Elliptic PDEs with distributional drift and backward SDEs driven by a càdlàg martingale with random terminal time

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July 11th, 2014

We introduce a generalized notion of semilinear elliptic partial differential equations where the corresponding second order partial differential operator L has a generalized drift. We investigate existence and uniqueness of generalized solutions of class C^1. The generator L is associated with a Markov process X which is the solution of a stochastic differential equation with distributional drift. If the semilinear PDE admits boundary conditions, its solution is naturally associated with a backward stochastic differential equation (BSDE) with random terminal time, where the forward process is X. Since X is a weak solution of the forward SDE, the BSDE appears naturally to be driven by a martingale. In the paper we also discuss the uniqueness of a BSDE with random terminal time when the driving process is a general càdlàg martingale.

KEY WORDS AND PHRASES: Backward stochastic differential equations, random terminal time, martingale problem, distributional drift, elliptic partial differential equations.

MSC 2010: 60H10; 60H30; 35H99.

1 Introduction

The paper involves three essential areas of study.

1. Elliptic semilinear PDEs with distributional drift.

2. Backward stochastic differential equations (BSDEs) driven by càdlàg martingales with terminal condition at random terminal time.

3. The representation of solutions of the above mentioned BSDEs through solutions of PDEs.

We consider a one-dimensional semilinear PDE of the type

$$Lu = F\left(x, u, u'\right), \tag{1.1}$$

on $[0,1]$ with boundary conditions, where L is the generator of a one-dimensional stochastic differential equation of the type $Lg = \frac{\sigma^2}{2}g'' + \beta' g'$, with σ, β being real continuous functions and σ is strictly positive. So the drift β' is the derivative of a continuous function β, therefore a distribution. A typical example of such β is the path of a fixed continuous process. F is a continuous real function defined on $[0,1] \times \mathbb{R}^2$. When F does not depend on u and u', and x varies on the real line, (1.1) was introduced in [13, 14], via the notion of C^1-solutions which appear as limit of solutions of elliptic problems with regularized coefficients. Indeed [13, 14] investigated the case of initial conditions.

One-dimensional stochastic differential equations with distributional drift were examined by several authors, see [13, 14, 2, 19] and references therein, with a recent contribution by [17]. Such an equation appears formally as

$$dX_t = \beta'(X_t)dt + \sigma(X_t)dW_t. \tag{1.2}$$

If β is the path of a two-sided Brownian motion and $\sigma = 1$, the solution X models a dynamical system in a random irregular medium context.

More recently some contributions also appeared in the multidimensional case, see [1], when the drift is a Kato class measure and in [12] for other type of time dependent drifts.

This paper is devoted to the following main objectives.

1. We study existence and uniqueness of a solution u of the semilinear equation (1.1) with prescribed initial conditions for $u(0)$ and $u'(0)$, see Proposition 3.6.

2. We show that the initial value problem allows to provide a solution to the boundary value problem on $[0,1]$ for (1.1), see Proposition 3.12.

3. We explore several assumptions on F which provide existence and/or uniqueness of the boundary value problem, see Corollary 3.11 and Propositions 3.12 and 3.14.

4. We study the uniqueness of solutions of BSDEs driven by a càdlàg martingale M such that $\langle M \rangle$ is continuous, see Theorem 5.3.

5. We show that a solution of the PDE (1.1) with Dirichlet boundary conditions on $[0,1]$ generates a solution to a special forward BSDE (see Theorem 6.2) with terminal condition at the random time τ, where τ is the exit time from $[0,1]$ of a solution X of an SDE with distributional drift.

6. Those solutions which are associated with (1.1) are the unique solution of the corresponding BSDE (in some reasonable class) whenever F fulfills in particular some strict monotonicity condition in the second variable (i. e. (3.18)) is fulfilled.

7. We illustrate situations where the BSDE admits no uniqueness in a reasonable class but the probabilistic representation still holds.

As we mentioned, a significant object of study is a backward SDE with random terminal time, which was studied and introduced by [10] when the driving martingale is a Brownian motion. BSDEs driven by a càdlàg martingale with fixed time terminal time were studied in [6, 11, 8, 5].

The paper is organized as follows. After the introduction in Section 2, we remind some preliminaries about linear elliptic PDEs with initial condition and the notion of martingale problem related to an SDE with distributional drift. In Section 3, we discuss existence and uniqueness of (1.1), in Section 4 we discuss the first exit time properties of a solution to equation (1.2). In Section 5 we investigate uniqueness for BSDEs with random terminal condition with related probabilistic representation. Finally Section 6 shows how a solution to (1.1) generates a solution to a special BSDE with terminal condition at random time.

2 Preliminaries

2.1 The linear elliptic PDE with distributional drift

If I is a real open interval, then $C(I)$ will be the space of continuous functions on I endowed with the topology of uniform convergence on compacts. For $k \geq 0$, $C^k(I)$ will be a similar space equipped with the topology of uniform convergence of the first k derivatives. If $I = \mathbb{R}$, then we will simply write C, C^k instead of $C(\mathbb{R})$, $C^k(\mathbb{R})$. If $I = [a, b]$ with $-\infty < a < b < +\infty$, then $u : I \to \mathbb{R}$ is said to be of class $C^1([a, b])$ if it is of class $C^1((a, b))$ and if the derivative extends continuously to $[a, b]$.

In this section we introduce the "generator" L of our diffusion with distributional drift adopting the notations and conventions of [13, 14].

Let σ, $\beta \in C^0$ such that $\sigma > 0$. We consider formally a PDE operator of the following type [13, section 2]:

$$Lg = \frac{\sigma^2}{2} g'' + \beta' g'. \tag{2.1}$$

By a mollifier, we intend a function Φ belonging to the Schwartz space $\mathcal{S}(\mathbb{R})$ with $\int \Phi(x)\, dx = 1$. We denote

$$\Phi_n(x) := n\Phi(nx), \quad \sigma_n^2 := \sigma^2 * \Phi_n, \quad \beta_n := \beta * \Phi_n.$$

We then consider

$$L_n g = \frac{\sigma_n^2}{2} g'' + \beta_n' g'. \tag{2.2}$$

A priori, σ_n^2, β_n and the operator L_n depend on the mollifier Φ.

Definition 2.1. Let $l \in C^0$. A function $f \in C^1(\mathbb{R})$ is said to be a C^1-*solution* to

$$Lf = l, \tag{2.3}$$

if, for any mollifier Φ, there are sequences (f_n) in C^2, (l_n) in C^0 such that

$$L_n f_n = l_n, \quad f_n \to f \text{ in } C^1, \quad l_n \to l \text{ in } C^0. \tag{2.4}$$

The following proposition gives conditions for the existence of a solution h to the homogeneous version of (2.3), see [13, prop. 2.3].

Proposition 2.2. *Let $a \in \mathbb{R}$ be fixed. There is a C^1-solution to $Lh = 0$ such that $h'(x) \neq 0$ for every $x \in \mathbb{R}$ if and only if*

$$\Sigma(x) := \lim_{n \to \infty} 2 \int_a^x \frac{\beta_n'}{\sigma_n^2}(y)\, dy \tag{2.5}$$

exists in C^0, independently from the mollifier. Moreover, in this case, any solution C^1-solution f to $Lf = 0$ fulfills

$$f'(x) = e^{-\Sigma(x)} f'(a), \ \forall x \in \mathbb{R}. \tag{2.6}$$

Remark 2.3. 1. In particular, this proves the uniqueness of the problem

$$Lf = l, \quad f \in C^1, \quad f(a) = x_0, \quad f'(a) = x_1, \tag{2.7}$$

for every $l \in C^0$, $x_0, x_1 \in \mathbb{R}$.

2. In most of the cases we will set $a = 0$.

In the sequel we will always suppose the existence of Σ as in (2.5). We will denote $h : \mathbb{R} \to \mathbb{R}$ such that $h(0) = 0$ and $h' = \exp(-\Sigma)$ and $h_n : \mathbb{R} \to \mathbb{R}$ so that $h_n = \exp(-\Sigma_n)$ with $\Sigma_n = 2\int_0^x \frac{\beta_n'}{\sigma_n^2}(y)\, dy$.

Lemma 2.4. *A solution to problem* (2.7) *is given by*

$$\begin{aligned} f(a) &= x_0, \\ f'(x) &= h'(x) \left(2 \int_a^x \frac{l(y)}{(\sigma^2 h')(y)}\, dy + x_1 \right). \end{aligned}$$

The proposition below was established in [13, Remark 2.7].

Proposition 2.5. *Let $a \in \mathbb{R}$ and $l \in C^0$ and $x_0, x_1 \in \mathbb{R}$. Then there is a unique C^1-solution to*

$$\begin{aligned} Lu &= l \\ u(a) = x_0, &\quad u'(a) = x_1. \end{aligned} \tag{2.8}$$

The solution satisfies

$$u'(x) = e^{-\Sigma(x)} \left(2 \int_a^x e^{\Sigma(x)} \frac{l(y)}{\sigma^2(y)}\, dy + x_1 \right). \tag{2.9}$$

We will denote by $\mathcal{D}_L$ the set of all $f \in C^1$ which are C^1-solutions of $Lf = l$ for some $l \in C^0$. This defines without ambiguity $L : \mathcal{D}_L \to C^0$.

2.2 Related martingale problem

For the moment we fix a probability space $(\Omega, \mathcal{G}, \mathbb{P})$. All processes will be considered with index in $\mathbb{R}_+$.

For convenience, we follow the framework of stochastic calculus introduced in [21] and developed in several papers. A survey of that calculus in finite dimension is given in [20]. We will fix a filtration $\mathcal{F} = (\mathcal{F}_t)$ which will fulfill the usual conditions.

The covariation of two continuous processes X and Y is defined as follows. Suppose that

$$A_t := \lim_{\varepsilon \to 0+} C_\varepsilon(X, Y)_t \tag{2.10}$$

exists for any $t \in [0, T]$ in probability, where

$$C_\varepsilon(X, Y)_t := \frac{1}{\varepsilon} \int_0^t (X_{s+\varepsilon} - X_s)(Y_{s+\varepsilon} - Y_s)\, ds.$$

We say that (X, Y) admit a covariation if the random function (A_t) admits a (necessarily unique) continuous version, which will be designated by $[X, Y]$. For $[X, X]$ we often shortly write $[X]$. All the covariation processes will be continuous.

Remark 2.6. In [20, Propositions 1, 9 and 11, Remarks 1 and 2] we can find the following.

a) If $[X, X]$ exists, then it is always an increasing process and X is called a *finite quadratic variation process.* If $[X, X] \equiv 0$, then X is said to be a *zero quadratic variation process.*

b) Let X and Y be continuous processes such that $[X, Y]$, $[X, X]$, $[Y, Y]$ exist. Then $[X, Y]$ is a bounded variation process. If $f, g \in C^1$, then

$$[f(X), g(Y)]_t = \int_0^t f'(X) g'(Y)\, d[X, Y].$$

c) If A is a zero quadratic variation process and X is a finite quadratic variation process, then $[X, A] \equiv 0$.

d) A bounded variation process is a zero quadratic variation process.

e) If X and Y are $\mathcal{F}$-semimartingales, then $[X, Y]$ is the usual covariation process $\langle M^X, M^Y \rangle$ of their martingale components.

An $\mathcal{F}$-*Dirichlet process* is the sum of an $\mathcal{F}$-local continuous martingale M and an $\mathcal{F}$-adapted zero quadratic variation process A, see [15, 4].

Remark 2.7. Let $X = M + A$ be an $\mathcal{F}$-Dirichlet process.

1. Remark 2.6c) and e) together with the bilinearity of the covariation operator imply that $[X] = \langle M \rangle$.

2. If $f \in C^1$, then $f(X) = M^f + A^f$ is an $\mathcal{F}$-Dirichlet process, where

$$M^f = \int_0^{\cdot} f'(X_s)\, dM_s$$

and $A^f := f(X) - M^f$ has zero quadratic variation. This easily follows from the bilinearity of covariation and Remark 2.6b), c) and e). See also [4] for a similar result and Proposition 17 in [20] for a generalization to weak Dirichlet processes.

Definition 2.8. Given a stopping time τ and a process X, we denote by X^τ the *stopped process*

$$X_t^\tau := X_{t\wedge\tau}, \qquad t \geq 0.$$

Remark 2.9. Let τ be an $\mathcal{F}$-stopping time. If X is an $\mathcal{F}$-semimartingale (resp. $\mathcal{F}$-Dirichlet process), then the stopped processes X^τ is also a semimartingale (resp. $\mathcal{F}$-Dirichlet process).

In the classical theory of Stroock and Varadhan, see e. g. [22], the solutions of martingale problems are probabilities on the canonical space $C([0,T])$ equipped with its Borel σ-field and the Wiener measure. Here the sense is a bit different since the solutions are considered to be processes.

Definition 2.10. A process X (defined on some probability space), is said to solve the *martingale problem* MP $(\sigma, \beta; x_0)$ related to L with initial condition $X_0 = x_0,\ x_0 \in \mathbb{R}$ if

$$f(X_t) - f(x_0) - \int_0^t Lf(X_s)ds$$

is a local martingale for any $f \in \mathcal{D}_L$.

In the sequel we will denote by $\mathcal{F}^X = (\mathcal{F}_t^X)$ the canonical filtration associated with X.

Definition 2.11. We say that the martingale problem MP$(\sigma, \beta; x_0)$ admits *uniqueness (in law)* if any processes X_1 and X_2, defined on some probability space and solving the martingale problem, have the same law.

The proposition below was the object of Proposition 3.13 of [13].

Proposition 2.12. *Let v be the unique solution to $Lv = 1$ in the C^1-sense such that $v(0) = v'(0) = 0$. Then there exists a unique (in law) solution to the martingale problem related to L with prescribed initial condition $x_0 \in \mathbb{R}$ if and only if*

$$v(-\infty) = v(+\infty) = +\infty. \tag{2.11}$$

In several contexts (see [13]) the solution of previous martingale problem appears to be a solution (in the proper sense) of (1.2), but it will not be used in this paper.

Proposition 2.12 implies the following.

Proposition 2.13. *Let $x_0 \in \mathbb{R}$. The martingale problem* $\mathrm{MP}(\sigma, \beta; x_0)$ *admits exactly one solution in law if and only if the function $v : \mathbb{R} \to \mathbb{R}$ defined by*

$$\begin{aligned} &v(0) = 0, \\ &v'(x) = e^{-\Sigma(x)} \left(2 \int_0^x \frac{1}{\sigma^2}(y)\, dy \right) \end{aligned} \tag{2.12}$$

fulfills

$$v(-\infty) = -\infty, \qquad\qquad v(+\infty) = +\infty. \tag{2.13}$$

Proof. This follows from Proposition 2.12, Proposition 2.5 and from the fact that v defined in (2.12) is the solution of the problem

$$Lv = 1, \qquad\qquad v(0) = v'(0) = 0.$$

□

From now on Assumption (2.13) for the function v defined by (2.12) will always be in force. Let then X be a solution to the martingale problem on a suitable probability space and $\mathcal{F}^X$ be its canonical filtration.

Remark 2.14. i) By Remark 3.3 of [13], choosing f as the identity function, X is an $\mathcal{F}^X$-Dirichlet process, whose local martingale part M^X is such that

$$\left[M^X\right]_t = \int_0^t \sigma^2(X_s) ds.$$

ii) Consequently by Remark 2.6c) and the bilinearity of covariation it follows $[X]_t = \int_0^t \sigma^2(X_s) ds$.

Proposition 2.15. *Let X be a solution of* $\mathrm{MP}(\sigma, \beta; x_0)$ *for some $x_0 \in \mathbb{R}$. For every $\varphi \in \mathcal{D}_L$ we have*

$$\varphi(X_t) = \varphi(X_0) + \int_0^t \varphi'(X_s) dM_s^X + \int_0^t (L\varphi)(X_s) ds.$$

Proof. By definition of the martingale problem there is an $\mathcal{F}^X$-local martingale M^φ such that

$$\varphi(X_t) = \varphi(X_0) + M_t^\varphi + \int_0^t (L\varphi)(X_s) ds. \tag{2.14}$$

On the other hand, by Remark 2.14i) and Remark 2.7 $\varphi(X_t)$ is an $\mathcal{F}^X$-Dirichlet process with decomposition

$$\varphi(X_t) = \varphi(X_0) + \int_0^t \varphi'(X_s) dM_s^X + A_t^\varphi, \tag{2.15}$$

where $[A^\varphi] \equiv 0$. By the uniqueness of Dirichlet decomposition and the identification of (2.14) and (2.15) the result follows. □

3 The semilinear elliptic PDE with distributional drift and boundary conditions

In this section we present the deterministic analytical framework that we will need in the paper.

3.1 The linear case

We explain here how to reduce the study of our initial problem to a boundary value problem.

Definition 3.1. Let $a, b, A, B \in \mathbb{R}$, such that $-\infty < a < b < \infty$. Additionally, let $g : [a, b] \to \mathbb{R}$ be continuous. We say that $u : [a, b] \to \mathbb{R}$ is a *solution of the boundary value problem*

$$\begin{cases} Lu = g, \\ u(a) = A, \\ u(b) = B \end{cases} \tag{3.1}$$

if there is a continuous extension $\tilde{g} : \mathbb{R} \to \mathbb{R}$ of g and a function $\tilde{u} \in \mathcal{D}_L$ fulfilling $\tilde{u}\big|_{[a,b]} = u$, such that $\tilde{u}$ is a solution of

$$L\tilde{u} = \tilde{g}, \tag{3.2}$$

in the sense of Definition 2.1, and $\tilde{u}(a) = A$, $\tilde{u}(b) = B$.

Proposition 3.2. *Let $g : [0, 1] \to \mathbb{R}$ be continuous, $A, B \in \mathbb{R}$, $a = 0$ and $b = 1$. Then there exists a unique solution u to* (3.1)*, given by*

$$u(x) = f(x) + \int_0^1 K(x, y)g(y)dy \tag{3.3a}$$

$$f(x) := \frac{B \int_0^x dy e^{-\Sigma(y)} + A \int_x^1 dy e^{-\Sigma(y)}}{\int_0^1 dy e^{-\Sigma(y)}} \tag{3.3b}$$

$$K(x, y) := \mathbb{1}_{y \le x} \frac{2e^{\Sigma(y)}}{\sigma^2(y)} \int_y^x dz e^{-\Sigma(z)} - 2\frac{\int_0^x dr e^{-\Sigma(r)}}{\int_0^1 dr e^{-\Sigma(r)}} \frac{e^{\Sigma(y)}}{\sigma^2(y)} \int_y^1 dz e^{-\Sigma(z)}. \tag{3.3c}$$

Remark 3.3. For every $y \in [0, 1], x \mapsto K(x, y)$ is absolutely continuous and $(x, y) \mapsto \partial_x K(x, y)$ belongs to $L^\infty([0, 1]^2)$.

Proof of Proposition 3.2. We start with existence. Let $\tilde{g}$ be a continuous extension of g and $x_1 \in \mathbb{R}$. Then, by Proposition 2.5, there exists a unique solution $\tilde{u}$ to the problem on the real line,

$$L\tilde{u}(x) = \tilde{g}(x), \ x \in \mathbb{R}, \tag{3.4a}$$

$$\tilde{u}(0) = A, \tag{3.4b}$$

$$\tilde{u}'(0) = x_1, \tag{3.4c}$$

given by

$$\tilde{u}(x) = A + \int_0^x e^{-\Sigma(y)} \left(2 \int_0^y e^{\Sigma(z)} \frac{\tilde{g}(z)}{\sigma^2(z)} dz + x_1 \right) dy. \tag{3.5}$$

We look for $x_1 \in \mathbb{R}$, so that $\tilde{u}(1) = B$. This gives

$$\begin{aligned} B &= A + x_1 \int_0^1 e^{-\Sigma(y)} dy + 2 \int_0^1 dy e^{-\Sigma(y)} \int_0^y dz e^{\Sigma(z)} \frac{\tilde{g}(z)}{\sigma^2(z)} \\ x_1 &= \frac{B - A - 2 \int_0^1 dz e^{\Sigma(z)} \frac{\tilde{g}(z)}{\sigma^2(z)} \int_z^1 dy e^{-\Sigma(y)}}{\int_0^1 e^{-\Sigma(y)} dy}. \end{aligned}$$

We insert x_1 into (3.5) and use the fact that $u = \tilde{u}|_{[0,1]}$ and $g = \tilde{g}|_{[0,1]}$. This gives (3.3), and we get $u(0) = A$ and $u(1) = B$.

To show uniqueness, let v^1 and v^2 be two solutions of (3.1), and set $v = v^1 - v^2$. Then there is $\tilde{v} \in \mathcal{D}_L$ with $\tilde{v}|_{[0,1]} = v$ and an $\tilde{l} \in C$ with $\tilde{l}|_{[0,1]} = 0$, so that

$$\begin{aligned} L\tilde{v}(x) &= \tilde{l}, \qquad \tilde{l}\,\big|_{[0,1]} = 0 \\ \tilde{v}(0) &= \tilde{v}(1) = 0. \end{aligned}$$

We need to show that $v \equiv 0$. By Lemma 2.4 we get

$$\tilde{v}'(x) = e^{-\Sigma(x)} \left(2 \int_0^x \frac{\tilde{l}(y)}{\sigma^2(y)} e^{\Sigma(y)} dy + \tilde{v}'(0) \right) \qquad \forall x \in \mathbb{R}.$$

In particular, since $\tilde{l}\,\big|_{[0,1]} = 0$, we get

$$\tilde{v}'(x) = e^{-\Sigma(x)} \tilde{v}'(0), \qquad \forall x \in [0,1].$$

Consequently, for $x \in [0,1]$,

$$\tilde{v}(x) = \left(\int_0^x dy\, e^{-\Sigma(y)} \right) \tilde{v}'(0).$$

Since $\tilde{v}(1) = 0$, it follows $\tilde{v}'(0) = 0$ and so $v(x) = \tilde{v}(x) = 0$ $\forall x \in [0,1]$. □

3.2 Solution of the semilinear problem on the real line

We extend here the notion of C^1-solution to the semilinear case.

Definition 3.4. Let $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. We say that $u \in C^1$ is a C^1-*solution* (on the real line) of

$$Lu = F\left(x, u, u'\right) \tag{3.6}$$

if u is a C^1-solution of $Lu = h$, with $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = F\left(x, u(x), u'(x)\right)$.

Definition 3.5. A function $F : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x, y, z) \mapsto F(x, y, z)$ will be said *globally Lipschitz* with respect to z (resp. (y, z)) if F is Lipschitz with respect to z (resp. (y, z)) uniformly on x varying in I and y in $\mathbb{R}$ (resp. uniformly on x varying in I).
More precisely, F is globally Lipschitz with respect to z if there exists some constant k, such that

$$|F(x, y, z) - F(x, y, \tilde{z})| \leq k\,|z - \tilde{z}|\,, \ \forall x \in I, \ \forall y, z, \tilde{z} \in \mathbb{R}. \tag{3.7}$$

k is called Lipschitz constant for F. Similarly we speak about Lipschitz constant k related to a function F which is globally Lipschitz with respect to (y, z).

Proposition 3.6. *Suppose that $F : \mathbb{R}^3 \to \mathbb{R}$, so that $(x, y, z) \mapsto F(x, y, z)$ restricted to $K \times \mathbb{R}^2$, for any compact interval K, is Lipschitz with respect to (y, z). Then there is a unique solution of*

$$\begin{aligned} Lu &= F\left(x, u(x), u'(x)\right), \qquad x \in \mathbb{R}, \\ u(0) &= x_0, \\ u'(0) &= x_1. \end{aligned} \tag{3.8}$$

Proof. By Proposition 2.5, $u : \mathbb{R} \to \mathbb{R}$ of class C^1 is a C^1-solution if and only if

$$\begin{aligned} u'(x) &= e^{-\Sigma(x)} \left(2 \int_0^x \frac{e^{\Sigma(y)}}{\sigma^2(y)} F\left(y, u(y), u'(y)\right) dy + x_1 \right), \quad \forall x \in \mathbb{R}, \\ u(0) &= x_0. \end{aligned} \tag{3.9}$$

We can reduce the well-posedness of (3.9) to the well-posedness of

$$\begin{aligned} u'(x) &= e^{-\Sigma(x)} \left(2 \int_0^x \frac{e^{\Sigma(y)}}{\sigma^2(y)} F\left(y, u(y), u'(y)\right) dy + x_1 \right), \forall x \in [-N, N], \\ u(0) &= x_0, \end{aligned} \tag{3.10}$$

for every $N \in \mathbb{N}^*$. In the sequel of the proof, since (3.10) depends on N, we will often denote it by (3.10)(N).

Indeed, if u_N is a solution of (3.10)(N), then any solution of (3.10)($N+1$), restricted to $[-N, N]$ is a solution (3.10)(N). In this way the yield of a solution of (3.9) is equivalent to the yield of a family (u_N) of functions which are respectively solutions of (3.10)(N). In the sequel we fix $N \in \mathbb{N}^*$ and we study existence and uniqueness for (3.10)(N), which is a PDE in a compact interval. We consider the map $T : C^1([-N, N]) \to C^1([-N, N])$ defined by

$$\begin{aligned} Tf(0) &= x_0 \\ (Tf)'(x) &= e^{-\Sigma(x)} \left(2 \int_0^x \frac{e^{\Sigma(y)}}{\sigma^2(y)} F\left(y, f(y), f'(y)\right) dy + x_1 \right). \end{aligned}$$

Clearly a function $u \in C^1([-N,N])$ is a strong solution of (3.10)(N) if and only if $Tu = u$. $C^1([-N,N])$ is a Banach space equipped with the norm

$$\|f\|_N = \sup_{|x|\le N} \left[|f(x)| + |f'(x)|\right].$$

The norm $\|\cdot\|_N$ is equivalent to

$$\|f\|_{N,\lambda} = \sup_{|x|\le N} \left(|f(x)| + |f'(x)|\right) e^{\Sigma(x)-\lambda|x|},$$

where $\lambda > 0$, has to be suitably chosen. It remains to show that T admits a unique fixed point. For this we will show that T is a contraction with respect to $\|\cdot\|_{N,\lambda}$. Let $u, v \in C^1([-N,N])$. Let us denote by $K/2$ a Lipschitz constant for F. We get

$$\begin{aligned}
&\left|(Tu - Tv)'(x)e^{\Sigma(x)}\right| \\
&\le 2\left|\int_0^x \left|F\left(y, u(y), u'(y)\right) - F\left(y, v(y), v'(y)\right)\right| \frac{e^{\Sigma(y)}}{\sigma^2(y)}dy\right| \le \\
&\le K \sup_{|z|\le N} \frac{1}{\sigma^2(z)} \left|\int_0^x \left(\left|u'(y) - v'(y)\right| + |u(y) - v(y)|\right) e^{\Sigma(y)}dy\right| \le \\
&\le K \sup_{|z|\le N} \frac{1}{\sigma^2(z)} \left|\int_0^x e^{\lambda|y|}dy\right| \|u - v\|_{N,\lambda} \\
&= K \sup_{|z|\le N} \frac{1}{\sigma^2(z)} \frac{e^{\lambda|x|} - 1}{\lambda} \|u - v\|_{N,\lambda}.
\end{aligned}$$

This implies that, for every $x \in [-N,N]$,

$$\left|(Tu - Tv)'(x)\right| e^{\Sigma(x)-\lambda|x|} \le \frac{K}{\lambda} \sup_{|z|\le N} \frac{1}{\sigma^2(z)} \|u - v\|_{N,\lambda}. \tag{3.11}$$

On the other hand, since $(Tu)(0) = (Tv)(0) = x_0$ we have

$$\begin{aligned}
|(Tu - Tv)(x)| &\le \left|\int_0^x \left|(Tu - Tv)'(y)\right| dy\right| = \\
&= \left|\int_0^x e^{\Sigma(y)-\lambda|y|} \left|(Tu - Tv)'(y)\right| e^{-\Sigma(y)+\lambda|y|}dy\right| \le \\
&\le \sup_{|s|\le N} e^{-\Sigma(s)} \sup_{|y|\le N} \left(e^{\Sigma(y)-\lambda|y|} \left|(Tu - Tv)'(y)\right|\right) \frac{e^{\lambda|x|} - 1}{\lambda}.
\end{aligned}$$

Finally, taking into account (3.11), we get

$$e^{\Sigma(x)-\lambda|x|} |(Tu - Tv)(x)| \le \frac{K}{\lambda^2} \sup_{|s|\le N} e^{-\Sigma(s)} \sup_{|y|\le N} e^{\Sigma(y)} \sup_{|z|\le N} \frac{1}{\sigma^2(z)} \|u - v\|_{N,\lambda}. \tag{3.12}$$

Summing up (3.11) and (3.12) we get

$$\|Tu - Tv\|_N \leq C(\lambda)\, \|u - v\|_{N,\lambda}\,, \tag{3.13}$$

where

$$C(\lambda) = \frac{K}{\lambda} \sup_{|z|\leq N} \frac{1}{\sigma^2(z)} + \frac{K}{\lambda^2} \sup_{s\leq N} e^{-\Sigma(s)} \sup_{|y|\leq N} e^{\Sigma(y)} \sup_{|x|\leq N} \frac{1}{\sigma^2(x)}.$$

If $C(\lambda) < 1$, (3.13) has shown that T is a contraction. The condition can be fulfilled by choosing λ sufficiently large. □

3.3 The semi-linear case with boundary conditions

Definition 3.7. Let

i) $a, b \in \mathbb{R}$, such that $0 < a < b < \infty$,

ii) $A, B \in \mathbb{R}$, and

iii) $F : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function.

We say that $u : [a, b] \to \mathbb{R}$ of class $C^1([a, b])$ is a *solution of the boundary value problem*

$$\begin{cases} Lu(x) = F(x, u, u'), \\ u(a) = A, \\ u(b) = B \end{cases} \tag{3.14}$$

if u is a solution of the boundary value problem

$$\begin{cases} Lu = \ell, \\ u(a) = A, \\ u(b) = B, \end{cases}$$

in the sense of Definition 3.1 with $\ell : [a, b] \to \mathbb{R}$ defined by $\ell(x) = F(x, u(x), u'(x))$.

In Section 5 we will observe that solving (3.14) is strongly related to the problem of solving BSDEs with random terminal time.

Lemma 3.8. *Suppose that the assumptions of Definition 3.7 are fulfilled. Then, u is a solution of the boundary value problem*

$$\begin{cases} Lu(x) = F(x, u, u'), \\ u(a) = A \\ u(b) = B \end{cases} \tag{3.15}$$

if and only if the functions $u_1, u_2 : [a, b] \to \mathbb{R}$, given by

$$\begin{aligned} u_1 &= u, \\ u_2 &= e^{\Sigma} u', \end{aligned}$$

belong to $C^1([a,b])$ *and fulfill*

$$\begin{aligned}
u_1'(x) &= e^{-\Sigma(x)}u_2(x), \\
u_2'(x) &= 2\frac{e^{\Sigma(x)}}{\sigma^2(x)}F\left(x, u_1(x), e^{-\Sigma(x)}u_2(x)\right), \\
u_1(a) &= A, \\
u_1(b) &= B.
\end{aligned} \tag{3.16}$$

Proof. Let u be a solution of the boundary value problem (3.15). This means, by Definition 3.7, that u is a solution of the boundary value problem

$$\begin{cases} Lu = \ell, \\ u(a) = A, \\ u(b) = B, \end{cases}$$

with

$$\ell(x) = F\left(x, u(x), u'(x)\right),$$

in the sense of Definition 3.1. By that definition, there are continuous extensions $\tilde{u}$ and $\tilde{\ell}$ such that

$$\begin{aligned} \tilde{u}|_{[a,b]} &= u, \\ \tilde{\ell}|_{[a,b]} &= \ell, \end{aligned}$$

and

$$L\tilde{u} = \tilde{\ell}$$

in the sense of Definition 2.1. Since $\tilde{u} \in C^1$, we can define

$$x_a := \tilde{u}'(a).$$

By Proposition 2.5 it follows that

$$\tilde{u}'(x) = e^{-\Sigma(x)}\left(2\int_a^x \frac{e^{\Sigma(y)}}{\sigma^2(y)}\tilde{\ell}(y)dy + x_a\right), \ \forall x \in \mathbb{R}.$$

By setting $\tilde{\ell}_1, \tilde{\ell}_2, \tilde{u}_1, \tilde{u}_2 : \mathbb{R} \to \mathbb{R}$ as

$$\begin{aligned}
\tilde{\ell}_1 &:= \tilde{u}', \\
\tilde{\ell}_2 &:= 2\frac{e^{\Sigma}}{\sigma^2}\tilde{\ell}, \\
\tilde{u}_1 &= \tilde{u}, \\
\tilde{u}_2 &= \tilde{u}'e^{\Sigma},
\end{aligned}$$

it yields that $\tilde{u}^1, \tilde{u}^2$ belong to C^1 and

$$\begin{aligned}
&\tilde{u}_1'(x) = \tilde{\ell}_1(x) \ \forall x \in \mathbb{R},\\
&\tilde{u}_2'(x) = \tilde{\ell}_2(x) \ \forall x \in \mathbb{R},\\
&\tilde{u}_1(a) = A,\\
&\tilde{u}_1(b) = B.
\end{aligned}$$

It follows now that $u_1, u_2 \in C^1([a,b],\mathbb{R})$, which are respectively restrictions of $\tilde{u}_1, \tilde{u}_2$, solve (3.16).

Concerning the converse, let $u_1, u_2 \in C^1([a,b];\mathbb{R}])$, so that (3.16) is fulfilled. We define $\tilde{\ell}_2 : \mathbb{R} \to \mathbb{R}$ as

$$\tilde{\ell}_2(x) = 2\frac{e^{\Sigma(x)}}{\sigma^2(x)}\tilde{\ell}(x),$$

where $\tilde{\ell} : \mathbb{R} \to \mathbb{R}$ is a continuous extension of

$$\ell(x) = F\left(x, u_1(x), u_2(x)e^{-\Sigma(x)}\right).$$

By (3.16), we have for some $x_a \in \mathbb{R}$

$$u_2(x) = 2\int_a^x \frac{e^{\Sigma(y)}}{\sigma^2(y)}\tilde{\ell}(y)dy + x_a, \tag{3.17}$$

for $x \in [a,b]$. We define $\tilde{u}_2 : \mathbb{R} \to \mathbb{R}$ as the right-hand side of (3.17) for all $x \in \mathbb{R}$. Clearly $\tilde{u}_2$ is a C^1 extension of u_2. We also define

$$\tilde{\ell}_1(x) = e^{-\Sigma(x)}\tilde{u}_2(x), \quad x \in \mathbb{R}.$$

(3.16) gives

$$u_1'(x) = \tilde{\ell}_1(x) = e^{-\Sigma(x)}\tilde{u}_2(x), \quad x \in [a,b].$$

We define $\tilde{u}_1(x) = \int_a^x \tilde{\ell}_1(y)dy + A$, $x \in \mathbb{R}$. $\tilde{u}_1$ is a C^1 extension of u_1. Consequently, setting $\tilde{u} = \tilde{u}_1$, we get

$$\begin{aligned}
&\tilde{u}'(x) = \tilde{u}_1'(x) = \tilde{\ell}_1(x) = e^{-\Sigma(x)}\tilde{u}_2(x) = e^{-\Sigma(x)}\left(2\int_a^x \frac{e^{\Sigma(y)}}{\sigma^2(y)}\tilde{\ell}(y)dy + x_a\right),\\
&\tilde{u}(a) = A,
\end{aligned}$$

taking into account (3.17) and the consideration below. We define $u : [a,b] \to \mathbb{R}$ as restriction of $\tilde{u}$ and get

$$\begin{aligned}
u(a) &= \tilde{u}(a) = A,\\
u(b) &= \tilde{u}(b) = \tilde{u}_1(b) = u_1(b) = B,
\end{aligned}$$

by (3.16). By Proposition 2.5, Definition 3.1 and Definition 3.4, u is a solution to the boundary value problem (3.15). □

The following result provides uniqueness under some monotonicity conditions.

Proposition 3.9. *Let*

$$F : [a, b] \times \mathbb{R}^2 \to \mathbb{R},$$
$$(x, y, z) \mapsto F(x, y, z),$$

be a continous function fulfilling the following assumptions.

1. *F is non-decreasing in y, i. e.*

$$(F(x, y, z) - F(x, \tilde{y}, z))\,(y - \tilde{y}) \geq 0,\ \ \forall y, \tilde{y}, z \in \mathbb{R},\ \ x \in [a, b]. \tag{3.18}$$

2. *F is globally Lipschitz (with respect to z).*

Then, for any $A, B \in \mathbb{R}$, the boundary value problem

$$\begin{cases} Lu(x) = F\left(x, u, u'\right), \\ u(a) = A, \\ u(b) = B, \end{cases} \tag{3.19}$$

has at most one C^1-solution.

Proof. Let u and v in $C^1([a, b])$ be two solutions of the boundary value problem (3.19) and define

$$x_a := u'(a),$$
$$y_a := v'(a).$$

Then, by Lemma 3.8, we get

$$u(x) = A + \int_a^x dz e^{-\Sigma(z)} \left(2 \int_a^z \frac{e^{\Sigma(y)}}{\sigma^2(y)} F\left(y, u(y), u'(y)\right) dy + x_a\right), \quad \forall x \in [a, b], \tag{3.20a}$$

$$v(x) = A + \int_a^x dz e^{-\Sigma(z)} \left(2 \int_a^z \frac{e^{\Sigma(y)}}{\sigma^2(y)} F\left(y, v(y), v'(y)\right) dy + y_a\right), \quad \forall x \in [a, b], \tag{3.20b}$$

$$u(a) = v(a) = A, \tag{3.20c}$$

$$u(b) = v(b) = B. \tag{3.20d}$$

Indeed, we are interested in the C^1-function

$$\phi : [a, b] \to \mathbb{R}$$
$$\phi = u - v,$$

which fulfills $\phi(a) = \phi(b) = 0$. We consider now the C^2-function χ, given by

$$\chi(a) = 0 \tag{3.21a}$$

$$\chi'(x) = e^{\Sigma(x)} \phi'(x), \tag{3.21b}$$

and we define

$$\psi := \chi'\phi.$$

By using (3.21b), the monotonicity and Lipschitz conditions, we get, on $[a,b]$,

$$\psi' = \chi''\phi + \chi'\phi' \geq \chi''\phi = 2\frac{e^{\Sigma}}{\sigma^2}\left(F\left(x,u,u'\right) - F\left(x,v,v'\right)\right)(u-v) \geq$$
$$\geq 2\frac{e^{\Sigma}}{\sigma^2}\left(F\left(x,\frac{u+v}{2},u'\right) - F\left(x,\frac{u+v}{2},v'\right)\right)(u-v) \geq -\frac{2k}{\sigma^2}\left|\chi'\phi\right|,$$

where k is the Lipschitz constant. So we get the differential inequality

$$\begin{aligned}\psi'(x) &\geq -\frac{2k}{\sigma^2(x)}\left|\psi(x)\right|,\ x \in [a,b],\\ \psi(a) &= 0,\\ \psi(b) &= 0.\end{aligned}$$

By some basic properties of differential inequalities (see e. g. [23, Preface]), we get

$$\psi(x) \geq 0,\ x \in [a,b]. \tag{3.22}$$

On the other hand,

$$\int_a^b \psi(x)e^{-\Sigma(x)}dx = \int_a^b \phi'(x)\phi(x)dx = \left.\frac{\phi^2(x)}{2}\right|_a^b = 0. \tag{3.23}$$

Finally, combining (3.22) and (3.23) leads to

$$\psi(x) = 0,\ \forall x \in [a,b].$$

By definition of ψ it follows that $(\phi^2)' = 0$ so that ϕ^2 is constantly equal to $\phi^2(0) = 0$.
$\square$

We consider now a classical boundary value problem of the type considered in (3.16). Let $f_1, f_2 : \mathbb{R}^3 \to \mathbb{R}$ be continuous and let $a, b, A, B \in \mathbb{R}$, $-\infty < a < b < \infty$. We are looking for solutions $u_1, u_2 : [a,b] \to \mathbb{R}$ of the system

$$u_1'(x) = f_1(x, u_1(x), u_2(x)), \tag{3.24a}$$
$$u_2'(x) = f_2(x, u_1(x), u_2(x)), \tag{3.24b}$$
$$u_1(a) = A, \tag{3.24c}$$
$$u_1(b) = B. \tag{3.24d}$$

Theorem 2.1.1 in [3] states the following.

Theorem 3.10. *Let $I = (\alpha,\beta]$, $-\infty \leq \alpha < \beta < \infty$, and $I^0 = (\alpha,\beta)$. Assume the following.*

i) For every $(x, y) \in I^0 \times \mathbb{R}$ $z \mapsto f_1(x, y, z)$ is an increasing function. Moreover we suppose

$$\lim_{z \to \pm\infty} f_1(x, y, z) = \pm\infty,$$

uniformly on compact sets in $I^0 \times \mathbb{R}$.

ii) All the local solutions defined on a subinterval of I of (3.24a) *and* (3.24b) *extend to a solution on the whole interval I.*

iii) There exists at most one solution of (3.24)*, for all $a = a_0, b = b_0 \in I^0$ and all $A = A_0, B = B_0 \in \mathbb{R}$.*

Then there exists exactly one solution of (3.24) *if $a \in I^0$ and $b \in I$.*

Previous theorem has an important consequence at the level of existence and uniqueness of boundary value problems.

Corollary 3.11. *Let $F : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$, $(x, y, z) \mapsto F(x, y, z)$ be a continuous function. We suppose the following.*

i) $y \mapsto F(x, y, 0)$ has linear growth uniformly with respect to x.

ii) F fulfills the monotonicity condition (3.18)*.*

iii) F is globally Lipschitz in z.

Then there exists exactly one solution to the boundary value problem

$$\begin{cases} Lu(x) = F(x, u, u'), \\ u(a) = A \\ u(b) = B. \end{cases} \tag{3.25}$$

Proof. Uniqueness follows immediately from Proposition 3.9. To show existence, we make use of Theorem 3.10. Let $\alpha < a$ and $\beta > b$. We extend F continuously on the entire $\mathbb{R}^3$ by introducing a new function $\tilde{F}$ in the following way:

$$\tilde{F}(x, y, z) := \begin{cases} F(a, y, z), & x < a, \\ F(x, y, z), & a \leq x \leq b, \\ F(b, y, z), & x > b. \end{cases} \tag{3.26}$$

F fulfills the assumptions of Lipschitz-continuity and monotonicity, and so does $\tilde{F}$. At this point we can show the existence of a unique solution $u_1, u_2 : [a, b] \to \mathbb{R}$ of the system

$$\begin{aligned} u_1'(x) &= e^{\Sigma(x)} u_2(x), \\ u_2'(x) &= 2\frac{e^{\Sigma(x)}}{\sigma^2(x)} \tilde{F}\left(x, u_1(x), e^{-\Sigma(x)} u_2(x)\right), \\ u_1(a) &= A, \\ u_1(b) &= B. \end{aligned} \tag{3.27}$$

That coincides with (3.24) setting

$$
\begin{aligned}
f_1(x,y,z) &= e^{\Sigma(x)}z,\\
f_2(x,y,z) &= 2\frac{e^{\Sigma(x)}}{\sigma^2(x)}\tilde{F}\left(x,y,ze^{-\Sigma(x)}\right).
\end{aligned}
$$

As the mentioned existence will be a consequence of Theorem 3.10, we check the validity of its assumptions. Clearly, i) is fulfilled. Furthermore, by assumption, $\tilde{F}:\mathbb{R}^3\to\mathbb{R}$ is continuous and has linear growth in the second and third variable. Therefore assumption ii) is fulfilled too. Indeed, by Peano theorem, we can continue (to the left and to the right) locally any solution of (3.27) to a possibly exploding solution. The linear growth condition and Gronwall's lemma imply that no solution explodes. Moreover, Assumption iii) of Theorem 3.10 holds. In fact, since $\tilde{F}$ fulfills the monotonicity condition (3.18) and is globally Lipschitz in z, uniqueness follows from Proposition 3.9. Finally, by Lemma 3.8, $u=u_1$ is a solution of (3.25). □

The proposition below shows existence and uniqueness in the Lipschitz case without the monotonicity condition.

Proposition 3.12. *Let $F:[0,1]\times\mathbb{R}^2\to\mathbb{R}$, $(x,y,z)\mapsto F(x,y,z)$ be bounded and globally Lipschitz with respect to (y,z). Then there exists a solution of the boundary value problem*

$$
\begin{cases}
Lu(x) = F\left(x,u,u'\right),\\
\quad u(0) = A,\\
\quad u(1) = B,
\end{cases}
$$

for any $A,B\in\mathbb{R}$.

Proof. We extend F to $\tilde{F}$ in the way of (3.26) with $a=0$ and $b=1$. Moreover, we define a real function $\Phi:\mathbb{R}\to\mathbb{R}$ in the following way: for $x_0=A$ and $x_1\in\mathbb{R}$ we denote the solution of (3.8) by u^{x_1}. Its existence follows from Proposition 3.6 since $\tilde{F}$ is Lipschitz with respect to (y,z). Now we set $\Phi(x_1)=u^{x_1}(1)$. Since Σ, F and σ are continuous, Φ can shown to be continuous as well. We leave this to the reader. By (3.9), we get then the following relation:

$$
\Phi(x_1)-x_0 = \int_0^1 dx e^{-\Sigma(x)}\left(2\int_0^x \frac{e^{\Sigma(y)}}{\sigma^2(y)}F\left(y,u^{x_1}(y),\left(u^{x_1}\right)'(y)\right)dy + x_1\right). \tag{3.28}
$$

Since F is bounded,

$$
\lim_{x_1\to\infty}\Phi(x_1) = \infty = -\lim_{x_1\to-\infty}\Phi(x_1). \tag{3.29}
$$

Consequently, by mean value theorem, for each $B\in\mathbb{R}$, there is an x_1 so that $\Phi(x_1)=B$. □

Remark 3.13. If F is not bounded, one cannot ensure existence in general. To give an example, we set $L = \frac{d^2}{dx^2}$ and $F(x, y, z) = -\pi^2 y$. Then the corresponding boundary value problem

$$\begin{cases} \quad u'' = -\pi^2 u, \\ u(0) = 0, \\ u(1) = 1, \end{cases}$$

has no solution.

Proposition 3.14. *Let $a = 0$, $b = 1$, and $F : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x, y, z) \mapsto F(x, y, z)$ be globally Lipschitz with respect to (y, z) and Lipschitz-constant k, fulfilling*

$$k < \frac{1}{\sup_{x\in[0,1]} \int_0^1 dy \left(|K(x, y)| + |\partial_x K(x, y)|\right)}, \tag{3.30}$$

where K was defined in (3.3c)*. Then,* (3.14) *has a unique solution for any $A, B \in \mathbb{R}$.*

Proof. We consider the map $T : C^1([0, 1]) \to C^1([0, 1])$ defined by

$$Th(x) = f(x) + \int_0^1 K(x, y)F(y, h(y), h'(y))dy,$$

with f is given by (3.3b). Taking into account Definition 3.7 and Proposition 3.2, (3.14) is well-posed if and only if T has a fixed point. We show the latter assertion. $C^1([0, 1])$ is a Banach space equipped with the norm

$$\|h\| = \sup_{x\in[0,1]} \left(|h(x)| + \left|h'(x)\right|\right).$$

To show that T admits a unique fixed point, we will show that T is a contraction with respect to $\|\cdot\|$. Let $u, v \in C^1([0, 1])$. We get

$$\begin{aligned} |(Tu - Tv)(x)| &= \left| \int_0^1 K(x, y) \left(F\left(y, u(y), u'(y)\right) - F\left(y, v(y), v'(y)\right)\right) dy \right| \\ &\le \int_0^1 dy\, |K(x, y)|\, k \left(|u(y) - v(y)| + \left|u'(y) - v'(y)\right|\right) \\ &\le \int_0^1 dy\, |K(x, y)|\, k\|u - v\| \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} \left|(Tu - Tv)'(x)\right| &= \left| \int_0^1 \partial_x K(x, y) \left(F\left(y, u(y), u'(y)\right) - F\left(y, v(y), v'(y)\right)\right) dy \right| \\ &\le \left| \int_0^1 dy \partial_x K(x, y) k \left(|u(y) - v(y)| + \left|u'(y) - v'(y)\right|\right) dy \right| \\ &\le \int_0^1 dy\, |\partial_x K(x, y)|\, k\|u - v\|. \end{aligned} \tag{3.32}$$

Summing up (3.31) and (3.32) and taking the supremum over x gives

$$\|Tu - Tv\| \leq \sup_{x\in[0,1]} \int_0^1 dy \left(|K(x,y)| + |\partial_x K(x,y)|\right) k\|u - y\|.$$

It follows, that T is a contraction if k fulfills (3.30). $\square$

4 Exit time of the solution to the forward martingale problem

We are interested in the nature of the first exit time τ from the interval $[0,1]$ of a solution $X = X^x$ to the martingale problem with respect to L and initial condition $x \in [0,1]$. So we define τ as

$$\tau := \begin{cases} \inf\left\{t \geq 0 \middle| X_t \notin [0,1]\right\} & : \quad \left\{t \geq 0 \middle| X_t \notin [0,1]\right\} \neq \emptyset \\ \infty & : \quad \text{otherwise.} \end{cases}$$

Proposition 4.1. *τ has finite expectation. In particular τ is finite almost surely.*

Proof. We consider $\Gamma : [0,1] \to \mathbb{R}$ as the unique solution of

$$\begin{aligned} L\Gamma &= -1 \\ \Gamma(0) = \Gamma(1) &= 0, \end{aligned}$$

in the sense of Definition 3.1, and we consider the associated function $\tilde{\Gamma} \in \mathcal{D}_L$. Since X is a solution to the martingale problem with respect to L and initial condition x, the process

$$N_t = \tilde{\Gamma}(X_t) - \tilde{\Gamma}(x) - \int_0^t L\tilde{\Gamma}(X_r)dr,$$

is a local martingale. By Proposition 2.15 we have $N_t = \int_0^t \tilde{\Gamma}'(X_s)dM_s^X$, which, by Remark 2.14, implies that

$$[N]_t = \int_0^t \sigma^2(X_s)\tilde{\Gamma}'(X_s)^2 ds.$$

Now, let (τ_n) be the family of stopping times defined as

$$\tau_n := \inf\left\{t \geq 0 \,\middle|\, \int_0^t \sigma^2(X_s)\tilde{\Gamma}'(X_s)^2 ds \geq n\right\}$$

with the assumption that $\inf(\emptyset) = \infty$.

The stopped processes N^{τ_n} are clearly square integrable martingales. By Doob's stopping theorem for martingales, the processes $(N^{\tau_n}_{t\wedge\tau})_{t\geq 0}$ are again martingales. Consequently,

$$E\left(\tilde{\Gamma}\left(X_{\tau_n\wedge t\wedge\tau}\right) - \tilde{\Gamma}(x) - \int_0^{\tau_n\wedge t\wedge\tau} \left(L\tilde{\Gamma}\right)(X_r)dr\right) = 0.$$

Since $L\tilde{\Gamma}$ restricted to $[0,1]$ equals -1, the previous expression gives

$$E\left(\tilde{\Gamma}\left(X_{\tau_n\wedge t\wedge\tau}\right)-\tilde{\Gamma}(x)\right)+E\left(\tau_n\wedge t\wedge\tau\right)=0.$$

Now we take the limit $n\to\infty$, and we can use the theorems of monotone and dominated convergence, since

$$\left|\tilde{\Gamma}\left(X_{\tau_n\wedge t\wedge\tau}\right)\right|\leq\sup_{x\in[0,1]}|\Gamma(x)|\,.$$

This gives, for every $x\in[0,1]$,

$$E\left(\Gamma\left(X_{t\wedge\tau}\right)\right)-\Gamma(x)+E\left(t\wedge\tau\right)=0. \tag{4.1}$$

Finally, we let $t\to\infty$, we then get

$$E\left(\tau\right)=\Gamma(x)-E\left(\Gamma\left(X_\tau\right)\right)=\Gamma(x),$$

by the same arguments as those used taking $n\to\infty$ above. □

As byproduct of the proof of Proposition 4.1 we get the following.

Proposition 4.2. *The expectation of the exit time* τ *is exactly* $\Gamma(x)$*, where* Γ *is the unique solution of*

$$L\Gamma=-1$$
$$\Gamma(0)=\Gamma(1)=0.$$

5 Martingale driven BSDEs with Random Terminal Time

5.1 Notion of solution

The present section does not aim at the greatest generality, which could be the object of future research. We consider the case of one-dimensional BSDEs driven by square integrable martingales with continuous predictable bracket.

Backward SDEs driven by martingales were investigated by several authors, see e. g. [6], [8], see also [11], [7] and [9] for recent developments. We are interested in such a BSDE with terminal condition at random time. This is motivated by the fact that the forward SDE (martingale problem) only admits weak solutions, therefore the reference filtration will only be the canonical one related to the solution and not the one associated with the underlying Brownian motion. We consider the following data.

i) An a. s. finite stopping time τ.

ii) An $\mathcal{F}$-local martingale $(M_t)_{t\geq 0}$ with an $\mathcal{F}$-predictable continuous quadratic variation process $\langle M\rangle$. We suppose moreover that M^τ is an $\mathcal{F}$-square integrable martingale, and we suppose the existence of a deterministic increasing function $\rho:\mathbb{R}_+\to\mathbb{R}_+$ with $\rho(0)=0$ and

$$\langle M^\tau\rangle_t\leq\rho(t),\ \forall t\geq 0.$$

iii) A terminal condition $\xi \in L^2(\Omega, \mathcal{F}_\tau, P; \mathbb{R})$.

iv) A coefficient $f : \Omega \times [0,T] \times \mathbb{R}^2 \to \mathbb{R}$, such that the process $f(\cdot, t, y, z)$, $t \geq 0$, is predictable for every y, z.

Our BSDE is the following.

$$Y_t = \xi - \int_t^\infty \mathbb{1}_{\{\tau \geq s\}} Z_s dM_s + \int_t^\infty \mathbb{1}_{\{\tau \geq s\}} f(\omega, s, Y_s, Z_s)\, d\langle M\rangle_s - (O_\tau - O_{t\wedge\tau}). \quad (5.1)$$

Without restriction of generality we suppose $Z_t = 0$ if $t > \tau$ and $O_t = O_\tau$ for $t \geq \tau$.

Definition 5.1. Let (Y, Z, O) be a triple of processes with the following properties.

i) Y is càdlàg $\mathcal{F}$-adapted.

ii) Z is $\mathcal{F}$-predictable such that $E\left(\int_0^\tau Z_s^2 d\langle M\rangle_s\right) < \infty$.

iii) O is a square integrable martingale such that $O_0 = 0$ and $E\left(O_\tau^2\right) < \infty$. Furthermore, O is strongly orthogonal to M, i. e. $\langle M, O\rangle = 0$.

iv) $Z_t = 0$ if $t > \tau$ and $O_t = O_\tau$ for $t \geq \tau$.

Such a triplet (Y, Z, O) is called *solution* of the BSDE (f, τ, ξ) if it fulfills (5.1).

Remark 5.2. i) If $t \geq \tau$ in (5.1) we get $Y_t = \xi = Y_\tau$, so in particular $Y_t = Y_\tau,\ t \geq \tau$.

ii) Indeed we will always suppose that $M = M^\tau$ so that (5.1) can be rewritten as

$$Y_t = \xi - \int_t^\infty Z_s dM_s + \int_t^\infty f(\omega, s, Y_s, Z_s)\, d\langle M\rangle_s - (O_\tau - O_{t\wedge\tau}). \quad (5.2)$$

When M is a Brownian motion, this was treated in [10] from which we inherit and adopt very close notations.

5.2 Uniqueness of Solutions

Theorem 5.3. *Let $a, b, \kappa \in \mathbb{R}$ and set $\gamma = b^2 - 2a$. We suppose the following.*

i) $(f(\omega, s, y_1, z) - f(\omega, s, y_2, z))(y_1 - y_2) \leq -a|y_1 - y_2|^2$, *for every* $\omega \in \Omega$, $s \in [0,T]$, $y_1, y_2, z \in \mathbb{R}$.

ii) $|f(\omega, s, y, z_1) - f(\omega, s, y, z_2)| \leq b|z_1 - z_2|$, *for every* $\omega \in \Omega$, $s \in [0,T]$, $y \in \mathbb{R}$.

iii) $|f(\omega, s, y, z) - f(\omega, s, 0, 0)| \leq \kappa(|y| + \kappa') + b|z|$, *where* $\kappa' \in \{1, 0\}$.

iv) $E\left(\int_0^\tau e^{\theta\langle M\rangle_t}\left(f(t,0,0)^2 + \kappa'\right) d\langle M\rangle_t\right) < \infty$ *for every* $\theta < \gamma$.

Let $\xi \in L^2(\Omega, \mathcal{F}_\tau)$. *Then the BSDE* (f, τ, ξ) *admits at most one solution* (Y, Z, O), *such that*

$$E\left(Y_0^2 + \int_0^\tau e^{\gamma\langle M\rangle_t}(Y_t^2 + Z_t^2) d\langle M\rangle_t + e^{\gamma\langle M\rangle_t} d\langle O\rangle_t\right) < \infty. \quad (5.3)$$

Remark 5.4. 1. In the proof of Theorem 5.3 that we develop below, we omit the dependence of f on ω in order to simplify the notations.

2. If we suppose $\mathcal{F}_0$ to be the trivial σ-field, then Y_0^2 can be deleted in (5.3).

Before the proof of Theorem 5.3, we start with a technical lemma, which is the generalization of Proposition 4.3 in [10].

Lemma 5.5. *Suppose the validity of hypotheses i), ii) and iii) of Theorem 5.3, and let (Y,Z,O) be a solution of BSDE (f,τ,ξ) such that for some θ,*

$$E\left(Y_0^2+\int_0^\tau e^{\theta\langle M\rangle_s}\left(|Y_s|^2+|Z_s|^2+f^2(s,0,0)+\kappa'\right)d\langle M\rangle_s+\int_0^\tau e^{\theta\langle M\rangle_s}d\langle O\rangle_s\right)<\infty. \tag{5.4}$$

Then

$$E\left(\sup_{s\le\tau}e^{\theta\langle M\rangle_s}|Y_s|^2\right)<\infty, \tag{5.5}$$

and

$$N_t=\int_0^{t\wedge\tau}e^{\theta\langle M\rangle_s}Y_{s-}\left(Z_sdM_s+dO_s\right)=\int_0^t e^{\theta\langle M\rangle_s}Y_{s-}\left(Z_sdM_s+dO_s\right) \tag{5.6}$$

is a uniformly integrable martingale.

Proof. Since Y solves the BSDE, by integration by parts we get

$$\begin{aligned}e^{\frac{\theta}{2}\langle M\rangle_{t\wedge\tau}}Y_{t\wedge\tau}&=Y_0+\int_0^{t\wedge\tau}e^{\frac{\theta}{2}\langle M\rangle_s}\left(Z_sdM_s+dO_s\right)-\\&-\int_0^{t\wedge\tau}e^{\frac{\theta}{2}\langle M\rangle_s}f\left(s,Y_s,Z_s\right)d\langle M\rangle_s+\frac{\theta}{2}\int_0^{t\wedge\tau}e^{\frac{\theta}{2}\langle M\rangle_s}Y_sd\langle M\rangle_s.\end{aligned} \tag{5.7}$$

By Assumption ii) of Section 5.1, $\langle M\rangle$ is continuous. Consequently,

$$\left[e^{\frac{\theta}{2}\langle M\rangle}Y\right]_{t\wedge\tau}=\left[N^\theta\right]_{t\wedge\tau}, \tag{5.8}$$

where

$$N_t^\theta:=\int_0^t e^{\frac{\theta}{2}\langle M\rangle_s}\left(Z_sdM_s+dO_s\right). \tag{5.9}$$

Remark 5.6. From (5.4) it follows that

$$E\left(\int_0^\tau e^{\theta\langle M\rangle_s}\left(Z_s^2d\langle M\rangle_s+d\langle O\rangle_s\right)\right)<\infty.$$

Consequently, N^θ is a square integrable martingale. So, by the proof of Proposition 4.50 in [16], there is a uniformly integrable martingale $\mathcal{M}^\theta$, so that

$$\left[N^\theta\right]=\left\langle N^\theta\right\rangle+\mathcal{M}^\theta.$$

We continue with the proof of Lemma 5.5 by using Itô's formula and (5.7) getting

$$
\begin{aligned}
e^{\theta\langle M\rangle_{t\wedge\tau}}Y^2_{t\wedge\tau} - Y_0^2 = \left(e^{\frac{\theta}{2}\langle M\rangle_{t\wedge\tau}}Y_{t\wedge\tau}\right)^2 - Y_0^2 = \\
&= 2\int_0^{t\wedge\tau} e^{\frac{\theta}{2}\langle M\rangle_s}Y_{s-}d\left(e^{\frac{\theta}{2}\langle M\rangle_s}Y_s\right) + \left[e^{\frac{\theta}{2}\langle M\rangle}Y\right]_{t\wedge\tau} = \\
&= 2\int_0^{t\wedge\tau} e^{\theta\langle M\rangle_s}Y_{s-}\left(Z_s dM_s + dO_s\right) - \\
&\quad - 2\int_0^{t\wedge\tau} e^{\theta\langle M\rangle_s}Y_s f\left(s, Y_s, Z_s\right) d\left\langle M\right\rangle_s + \\
&\quad + 2\frac{\theta}{2}\int_0^{t\wedge\tau} e^{\theta\langle M\rangle_s}Y_s^2 d\left\langle M\right\rangle_s + \left[N^\theta\right]_{t\wedge\tau},
\end{aligned}
\tag{5.10}
$$

where in the latter equality we have taken into account (5.8). Since $\langle M\rangle$ is continuous we have been allowed to replace Y_{s-} with Y_s in the two lines above. By use of Cauchy-Schwarz, the inequality $2\alpha\beta \leq \alpha^2 + \beta^2$ and assumption iii) of Theorem 5.3, there is a constant c, depending on κ, b and θ, such that

$$
\begin{aligned}
e^{\theta\langle M\rangle_{t\wedge\tau}}Y^2_{t\wedge\tau} - Y_0^2 \leq c\int_0^{t\wedge\tau} e^{\theta\langle M\rangle_s}\big(Y_s^2 + Z_s^2 + \\
+ f^2(s,0,0) + \kappa'\big)\, d\left\langle M\right\rangle_s + 2\int_0^{t\wedge\tau} e^{\frac{\theta}{2}\langle M\rangle_s}Y_{s-}dN_s^\theta + \left[N^\theta\right]_{t\wedge\tau}.
\end{aligned}
\tag{5.11}
$$

Now we continue with a localization of (5.11). For that we define for each $n \in \mathbb{N}$ a stopping time $\tau(n)$ by

$$\tau(n) := \inf\left\{t | Y_t \geq n\right\} \wedge n.$$

Replacing t with $t \wedge \tau(n)$ in (5.11) gives

$$
\begin{aligned}
e^{\theta\langle M\rangle_{t\wedge\tau(n)\wedge\tau}}Y^2_{t\wedge\tau(n)\wedge\tau} - Y_0^2 \leq c\int_0^{t\wedge\tau(n)\wedge\tau} e^{\theta\langle M\rangle_s}\big(Y_s^2 + Z_s^2 + \\
+ f^2(s,0,0) + \kappa'\big)\, d\left\langle M\right\rangle_s + 2\int_0^{t\wedge\tau(n)\wedge\tau} e^{\frac{\theta}{2}\langle M\rangle_s}Y_{s-}dN_s^\theta + \left[N^\theta\right]_{t\wedge\tau(n)\wedge\tau}.
\end{aligned}
\tag{5.12}
$$

We take the supremum over t in the left-hand side and afterwards the expectation. Reminding that

$$N_t = \int_0^t e^{\frac{\theta}{2}\langle M\rangle_s}Y_{s-}dN_s^\theta, \tag{5.13}$$

this yields

$$
\begin{aligned}
E\left(\sup_{t\leq\tau(n)\wedge\tau}\left(e^{\theta\langle M\rangle_{t\wedge\tau(n)\wedge\tau}}Y^2_{t\wedge\tau(n)\wedge\tau}\right)\right) \leq \\
\leq E\left(Y_0^2\right) + c\,E\left(\mathcal{D}\right) + 2E\left(\sup_{t\geq 0}\left|N_t^{\tau(n)\wedge\tau}\right|\right) + E\left(\left[N^\theta\right]_{\tau(n)\wedge\tau}\right),
\end{aligned}
\tag{5.14}
$$

where

$$\mathcal{D}=\int_0^\tau e^{\theta\langle M\rangle_s}\left(Y_s^2+Z_s^2+f^2(s,0,0)+\kappa'\right)d\left\langle M\right\rangle_s, \tag{5.15}$$

which has finite expectation because of (5.4). By Remark 5.6,

$$\begin{aligned}E\left(\left[N^\theta\right]_{\tau(n)\wedge\tau}\right)=E\left(\left\langle N^\theta\right\rangle_{\tau(n)\wedge\tau}\right)&=\\ =E\left(\int_0^{\tau(n)\wedge\tau}e^{\theta\langle M\rangle_s}\left(Z_s^2d\left\langle M\right\rangle_s+d\left\langle O\right\rangle_s\right)\right)&.\end{aligned} \tag{5.16}$$

We show now that $N^{\tau(n)\wedge\tau}$ is a square integrable martingale. This happens because by (5.6) we have

$$\begin{aligned}E\left(\left\langle N\right\rangle_{\tau(n)\wedge\tau}\right)&=E\left(\int_0^{\tau(n)\wedge\tau}e^{2\theta\langle M\rangle_s}Y_s^2\left(Z_s^2d\left\langle M\right\rangle_s+d\left\langle O\right\rangle_s\right)\right)\le\\ &\le n^2E\left(\int_0^{\tau\wedge n}e^{2\theta\langle M\rangle_s}\left(Z_s^2d\left\langle M\right\rangle_s+d\left\langle O\right\rangle_s\right)\right)\le\\ &\le n^2e^{\theta\rho(n)}E\left(\int_0^{\tau}e^{\theta\langle M\rangle_s}\left(Z_s^2d\left\langle M\right\rangle_s+d\left\langle O\right\rangle_s\right)\right)<\infty,\end{aligned}$$

taking into account Assumption ii) at the beginning of Section 5.1. So by Proposition 4.50 of [16], there is a uniformly integrable martingale $\tilde{\mathcal{M}}$ so that

$$\left[N^{\tau(n)\wedge\tau}\right]=\left\langle N^{\tau(n)\wedge\tau}\right\rangle+\tilde{\mathcal{M}}.$$

Due to the Burkholder-Davis-Gundy (BDG) inequalities (see e. g. [18, Theorem IV.48]), there is a universal constant c_0 such that

$$E\left(\sup_{t\ge0}\left|N_t^{\tau(n)\wedge\tau}\right|\right)\le c_0E\left([N,N]^{\frac12}_{\tau(n)\wedge\tau}\right). \tag{5.17}$$

We denote by $\mathcal{N}$ the local martingale

$$\mathcal{N}_t=\int_0^t Z_sdM_s+O_t.$$

By Theorem 29 in Chapter II of [18] the right-hand side of (5.17) equals

$$\begin{aligned}&c_0E\left(\left(\int_0^{\tau(n)\wedge\tau}e^{\theta\langle M\rangle_s}e^{\theta\langle M\rangle_s}Y_s^2d\left[\mathcal{N}\right]_s\right)^{\frac12}\right)\le\\ &\qquad\le c_0E\left(\left(\sup_{t\le\tau(n)\wedge\tau}\left(e^{\theta\langle M\rangle_t}Y_t^2\right)\right)^{\frac12}\left(\int_0^{\tau(n)\wedge\tau}e^{\theta\langle M\rangle_s}d\left[\mathcal{N}\right]_s\right)^{\frac12}\right).\end{aligned} \tag{5.18}$$

By $2\alpha\beta \leq \frac{\alpha^2}{c_3} + c_3\beta^2$, for any $c_3 > 0$, the right-hand side of (5.18) is bounded by

$$\frac{c_0}{2c_3} E\left(\sup_{t\leq\tau(n)\wedge\tau}\left(e^{\theta\langle M\rangle_t} Y_t^2\right)\right) + \frac{c_0 c_3}{2} E\left(\int_0^{\tau(n)\wedge\tau} e^{\theta\langle M\rangle_s} d[\mathcal{N}]_s\right) =$$
$$= \frac{c_0}{2c_3} E\left(\sup_{t\leq\tau(n)\wedge\tau}\left(e^{\theta\langle M\rangle_t} Y_t^2\right)\right) + \frac{c_0 c_3}{2} E\left(\left[N^\theta\right]_{\tau(n)\wedge\tau}\right), \quad (5.19)$$

also using (5.9) and [18], Theorem 29, chapter II. This gives, by (5.17), (5.16) and (5.18),

$$E\left(\sup_{t\geq 0}\left|N_t^{\tau(n)\wedge\tau}\right|\right) \leq \frac{c_0}{2c_3} E\left(\sup_{t\leq\tau(n)\wedge\tau}\left(e^{\theta\langle M\rangle_t} Y_t^2\right)\right) +$$
$$+ \frac{c_0 c_3}{2} E\left(\int_0^{\tau(n)\wedge\tau} e^{\theta\langle M\rangle_s}\left(Z_s^2 d\left\langle M\right\rangle_s + d\left\langle O\right\rangle_s\right)\right). \quad (5.20)$$

Plugging (5.16) and (5.20) in (5.14) gives

$$E\left(\sup_{t\leq\tau(n)\wedge\tau}\left(e^{\theta\langle M\rangle_t} Y_t^2\right)\right) \leq$$
$$\leq E\left(Y_0^2\right) + E\left(\mathcal{D}\right)\left(1 + c + c_0 c_3\right) + \frac{c_0}{c_3} E\left(\sup_{t\leq\tau(n)\wedge\tau}\left(e^{\theta\langle M\rangle_t} Y_t^2\right)\right).$$

Choosing $c_3 = 2c_0$, we get

$$E\left(\sup_{t\leq\tau(n)\wedge\tau}\left(e^{\theta\langle M\rangle_t} Y_t^2\right)\right) \leq 2E\left(Y_0^2\right) + 2E\left(\mathcal{D}\right)\left(1 + c + 2c_0^2\right).$$

By monotone convergence theorem, letting $n \to \infty$, we get

$$E\left(\sup_{t\leq\tau}\left(e^{\theta\langle M\rangle_t} Y_t^2\right)\right) \leq 2E\left(Y_0^2\right) + 2E(\mathcal{D})\left(1 + c + 2c_0^2\right),$$

which shows (5.5).

We go on with the second part, i. e. the fact that N defined in (5.6) is a uniformly integrable martingale. By BDG and Cauchy-Schwarz inequalities,

$$E\left(\sup_{t\geq 0}|N_t|\right) \leq c_0 E\left([N,N]^{\frac{1}{2}}\right) \leq c_0 E\left(\left(\int_0^{\cdot} e^{2\theta\langle M\rangle_s} Y_{s-}^2 d[\mathcal{N}]_s\right)^{\frac{1}{2}}\right) \leq$$
$$\leq c_0 E\left(\left(\sup_{t\leq\tau} e^{\theta\langle M\rangle_t} Y_t^2\right)^{\frac{1}{2}}\left(\int_0^{\tau} e^{\theta\langle M\rangle_s} d[\mathcal{N}]_s\right)^{\frac{1}{2}}\right) \leq$$
$$\leq c_0\left(E\left(\sup_{t\leq\tau} e^{\theta\langle M\rangle_t} Y_t^2\right)\right)^{\frac{1}{2}}\left(E\left(\int_0^{\tau} e^{\theta\langle M\rangle_s} d[\mathcal{N}]_s\right)\right)^{\frac{1}{2}} =$$
$$= c_0\left(E\left(\sup_{t\leq\tau} e^{\theta\langle M\rangle_t} Y_t^2\right)\right)^{\frac{1}{2}}\left(E\left(\left[N^\theta\right]_\tau\right)\right)^{\frac{1}{2}}. \quad (5.21)$$

By Remark 5.6

$$E\left(\left[N^{\theta}\right]_{\tau}\right)=E\left(\left\langle N^{\theta}\right\rangle_{\tau}\right)=E\left(\int_{0}^{\tau}e^{\theta\langle M\rangle_{s}}\left(Z_{s}^{2}d\left\langle M\right\rangle_{s}+d\left\langle O\right\rangle_{s}\right)\right)<\infty.$$

This shows that N is a uniformly integrable martingale and finally, Lemma 5.5 is established. $\square$

Proof of Theorem 5.3. We start with some a priori bounds. Let $\theta<\gamma$. By assumptions i) and ii), for any $\varepsilon\geq 0$, using $2\alpha\beta\leq\frac{\alpha^{2}}{1+\varepsilon}+(1+\varepsilon)\beta^{2}$, we can easily show that

$$2\left(y-\bar{y}\right)\left(f(s,y,z)-f\left(s,\bar{y},\bar{z}\right)\right)\leq-2a\left|y-\bar{y}\right|^{2}+b^{2}(1+\varepsilon)\left|y-\bar{y}\right|^{2}+\frac{\left|z-\bar{z}\right|^{2}}{1+\varepsilon}. \quad (5.22)$$

Let $\left(Y^{i},Z^{i},O^{i}\right)$, $i=1,2$ be two solutions fulfilling (5.3) of the statement. By similar arguments as (5.10) and in the lines before, for $Y=Y^{1}-Y^{2}$, $Z=Z^{1}-Z^{2}$, $O=O^{1}-O^{2}$ we have

$$\begin{aligned}e^{\theta\langle M\rangle_{\tau}}Y_{\tau}^{2}-e^{\theta\langle M\rangle_{t\wedge\tau}}Y_{t\wedge\tau}^{2}&=\int_{t\wedge\tau}^{\tau}\theta e^{\theta\langle M\rangle_{s}}Y_{s}^{2}d\left\langle M\right\rangle_{s}+2\int_{t\wedge\tau}^{\tau}e^{\frac{\theta}{2}\langle M\rangle_{s}}Y_{s-}dN_{s}^{\theta}-\\&-2\int_{t\wedge\tau}^{\tau}e^{\theta\langle M\rangle_{s}}Y_{s}\left(f\left(s,Y_{s}^{1},Z_{s}^{1}\right)-f\left(s,Y_{s}^{2},Z_{s}^{2}\right)\right)d\left\langle M\right\rangle_{s}+\left[N^{\theta}\right]_{\tau}-\left[N^{\theta}\right]_{t\wedge\tau}, \quad (5.23)\end{aligned}$$

where N^{θ} was defined in (5.9). By (5.22) we get

$$\begin{aligned}&2\int_{t\wedge\tau}^{\tau}e^{\theta\langle M\rangle_{s}}Y_{s}\left(f\left(s,Y_{s}^{1},Z_{s}^{1}\right)-f\left(s,Y_{s}^{2},Z_{s}^{2}\right)\right)d\left\langle M\right\rangle_{s}\leq\\&\leq\int_{t\wedge\tau}^{\tau}e^{\theta\langle M\rangle_{s}}\left(b^{2}(1+\varepsilon)-2a\right)Y_{s}^{2}d\left\langle M\right\rangle_{s}+\int_{t\wedge\tau}^{\tau}e^{\theta\langle M\rangle_{s}}\frac{\left|Z_{s}\right|^{2}}{1+\varepsilon}d\left\langle M\right\rangle_{s}. \quad (5.24)\end{aligned}$$

$(Y^{i},Z^{i},O^{i}),i=1,2$ fulfills (5.4) by (5.3) and Assumption iv) of Theorem 5.3. Consequently (Y,Z,O) also fulfills (5.4). By Remark 5.6, since $\theta<\gamma$, N^{θ} is a square integrable martingale and

$$\left[N^{\theta}\right]=\left\langle N^{\theta}\right\rangle+\mathcal{M}^{\theta},$$

where $\mathcal{M}^{\theta}$ is a uniformly integrable martingale. So

$$E\left(\left[N^{\theta}\right]_{\tau}-\left[N^{\theta}\right]_{t\wedge\tau}\right)=E\left(\int_{t\wedge\tau}^{\tau}e^{\theta\langle M\rangle_{s}}\left(Z_{s}^{2}d\left\langle M\right\rangle_{s}+d\left\langle O\right\rangle_{s}\right)\right). \quad (5.25)$$

(5.23), (5.24) and the fact that $Y_{\tau}=0$, gives

$$\begin{aligned}&e^{\theta\langle M\rangle_{t\wedge\tau}}Y_{t\wedge\tau}^{2}+\left[N^{\theta}\right]_{\tau}-\left[N^{\theta}\right]_{t\wedge\tau}+2\int_{t\wedge\tau}^{\tau}e^{\frac{\theta}{2}\langle M\rangle_{s}}Y_{s-}dN_{s}^{\theta}\leq\\&\leq\int_{t\wedge\tau}^{\tau}e^{\theta\langle M\rangle_{s}}\left(b^{2}(1+\varepsilon)-2a-\theta\right)Y_{s}^{2}d\left\langle M\right\rangle_{s}+\int_{t\wedge\tau}^{\tau}e^{\theta\langle M\rangle_{s}}\frac{Z_{s}^{2}}{1+\varepsilon}d\left\langle M\right\rangle_{s}. \quad (5.26)\end{aligned}$$

By Lemma 5.5, since $\theta < \gamma$,

$$\left(\int_0^{t\wedge\tau} e^{\frac{\theta}{2}\langle M\rangle_s} Y_{s-} dN_s^\theta\right)_{t\geq 0}$$

is a uniformly integrable martingale. So its expectation is zero. By previous considerations, (5.24) and (5.25), we take the expectation in (5.26) to get

$$E\left(e^{\theta\langle M\rangle_{t\wedge\tau}} Y_{t\wedge\tau}^2 + \int_{t\wedge\tau}^{\tau} e^{\theta\langle M\rangle_s}\left(\frac{\varepsilon Z_s^2}{1+\varepsilon} d\langle M\rangle_s + d\langle O\rangle_s\right)\right) \leq$$
$$\leq E\left(\int_{t\wedge\tau}^{\tau} e^{\theta\langle M\rangle_s}\left(b^2(1+\varepsilon) - 2a - \theta\right) Y_s^2 d\langle M\rangle_s\right). \quad (5.27)$$

Since $\theta < \gamma = b^2 - 2a$, then $b^2(1+\varepsilon) - 2a - \theta > 0$, $\forall \varepsilon \geq 0$. We let $\varepsilon \to 0$ so that (5.27) becomes

$$E\left(e^{\theta\langle M\rangle_{t\wedge\tau}} Y_{t\wedge\tau}^2 + \int_{t\wedge\tau}^{\tau} e^{\theta\langle M\rangle_s} d\langle O\rangle_s\right) \leq$$
$$\leq E\left(\int_{t\wedge\tau}^{\tau} e^{\theta\langle M\rangle_s}\left(b^2 - 2a - \theta\right) Y_s^2 d\langle M\rangle_s\right). \quad (5.28)$$

Equation (5.28) holds for every $\theta < \gamma$. We let $\theta \nearrow \gamma$. By monotone convergence theorem we get

$$E\left(e^{\gamma\langle M\rangle_{t\wedge\tau}} Y_{t\wedge\tau}^2 + \int_{t\wedge\tau}^{\tau} e^{\gamma\langle M\rangle_s} d\langle O\rangle_s\right) \leq 0. \quad (5.29)$$

Equation (5.29) finally shows that $Y \equiv 0$ and $\langle O\rangle \equiv 0$. Coming back to (5.27), it easily follows that $Z \equiv 0$ $d\langle M\rangle$ a.s. $\square$

Remark 5.7. Adapting the results of [10] Proposition 3.3, it is possible to state and prove also an existence theorem. We have decided not to do it for two reasons.

1. The techniques can be adapted from the proof of Proposition 3.3 by the same techniques as in the proof of Theorem 5.3.
2. For our applications to the probabilistic representation of semilinear PDEs, we already provide an *existence* theorem through the resolution of the PDE.

6 Solutions for BSDEs via solutions of elliptic PDEs

In this final section we will make the assumption of Section 2.1 which guarantee existence and uniqueness in law of the martingale problem with respect to L. In particular we will suppose that $\sigma > 0$, Σ as defined in (2.5) exists and Assumption (2.13) for the function v defined in (2.12). Let $x_0 \in \mathbb{R}$.

Let X solving a martingale problem $\mathrm{MP}(\sigma, \beta; x_0)$ (2.2). In this section we are interested in a BSDE with terminal condition at the random time τ, which is the exit time of

X from interval $[0,1]$. Since X solves the martingale problem, by Remark 2.14, X is an $\mathcal{F}$-Dirichlet process with $\mathcal{F}$-local martingale component M^X. M will be M^X equipped with the canonical filtration $\mathcal{F}^X$ of X.

Let $F : \mathbb{R}^3 \to \mathbb{R}$ continuous. We set

$$f(\omega, t, y, z) = -F(X_t(\omega), y, z), \qquad t \geq 0,\ y, z \in \mathbb{R}. \tag{6.1}$$

We define

$$\tau = \inf\{t \geq 0 |\, X_t \notin I\}. \tag{6.2}$$

Let $u_0, u_1 \in \mathbb{R}$ and set $\xi = \mathbb{1}_{\{X_\tau = 0\}} u_0 + \mathbb{1}_{\{X_\tau = 1\}} u_1$, so

$$\xi = u(X_\tau), \tag{6.3}$$

for a function $u : [0,1] \to \mathbb{R}$ such that $u(0) = u_0, u(1) = u_1$.

Our method allows to construct solutions of BSDE(f, ξ, τ) even in cases that f does not fulfill necessarily Lipschitz or monotonicity assumptions.

We need to check that we are in the framework of the hypotheses at the beginning of Section 5.1.

- i) is verified because of Proposition 4.1.
- ii) holds because
$$\langle M^\tau \rangle_t = \int_0^{t \wedge \tau} \sigma^2(X_s) ds \leq \rho(t),$$
where $\rho(t) = t \sup_{x \in [0,1]} \sigma^2(x)$.
- iii) is fulfilled since ξ is a bounded random variable, of course $\mathcal{F}_\tau$-measurable.
- iv) is verified, by construction and because X is a continuous adapted process.

The aim of this section is to show that the C^1 type solutions of elliptic PDEs in the sense of Definition 3.7 produce solutions to a BSDE of the type defined in Definition 5.1.

Remark 6.1.

1. $\mathcal{F}^X$ is generally not a Brownian filtration, so that the theory of [10] for existence and uniqueness of BSDEs with random terminal time cannot directly be applied.
2. Even for a simple equation of the type
$$dX_t = \sigma_0(X_t) dW_t,$$
where σ_0 is only a continuous bounded non-degenerate function, $\mathcal{F}^X$ is not necessarily equal to $\mathcal{F}^W$ even though W is an $\mathcal{F}^X$-Brownian motion.
3. In general, the solution of a semilinear PDE of the type (3.14) can associated with the solution of a BSDE driven by the martingale M^X which is the martingale component of the $\mathcal{F}^X$-Dirichlet process X.

4. In Section 5 we have investigated BSDEs driven by (even not continuous) martingales, which has an independent interest.

Theorem 6.2. *Let $I = [0,1]$ and $u : I \to \mathbb{R}$ be a C^1-solution of*

$$\begin{aligned} Lu(x) &= F\left(x, u(x), u'(x)\right) \\ u(0) &= u_0 \\ u(1) &= u_1. \end{aligned} \tag{6.4}$$

Let $x_0 \in [0,1]$. Let $(X_t) = X^{x_0}$ be a solution of $\mathrm{MP}(\sigma, \beta; x)$ *on some probability space $(\Omega, \mathcal{G}, P)$. We set, for $t \in [0,T]$,*

$$\begin{aligned} Y_t &= u(X_t^{\tau}) \\ Z_t &= u'(X_t)\sigma(X_t)\mathbb{1}_{[0,\tau]}(t) \\ O_t &= 0. \end{aligned}$$

Then (Y, Z, O) is a solution on $(\Omega, \mathcal{G}, P)$ to the BSDE (f, ξ, τ), where f, τ, ξ were defined in (6.1), (6.2), (6.3).

Proof. We remind that, by Proposition 4.1, $\tau < \infty$ almost surely.

By Definitions 3.7 and 3.1, there exists $\tilde{u} \in \mathcal{D}_L$ which extends u to the real line and $L\tilde{u} = \tilde{\ell}$ and $\tilde{\ell} : \mathbb{R} \to \mathbb{R}$ is a continuous function extending $\ell(x) = F(x, u(x), u'(x))$. By definition of martingale problem, we have

$$M_t^{\tilde{u}} := \tilde{u}(X_t) - \tilde{u}(X_0) - \int_0^t L\tilde{u}(X_s)ds, t \in [0,T], \tag{6.5}$$

is an $\mathcal{F}^X$-local martingale. We define

$$W_t^X := \int_0^t \frac{dM_s^X}{\sigma(X_s)},$$

where M^X is the martingale part of the Dirichlet process X. Taking into account Remark 2.7, we have

$$\left[W^X\right]_t = \left\langle W^X \right\rangle_t = \int_0^t \frac{1}{\sigma^2(X_s)} d\left\langle M^X \right\rangle_s = t, \qquad \forall t \geq 0. \tag{6.6}$$

By Lévy's characterization theorem W^X is an $\left(\mathcal{F}^X\right)$-Brownian motion. By Kunita-Watanabe theorem, there is an $\left(\mathcal{F}^X\right)$-predictable process (R_t) such that

$$\int_0^T R_s^2 ds < \infty, \text{ a.s.},$$

and an $\left(\mathcal{F}^X\right)$-local martingale (O_t) such that $\left\langle W^X, O\right\rangle = 0$ and

$$M_t^{\tilde{u}} = \int_0^t R_s dW_s^X + O_t. \tag{6.7}$$

Since W^X and O are continuous we get

$$\left[W^X, O\right] = \left\langle W^X, O\right\rangle = 0. \tag{6.8}$$

By Remark 2.6 b) and Remark 2.14 (ii) we get

$$[\tilde{u}(X), X]_t = \int_0^t \tilde{u}'(X_s) d\,[X, X]_s = \int_0^t \tilde{u}'(X_s)\sigma^2(X_s) ds.$$

By Remark 2.7 $\tilde{u}(X)$ is an $\mathcal{F}^X$-Dirichlet process, By Remark 2.6c), by (6.7) and (6.8), we have

$$[\tilde{u}(X), X)]_t = \left[M^{\tilde{u}}, X\right]_t = \left[M^{\tilde{u}}, W^X\right]_t = \int_0^t R_s ds.$$

By differentiation we get

$$R_t = \tilde{u}'(X_t)\sigma(X_t)\, dt\, dP \text{ a. e.}$$

By Remark 2.7 $\tilde{Y}_t = \tilde{u}(X_t)$ is an $\left(\mathcal{F}^X\right)$-Dirichlet process with martingale component $\int_0^t u'(X_s) dM_s^X$. On the other hand, by (6.5) and (6.7), $\left(\tilde{Y}_t\right)$ is an $\left(\mathcal{F}^X\right)$-semimartingale with martingale component

$$\int_0^\cdot \tilde{u}'(X_s)\sigma(X_s) dW^X + O = \int_0^\cdot \tilde{u}'(X_s) dM_s^X + O.$$

By uniqueness of decomposition of Dirichlet processes O vanishes. We set now

$$\begin{aligned} Y_t &= \tilde{u}(X_{t\wedge\tau}) \\ Z_t &= \partial_x \tilde{u}(X_t)\sigma(X_t)\mathbb{1}_{[0,\tau]}(t). \end{aligned}$$

(6.5) gives

$$\tilde{u}(X_t) - \tilde{u}(X_0) = \int_0^t L\tilde{u}(X_s) ds + \int_0^t \tilde{u}'(X_s) dM_s^X.$$

Stopping previous identity at time τ, for every $T > 0$, implies

$$Y_{T\wedge\tau} - Y_{t\wedge\tau} = \int_{t\wedge\tau}^{T\wedge\tau} Lu(X_s) ds + \int_{t\wedge\tau}^{T\wedge\tau} u'(X_s)\sigma(X_s) dW_s^X.$$

Letting $T \to \infty$, since $\tau < \infty$ a. s. gives

$$Y_t = Y_\tau - \int_{t\wedge\tau}^{\tau} Lu(X_s) ds - \int_{t\wedge\tau}^{\tau} Z_s dW_s^X.$$

Since u solves equation (6.4), (Y, Z, O) solves BSDE (f, τ, ξ) with $O \equiv 0$. In particular the conditions of Definition 5.1 are fulfilled. In fact i), iii) and iv) are trivial. ii) holds since $\partial_x u$ is bounded on $[0, 1]$.

□

Remark 6.3. Since u is bounded, that we also have $E\left(\sup_{t\leq\tau} Y_t^2\right) < \infty$.

By Theorem 6.2, Corollary 3.11 and the Propositions 3.12 and 3.14, we conclude the following.

Corollary 6.4. *Let $F : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Suppose that at least one of the following assumptions holds.*

a) $F(x, y, z)$ has linear growth in y, it is non-decreasing in y (i. e. (3.18) is fulfilled) and it is globally Lipschitz in z.

b) $F(x, y, z)$ is bounded and globally Lipschitz with respect to (y, z).

c) $F(x, y, z)$ is globally Lipschitz with respect to (y, z) and Lipschitz-constant k, fulfilling

$$k < \frac{1}{\sup_{x\in[0,1]} \int_0^1 dy \left(|K(x, y)| + |\partial_x K(x, y)|\right)},$$

K being the kernel introduced in (3.3c). Then there is a solution (Y, Z, O) of BSDE (f, τ, ξ), given by (5.1), where f, τ, ξ were defined in (6.1), (6.2), (6.3).

Remark 6.5. The solution is provided in the statement of Theorem 6.2.

Corollary 6.6. *Let $F : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ with the following assumptions.*

i) F has linear growth in y uniformly in x,

ii) $\left(F\left(x, y_1, z\right) - F\left(x, y_2, z\right)\right)\left(y_1 - y_2\right) \geq a\left(y_1 - y_2\right)^2$ for some a,

iii) F is globally Lipschitz in z with constant b.

iv) $\gamma = b^2 - 2a \leq 0$.

Then the solution (Y, Z, O) provided by Corollary 6.4 is unique in the class of

$$E\left(\int_0^\tau e^{\gamma\langle M\rangle_s} Y_s^2 d\left\langle M\right\rangle_s + \int_0^\tau e^{\gamma\langle M\rangle_s} Z_s^2 d\left\langle M\right\rangle_s + \int_0^\tau e^{\gamma\langle M\rangle_s} d\left\langle O\right\rangle_s\right) < \infty. \tag{6.9}$$

Remark 6.7. a) Condition iv) implies that $a < 0$. In particular F is increasing in y.

b) The validity of hypotheses i), ii), iii) imply Hypothesis a) in Corollary 6.4.

c) The solution provided by Corollary 6.4 fulfills (6.9) since $\gamma \leq 0$, u, u' are bounded and $O \equiv 0$.

d) If $\gamma > 0$, then the solutions provided by Corollary 6.4 do not necessarily fulfill (6.9). Indeed, even when X is a Brownian motion, τ has no exponential moments.

Remark 6.8. When γ is strictly positive, i.e. Assumption iv) is not fulfilled, then BSDE(f, τ, ξ) is generally not well-posed in the class of (Y, Z, O) fulfilling (5.3) with $\gamma = 0$, i.e.

$$E\left(Y_0^2 + \int_0^\tau (Y_t^2 + Z_t^2) d\langle M\rangle_t + d\langle O\rangle_t\right) < \infty. \tag{6.10}$$

This can be illustrated by the example below.
Consider $F(x, y, z) = -\pi^2 y$. The PDE

$$\begin{cases} u''(x) = F(x, u, u'(x)) \\ u(0) = u(1) = 0 \end{cases} \tag{6.11}$$

is not well-posed, since $u(x) = \gamma \sin(\pi x), \gamma \in \mathbb{R}$, provide a class of solutions of (6.11), and so by Theorem 6.2 provides a family of solutions of BSDE (f, τ, ξ), $\xi \equiv 0$, f, τ, ξ being defined in (6.1), (6.2), (6.3). We observe that $a = -\pi^2$, so $b^2 - 2a > 0$.

ACKNOWLEDGEMENTS: The research was partially supported by the ANR Project MASTERIE 2010 BLAN-0121-01. The research of the second named author was supported by a " Marietta-Blau-Stipendium" coming from the Austrian federal Ministry of science, research and economy (BMWF)

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