb{X}.$$

In fact, since $\mathbb{X} \subset V$ continuously, we have that for $u \in H_0 \subset V^* \subset \mathbb{X}^*, v \in \mathbb{X}$

$$|{}_{\mathbb{X}^*}\langle u, v \rangle_{\mathbb{X}}| \leq \|u\|_{\mathbb{X}^*} \|v\|_{\mathbb{X}} \leq C \|u\|_{V^*} \|v\|_{\mathbb{X}} \leq C \|u\|_{H_0} \|v\|_{\mathbb{X}},$$

which implies that there exists some $\Lambda_0 v \in H_0$ such that for $u \in H_0$

$$\langle u, \Lambda_0 v \rangle_{H_0} = |{}_{\mathbb{X}^*}\langle u, v \rangle_{\mathbb{X}}|,$$

hence $v \mapsto \Lambda_0 v$ is linear. Moreover since

$$|\langle u, \Lambda_0 v \rangle_{H_0}| = |{}_{\mathbb{X}^*}\langle u, v \rangle_{\mathbb{X}}| \leq C \|u\|_{V^*},$$

by extension (4.1) follows.

Suppose that for $p \geq 2$ as in Section 2, $A_1 : \mathbb{R}^+ \times V \rightarrow V^*$ is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(V)$ -measurable; $A_2 : \mathbb{R}^+ \times (L_V^p \cap \mathfrak{C}) \rightarrow V^*$ is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(L_V^p \cap \mathfrak{C})$ -measurable; $B : \mathbb{R}^+ \times (L_V^p \cap \mathfrak{C}) \rightarrow L_2(U; H_0)$ is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(L_V^p \cap \mathfrak{C})$ -measurable.

There exist locally bounded measurable functions $c_2, c_3 : \mathbb{R} \rightarrow \mathbb{R}^+$ and $\rho_1 : \mathbb{R}^+ \times (\mathfrak{C} \cap L_V^p) \rightarrow [0, +\infty)$ such that for $u, \eta \in \mathbb{S}_1$, $A_1(\cdot, u(\cdot)), A_2(\cdot, u(\cdot)) \in L_{\text{loc}}^1([0, \infty); V^*)$, $B(\cdot, u(\cdot)) \in L_{\text{loc}}^2([0, \infty); L_2(U; H_0))$ and every $t \geq 0$:

(H4) (Local monotonicity)

$$\begin{aligned} & 2 \int_0^t {}_{V^*}\langle A_1(s, u(s)) + A_2(s, u_s) - A_1(s, \eta(s)) - A_2(s, \eta_s), u(s) - \eta(s) \rangle_V ds \\ & \leq \int_0^t [\rho_1(s, \eta_s) + \rho_1(s, u_s)] \|u_s - \eta_s\|_{C([-h, 0]; H_0)}^2 ds + c_2(t) \int_{-h}^0 \mathcal{N}_1(u(s) - \eta(s)) ds, \\ & \int_0^t \|B(s, u_s) - B(s, \eta_s)\|_{L_2(U; H_0)}^2 ds \\ & \leq \int_0^t [\rho_1(s, \eta_s) + \rho_1(s, u_s)] \|u_s - \eta_s\|_{C([-h, 0]; H_0)}^2 ds + c_2(t) \int_{-h}^0 \mathcal{N}_1(u(s) - \eta(s)) ds, \end{aligned}$$

and

$$\int_0^t \rho_1(s, \eta_s) ds \leq c_3(t) \left[\int_{-h}^t \mathcal{N}_1(\eta(s)) ds \right]^q + c_3(t) \left(1 + \sup_{s \in [-h, t]} \|\eta(s)\|_{\mathbb{H}}^{2q} \right),$$

for some $q \in \mathbb{N}$.

(H5) (Growth condition) There exist locally bounded measurable functions $\lambda_5, \lambda_6 : \mathbb{R} \mapsto \mathbb{R}^+$ such that for $u \in \mathbb{S}_1$, we have for every $t \geq 0$,

$$\int_0^t [\|A_1(s, u(s))\|_{V^*}^{\frac{p}{p-1}} + \|A_2(s, u_s)\|_{V^*}^{\frac{p}{p-1}}] ds \leq \lambda_5(t) \left[\int_{-h}^t \mathcal{N}_1(u(s)) ds \right]^q + \lambda_5(t) \left(\sup_{s \in [-h, t]} \|u(s)\|_{\mathbb{H}}^{2q} + 1 \right),$$

$$\|B(t, u_t)\|_{L_2(U; H_0)}^2 \leq \lambda_6(t) (1 + \|u_t\|_{\mathfrak{C}}^2),$$

for some $q \in \mathbb{N}$.

By (H5) and Theorem 2.2 we obtain that

Theorem 4.1. *Suppose (H1) – (H3), (H5) hold for some $p \geq 2$. Then for every $\psi \in C([-h, 0]; H_0) \cap \mathfrak{C}$ with $\mathcal{N}(\psi) < \infty$, (2.1) has a weak solution $X \in C([-h, \infty); H_0)$ such that for every $q \in \mathbb{N}$, $T > 0$*

$$\mathbb{E}^P \left(\sup_{t \in [-h, T]} \|X(t)\|_{\mathbb{H}}^{2q} + \left[\int_{-h}^T \mathcal{N}_1(X(t)) dt \right]^q \right) < \infty,$$

and

$$\mathbb{E}^P \left(\sup_{t \in [-h, T]} \|X(t)\|_{H_0}^2 \right) \leq C.$$

Proof. By Theorem 2.2 we have that for every $l \in \mathbb{X}$, $t \in [0, T]$

$${}_{\mathbb{X}^*} \langle X(t), l \rangle_{\mathbb{X}} - \int_0^t {}_{\mathbb{X}^*} \langle A_1(s, X(s)) + A_2(s, X_s), l \rangle_{\mathbb{X}} ds = \langle X(0), l \rangle_{\mathbb{H}} + \left\langle \int_0^t B(s, X_s) dW(s), l \right\rangle_{\mathbb{H}}.$$

By (H5) and Theorem 2.2 we know that for every $T > 0$

$$(4.2) \quad \mathbb{E}^P \int_0^T [\|A_1(s, X(s))\|_{V^*}^{\frac{p}{p-1}} + \|A_2(s, X_s)\|_{V^*}^{\frac{p}{p-1}}] ds \leq C,$$

$$(4.3) \quad \mathbb{E}^P \int_0^T \|B(s, X_s)\|_{L_2(U; H_0)}^2 ds \leq C,$$

for some constant C . We define

$$\bar{X}(t) := X(0) + \int_0^t [A_1(s, X(s)) + A_2(s, X_s)] ds + \int_0^t B(s, X_s) dW(s),$$

in V^* . By (4.1) we obtain that for every $l \in \mathbb{X}$, $t \in [0, T]$

$${}_{\mathbb{X}^*} \langle \bar{X}(t), l \rangle_{\mathbb{X}} - \int_0^t {}_{\mathbb{X}^*} \langle A_1(s, X(s)) + A_2(s, X_s), l \rangle_{\mathbb{X}} ds = \langle X(0), l \rangle_{\mathbb{H}} + \left\langle \int_0^t B(s, X_s) dW(s), l \right\rangle_{\mathbb{H}},$$

which implies that

$$X(t) = \bar{X}(t) \quad \forall t \in [0, \infty), P - a.s..$$

Moreover, by (2.2) and Theorem 2.2 we obtain

$$\mathbb{E}^P \int_0^T \|X(s)\|_V^p ds \leq C,$$

which combined with (4.2), (4.3) and [PR07, Theorem 4.2.5] implies that $X \in C([-h, \infty), H_0)$ and

$$\mathbb{E}^P \left[\sup_{t \in [0, T]} \|X(t)\|_{H_0}^2 \right] \leq C.$$

□

Definition 4.1. We say that there exists a (probabilistically) strong solution to (2.1) if for every probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$ with an (\mathcal{F}_t) -Wiener process W and given an initial value $\psi \in L^p(-h, 0; V) \cap C([-h, 0]; H_0) \cap \mathfrak{C}$ there exists an (\mathcal{F}_t) -adapted process $X : [0, T] \times \Omega \rightarrow H_0$ such that for P -a.e. $\omega \in \Omega$,

$$X(\cdot, \omega) \in \mathbb{S}_1 \cap C([-h, \infty); H_0),$$

and such that P -a.s.

$$\begin{aligned} X(t) &= \psi(t), \quad -h \leq t \leq 0, \\ X(t) &= X(0) + \int_0^t [A_1(s, X(s)) + A_2(s, X_s)] ds + \int_0^t B(s, X_s) dW(s), \end{aligned}$$

holds in V^* for all $t \in [0, \infty)$.

Theorem 4.2. *Suppose that (H1) – (H5) hold. Then for every $\psi \in C([-h, 0]; H_0) \cap \mathfrak{C}$ with $\mathcal{N}(\psi) < \infty$, (2.1) has a unique (probabilistically) strong solution $X \in C([-h, \infty); H_0)$ satisfying for every $T > 0$ and $q \in \mathbb{N}$*

$$(4.4) \quad \mathbb{E}^P \left(\sup_{t \in [-h, T]} \|X(t)\|_{\mathbb{H}}^{2q} + \left[\int_0^T \mathcal{N}_1(X(t)) dt \right]^q \right) < \infty.$$

Moreover,

$$\mathbb{E}^P \left(\sup_{t \in [-h, T]} \|X(t)\|_{H_0}^2 \right) < \infty.$$

Proof. Suppose X, Y are the solutions of (2.1), both with initial conditions ψ respectively, i.e.

$$(4.5) \quad \begin{aligned} X(t) &= \psi(0) + \int_0^t [A_1(s, X(s)) + A_2(s, X_s)] ds + \int_0^t B(s, X_s) dW(s), \quad t \in [0, T]; \\ Y(t) &= \psi(0) + \int_0^t [A_1(s, Y(s)) + A_2(s, Y_s)] ds + \int_0^t B(s, Y_s) dW(s), \quad t \in [0, T]. \end{aligned}$$

Then by Itô's formula we have

$$\begin{aligned}
& \|X(t) - Y(t)\|_{H_0}^2 \\
& \leq 2 \int_0^t \langle A_1(s, X(s)) + A_2(s, X_s) - A_1(s, Y(s)) - A_2(s, Y_s), X(s) - Y(s) \rangle_V ds \\
& \quad + 2 \int_0^t \langle X(s) - Y(s), (B(s, X_s) - B(s, Y_s)) dW(s) \rangle_{H_0} \\
& \quad + \int_0^t \|B(s, X_s) - B(s, Y_s)\|_{L_2(U; H_0)}^2 ds, \quad t \in [0, T].
\end{aligned}$$

Define

$$\tau_n := \inf\{t \geq 0 \mid \int_0^t [\rho_1(s, Y_s) + \rho_1(s, X_s)] ds > n\} \wedge T.$$

By (4.4) and (H4) we have

$$\lim_{n \rightarrow \infty} \tau_n = T, \quad P - a.s..$$

By (H4) and the B-D-G inequality we have for every stopping time τ

$$\begin{aligned}
& \mathbb{E} \sup_{t \in [0, \tau \wedge \tau_n]} \|X(t) - Y(t)\|_{H_0}^2 \\
& \leq C \mathbb{E} \int_0^{\tau \wedge \tau_n} (\rho_1(s, X_s) + \rho_1(s, Y_s)) \|X_s - Y_s\|_{C([-h, 0]; H_0)}^2 ds \\
& \quad + 2 \mathbb{E} \sup_{0 \leq t \leq \tau \wedge \tau_n} \int_0^t \langle X(s) - Y(s), (B(s, X_s) - B(s, Y_s)) dW(s) \rangle_{H_0} \\
& \leq C \mathbb{E} \int_0^{\tau \wedge \tau_n} (\rho_1(s, X_s) + \rho_1(s, Y_s)) \|X_s - Y_s\|_{C([-h, 0]; H_0)}^2 ds + \frac{1}{2} \mathbb{E} \sup_{t \in [0, \tau \wedge \tau_n]} \|X(t) - Y(t)\|_{H_0}^2.
\end{aligned}$$

By [GM83 Lemma 2] we have for every $n \in \mathbb{N}$

$$X(t) = Y(t), \quad t \in [0, \tau_n], \quad \mathbb{P} - a.s..$$

Letting $n \rightarrow \infty$ we have

$$X(t) = Y(t), \quad t \in [0, \infty), \quad \mathbb{P} - a.s..$$

Then by using the Yamada-Watanabe Theorem (see e.g. [Ku07], [RSZ08]) the results follow. \square

5 Application to examples

In this section we describe some examples for (2.1) satisfying conditions (H1)-(H5) imposed above. First, we recall some useful estimates which will be used later.

Let O be a bounded open domain in \mathbb{R}^d with smooth boundary and let $C_0^\infty(O)$ denote the set of all smooth functions from O to \mathbb{R} with compact support. For $p > 1$, let $L^p(O)$ be the

L^p -space with norm denoted by $\|\cdot\|_{L^p}$. If A is $-\Delta$ on the domain O with Dirichlet boundary condition, then we have the following estimates which will be used later. For $s \geq 0, p \in [1, \infty]$ we use $W_0^{s,p}(O)$ to denote the Sobolev space of all $f \in L^2$ for which $\|A^{s/2}f\|_{L^p}$ is finite.

Remark 5.1. Here we can also consider the equation on the torus \mathbb{T}^d and the results still can be applied in this case.

We shall also use the following standard Sobolev inequality (cf. [St70, Chapter V]):

Lemma 5.1. *Suppose that $q > 1, p \in [q, \infty)$ and*

$$\frac{1}{p} + \frac{\sigma}{d} = \frac{1}{q}.$$

Suppose that $f \in W_0^{\sigma,q}$, then $f \in L^p$ and there is a constant $C \geq 0$ such that

$$\|f\|_{L^p} \leq C\|f\|_{W_0^{\sigma,q}}.$$

In the following, let $\{W^k(t); t \geq 0, k \in \mathbb{N}\}$ be a sequence of independent standard Brownian motions and l^2 be the Hilbert space of all square summable sequences of real number sequences. We will use $\partial_i := \partial_{\xi_i}, \partial_{ij}^2 := \partial_{\xi_i \xi_j}^2$ and the usual Einstein summation convention.

5.1 Fractional Navier-Stokes equation

Now we apply Theorem 2.1 to the stochastic d-dimensional fractional Navier-Stokes equation in a bounded domain O in \mathbb{R}^d with smooth boundary:

$$\begin{aligned} (5.1) \quad du(t) = & [-(-\Delta)^\alpha u(t) - u(t) \cdot \nabla u(t) + \nabla p(t) \\ & + c_1 u(t - r_1(t)) \cdot \nabla u(t) + c_2 \int_{-h}^0 u(t+r) \cdot \nabla u(t) dr \\ & + \int_{-h}^0 f_1(r, \xi, u(t+r)) dr + f(\xi, u(t - r_2(t)))] dt \\ & + [\nabla \tilde{p}_i(t) + g_i(\xi, u(t - r_3(t)))] dW^i(t), \\ & \operatorname{div} u(t) = 0, \\ & u(t, \xi) = 0, (t, \xi) \in [-h, \infty) \times \partial O, \\ & u(t) = \psi(t), -h \leq t \leq 0, \end{aligned}$$

where $p(t, \xi)$ and $\tilde{p}_i(t, \xi), i \in \mathbb{N}$, are unknown scalar functions, u is the velocity vector, $c_i \in \mathbb{R}, i = 1, 2, r_i, i = 1, 2, 3$, are continuous differentiable functions on \mathbb{R} with $0 \leq r_i(t) < h, i = 1, 2, 3, \sup r_i'(t) < 1, i = 1, 2, f$ and g are functions from $O \times \mathbb{R}^d$ to \mathbb{R}^d and $\mathbb{R}^d \times l^2$, respectively, continuous with respect to the second variable, f_1 is a function from $[-h, 0] \times O \times \mathbb{R}^d$ to \mathbb{R}^d , continuous with respect to the third variable, satisfying for some $\kappa_0 > 0$ and $q \in L^2(O)$

$$|f_1(r, \xi, u)| + |f(\xi, u)| + \|g(\xi, u)\|_{l^2} \leq \kappa_0 \cdot |u| + q(\xi), \quad \forall (r, \xi, u) \in [-h, 0] \times O \times \mathbb{R}^d.$$

When $\alpha = 1, g = c_1 = c_2 = f = f_1 = 0$, (5.1) reduces to the usual Navier-Stokes equation. In the case without delay for general α this equation has been studied by many authors (see [W05], [Z12], [RZZ14b] and the references therein). In [W05], the author obtains global existence and uniqueness of solutions for small initial values when $\alpha > \frac{1}{2}$. In [Z12], the author obtains local existence and uniqueness of solutions by using a stochastic Lagrangian particle trajectories approach for $\alpha = \frac{1}{2}$ and global existence and uniqueness when $d = 2$. In [RZZ14b] we obtain local existence and uniqueness of solutions of stochastic fractional Navier-Stokes equations for $\alpha \in (0, 1]$ and every $d \in \mathbb{N}$.

Let $C_{0,\sigma}^\infty(O)^d$ be the space of all smooth d -dimensional divergence free vector fields on O with compact supports in O . For $s \geq 0, p > 1$, the completion of $C_{0,\sigma}^\infty(O)^d$ in $W_{0,\sigma}^{s,p}(O)^d$ is denoted by $W_{0,\sigma}^{s,p}(O)$. Below we choose

$$\mathbb{Y} = \mathbb{H} = W_{0,\sigma}^{0,2}(O), V = W_{0,\sigma}^{\alpha,2}(O)$$

and

$$\mathbb{X}^* = (W_{0,\sigma}^{2+d,2}(O))^*, \mathbb{X} = W_{0,\sigma}^{2+d,2}(O).$$

Let \mathcal{P} be the orthogonal projection operator from $L^2(O)^d$ onto \mathbb{H} . We define the operators A_1, A_2 and B as follows: for $x \in C_{0,\sigma}^\infty(O)$

$$A_1(x) := -\mathcal{P}[(-\Delta)^\alpha x] - \mathcal{P}[(x \cdot \nabla)x],$$

and for $u \in C([-h, 0]; C_{0,\sigma}^\infty(O))$

$$\begin{aligned} A_2(t, u) &:= c_1 \mathcal{P}[u(-r_1(t)) \cdot \nabla u(0)] + c_2 \mathcal{P}\left[\int_{-h}^0 u(r) \cdot \nabla u(0) dr\right] \\ &\quad + \mathcal{P}\left[\int_{-h}^0 f_1(r, u(r)) dr\right] + \mathcal{P}f(u(-r_2(t))). \\ B(t, u) &:= \mathcal{P}[g(u(-r_3(t)))]. \end{aligned}$$

Then by similar arguments as in [GRZ09] we have that for $x_1, y_1, x, y \in C_{0,\sigma}^\infty(O)$,

$$(5.2) \quad \|\mathcal{P}(-\Delta)^\alpha x - \mathcal{P}(-\Delta)^\alpha y\|_{\mathbb{X}^*} \leq C\|x - y\|_{\mathbb{H}},$$

$$(5.3) \quad \|\mathcal{P}[x_1 \cdot \nabla x] - \mathcal{P}[y_1 \cdot \nabla y]\|_{\mathbb{X}^*} \leq C\|x_1\|_{\mathbb{H}}\|x - y\|_{\mathbb{H}} + C\|y\|_{\mathbb{H}}\|x_1 - y_1\|_{\mathbb{H}}.$$

We can extend the operators A_1 and A_2, B to \mathbb{H} and \mathfrak{C} such that for $x \in \mathbb{H}, u \in \mathfrak{C}, A_1(x) \in \mathbb{X}^*$ and $A_2(t, u) \in \mathbb{X}^*, B(t, u) \in L_2(l_2, \mathbb{H})$. Thus, we can write the system in the following abstract form

$$du(t) = [A_1(t, u(t)) + A_2(t, u_t)]dt + B(t, u_t)dW(t), \quad u(t) = \psi(t), -h \leq t \leq 0.$$

Moreover, by (5.2), (5.3) we have that for $u \in \mathfrak{C}_\infty$

$$A_1(u(\cdot)) \in L_{\text{loc}}^1([0, \infty); \mathbb{X}^*), A_2(\cdot, u) \in L_{\text{loc}}^1([0, \infty); \mathbb{X}^*).$$

Let $e_n, n \in \mathbb{N}$, Π_n, \mathbb{H}_n be as in Section 2. We can easily deduce that $\|\Pi_n A_1(\cdot)\|_{\mathbb{X}^*}$ is locally bounded and continuous on \mathbb{H}_n and $\|\Pi_n A_2(\cdot, \cdot)\|_{\mathbb{X}^*}, \|\Pi_n B(\cdot, \cdot)\|_{L_2(U; \mathbb{H})}$ are bounded on balls in $\mathbb{R}^+ \times C([-h, 0]; \mathbb{H}_n)$. Furthermore, for every $t \in [0, T]$, we have for $u, v \in C([-h, \infty); \mathbb{X})$

$$\begin{aligned} & \|\mathcal{P}[u(t - r_1(t)) \cdot \nabla u(t)] - \mathcal{P}[v(t - r_1(t)) \cdot \nabla v(t)]\|_{\mathbb{X}^*} \\ & \leq C \left[\sup_{s \in [-h, T]} \|u(s)\|_{\mathbb{H}} + \sup_{s \in [-h, T]} \|v(s)\|_{\mathbb{H}} \right] \sup_{s \in [-h, T]} \|u(s) - v(s)\|_{\mathbb{H}}, \end{aligned}$$

and

$$\begin{aligned} & \left\| \int_{-h}^0 \mathcal{P}[u(t+r) \cdot \nabla u(s)] - \mathcal{P}[v(t+r) \cdot \nabla v(t)] dr \right\|_{\mathbb{X}^*} \\ & \leq C \left[\sup_{s \in [-h, T]} \|u(s)\|_{\mathbb{H}} + \sup_{s \in [-h, T]} \|v(s)\|_{\mathbb{H}} \right] \sup_{s \in [-h, T]} \|u(s) - v(s)\|_{\mathbb{H}}, \end{aligned}$$

which implies that for every $t > 0$, $\Pi_n A_2(t, \cdot), \Pi_n B(t, \cdot)$ are continuous on $C([-h, 0]; \mathbb{H}_n)$.

We define the functional \mathcal{N}_1 on \mathbb{H} as follows:

$$\mathcal{N}_1(u) = \begin{cases} \|(-\Delta)^{\frac{\alpha}{2}} u\|_{L^2(O)}^2 & \text{if } u \in W_{0,\sigma}^{\alpha,2}(O), \\ 0 & \text{otherwise.} \end{cases}$$

We easily deduce that $\mathcal{N}_1 \in \mathfrak{A}^2$. In this case $\mathcal{N}(\psi) < \infty$ is equivalent to $\psi \in L^2([-h, 0]; V)$.

Theorem 5.2. *For every $\psi \in \mathfrak{C} \cap L_V^2$ (5.1) has a weak solution $X : [-h, \infty) \times \Omega \rightarrow \mathbb{H}$ such that for every $q \in \mathbb{N}, T > 0$*

$$\mathbb{E}^P \left(\sup_{t \in [-h, T]} \|X(t)\|_{\mathbb{H}}^{2q} + \left[\int_{-h}^T \mathcal{N}_1(X(t)) dt \right]^q \right) < \infty.$$

Proof. First we check (H3): by (5.2), (5.3) and the linear growth of f, f_1, g we have for $u \in \mathfrak{C}_\infty, t \geq 0$,

$$\|A_1(u(t))\|_{\mathbb{X}^*} \leq C(1 + \|u_t\|_{\mathfrak{C}}^2),$$

$$\begin{aligned} \|A_2(t, u_t)\|_{\mathbb{X}^*} & \leq C[\|u(t - r_1(t))\|_{\mathbb{H}} \|u(t)\|_{\mathbb{H}} + \int_{-h}^0 \|u(t+r)\|_{\mathbb{H}} \|u(t)\|_{\mathbb{H}} dr] \\ & \quad + C[1 + \|u(t - r_2(t))\|_{\mathbb{H}} + \int_{-h}^0 \|u(t+r)\|_{\mathbb{H}} dr] \\ & \leq C(1 + \|u_t\|_{\mathfrak{C}}^2), \end{aligned}$$

$$\|B(t, u_t)\|_{L_2(l^2; \mathbb{H})}^2 \leq C(1 + \|u_t\|_{\mathfrak{C}}^2),$$

where C is a constant changing from line to line. Now (H3) follows immediately by integrating with respect to t .

Now we check (H1): by Remark 2.2, (H3), (5.2) and (5.3) it follows that (2.3) holds in this case. Now by (5.3) we have for $u, v \in L^2_{\text{loc}}([-h, \infty); \mathbb{H})$ and $t > 0$

$$\begin{aligned}
& \int_0^t \|\mathcal{P}[u(s - r_1(s)) \cdot \nabla u(s)] - \mathcal{P}[v(s - r_1(s)) \cdot \nabla v(s)]\|_{\mathbb{X}^*} ds \\
& \leq C \left(\int_0^t \|u(s)\|_{\mathbb{H}}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|u(s - r_1(s)) - v(s - r_1(s))\|_{\mathbb{H}}^2 ds \right)^{\frac{1}{2}} \\
& \quad + \left(\int_0^t \|v(s - r_1(s))\|_{\mathbb{H}}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|u(s) - v(s)\|_{\mathbb{H}}^2 ds \right)^{\frac{1}{2}} \\
& \leq C \left(\left(\int_{-h}^t \|u(s)\|_{\mathbb{H}}^2 ds \right)^{\frac{1}{2}} + \left(\int_{-h}^t \|v(s)\|_{\mathbb{H}}^2 ds \right)^{\frac{1}{2}} \right) \left(\int_{-h}^t \|u(s) - v(s)\|_{\mathbb{H}}^2 ds \right)^{\frac{1}{2}},
\end{aligned}$$

where in the last inequality we used $\sup_t r'_i(t) < 1$. Similarly, for $u, v \in L^2_{\text{loc}}([-h, \infty); \mathbb{H})$ and $t > 0$

$$\begin{aligned}
& \int_0^t \left\| \int_{-h}^0 (\mathcal{P}[u(s+r) \cdot \nabla u(s)] - \mathcal{P}[v(s+r) \cdot \nabla v(s)]) dr \right\|_{\mathbb{X}^*} ds \\
& \leq \int_0^t \int_{-h}^0 \|\mathcal{P}[u(s+r) \cdot \nabla u(s)] - \mathcal{P}[v(s+r) \cdot \nabla v(s)]\|_{\mathbb{X}^*} dr ds \\
& \leq C \left(\left(\int_{-h}^t \|u(s)\|_{\mathbb{H}}^2 ds \right)^{\frac{1}{2}} + \left(\int_{-h}^t \|v(s)\|_{\mathbb{H}}^2 ds \right)^{\frac{1}{2}} \right) \left(\int_{-h}^t \|u(s) - v(s)\|_{\mathbb{H}}^2 ds \right)^{\frac{1}{2}}.
\end{aligned}$$

Let u_n converge to u in $L^2_{\text{loc}}([-h, \infty), \mathbb{H})$. By Lebesgue's dominated convergence theorem we have

$$\begin{aligned}
& \int_0^t \|\mathcal{P}[\int_{-h}^0 f_1(r, \cdot, u_n(s+r)) dr] - \mathcal{P}[\int_{-h}^0 f_1(r, \cdot, u(s+r)) dr]\|_{\mathbb{X}^*} ds \\
& \leq C \int_0^t \int_{-h}^0 \|f_1(r, \cdot, u_n(s+r)) - f_1(r, \cdot, u(s+r))\|_{L^1(O)} dr ds \rightarrow 0, n \rightarrow \infty.
\end{aligned}$$

Similarly, the convergence for f can be obtained. Then the above estimates imply that (2.4) in (H1) holds in this case. The convergence for g can be obtained similarly.

For (H2), since

$${}_{\mathbb{X}^*} \langle \mathcal{P}[y \cdot \nabla x], x \rangle_{\mathbb{X}} = \langle y \cdot \nabla x, x \rangle_{\mathbb{H}} = 0, x, y \in \mathbb{X},$$

we have that for $u \in C([-h, \infty); \mathbb{X})$,

$$\begin{aligned}
{}_{\mathbb{X}^*} \langle A_1(u(t)) + A_2(u_t), u(t) \rangle_{\mathbb{X}} &= -\mathcal{N}_1(u(t)) + \int_{-h}^0 \langle f_1(r, \cdot, u(t+r)), u(t) \rangle_{\mathbb{H}} dr \\
&\quad + \langle f(\cdot, u(t - r_2(t))), u(t) \rangle_{\mathbb{H}} \\
&\leq -\mathcal{N}_1(u(t)) + C(1 + \|u_t\|_{\mathcal{C}}^2).
\end{aligned}$$

Now (H2) follows by taking integration w.r.t. t .

By Theorem 2.2 the results follow. \square

Stochastic 2-D Navier-Stokes equation

Now we come to the case $d = 2$, $\alpha = 1$. Define

$$H_0 = \mathbb{H}, \quad V = W_{0,\sigma}^{1,2}(O)$$

It is standard that using the Gelfand triple

$$V \subseteq H_0 \equiv H_0^* \subseteq V^*,$$

we see that the following mappings

$$A_1 : V \rightarrow V^*, \quad A_2 : \mathbb{R}^+ \times (L_V^2 \cap \mathfrak{C}) \rightarrow V^*$$

are well defined. For $u \in \mathbb{S}_1$, we easily deduce that $A_1(\cdot, u(\cdot)), A_2(\cdot, u) \in L_{\text{loc}}^1([0, \infty); V^*), B(\cdot, u) \in L_{\text{loc}}^2([0, \infty); L_2(l_2; H_0))$.

Suppose that f_1, f, g satisfy

$$(5.4) \quad |f_1(r, \xi, u_1) - f_1(r, \xi, u_2)| + |f(\xi, u_1) - f(\xi, u_2)| \leq C(1 + |u_1|^\gamma + |u_2|^\gamma)|u_1 - u_2|, \quad 1 \leq \gamma \leq 2$$

$$(5.5) \quad \|g(\xi, u_1) - g(\xi, u_2)\|_{l^2} \leq C|u_1 - u_2|.$$

Theorem 5.3. *Fix $d = 2, \alpha = 1$. Suppose that f_1, f and g satisfy (5.4), (5.5) respectively. Then for every $\psi \in L^2(-h, 0; V) \cap C([-h, 0]; H_0)$, (5.1) has a unique (probabilistically) strong solution $X \in C([-h, \infty), H_0)$ satisfying for every $T > 0$ and $q \in \mathbb{N}$*

$$\mathbb{E}^P \left(\sup_{t \in [-h, T]} \|X(t)\|_{\mathbb{H}}^{2q} + \left[\int_0^T \mathcal{N}_1(X(t)) dt \right]^q \right) < \infty.$$

Proof. Now we check (H4): we have for $u, v \in L_{\text{loc}}^2([-h, \infty); V) \cap C([-h, \infty); \mathbb{H})$

$$\begin{aligned} & \left| \int_0^T \langle u(s - r_1(s)) \cdot \nabla u(s) - v(s - r_1(s)) \cdot \nabla v(s), u(s) - v(s) \rangle_V ds \right| \\ & \leq C \int_0^T \|\nabla u(s)\|_{\mathbb{H}} \|u(s - r_1(s)) - v(s - r_1(s))\|_{L^4} \|u(s) - v(s)\|_{L^4} ds \\ & \leq C \int_0^T \|\nabla u(s)\|_{\mathbb{H}} \|u(s - r_1(s)) - v(s - r_1(s))\|_V^{1/2} \|u(s - r_1(s)) - v(s - r_1(s))\|_{\mathbb{H}}^{1/2} \\ & \quad \|u(s) - v(s)\|_V^{1/2} \|u(s) - v(s)\|_{\mathbb{H}}^{1/2} ds \\ & \leq C \int_0^T \|\nabla u(s)\|_{\mathbb{H}}^2 \|u_s - v_s\|_{\mathfrak{C}}^2 ds + \varepsilon \left[\int_0^T \|u(s) - v(s)\|_V^2 ds \right]^{1/2} \left[\int_0^T \|u(s - r_1(s)) - v(s - r_1(s))\|_V^2 ds \right]^{1/2} \\ & \leq C \int_0^T \|\nabla u(s)\|_{\mathbb{H}}^2 \|u_s - v_s\|_{\mathfrak{C}}^2 ds + \varepsilon \int_{-h}^T \|\nabla u(s) - \nabla v(s)\|_{\mathbb{H}}^2 ds, \end{aligned}$$

where we used $\text{div} v = 0$ in the first inequality, Lemma 5.1 and the interpolation inequality in the second inequality, Young's inequality in the third inequality.

For $u(s) \cdot \nabla u(s)$ and $\int_{-h}^0 u(s+r) \cdot \nabla u(s) dr$ we have similar results.

Now for f we have for $u, v \in L_{\text{loc}}^2([-h, \infty); V) \cap C([-h, \infty), \mathbb{H})$

$$\begin{aligned}
& \left| \int_0^T \int_{V^*} \langle f(u(s-r_2(s))) - f(v(s-r_2(s))), u(s) - v(s) \rangle_V ds \right| \\
& \leq C \int_0^T \int \left[|u(s-r_2(s))|^\gamma + |v(s-r_2(s))|^\gamma + 1 \right] |u(s-r_2(s)) - v(s-r_2(s))| |u(s) - v(s)| d\xi ds \\
& \leq C \int_0^T \left[1 + \|u(s-r_2(s))\|_{L^{\gamma p_1}}^\gamma + \|v(s-r_2(s))\|_{L^{\gamma p_1}}^\gamma \right] \|u(s-r_2(s)) - v(s-r_2(s))\|_{L^{2q_1}} \\
& \quad \|u(s) - v(s)\|_{L^{2q_1}} ds \\
& \leq C \int_0^T \left[1 + \|u(s-r_2(s))\|_{L^{\gamma p_1}}^\gamma + \|v(s-r_2(s))\|_{L^{\gamma p_1}}^\gamma \right] \|u(s-r_2(s)) - v(s-r_2(s))\|_V^\sigma \|u(s) - v(s)\|_V^\sigma \\
& \quad \|u(s-r_2(s)) - v(s-r_2(s))\|_{\mathbb{H}}^{1-\sigma} \|u(s) - v(s)\|_{\mathbb{H}}^{1-\sigma} ds \\
& \leq C \int_0^T \left[1 + \|u(s-r_2(s))\|_V^{\sigma_0 \gamma \frac{1}{1-\sigma}} \|u(s-r_2(s))\|_{\mathbb{H}}^{(1-\sigma_0) \gamma \frac{1}{1-\sigma}} \right. \\
& \quad \left. + \|v(s-r_2(s))\|_V^{\sigma_0 \gamma \frac{1}{1-\sigma}} \|v(s-r_2(s))\|_{\mathbb{H}}^{(1-\sigma_0) \gamma \frac{1}{1-\sigma}} \right] \|u_s - v_s\|_{\mathfrak{E}}^2 ds \\
& \quad + \varepsilon \int_{-h}^T \|u(s) - v(s)\|_V^2 ds,
\end{aligned}$$

where $\frac{1}{p_1} + \frac{1}{q_1} = 1$, $\sigma = 1 - \frac{1}{q_1} < 1$, $\sigma_0 = 1 - \frac{2}{\gamma p_1} < 1$, $\frac{\sigma_0 \gamma}{1-\sigma} \leq 2$ and we used (5.4) in the first inequality, Lemma 5.1 and the interpolation inequality in the third inequality. The term $\int_{-h}^0 f_1(r, u(t+r)) dr$ can be treated similarly. Since g satisfies Lipschitz conditions, we easily deduce that

$$\|B(t, u_t) - B(t, v_t)\|_{L_2(l_2; H_0)}^2 \leq C \|u_t - v_t\|_{\mathfrak{E}}^2.$$

Now (H4) follows.

Now for (H5), we have for $u \in L_{\text{loc}}^2([-h, \infty); V) \cap L_{\text{loc}}^\infty([-h, \infty); \mathbb{H}) \cap C([-h, \infty); \mathbb{X}^*)$

$$\begin{aligned}
& \int_0^T \|u(s-r_1(s)) \cdot \nabla u(s)\|_{V^*}^2 ds \leq \int_0^T \|u(s-r_1(s))\|_{L^4}^2 \|u(s)\|_{L^4}^2 ds \\
& \leq C \int_0^T \|u(s-r_1(s))\|_V \|u(s-r_1(s))\|_{\mathbb{H}} \|u(s)\|_V \|u(s)\|_{\mathbb{H}} ds \\
& \leq C \left[\int_0^T \|u(s-r_1(s))\|_V^2 \|u(s-r_1(s))\|_{\mathbb{H}}^2 ds \right]^{1/2} \left[\int_0^T \|u(s)\|_V^2 \|u(s)\|_{\mathbb{H}}^2 ds \right]^{1/2} \\
& \leq C \int_{-h}^T \|u(s)\|_V^2 \|u(s)\|_{\mathbb{H}}^2 ds \\
& \leq C \left[\int_{-h}^T \|u(s)\|_V^2 ds \right]^2 + C \sup_{s \in [-h, T]} \|u(s)\|_{\mathbb{H}}^4.
\end{aligned}$$

The other terms can be checked easily since f, f_1, g satisfy linear growth conditions. Therefore, existence and uniqueness of solutions follow from Theorem 4.2. \square

Remark 5.2. The existence results in Section 2 can also be applied to other stochastic equations with delays from fluid dynamics such as the stochastic quasi-geostrophic equation with delays, which has been studied in [RZZ14] without delays. However, to prove the uniqueness of the solutions, we need L^p -estimates for the solutions, which cannot be obtained by Theorem 4.2. This will be studied in the future.

5.2 Stochastic semilinear equation

Consider the following stochastic semilinear equation in a smooth domain $O \subset \mathbb{R}^d$:

$$\begin{aligned}
(5.6) \quad dX(t) = & \Delta[X(t) + a_1(X(t - r_0(t))) + \int_{-h}^0 a_2(r, X(t+r))dr]dt \\
& + (f_1(X(t)) + f_2(X(t - r_1(t))) + \int_{-h}^0 f_3(r, X(t+r))dr) \\
& \cdot [\nabla X(t) + \nabla X(t - r_2(t)) + \int_{-h}^0 \nabla X(t+r)dr]dt \\
& + [\partial_i b_1^i(X(t)) + \partial_i b_2^i(X(t - r_3(t))) + \int_{-h}^0 \partial_i b_3^i(r, X(t+r))dr]dt \\
& + [g_1(X(t)) + g_2(X(t - r_4(t))) + \int_{-h}^0 g_3(r, X(t+r))dr]dt \\
& + c_i(X(t - r_5(t)))dW^i(t),
\end{aligned}$$

$$X(t, \xi) = 0, (t, \xi) \in [-h, \infty) \times \partial O,$$

$$X(t) = \psi(t), -h \leq t \leq 0,$$

where $r_i, i = 0, 1, \dots, 5$, are continuous differentiable functions on \mathbb{R} with $0 \leq r_i(t) < h, i = 0, 1, \dots, 5, \sup r_i'(t) < 1, i = 0, 1, 2, 3, 4$, $a_1, f_i, b_i, g_i, i = 1, 2$, and c are functions from $O \times \mathbb{R}$ to $\mathbb{R}, \mathbb{R}^d, \mathbb{R}^d, \mathbb{R}$ and $\mathbb{R} \times l^2$, respectively, continuous with respect to the second variable, a_2, f_3, b_3, g_3 are functions from $[-h, 0] \times O \times \mathbb{R}$ to $\mathbb{R}, \mathbb{R}^d, \mathbb{R}^d, \mathbb{R}$, continuous with respect to the third variable, satisfying

$$(5.7) \quad |g_1(\xi, u)| \leq C[|u|^{\gamma_1} + 1], \quad g_1(\xi, u)u \leq C|u|^2, (\xi, u) \in O \times \mathbb{R},$$

for $2 \leq \gamma_1 < 2 + \frac{4}{d}$, and for some $\kappa_0 > 0$ and $q \in L^2(O)$

$$\begin{aligned}
(5.8) \quad & \|b_3(r, \xi, u)\|_{\mathbb{R}^d} + \|b_1(\xi, u)\|_{\mathbb{R}^d} + \|b_2(\xi, u)\|_{\mathbb{R}^d} + \|c(\xi, u)\|_{l^2} \\
& + |g_2(\xi, u)| + |g_3(r, \xi, u)| \\
& \leq \kappa_0 \cdot |u| + q(\xi), \quad \forall (r, \xi, u) \in [-h, 0] \times O \times \mathbb{R},
\end{aligned}$$

$$(5.9) \quad \sup_s \frac{1}{1 - r_0'(s)} |\partial_u a_1(\xi, u)| + h |\partial_u a_2(r, \xi, u)| \leq \frac{1}{4}, \quad \forall (r, \xi, u) \in [-h, 0] \times O \times \mathbb{R},$$

$$(5.10) \quad \|\partial_\xi a_1(\xi, u)\|_{\mathbb{R}^d} + \|\partial_\xi a_2(r, \xi, u)\|_{\mathbb{R}^d} \leq C(1 + |u|), \quad \forall (r, \xi, u) \in [-h, 0] \times O \times \mathbb{R}.$$

Moreover, $f_i, i = 1, 2$, and f_3 are bounded and Lipschitz continuous with respect to the second variable and the third variable respectively.

When $f_i, a_i, b_i = 0, i = 1, 2, 3$, then the above equation is the stochastic reaction-diffusion equation which has also attracted a lot of attention (see e.g. [DPZ92], [RZZ14a] and the references therein).

Below we choose

$$\mathbb{Y} = \mathbb{H} = L^2(O), \quad V = W_0^{1,2}(O),$$

and

$$\mathbb{X}^* = (W_0^{2+d,2}(O))^*, \quad \mathbb{X} = W_0^{2+d,2}(O).$$

We define the operators A and B as follows: for $x \in C_0^\infty(O)$

$$A_1(x) := \Delta x + \partial_i b_1^i(x) + g_1(x),$$

and for $u \in C([-h, 0]; C_0^\infty(O))$

$$\begin{aligned} A_2(t, u) := & \Delta[a_1(u(-r_0(t))) + \int_{-h}^0 a_2(r, u(r))dr] + [f_1(u(0)) + f_2(u(-r_1(t))) + \int_{-h}^0 f_3(r, u(r))dr] \\ & \cdot [\nabla u(0) + \nabla u(-r_2(t)) + \int_{-h}^0 \nabla u(r)dr] \\ & + \partial_i b_2^i(u(-r_3(t))) + \int_{-h}^0 \partial_i b_3^i(r, u(r))dr + g_2(u(-r_4(t))) + \int_{-h}^0 g_3(r, u(r))dr. \\ B(t, u) := & c(u(-r_5(t))). \end{aligned}$$

Let $e_n, n \in \mathbb{N}$, Π_n, \mathbb{H}_n be as in Section 2. We define the functional \mathcal{N}_1 on \mathbb{H} as follows:

$$\mathcal{N}_1(u) = \begin{cases} \|\nabla u\|_{L^2(O)}^2 & \text{if } u \in W_{0,\sigma}^{1,2}(O), \\ 0 & \text{otherwise,} \end{cases}$$

It is obvious that $\mathcal{N}_1 \in \mathfrak{U}^2$. In this case $\mathcal{N}(\psi) < \infty$ is equivalent to $\psi \in L^2(-h, 0; V)$.

Then we have for $x, y \in C_0^\infty(O)$, $y_n \rightarrow y$ in \mathbb{H} ,

$$(5.11) \quad \|\Delta x - \Delta y\|_{\mathbb{X}^*} \leq C\|x - y\|_{\mathbb{H}},$$

$$\begin{aligned} |\mathbb{X}^* \langle f_1(x) \cdot \nabla x - f_1(y) \cdot \nabla y, \varphi \rangle_{\mathbb{X}}| & \leq C |\mathbb{X}^* \langle f_1(x) \cdot (\nabla x - \nabla y) + (f_1(x) - f_1(y)) \cdot \nabla y, \varphi \rangle_{\mathbb{X}}| \\ & \leq C \|x - y\|_{\mathbb{H}} [\|\nabla x\|_{\mathbb{H}} + \|\nabla y\|_{\mathbb{H}} + 1] \|\varphi\|_{\mathbb{X}}, \end{aligned}$$

$$(5.12) \quad |\mathbb{X}^* \langle \partial_i b_1^i(y_n) - \partial_i b_1^i(y), \varphi \rangle_{\mathbb{X}}| \leq C \|\varphi\|_{C^1(O)} \int |b_1(\xi, y_n(\xi)) - b_1(\xi, y(\xi))| d\xi \rightarrow 0.$$

Moreover, if $y_n \rightarrow y$ in \mathbb{H} and weakly in V , we obtain

$$|\mathbb{X}^* \langle g_1(y_n) - g_1(y), \varphi \rangle_{\mathbb{X}}| \leq C \|\varphi\|_{C(O)} \int |g_1(\xi, y_n(\xi)) - g_1(\xi, y(\xi))| d\xi \rightarrow 0,$$

where we used (5.7) to deduce that

$$\|g_1(\cdot, y_n(\cdot))\|_{L^{1+\varepsilon}(O)}^{1+\varepsilon} \leq C[1 + \|y_n\|_{L^{(1+\varepsilon)\gamma_1}}^{(1+\varepsilon)\gamma_1}] \leq C[1 + \|y_n\|_V^{d(\frac{(1+\varepsilon)\gamma_1}{2}-1)} \|y_n\|_{\mathbb{H}}^{(1+\varepsilon)\gamma_1 - d(\frac{(1+\varepsilon)\gamma_1}{2}-1)}],$$

where $d(\frac{(1+\varepsilon)\gamma_1}{2}-1) < 2$. By (5.7)-(5.10), we can extend the operators A_1 and A_2, B to V and $\mathfrak{C} \cap L_V^2$ such that for $x \in V, u \in \mathfrak{C} \cap L_V^2$, $A_1(x) \in \mathbb{X}^*$ and $A_2(t, u) \in \mathbb{X}^*$, $B(t, u) \in L_2(l_2, \mathbb{H})$. Thus, we can write the system in the following abstract form

$$du(t) = [A_1(u(t)) + A_2(t, u_t)]dt + B(t, u_t)dW(t), \quad u(t) = \psi(t), -h \leq t \leq 0.$$

We can easily deduce that $\|\Pi_n A_1(\cdot)\|_{\mathbb{X}^*}$ is locally bounded on \mathbb{H}_n and $\|\Pi_n A_2(\cdot, \cdot)\|_{\mathbb{X}^*}, \|\Pi_n B(\cdot, \cdot)\|_{L_2(l_2; \mathbb{H})}$ are bounded on balls in $\mathbb{R}^+ \times C([-h, 0]; \mathbb{H}_n)$. Furthermore, we easily deduce that for every $t > 0$, $\Pi_n A_1(t, \cdot)$ is continuous on \mathbb{H}_n , $\Pi_n A_2(t, \cdot), \Pi_n B(t, \cdot)$ are continuous on $C([-h, 0]; \mathbb{H}_n)$.

Theorem 5.4. *For every $\psi \in \mathfrak{C} \cap L_V^2$ (5.6) has a weak solution $X : [-h, \infty) \times \Omega \rightarrow \mathbb{H}$ such that for every $q \in \mathbb{N}, T > 0$*

$$\mathbb{E}^P \left(\sup_{t \in [-h, T]} \|X(t)\|_{\mathbb{H}}^{2q} + \left[\int_0^T \mathcal{N}_1(X(t))dt \right]^q \right) < \infty.$$

Proof. Now we check (H1): Let u_n converge to u in $L_{\text{loc}}^2([-h, \infty); \mathbb{H})$ and weakly in $L_{\text{loc}}^2([-h, \infty); V)$. We prove the convergence required in (H1) for each term. By (5.9) we have for $\varphi \in \mathbb{X}$,

$$\begin{aligned} & \left| \int_0^t \mathbb{X}^* \langle \Delta[a_1(u_n(s - r_0(s))) - a_1(u(s - r_0(s)))] , \varphi \rangle_{\mathbb{X}} ds \right| \\ & \leq C \left[\int_0^t \|u_n(s - r_0(s)) - u(s - r_0(s))\|_{\mathbb{H}}^2 ds \right]^{\frac{1}{2}} \\ & \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

The convergence for a_2 can be obtained similarly. Since f_2 is bounded and Lipschitz continuous with respect to the second variable, we have

$$\begin{aligned} & \left| \int_0^t \mathbb{X}^* \langle f_2(u_n(s - r_1(s))) \cdot \nabla u_n(s) - f_2(u(s - r_1(s))) \cdot \nabla u(s), \varphi \rangle_{\mathbb{X}} ds \right| \\ & \leq C \left[\int_0^t \|\nabla u_n(s)\|_{\mathbb{H}}^2 ds \right]^{\frac{1}{2}} \left[\int_0^t \|u_n(s - r_1(s)) - u(s - r_1(s))\|_{\mathbb{H}}^2 ds \right]^{\frac{1}{2}} \\ & \quad + \int_0^t \mathbb{X}^* \langle f_2(u(s - r_1(s))) [\nabla u_n(s) - \nabla u(s)], \varphi \rangle_{\mathbb{X}} ds \\ & \leq C \left[\int_0^t \|\nabla u_n(s)\|_{\mathbb{H}}^2 ds \right]^{\frac{1}{2}} \left[\int_{-h}^t \|u_n(s) - u(s)\|_{\mathbb{H}}^2 ds \right]^{\frac{1}{2}} \\ & \quad + \int_0^t \mathbb{X}^* \langle f_2(u(s - r_1(s))) [\nabla u_n(s) - \nabla u(s)], \varphi \rangle_{\mathbb{X}} ds \\ & \rightarrow 0, \quad n \rightarrow \infty, \end{aligned}$$

where we used that u_n converge to u weakly in $L^2_{\text{loc}}([-h, \infty); V)$ to deduce $\int_0^T \|u_n(s)\|_V^2 ds \leq M$ in the last convergence. The convergence for $f_1(u(s)) \cdot \nabla u(s)$, $\int_{-h}^0 f_3(r, u(s+r)) dr \cdot \nabla u(s)$ can be obtained similarly. Moreover, we have

$$\begin{aligned}
& \left| \int_0^t \mathbb{X}^* \langle f_2(u_n(s - r_1(s))) \cdot \nabla u_n(s - r_2(s)) - f_2(u(s - r_1(s))) \cdot \nabla u(s - r_2(s)), \varphi \rangle_{\mathbb{X}} ds \right| \\
& \leq C \left[\int_{-h}^t \|\nabla u_n(s)\|_{\mathbb{H}}^2 ds \right]^{\frac{1}{2}} \left[\int_0^t \|u_n(s - r_1(s)) - u(s - r_1(s))\|_{\mathbb{H}}^2 ds \right]^{\frac{1}{2}} \\
& \quad + \int_0^t \mathbb{X}^* \langle f_2(u(s - r_1(s))) \cdot [\nabla u_n(s - r_2(s)) - \nabla u(s - r_2(s))], \varphi \rangle_{\mathbb{X}} ds \\
& \rightarrow 0, \quad n \rightarrow \infty,
\end{aligned}$$

where we used that u_n converge to u weakly in $L^2_{\text{loc}}([-h, \infty); V)$ in the last convergence. Furthermore, we get that

$$\begin{aligned}
& \left| \int_0^t \int_{-h}^0 \mathbb{X}^* \langle f_2(u_n(s - r_1(s))) \cdot \nabla u_n(s + r) - f_2(u(s - r_1(s))) \cdot \nabla u(s + r), \varphi \rangle_{\mathbb{X}} dr ds \right| \\
& \leq Ch \left[\int_{-h}^t \|\nabla u_n(s)\|_{\mathbb{H}}^2 ds \right]^{\frac{1}{2}} \left[\int_0^t \|u_n(s - r_1(s)) - u(s - r_1(s))\|_{\mathbb{H}}^2 ds \right]^{\frac{1}{2}} \\
& \quad + \int_0^t \int_{-h}^0 \mathbb{X}^* \langle f_2(u(s - r_1(s))) [\nabla u_n(s + r) - \nabla u(s + r)], \varphi \rangle_{\mathbb{X}} dr ds \\
& \leq Ch \left[\int_{-h}^t \|\nabla u_n(s)\|_{\mathbb{H}}^2 ds \right]^{\frac{1}{2}} \left[\int_0^t \|u_n(s - r_1(s)) - u(s - r_1(s))\|_{\mathbb{H}}^2 ds \right]^{\frac{1}{2}} \\
& \quad + \int_{-h}^0 \int_0^t \mathbb{X}^* \langle f_2(u(s - r_1(s))) [\nabla u_n(s + r) - \nabla u(s + r)], \varphi \rangle_{\mathbb{X}} ds dr \\
& \rightarrow 0, \quad n \rightarrow \infty,
\end{aligned}$$

where we used u_n converge to u weakly in $L^2_{\text{loc}}([-h, \infty); V)$ and the Lebesgue's dominated convergence theorem in the last convergence. For the other terms in $[f_1(u(t)) + f_2(u(t - r_1(t))) + \int_{-h}^0 f_3(r, u(t+r)) dr] \cdot [\nabla u(t - r_2(t)) + \int_{-h}^0 \nabla u(t+r) dr]$ we have similar estimates and obtain similar convergence required in (H1). Now for b_2 , by (5.8) we have

$$\begin{aligned}
& \left| \int_0^t \mathbb{X}^* \langle \partial_i b_2^i(u_n(s - r_3(s))) - \partial_i b_2^i(u(s - r_3(s))), \varphi \rangle_{\mathbb{X}} ds \right| \\
& \leq C \|\varphi\|_{C^1(O)} \int_0^t \|b_2(u_n(s - r_3(s))) - b_2(u(s - r_3(s)))\|_{L^1(O)} ds \\
& \rightarrow 0, \quad n \rightarrow \infty,
\end{aligned}$$

where the last convergence follows from the Lebesgue's dominated convergence theorem. The convergence for b_1, b_3 can be obtained similarly. For cut-off function χ_R as in (H1), we

also have

$$\begin{aligned}
& \int_0^t \mathbb{X}^* \int |g_1(u_n(s))\chi_R(\|u_n(s)\|_{\mathbb{H}}) - g_1(u(s))\chi_R(\|u(s)\|_{\mathbb{H}})|^{1+\varepsilon} |\varphi|^{1+\varepsilon} d\xi ds \\
& \leq C \|\varphi\|_{C(O)}^{1+\varepsilon} \int_0^t \|g_1(u_n(s))\chi_R(\|u_n(s)\|_{\mathbb{H}}) - g_1(u(s))\chi_R(\|u(s)\|_{\mathbb{H}})\|_{L^{1+\varepsilon}(O)}^{1+\varepsilon} ds \\
& \leq C \|\varphi\|_{C(O)}^{1+\varepsilon} \int_0^t [1 + \|u_n(s)\|_{L^{\gamma_1(1+\varepsilon)}(\mathbb{H})}^{\gamma_1(1+\varepsilon)} \chi_R(\|u_n(s)\|_{\mathbb{H}}) + \|u(s)\|_{L^{\gamma_1(1+\varepsilon)}(\mathbb{H})}^{\gamma_1(1+\varepsilon)} \chi_R(\|u(s)\|_{\mathbb{H}})] ds \\
& \leq C \int_0^t [1 + \|u_n(s)\|_V^{d(\frac{\gamma_1(1+\varepsilon)}{2}-1)} \chi_R(\|u_n(s)\|_{\mathbb{H}}) \|u_n(s)\|_{\mathbb{H}}^{\gamma_1(1+\varepsilon)-d(\frac{\gamma_1(1+\varepsilon)}{2}-1)} \\
& \quad + \|u(s)\|_V^{d(\frac{\gamma_1(1+\varepsilon)}{2}-1)} \chi_R(\|u(s)\|_{\mathbb{H}}) \|u(s)\|_{\mathbb{H}}^{\gamma_1(1+\varepsilon)-d(\frac{\gamma_1(1+\varepsilon)}{2}-1)}] ds \\
& \leq C \int_0^t [1 + \|u_n(s)\|_V^{d(\frac{\gamma_1(1+\varepsilon)}{2}-1)} + \|u(s)\|_V^{d(\frac{\gamma_1(1+\varepsilon)}{2}-1)}] ds.
\end{aligned}$$

Since $d(\frac{\gamma_1(1+\varepsilon)}{2} - 1) \leq 2$ we obtain that

$$\int_0^t \mathbb{X}^* \langle g_1(u_n(s))\chi_R(\|u_n(s)\|_{\mathbb{H}}) - g_1(u(s))\chi_R(\|u(s)\|_{\mathbb{H}}), \varphi \rangle_{\mathbb{X}} ds \rightarrow 0, n \rightarrow \infty.$$

For g_2 we have

$$\begin{aligned}
& \int_0^t \mathbb{X}^* \langle g_2(u_n(s - r_4(s))) - g_2(u(s - r_4(s))), \varphi \rangle_{\mathbb{X}} ds \\
& \leq C \|\varphi\|_{C(O)} \int_0^t \|g_2(u_n(s - r_4(s))) - g_2(u(s - r_4(s)))\|_{L^1(O)} ds \\
& \rightarrow 0, n \rightarrow \infty,
\end{aligned}$$

where the last convergence follows from the Lebesgue's dominated convergence theorem. The convergence for $\int_{-h}^0 g_3(r, u(t+r)) dr$ can be obtained similarly. By Remark 2.2 (i), (5.11) and (5.12), we obtain

$$\begin{aligned}
& \int_0^t \mathbb{X}^* \langle \Delta u_n(s)\chi_R(\|u_n(s)\|_{\mathbb{H}}) - \Delta u(s)\chi_R(\|u(s)\|_{\mathbb{H}}), \varphi \rangle_{\mathbb{X}} ds \rightarrow 0, n \rightarrow \infty. \\
& \int_0^t \mathbb{X}^* \langle \partial_i b_1^i(u_n(s))\chi_R(\|u_n(s)\|_{\mathbb{H}}) - \partial_i b_1^i(u(s))\chi_R(\|u(s)\|_{\mathbb{H}}), \varphi \rangle_{\mathbb{X}} ds \rightarrow 0, n \rightarrow \infty.
\end{aligned}$$

Then combining the estimates above, (2.3) and (2.4) in (H1) follow. By (5.8) we obtain that if u_n converge to u in $L_{loc}^2([-h, \infty); \mathbb{H})$, then

$$\lim_{n \rightarrow \infty} \int_0^t \|B^*(s, u_s^n)(v) - B^*(s, u_s)(v)\|_{t_2} ds = 0.$$

Now (H1) follows.

For (H2); by (5.7)-(5.10) we have for $u \in C([-h, \infty), \mathbb{X})$ and $T > 0$

$$\begin{aligned}
& \int_0^T \mathbb{X}^* \langle A_1(u(t)) + A_2(u_t), u(t) \rangle_{\mathbb{X}} dt \\
&= \int_0^T [-\mathcal{N}_1(u(t)) + \mathbb{X}^* \langle \Delta[a_1(u(t - r_0(t))) + \int_{-h}^0 a_2(u(t+r))dr], u(t) \rangle_{\mathbb{X}} + \langle (f_1(u(t)) + f_2(u(t - r_1(t))) \\
&\quad + \int_{-h}^0 f_3(r, u(t+r))dr) \cdot (\nabla u(t) + \nabla u(t - r_2(t)) + \int_{-h}^0 \nabla u(t+r)dr), u(t) \rangle \\
&\quad - \langle b_1^i(u(t)) + b_2^i(u(t - r_3(t))) + \int_{-h}^0 b_3^i(r, u(t+r))dr, \partial_i u(t) \rangle \\
&\quad + \langle g_1(u(t)) + g_2(u(t - r_4(t))) + \int_{-h}^0 g_3(r, u(t+r))dr, u(t) \rangle] dt \\
&\leq \int_0^T [-\frac{7}{8}\mathcal{N}_1(u(t)) + C(1 + \|u_t\|_{\mathfrak{E}}^2)] dt + \int_{-h}^0 \mathcal{N}_1(u(t)) dt \\
&\quad + \int_0^T \int [|\partial_u a_1(u(t - r_0(t)))| |\nabla u(t - r_0(t))| + |\partial_\xi a_1(u(t - r_0(t)))|] |\nabla u(t)| \\
&\quad + \int_{-h}^0 [|\partial_u a_2(u(t+r))| |\nabla u(t+r)| + |\partial_\xi a_2(u(t+r))|] |\nabla u(t)| dr d\xi dt \\
&\leq \int_0^T [-\frac{3}{4}\mathcal{N}_1(u(t)) + C(1 + \|u_t\|_{\mathfrak{E}}^2)] dt + \int_{-h}^0 \mathcal{N}_1(u(t)) dt + \left(\int_0^T \int |\nabla u(t)|^2 d\xi dt \right)^{1/2} \\
&\quad \left[\left(\int_0^T \int [|\partial_u a_1(u(t - r_0(t)))|^2 |\nabla u(t - r_0(t))|^2] d\xi dt \right)^{1/2} \right. \\
&\quad \left. + h^{1/2} \left(\int_0^T \int \int_{-h}^0 |\partial_u a_2(u(t+r))|^2 |\nabla u(t+r)|^2 dr d\xi dt \right)^{1/2} \right] \\
&\leq \int_0^T [-\frac{1}{2}\mathcal{N}_1(u(t)) + C(1 + \|u_t\|_{\mathfrak{E}}^2)] dt + 2 \int_{-h}^0 \mathcal{N}_1(u(t)) dt.
\end{aligned}$$

For (H3); Similar as the computation above we have $u \in L_{loc}^2([-h, \infty); V) \cap L_{loc}^\infty([-h, \infty); \mathbb{H}) \cap C([-h, \infty); \mathbb{X}^*)$ and every $t > 0$

$$\|A_1(u(t))\|_{\mathbb{X}^*} \leq C(1 + \|u_t\|_{\mathfrak{E}} + \|u(t)\|_V^{d(\frac{\gamma_1}{2}-1)} \|u(t)\|_{\mathbb{H}}^{(\gamma_1-d(\frac{\gamma_1}{2}-1))}),$$

$$\begin{aligned}
\|A_2(t, u_t)\|_{\mathbb{X}^*} &\leq C[1 + \|u(t - r_0(t))\|_{\mathbb{H}} + \|u_t\|_{\mathfrak{E}} + \|\nabla u(t)\|_{\mathbb{H}} \\
&\quad + \|\nabla u(t - r_2(t))\|_{\mathbb{H}} + \int_{-h}^0 \|\nabla u(t+r)\|_{\mathbb{H}} dr],
\end{aligned}$$

$$\|B(t, u_t)\|_{L^2(I^2; \mathbb{H})}^2 \leq C(1 + \|u_t\|_{\mathfrak{E}}^2),$$

where C is a constant changing from line to line. Now for $1 < \gamma < 2$ with $d\gamma(\frac{\gamma_1}{2} - 1) < 2$ we

have for every $T > 0$

$$\begin{aligned} & \int_0^T [\|A_1(u(t))\|_{\mathbb{X}^*}^\gamma + \|A_2(t, u_t)\|_{\mathbb{X}^*}^\gamma] dt \\ & \leq \int_0^T C(1 + \|u(t)\|_V^{d\gamma(\frac{\gamma_1}{2}-1)}) \|u(t)\|_{\mathbb{H}}^{\gamma(\gamma_1-d(\frac{\gamma_1}{2}-1))} + \|u_t\|_{\mathfrak{C}}^2 + \|u(t)\|_V^2 dt + \int_{-h}^0 C\|u(t)\|_V^2 dt, \\ & \|B(t, u_t)\|_{L_2(l^2; \mathbb{H})}^2 \leq C(1 + \|u_t\|_{\mathfrak{C}}^2). \end{aligned}$$

Now (H3) follows. Then the results follow from Theorem 2.2. \square

Now we assume that

$$(5.13) \quad \begin{aligned} & \sum_{i=1}^2 |g_i(\xi, u_1) - g_i(\xi, u_2)| + |g_3(r, \xi, u_1) - g_3(r, \xi, u_2)| \\ & \leq C[|u_1|^{\gamma_3} + |u_2|^{\gamma_3} + 1]|u_1 - u_2|, \forall(r, \xi, u_j) \in [-h, 0] \times O \times \mathbb{R}, j = 1, 2, \end{aligned}$$

for $1 \leq \gamma_3 \leq \frac{4}{d}$.

$$\|c(\xi, u_1) - c(\xi, u_2)\|_{l^2} \leq C|u_1 - u_2|.$$

$\partial_u a_1, \partial_u a_2$ do not depend on u .

If $d \leq 2$, suppose that

$$(5.14) \quad \begin{aligned} & \sum_{i=1}^2 \|b_i(\xi, u_1) - b_i(\xi, u_2)\|_{\mathbb{R}^d} + \|b_3(r, \xi, u_1) - b_3(r, \xi, u_2)\|_{\mathbb{R}^d} \\ & \leq C(1 + |u_1| + |u_2|)|u_1 - u_2|, \forall(r, \xi, u_j) \in [-h, 0] \times O \times \mathbb{R}, j = 1, 2, \end{aligned}$$

$$(5.15) \quad \begin{aligned} & \|\partial_\xi a_1(\xi, u_1) - \partial_\xi a_1(\xi, u_2)\|_{\mathbb{R}^d} + \|\partial_\xi a_2(r, \xi, u_1) - \partial_\xi a_2(r, \xi, u_2)\|_{\mathbb{R}^d} \\ & \leq C(1 + |u_1| + |u_2|)|u_1 - u_2|, \forall(r, \xi, u_j) \in [-h, 0] \times O \times \mathbb{R}, j = 1, 2, \end{aligned}$$

If $d \geq 3$, f_1, f_2, f_3 do not depend on u , $b_1, b_2, \partial_\xi a_1$ and $b_3, \partial_\xi a_2$ are Lipschitz continuous with respect to the second variable and the third variable, respectively.

Choose $H_0 = \mathbb{H}$.

If $\gamma_1 \leq (1 + \frac{4}{d}) \wedge 2(1 + \frac{1}{d}) - \varepsilon$ for some $\varepsilon > 0$, we have that for $u \in L_V^2 \cap \mathfrak{C}$,

$$\|g_1(u)\|_{V^*} \leq \|g_1(u)\|_{L^q} \leq C(1 + \|u\|_V^{d(\frac{\gamma_1}{2}-\frac{1}{q})}) \|u\|_{\mathbb{H}}^{\gamma_1-d(\frac{\gamma_1}{2}-\frac{1}{q})},$$

where $q = \frac{2d}{d+2} \vee (1 + \varepsilon_0)$ for some ε_0 and we used Lemma 5.1 in the first inequality. From this we see that the following mappings

$$A_1 : V \rightarrow V^*, \quad A_2 : \mathbb{R}^+ \times (L_V^2 \cap \mathfrak{C}) \rightarrow V^*$$

are well defined. For $u \in \mathbb{S}_1$, we easily deduce that $A_1(\cdot, u(\cdot)), A_2(\cdot, u) \in L_{\text{loc}}^1([0, \infty); V^*), B(\cdot, u) \in L_{\text{loc}}^2([0, \infty); L_2(l^2; H_0))$.

Theorem 5.5. *Suppose that all the assumptions above are satisfied with $\gamma_1 \leq (1 + \frac{4}{d}) \wedge 2(1 + \frac{1}{d}) - \varepsilon$ for some $\varepsilon > 0$. For every $\psi \in L^2(-h, 0; V) \cap C([-h, 0]; H_0)$, (5.6) has a unique (probabilistically) strong solution $X \in C([-h, \infty); H_0)$ satisfying for every $T > 0, q \in \mathbb{N}$*

$$\mathbb{E}^P \left(\sup_{t \in [-h, T]} \|X(t)\|_{\mathbb{H}}^{2q} + \left[\int_0^T \mathcal{N}_1(X(t)) dt \right]^q \right) < \infty.$$

Proof. Now we check (H4):

When $d \geq 3$ for f_1, f_2, f_3 it is obvious that (H4) holds. Now we only check the case when $d \leq 2$: for $u, v \in L^2_{\text{loc}}([-h, \infty); V) \cap C([-h, \infty); \mathbb{H})$, $T > 0$, we have

$$\begin{aligned} & \int_0^T \langle V^* \langle f_2(u(s - r_1(s))) \cdot \nabla u(s - r_2(s)) - f_2(v(s - r_1(s))) \cdot \nabla v(s - r_2(s)), u(s) - v(s) \rangle_V \rangle_V ds \\ & \leq C \int_0^T \|\nabla u(s - r_2(s))\|_{\mathbb{H}} \|u(s - r_1(s)) - v(s - r_1(s))\|_{L^4} \|u(s) - v(s)\|_{L^4} ds \\ & \quad + \varepsilon \int_0^T \|\nabla u(s - r_2(s)) - \nabla v(s - r_2(s))\|_{\mathbb{H}}^2 ds + C \int_0^T \|u(s) - v(s)\|_{\mathbb{H}}^2 ds \\ & \leq C \int_0^T \|\nabla u(s - r_2(s))\|_{\mathbb{H}} \|u(s - r_1(s)) - v(s - r_1(s))\|_V^{1/2} \|u(s - r_1(s)) - v(s - r_1(s))\|_{\mathbb{H}}^{1/2} \\ & \quad \|u(s) - v(s)\|_V^{1/2} \|u(s) - v(s)\|_{\mathbb{H}}^{1/2} ds + \varepsilon \int_{-h}^T \|\nabla u(s) - \nabla v(s)\|_{\mathbb{H}}^2 ds + C \int_0^T \|u(s) - v(s)\|_{\mathbb{H}}^2 ds \\ & \leq C \int_0^T \|\nabla u(s - r_2(s))\|_{\mathbb{H}}^2 \|u_s - v_s\|_{\mathfrak{C}}^2 ds + \varepsilon \left[\int_0^T \|u(s) - v(s)\|_V^2 ds \right]^{1/2} \\ & \quad \left[\int_0^T \|u(s - r_1(s)) - v(s - r_1(s))\|_V^2 ds \right]^{1/2} + \varepsilon \int_{-h}^T \|\nabla u(s) - \nabla v(s)\|_{\mathbb{H}}^2 ds + C \int_0^T \|u(s) - v(s)\|_{\mathbb{H}}^2 ds \\ & \leq C \int_0^T \|\nabla u(s - r_2(s))\|_{\mathbb{H}}^2 \|u_s - v_s\|_{\mathfrak{C}}^2 + \varepsilon \int_{-h}^T \|\nabla u(s) - \nabla v(s)\|_{\mathbb{H}}^2 ds + C \int_0^T \|u(s) - v(s)\|_{\mathbb{H}}^2 ds. \end{aligned}$$

We also have for $u, v \in L^2_{\text{loc}}([-h, \infty); V) \cap C([-h, \infty); \mathbb{H})$, $T > 0$,

$$\begin{aligned} & \int_0^T \langle V^* \langle f_2(u(s - r_1(s))) \cdot \int_{-h}^0 \nabla u(s + r) dr - f_2(v(s - r_1(s))) \cdot \int_{-h}^0 \nabla v(s + r) dr, u(s) - v(s) \rangle_V \rangle_V ds \\ & \leq C \int_0^T \int_{-h}^0 \|\nabla u(s + r)\|_{\mathbb{H}} dr \|u(s - r_1(s)) - v(s - r_1(s))\|_{L^4} \|u(s) - v(s)\|_{L^4} ds \\ & \quad + \varepsilon \int_0^T \int_{-h}^0 \|\nabla u(s + r) - \nabla v(s + r)\|_{\mathbb{H}}^2 dr ds + C \int_0^T \|u(s) - v(s)\|_{\mathbb{H}}^2 ds \\ & \leq C \int_0^T \int_{-h}^0 \|\nabla u(s + r)\|_{\mathbb{H}}^2 dr \|u_s - v_s\|_{\mathfrak{C}}^2 ds + \varepsilon \left[\int_0^T \|u(s) - v(s)\|_V^2 ds \right]^{1/2} \\ & \quad \left[\int_0^T \|u(s - r_1(s)) - v(s - r_1(s))\|_V^2 ds \right]^{1/2} + \varepsilon h \int_{-h}^T \|\nabla u(s) - \nabla v(s)\|_{\mathbb{H}}^2 ds + C \int_0^T \|u(s) - v(s)\|_{\mathbb{H}}^2 ds \\ & \leq C \int_0^T \int_{-h}^0 \|\nabla u(s + r)\|_{\mathbb{H}}^2 dr \|u_s - v_s\|_{\mathfrak{C}}^2 ds + \varepsilon \int_{-h}^T \|\nabla u(s) - \nabla v(s)\|_{\mathbb{H}}^2 ds + C \int_0^T \|u(s) - v(s)\|_{\mathbb{H}}^2 ds. \end{aligned}$$

We obtain similar bounds for the other terms including f_1, f_3 . Now when $d \geq 3$ we have for $u, v \in L^2_{\text{loc}}([-h, \infty); V) \cap C([-h, \infty); \mathbb{H})$, $s > 0$

$$\begin{aligned} & |_{V^*} \langle b_2^i(u(s - r_3(s))) - b_2^i(v(s - r_3(s))), \partial_i u(s) - \partial_i v(s) \rangle_V | \\ & \leq C \|u_s - v_s\|_{\mathfrak{E}}^2 + \varepsilon \|\nabla u(s) - \nabla v(s)\|_{\mathbb{H}}^2. \end{aligned}$$

We obtain similar bounds for b_3 .

When $d \leq 2$ we have for $u, v \in L^2_{\text{loc}}([-h, \infty); V) \cap C([-h, \infty); \mathbb{H})$, $T > 0$,

$$\begin{aligned} & \int_0^T |_{V^*} \langle b_2^i(u(s - r_3(s))) - b_2^i(v(s - r_3(s))), \partial_i u(s) - \partial_i v(s) \rangle_V | ds \\ & \leq C \int_0^T \|b_2(u(s - r_3(s))) - b_2(v(s - r_3(s)))\|_{\mathbb{H}}^2 ds + \varepsilon \int_0^T \|\nabla u(s) - \nabla v(s)\|_{\mathbb{H}}^2 ds \\ & \leq C \int_0^T [1 + \|u(s - r_3(s))\|_{L^4}^2 + \|v(s - r_3(s))\|_{L^4}^2] \|u(s - r_3(s)) - v(s - r_3(s))\|_{L^4}^2 ds \\ & \quad + \varepsilon \int_0^T \|\nabla u(s) - \nabla v(s)\|_{\mathbb{H}}^2 ds \\ & \leq C \int_0^T [1 + \|u(s - r_3(s))\|_V \|u(s - r_3(s))\|_{\mathbb{H}} + \|v(s - r_3(s))\|_V \|v(s - r_3(s))\|_{\mathbb{H}}] \\ & \quad \|u(s - r_3(s)) - v(s - r_3(s))\|_V \|u(s - r_3(s)) - v(s - r_3(s))\|_{\mathbb{H}} ds + \varepsilon \int_0^T \|\nabla u(s) - \nabla v(s)\|_{\mathbb{H}}^2 ds \\ & \leq C \int_0^T [1 + \|u(s - r_3(s))\|_V^2 \|u(s - r_3(s))\|_{\mathbb{H}}^2 + \|v(s - r_3(s))\|_V^2 \|v(s - r_3(s))\|_{\mathbb{H}}^2] \|u_s - v_s\|_{\mathfrak{E}}^2 ds \\ & \quad + \varepsilon \int_{-h}^T \|u(s) - v(s)\|_V^2 ds. \end{aligned}$$

The other terms including b_1, b_3 can be estimated similarly. For a_1 we have that for $u, v \in L^2_{\text{loc}}([-h, \infty); V) \cap C([-h, \infty); \mathbb{H})$, $T > 0$,

$$\begin{aligned} & \int_0^T |_{V^*} \langle \Delta(a_1(u(s - r_0(s))) - a_1(v(s - r_0(s))), u(s) - v(s) \rangle_V | ds \\ & \leq C \int_0^T \|\partial_\xi a_1(u(s - r_0(s))) - \partial_\xi a_1(v(s - r_0(s)))\|_{\mathbb{H}}^2 ds + \left(\varepsilon + \frac{1}{4}\right) \int_{-h}^T \|\nabla u(s) - \nabla v(s)\|_{\mathbb{H}}^2 ds. \end{aligned}$$

Then by the above estimates for b_2 , a_1 can be estimated similarly. We also have similar estimates for a_2 .

For g we have for $u, v \in L_{\text{loc}}^2([-h, \infty); V) \cap C([-h, \infty); \mathbb{H})$, $s > 0$

$$\begin{aligned}
& v^* \langle g_1(u(s)) - g_1(v(s)), u(s) - v(s) \rangle_V \\
& \leq C \int [|u(s)|^{\gamma_3} + |v(s)|^{\gamma_3} + 1] |u(s) - v(s)|^2 d\xi \\
& \leq C [1 + \|u\|_{L^{\gamma_3 p}}^{\gamma_3} + \|v\|_{L^{\gamma_3 p}}^{\gamma_3}] \|u - v\|_{L^{2q}}^2 \\
& \leq C [1 + \|u\|_{L^{\gamma_3 p}}^{\gamma_3} + \|v\|_{L^{\gamma_3 p}}^{\gamma_3}] \|u - v\|_V^{2\sigma} \|u - v\|_{\mathbb{H}}^{2(1-\sigma)} \\
& \leq C [1 + \|u\|_{L^{\gamma_3 p}}^{\gamma_3 \frac{1}{1-\sigma}} + \|v\|_{L^{\gamma_3 p}}^{\gamma_3 \frac{1}{1-\sigma}}] \|u - v\|_{\mathbb{H}}^2 + \varepsilon \|u - v\|_V^2 \\
& \leq C [1 + \|u\|_V^{\sigma_0 \gamma_3 \frac{1}{1-\sigma}} \|u\|_{\mathbb{H}}^{(1-\sigma_0) \gamma_3 \frac{1}{1-\sigma}} + \|v\|_V^{\sigma_0 \gamma_3 \frac{1}{1-\sigma}} \|v\|_{\mathbb{H}}^{(1-\sigma_0) \gamma_3 \frac{1}{1-\sigma}}] \|u - v\|_{\mathbb{H}}^2 + \varepsilon \|u - v\|_V^2,
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $\sigma = (\frac{1}{2} - \frac{1}{2q})d < 1$, $\sigma_0 = (\frac{1}{2} - \frac{1}{\gamma_3 p})d < 1$, $\frac{\sigma_0 \gamma_3}{1-\sigma} \leq 2$. Similarly we have for $u, v \in L_{\text{loc}}^2([-h, \infty); V) \cap C([-h, \infty); \mathbb{H})$, $T > 0$

$$\begin{aligned}
& \int_0^T v^* \langle g_2(u(s - r_4(s))) - g_2(v(s - r_4(s))), u(s) - v(s) \rangle_V \\
& \leq C \int_0^T \int [|u(s - r_4(s))|^{\gamma_3} + |v(s - r_4(s))|^{\gamma_3} + 1] |u(s - r_4(s)) - v(s - r_4(s))| |u(s) - v(s)| d\xi \\
& \leq C \int_0^T [1 + \|u(s - r_4(s))\|_{L^{\gamma_3 p}}^{\gamma_3} + \|v(s - r_4(s))\|_{L^{\gamma_3 p}}^{\gamma_3}] \|u(s - r_4(s)) - v(s - r_4(s))\|_{L^{2q}} \\
& \quad \|u(s) - v(s)\|_{L^{2q}} ds \\
& \leq C \int_0^T [1 + \|u(s - r_4(s))\|_{L^{\gamma_3 p}}^{\gamma_3} + \|v(s - r_4(s))\|_{L^{\gamma_3 p}}^{\gamma_3}] \|u(s - r_4(s)) - v(s - r_4(s))\|_V^\sigma \|u(s) - v(s)\|_V^\sigma \\
& \quad \|u(s - r_4(s)) - v(s - r_4(s))\|_{\mathbb{H}}^{1-\sigma} \|u(s) - v(s)\|_{\mathbb{H}}^{1-\sigma} ds \\
& \leq C \int_0^T [1 + \|u(s - r_4(s))\|_V^{\sigma_0 \gamma_3 \frac{1}{1-\sigma}} \|u(s - r_4(s))\|_{\mathbb{H}}^{(1-\sigma_0) \gamma_3 \frac{1}{1-\sigma}} \\
& \quad + \|v(s - r_4(s))\|_V^{\sigma_0 \gamma_3 \frac{1}{1-\sigma}} \|v(s - r_4(s))\|_{\mathbb{H}}^{(1-\sigma_0) \gamma_3 \frac{1}{1-\sigma}}] \|u_s - v_s\|_{\mathbb{C}}^2 ds \\
& \quad + \varepsilon \int_{-h}^T \|u(s) - v(s)\|_V^2 ds.
\end{aligned}$$

g_3 can be estimated similarly.

For (H5); for a we have for $u \in L_{\text{loc}}^2([-h, \infty); V) \cap C([-h, \infty); \mathbb{X}^*) \cap L_{\text{loc}}^\infty(-h, \infty; \mathbb{H})$, $T > 0$

$$\begin{aligned}
& \int_0^T \|\Delta a_1(u(t - r_0(t)))\|_{V^*}^2 dt \\
& \leq C \int_0^T [\|\partial_u a_1(u(t - r_0(t))) \nabla u(t - r_0(t))\|_{\mathbb{H}}^2 + \|\partial_\xi a_1(u(t - r_0(t)))\|_{\mathbb{H}}^2] dt \\
& \leq C \int_0^T [\|\nabla u(t - r_0(t))\|_{\mathbb{H}}^2 + \|u(t - r_0(t))\|_{\mathbb{H}}^2 + 1] dt \\
& \leq C \int_{-h}^T [\|\nabla u(t)\|_{\mathbb{H}}^2 + \|u(t)\|_{\mathbb{H}}^2 + 1] dt.
\end{aligned}$$

Since $\gamma_1 \leq (1 + \frac{4}{d}) \wedge 2(1 + \frac{1}{d}) - \varepsilon$, by Lemma 5.1 and the interpolation inequality, we have for $u \in L_{\text{loc}}^2([-h, \infty); V) \cap L_{\text{loc}}^\infty([-h, \infty); \mathbb{H}) \cap C([-h, \infty); \mathbb{X}^*)$ and every $T > 0$

$$\|A_1(u(t))\|_{V^*} \leq C(1 + \|u_t\|_{\mathfrak{E}} + \|u(t)\|_V^{d(\frac{\gamma_1}{2} - \frac{1}{q})} \|u(t)\|_{\mathbb{H}}^{(\gamma_1 - d(\frac{\gamma_1}{2} - \frac{1}{q}))} + \|\nabla u(t)\|_{\mathbb{H}}),$$

$$\begin{aligned} \|A_2(t, u_t)\|_{V^*} &\leq C[1 + \|u(t - r_0(t))\|_V + \|u_t\|_{\mathfrak{E}} + \|\nabla u(t)\|_{\mathbb{H}} \\ &\quad + \|\nabla u(t - r_2(t))\|_{\mathbb{H}} + \int_{-h}^0 \|\nabla u(t + r)\|_{\mathbb{H}} dr], \end{aligned}$$

$$\|B(t, u_t)\|_{L_2(l^2; \mathbb{H})}^2 \leq C(1 + \|u_t\|_{\mathfrak{E}}^2),$$

where $q = \frac{2d}{d+2} \vee (1 + \varepsilon)$, $W^{-1,2} \subset L^q$ continuously and C is a constant changing from line to line. Now we have for every $T > 0$

$$\begin{aligned} &\int_0^T [\|A_1(u(t))\|_{V^*}^2 + \|A_2(t, u_t)\|_{V^*}^2] dt \\ &\leq \int_0^T C(1 + \|u(t)\|_V^{2d(\frac{\gamma_1}{2} - \frac{1}{q})} \|u(t)\|_{\mathbb{H}}^{2(\gamma_1 - d(\frac{\gamma_1}{2} - \frac{1}{q}))} + \|u(t)\|_V^2 + \|u_t\|_{\mathfrak{E}}^2) dt + \int_{-h}^0 C\|u(t)\|_V^2 dt \\ &\quad + \int_0^T (\|\nabla u(t)\|_{\mathbb{H}}^2 + \|\nabla u(t - r_2(t))\|_{\mathbb{H}}^2 + \int_{-h}^0 \|\nabla u(t + r)\|_{\mathbb{H}}^2 dr) dt \\ &\leq [\int_0^T C(1 + \|u(t)\|_V^2) dt]^2 + \sup_{t \in [-h, T]} \|u(t)\|_{\mathbb{H}}^{4(\gamma_1 - d(\frac{\gamma_1}{2} - \frac{1}{q}))} + \int_{-h}^0 C\|u(t)\|_V^2 dt \\ &\quad + C \int_{-h}^T \|\nabla u(t)\|_{\mathbb{H}}^2 dt + C(1 + \sup_{t \in [-h, T]} \|u(t)\|_{\mathbb{H}}^2), \end{aligned}$$

and

$$\|B(t, u_t)\|_{L_2(l^2; \mathbb{H})}^2 \leq C(1 + \|u_t\|_{\mathfrak{E}}^2).$$

Now (H5) follows. Then the results follow from Theorem 4.2. \square

5.3 Stochastic generalized Porous Medium Equations

For $k \geq 0, p > 1$, the dual space of $W_0^{k,p}(O)$ is given by $W^{-k,p'}(O)$, where $p' = \frac{p}{p-1}$. The following Sobolev embeddings hold:

$$W_0^{k,p}(O) \subset C^m(\bar{O}), 0 \leq m < k - \frac{d}{p}.$$

By Poincaré's inequality we have for $x \in W_0^{1,2}(O)$

$$\int_O |x(\xi)|^2 d\xi \leq \rho_O \int_O |\nabla x(\xi)|^2 d\xi.$$

Consider the following quasi linear SPDE with Dirichlet boundary conditions:

(5.16)

$$\begin{aligned} dX(t) = & [\partial_{ij}^2 a^{ij}(\xi, X(t)) + \partial_i b^i(\xi, X(t)) + c(\xi, X(t))]dt \\ & + [\partial_i b_1^i(\xi, X(t - r_1(t)), X(t)) + \int_{-h}^0 \partial_i b_2^i(r, \xi, X(t+r), X(t))dr + c_1(\xi, X(t - r_2(t))) \\ & + \int_{-h}^0 c_2(r, \xi, X(t+r))dr]dt + \sigma_i(\xi, X(t - r_3(t)))dW^i(t), \end{aligned}$$

$$X(t, \xi) = 0, \quad (t, \xi) \in \mathbb{R}^+ \times \partial O,$$

$$X(t) = \psi(t), \quad -h \leq t \leq 0,$$

where $r_i, i = 1, 2, 3$, are continuous differentiable functions on \mathbb{R} satisfying $-h \leq r_i \leq 0, i = 1, 2, 3$ and $\sup r_i'(t) < 1, i = 1, 2$, a, b, c, c_1 and σ are functions from $O \times \mathbb{R}$ to $\mathbb{R}^{d^2}, \mathbb{R}^d, \mathbb{R}, \mathbb{R}$ and l^2 respectively, continuous with respect to the second variable; $b_1(\xi, u_1, u_2)$ is a function from $O \times \mathbb{R} \times \mathbb{R}$ to \mathbb{R}^d , continuous with respect to (u_1, u_2) ; $b_2(r, \xi, u_1, u_2)$ is a function from $[-h, 0] \times O \times \mathbb{R} \times \mathbb{R}$ to \mathbb{R}^d , continuous with respect to (u_1, u_2) ; $c_1(\xi, u_1)$ is a function from $O \times \mathbb{R}$ to \mathbb{R} , $c_2(r, \xi, u_1)$ is a function from $[-h, 0] \times O \times \mathbb{R}$ to \mathbb{R} , continuous with respect to the third variable, and satisfy for some fixed $q \geq 2$ and all $\xi \in O, u_1, u_2 \in \mathbb{R}, r \in [-h, 0]$:

$$\partial_u a^{ij}(\xi, u_1)x_i x_j \geq \kappa_{a,0}|u_1|^{q-2}|x|^2, \quad x \in \mathbb{R}^d,$$

$$\|a(\xi, u_1)\|_{\mathbb{R}^{d^2}} \leq \kappa_{a,1}(|u_1|^{q-1} + 1),$$

$$\|\partial_j a^{ij}(\xi, u_1)\|_{\mathbb{R}^d} + \|b(\xi, u_1)\|_{\mathbb{R}^d} \leq \kappa_{a,b}|u_1|^{q-1} + \kappa'_{a,b}|u_1|^{\frac{q}{2}},$$

$$|c(\xi, u_1)| \leq \kappa_{c,1}|u_1|^{q-1} + \kappa_{c,2}(|u_1| + 1),$$

$$\|b_1(\xi, u_1, u_2)\|_{\mathbb{R}^d} \leq C|u_1||u_2|^{\frac{q}{2}-1},$$

$$\|b_2(r, \xi, u_1, u_2)\|_{\mathbb{R}^d} \leq C|u_1||u_2|^{\frac{q}{2}-1},$$

$$|c_1(\xi, u_1)| \leq \kappa_{c,2}(|u_1| + 1),$$

$$|c_2(r, \xi, u_1)| \leq \kappa_{c,2}(|u_1| + 1),$$

$$\|\sigma(\xi, u_1)\|_{l^2} \leq \kappa_\sigma(|u_1| + 1),$$

where all κ with subscripts are strictly positive constants, and

$$\frac{\kappa_{a,b}}{2}\left(1 + \frac{q^2 \rho_O}{4}\right) + \frac{\kappa_{c,1} \cdot q^2 \rho_O}{4} \leq \frac{\kappa_{a,0}}{2}.$$

We choose

$$V = \mathbb{Y} := L^q(O), \quad \mathbb{H} := L^2(O),$$

and

$$\mathbb{X} := W_0^{d+2,2}(O), \quad \mathbb{X}^* := W^{-d-2,2}(O).$$

Let $e_n, n \in \mathbb{N}$, Π_n, \mathbb{H}_n be as in Section 2. Define the functional \mathcal{N}_1 on \mathbb{Y} as follows:

$$\mathcal{N}_1(y) := \begin{cases} \frac{\kappa_0}{q^2} \int |\nabla(|y(\xi)|^{\frac{q}{2}-1}y(\xi))|^2 d\xi & \text{if } |y|^{\frac{q}{2}-1}y \in W_0^{1,2}(O), \\ +\infty & \text{otherwise.} \end{cases}$$

Then by [GRZ09 Lemma 5.1] we know $\mathcal{N}_1 \in \mathfrak{U}^q$. In this case, if $\psi \in L^q(-h, 0; W_0^{1V, \frac{d+\varepsilon}{2}, 2})$ for some $\varepsilon > 0$, we have $\mathcal{N}_1(\psi) \leq C\|\psi\|_{W_0^{1V, \frac{d+\varepsilon}{2}, 2}}$, then $\mathcal{N}(\psi) < \infty$.

Define for $x \in V = L^q(O)$

$$A_1(x) := \partial_{ij}^2 a^{ij}(\cdot, x(\cdot)) + \partial_i b^i(\cdot, x(\cdot)) + c(\cdot, x(\cdot)) \in \mathbb{X}^*,$$

and for $u \in L_V^q \cap \mathfrak{C}$,

$$\begin{aligned} A_2(t, u) := & \partial_i b_1^i(\cdot, u(-r_1(t)), u(0)) + \int_{-h}^0 \partial_i b_2^i(r, \cdot, u(r), u(0)) dr + c_1(\cdot, u(-r_2(t))) \\ & + \int_{-h}^0 c_2(r, \cdot, u(r)) dr \in \mathbb{X}^*, \end{aligned}$$

$$B(t, u) := \sigma(\cdot, u(-r_3(t))) \in L_2(l^2; \mathbb{H}).$$

Thus, we can write the system in the following abstract form

$$dX(t) = [A_1(X(t)) + A_2(t, X_t)]dt + B(t, X_t)dW(t), \quad u(0) = u_0.$$

We can easily deduce that $\|\Pi_n A_1(\cdot)\|_{\mathbb{X}^*}$ is locally bounded on \mathbb{H}_n and $\|\Pi_n A_2(\cdot, \cdot)\|_{\mathbb{X}^*}$, $\|\Pi_n B(\cdot, \cdot)\|_{L_2(l_2; \mathbb{H})}$ are bounded on balls in $\mathbb{R}^+ \times C([-h, 0]; \mathbb{H}_n)$. Furthermore, we easily deduce that for every $t > 0$ $\Pi_n A_1(t, \cdot)$ is continuous on \mathbb{H}_n and $\Pi_n A_2(t, \cdot)$, $\Pi_n B(t, \cdot)$ are continuous on $C([-h, 0]; \mathbb{H}_n)$.

Theorem 5.6. *For every $\psi \in L^q(-h, 0; V) \cap \mathfrak{C}$ with $\mathcal{N}(\psi) < \infty$ (5.16) has a weak solution $X : [-h, \infty) \times \Omega \rightarrow \mathbb{H}$ such that for every $q_1 \in \mathbb{N}$, $T > 0$*

$$\mathbb{E}^P \left(\sup_{t \in [-h, T]} \|X(t)\|_{\mathbb{H}}^{2q_1} + \left[\int_0^T \mathcal{N}_1(X(t)) dt \right]^{q_1} \right) < \infty.$$

Proof. Now for (H1); let u_n converge to u in $L_{\text{loc}}^2([-h, \infty); \mathbb{H})$ and weakly in $L_{\text{loc}}^q([-h, \infty); V)$. We have for cut-off function χ_R and every $t > 0$

$$\begin{aligned} & \left| \int_0^t \mathbb{X}^* \langle \partial_{ij}^2 a(u_n(s)), \varphi \rangle_{\mathbb{X}} \chi_R(\|u_n(s)\|_{\mathbb{H}}) - \mathbb{X}^* \langle \partial_{ij}^2 a(u(s)), \varphi \rangle_{\mathbb{X}} \chi_R(\|u(s)\|_{\mathbb{H}}) ds \right| \\ = & \left| \int_0^t \int [a(u_n(s)) \chi_R(\|u_n(s)\|_{\mathbb{H}}) - a(u(s)) \chi_R(\|u(s)\|_{\mathbb{H}})] \partial_{ij}^2 \varphi d\xi ds \right|. \end{aligned}$$

Moreover, we obtain for all $n \in \mathbb{N}$

$$\begin{aligned} & \int_0^t \int |a(u_n(s)) \chi_R(\|u_n(s)\|_{\mathbb{H}}) - a(u(s)) \chi_R(\|u(s)\|_{\mathbb{H}})|^{\frac{q}{q-1}} d\xi ds \\ \leq & C \|\varphi\|_{C^2(O)} \int_0^t \int [|u_n(s)|^q + |u(s)|^q + 1] d\xi ds \leq M, \end{aligned}$$

which implies that for every $t > 0$

$$\int_0^t \mathbb{X}^* \langle \partial_{ij}^2 a(u_n(s)), \varphi \rangle_{\mathbb{X}\chi_R(\|u_n(s)\|_{\mathbb{H}})} - \mathbb{X}^* \langle \partial_{ij}^2 a(u(s)), \varphi \rangle_{\mathbb{X}\chi_R(\|u(s)\|_{\mathbb{H}})} ds \rightarrow 0.$$

Similar convergence also hold for the terms including b and c . Then (2.3) follows. Now for (2.4) we have for every $t > 0$

$$\begin{aligned} & \left| \int_0^t \mathbb{X}^* \langle \partial_i b_1^i(u_n(s - r_1(s)), u_n(s)) - \partial_i b_1^i(u(s - r_1(s)), u(s)), \varphi \rangle_{\mathbb{X}} ds \right| \\ & \leq C \|\varphi\|_{C^1(O)} \int_0^t \int |b_1(u_n(s - r_1(s)), u_n(s)) - b_1(u(s - r_1(s)), u(s))| d\xi ds \\ & \leq C \|\varphi\|_{C^1(O)} \int_0^t \int [|u_n(s - r_1(s))| |u_n(s)|^{q/2-1} + |u(s - r_1(s))| |u(s)|^{q/2-1}] d\xi ds. \end{aligned}$$

Now for $1 < \gamma < \frac{q}{q-1}$ we have for every $t > 0$

$$\begin{aligned} & \int_0^t \int [|u_n(s - r_1(s))|^\gamma |u_n(s)|^{(q/2-1)\gamma} + |u(s - r_1(s))|^\gamma |u(s)|^{(q/2-1)\gamma}] d\xi ds \\ & \leq \int_0^t \int [|u_n(s - r_1(s))|^2 + |u(s - r_1(s))|^2 + C[|u_n(s)|^q + |u(s)|^q + 1]] d\xi ds \\ & \leq C \int_{-h}^t [\|u_n(s)\|_{\mathbb{H}}^2 + \|u(s)\|_{\mathbb{H}}^2 + \|u_n(s)\|_V^q + \|u(s)\|_V^q + 1] ds. \end{aligned}$$

Then we obtain for every $t > 0$

$$\int_0^t \mathbb{X}^* \langle \partial_i b_1^i(u_n(s - r_1(s)), u_n(s)) - \partial_i b_1^i(u(s - r_1(s)), u(s)), \varphi \rangle_{\mathbb{X}} ds \rightarrow 0, n \rightarrow \infty.$$

The convergence for b_2 can be obtained similarly. Now for c_1 we have for every $t > 0$

$$\begin{aligned} & \int_0^t \mathbb{X}^* \langle c_1(u_n(s - r_2(s))) - c_1(u(s - r_2(s))), \varphi \rangle_{\mathbb{X}} ds \\ & = \int_0^t \int (c_1(u_n(s - r_2(s))) - c_1(u(s - r_2(s)))) \varphi d\xi ds \end{aligned}$$

Similarly, we obtain that

$$\begin{aligned} & \int_0^t \int |c_1(u_n(s - r_2(s))) - c_1(u(s - r_2(s)))| d\xi ds \\ & \leq C \|\varphi\|_{C(O)} \int_0^t \int [|u_n(s - r_2(s))| + |u(s - r_2(s))| + 1] d\xi ds. \end{aligned}$$

Similarly as above we obtain for every $t > 0$

$$\int_0^t \mathbb{X}^* \langle c_1(u_n(s - r_2(s))) - c_1(u(s - r_2(s))), \varphi \rangle_{\mathbb{X}} ds \rightarrow 0, n \rightarrow \infty.$$

We obtain the similar convergence results for c_2 . Then combining the estimates above, (2.4) in (H1) follows. By the assumptions for σ we obtain that if u_n converge to u in $L^2_{\text{loc}}([-h, \infty); \mathbb{H})$ then

$$\lim_{n \rightarrow \infty} \int_0^t \|B^*(s, u_s^n)(v) - B^*(s, u_s)(v)\|_{l_2} ds = 0.$$

So (H1) follows.

Now for (H2); by a similar calculation in [GRZ09, Section 5] we obtain that

$$\begin{aligned} & \int_0^T \mathbb{X}^* \langle A_1(u(t)) + A_2(u_t), u(t) \rangle_{\mathbb{X}} dt \\ & \leq \int_0^T [-\mathcal{N}_1(u(t)) + C(1 + \|u_t\|_{\mathfrak{E}}^2)] dt \\ & \quad - \int_0^T [\mathbb{X}^* \langle b_1^i(u(t - r_1(t)), u(t)) + \int_{-h}^0 b_2^i(r, u(t+r), u(t)) dr, \partial_i u(t) \rangle_{\mathbb{X}} \\ & \quad + \mathbb{X}^* \langle c_1(u(t - r_2(t))) + \int_{-h}^0 c_2(r, u(t+r)) dr, u(t) \rangle_{\mathbb{X}}] dt \\ & \leq \int_0^T [-\mathcal{N}_1(u(t)) + C(1 + \|u_t\|_{\mathfrak{E}}^2) \\ & \quad + C \int \left(|u(t - r_1(t))| |u(t)|^{\frac{q}{2}-1} |\nabla u(t)| + \int_{-h}^0 |u(t+r)| |u(t)|^{\frac{q}{2}-1} |\nabla u(t)| dr \right) d\xi \\ & \quad + \mathbb{X}^* \langle c_1(u(t - r_2(t))) + \int_{-h}^0 c_2(r, u(t+r)) dr, u(t) \rangle_{\mathbb{X}}] dt \\ & \leq \int_0^T \left[-\frac{1}{2} \mathcal{N}_1(u(t)) + C(1 + \|u_t\|_{\mathfrak{E}}^2) \right] dt + c \int_{-h}^0 \mathcal{N}_1(u(t)) dt. \end{aligned}$$

Now for (H3); for $u \in C([-h, \infty); \mathbb{X})$ we have

$$\|\partial_{ij}^2 a(u(s))\|_{\mathbb{X}^*} \leq C \int [|u(s)|^{q-1} + 1] d\xi \leq C(\mathcal{N}_1(u(s)) + 1).$$

Now for b we have

$$\begin{aligned} & \|\partial_i b_1^i(u(s - r_1(s)), u(s))\|_{\mathbb{X}^*} \leq C \|b_1(u(s - r_1(s)), u(s))\|_{L^1(O)} \\ & \leq C \int |u(s - r_1(s))| |u(s)|^{q/2-1} d\xi. \end{aligned}$$

Now for $1 < \gamma < \frac{q}{q-1}$ we have

$$\begin{aligned} & \int |u(s - r_1(s))|^\gamma |u(s)|^{(q/2-1)\gamma} d\xi \\ & \leq \int |u(s - r_1(s))|^2 + C[|u(s)|^q + 1] d\xi \\ & \leq C[\|u_s\|_{\mathfrak{E}}^2 + \mathcal{N}_1(u(s)) + 1]. \end{aligned}$$

Similarly, we obtain that

$$\left\| \int_{-h}^0 \partial_i b_2^i(u(s+r), u(s)) \right\|_{\mathbb{X}^*}^{\gamma} \leq C[\|u_s\|_{\mathfrak{C}}^2 + \mathcal{N}_1(u(s)) + 1].$$

Now for c_1, c_2 we also have

$$\begin{aligned} & \|c_1(u(s - r_2(s)))\|_{\mathbb{X}^*} + \left\| \int_{-h}^0 c_2(r, u(r)) dr \right\|_{\mathbb{X}^*} \\ & \leq C[\|u_s\|_{\mathfrak{C}} + 1]. \end{aligned}$$

$$\|B(t, u_t)\|_{L_2(l^2; \mathbb{H})}^2 \leq C(1 + \|u_t\|_{\mathfrak{C}}^2).$$

Then (H3) follows. Now the results follow from Theorem 2.2. \square

Now we prove the uniqueness. We consider the case where

$$(5.17) \quad \begin{aligned} dX(t) = & [\Delta\Phi(X(t)) + c(\xi)X(t) + c_1(\xi)X(t - r_2(t)) \\ & + \int_{-h}^0 c_2(r, \xi)X(t+r)dr]dt + \sigma_i(\xi)X(t - r_3(t))dW^i(t), \end{aligned}$$

$$X(t, \xi) = 0, \quad (t, \xi) \in \mathbb{R}^+ \times \partial O,$$

$$X(t) = \psi(t), \quad -h \leq t \leq 0,$$

where $c, c_1, c_2, \|\sigma\|_{l^2} \in W_0^{1,2}(O)$ and $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function satisfying $(\Phi(s) - \Phi(t))(s - t) \geq 0$, $\Psi(s)s \geq m_1|s|^q - m_2$, $|\Psi(s)| \leq m_3|s|^{q-1} + m_4$, $s, t \in \mathbb{R}$, $m_1, m_3, m_4 > 0$, $m_2 \geq 0$. In this case, we choose the following Gelfand triple:

$$V = L^q(O) \subset W^{-1,2}(O) =: H_0 \cong H_0^* =: W_0^{1,2}(O) \subset (L^q(O))^* = V^*.$$

From [PR07, Chapter 4], we only need to check (H4) for c : we have

$$\begin{aligned} & v^* \langle c_1(\xi)u(t - r_2) - c_1(\xi)v(t - r_2), u(t) - v(t) \rangle_V \\ & \leq C \|c_1(\xi)(u(t - r_2) - v(t - r_2))\|_{H_0} \|u(t) - v(t)\|_{H_0} \\ & \leq C \|u_t - v_t\|_{C([-h,0]; H_0)}^2. \end{aligned}$$

For c, c_2, σ we have similar estimates. Then we obtain:

Theorem 5.7. *For every $\psi \in L^q(-h, 0; V) \cap C([-h, 0]; H_0)$ with $\mathcal{N}(\psi) < \infty$, (5.17) has a unique (probabilistically) strong solution $X \in C([-h, \infty), H_0)$ satisfying for every $T > 0$, $q_1 \in \mathbb{N}$*

$$\mathbb{E}^P \left(\sup_{t \in [-h, T]} \|X(t)\|_{\mathbb{H}}^{2q_1} + \left[\int_0^T \mathcal{N}_1(X(t)) dt \right]^{q_1} \right) < \infty.$$

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