Three-dimensional Navier-Stokes equations driven by space-time white noise *

Rongchan Zhu[a], **Xiangchan Zhu**[b,† ‡]

[a]Department of Mathematics, Beijing Institute of Technology, Beijing 100081, China
[b]School of Science, Beijing Jiaotong University, Beijing 100044, China

Abstract
In this paper we study 3D Navier-Stokes (NS) equation driven by space-time white noise by using regularity structure theory introduced in [Hai14] and paracontrolled distribution proposed in [GIP13]. We obtain local existence and uniqueness of solutions to the 3D Navier-Stokes equation driven by space-time white noise.

2000 Mathematics Subject Classification AMS: 60H15, 82C28

Keywords: stochastic Navier-Stokes equation, regularity structure, paracontrolled distribution, space-time white noise, renormalisation

1 Introduction

In this paper, we consider 3D Navier-Stokes equation driven by space-time white noise: Recall that the Navier-Stokes equations describe the time evolution of an incompressible fluid and are given by

$$\begin{aligned} \partial_t u + u\cdot\nabla u =& \nu\Delta u - \nabla p + \xi \\ u(0) =& u_0, \quad divu = 0 \end{aligned} \tag{1.1}$$

where $u(x,t)\in\mathbb{R}^3$ denotes the value of the velocity field at time t and position x, $p(x,t)$ denotes the pressure, and $\xi(x,t)$ is an external force field acting on the fluid. We will consider the case when $x\in\mathbb{T}^3$, the three-dimensional torus. Our mathematical model for the driving force ξ is a Gaussian field which is white in time and space.

Random Navier-Stokes equations, especially stochastic 2D Navier-Stokes equation driven by trace-class noise, have been studied in many articles (see e.g. [FG95], [HM06], [De13], [RZZ14] and the reference therein). For two dimensional case: existence and uniqueness of the strong solutions have been obtained if the noisy forcing term is white in time and colored in space. For three dimensional case, existence of Markov solutions for stochastic 3D Navier-Stokes equations

*Research supported in part by NSFC (No.11301026) and DFG through IRTG 1132 and CRC 701
†Corresponding author
‡E-mail address: zhurongchan@126.com(R. C. Zhu), zhuxiangchan@126.com(X. C. Zhu)

driven by trace-class noise has been obtained in [FR08], [DD03], [GRZ09]. Furthermore, the ergodicity has been obtained for every Markov selections of the martingale solutions if driven by non-degenerate trace-class noise (see [FR08]).

This paper aims at giving a meaning of the equation (1.1) when ξ is space-time white noise and obtain local (in time) solution. Such a noise might not be relevant for the study of turbulence. However, in other cases, when a flow is subjected to an external forcing with very small time and space correlation length, a space-time white noise can be considered. The main difficulty in this case is that ξ is so singular that the non-linear term is not well-defined.

In two dimensional case, Navier-Stokes equation driven by space-time white noise has been studied in [DD02], where a unique global solution in (probabilistically) strong sense has been obtained by using the Gaussian invariant measure for this equation. Thanks to the incompressibility condition, we can write $u\cdot\nabla u=\frac{1}{2}div(u\otimes u)$. The authors split the unknown into the solution to the linear equations and of the solution to modified Navier-Stokes equations:

$$\partial_t z=\nu\Delta z-\nabla\pi+\xi,\quad divz=0;$$

$$\partial_t v=\nu\Delta v-\nabla q-\frac{1}{2}div(v+z)\otimes(v+z),\quad divv=0. \tag{1.2}$$

The first part z is a Gaussian process with non-smooth paths and v is smoother and the nonlinear terms can be defined even though z is only a distribution in this case. By a fixed point argument they obtain existence and uniqueness of the local solutions in the two dimensional case. Then by using Gaussian invariant measure for 2D Navier-Stokes equation driven by space-time white noise, existence and uniqueness of the (probabilistically) strong solutions starting from almost every initial value has been obtained. (For one-dimensional case we refer to [DDT94]).

However, in the three dimensional case, the trick in two dimensional case breaks down here since v and z in (1.2) are so singular that the nonlinear term cannot be well-defined. As a result, we cannot make sense of (1.2) and obtain existence and uniqueness of the local solutions as in the two dimensional case. If we iterate the above trick as follows: $v=v_2+v_3$ with v_2,v_3 are solutions to the following equations:

$$\partial_t v_2=\nu\Delta v_2-\nabla q_2-\frac{1}{2}div(z\otimes z),\quad divv_2=0.$$

$$\partial_t v_3=\nu\Delta v_3-\nabla q_3-\frac{1}{2}div[(v_3+v_2)\otimes(v_3+v_2)]-\frac{1}{2}div((v_3+v_2)\otimes z)-\frac{1}{2}div(z\otimes(v_3+v_2)),\quad divv_3=0. \tag{1.3}$$

Now we can make sense of the terms without v_3 in the right hand side of (1.3), hope v_3 become smoother such that the nonlinear terms including v_3 are well-defined and try to obtain a well-posed equation. However, this is not the case. For the unknown v_3 the nonlinear term $v_3\otimes z$ is still not well-defined. No matter how many times we modify this equation again as above, the equation always contains the multiplication for the unknown and z, which is not well-defined. Hence, this equation is ill-posed in the traditionally sense.

Thanks to the regularity structure theory introduced by Martin Hairer in [Hai14] and the paracontrolled distribution proposed by Gubinelli, Imkeller and Perkowski in [GIP13] we can solve this problem and obtain existence and uniqueness of the local solutions to the three dimensional Navier-Stokes equations driven by space-time white noise. Recently, these two approaches have been successful in giving a meaning to a lot of ill-posed stochastic PDEs like the KPZ equation ([KPZ86], [BG97], [Hai13]), the dynamical Φ_3^4 model ([Hai14], [CC13]) and so on. From a philosophical perspective, the theory of regularity structures and the paracontrolled distribution are inspired by the theory of controlled rough paths [Lyo98, Gub04]. The main difference is that the regularity structure theory consider the problem locally, while the paracontrolled distribution method is a global approach using Fourier analysis.

In the theory of regularity structures, the right objects, e.g. regularity structure that could possibly take the place of Taylor polynomials can be constructed. The regularity can also be endowed with a model, which is a concrete way of associating every distribution to the abstract regularity structure. Multiplication, differentiation, the living space of the solutions, and the convolution with singular kernel can be defined on this regularity structure and then the equation has been lifted on the regularity structure. On this regularity structure, the fixed point argument can be applied to obtain local existence and uniqueness of the solutions. Furthermore, we can go back to the real world with the help of another central tool of the theory the reconstruction operator $\mathcal{R}$. If ξ is a smooth process, $\mathcal{R}u$ coincides with the classic solution of the equation.

In this paper we first apply Martin Hairer's regularity structure theory to solve three dimensional Navier-Stokes equations driven by space-time white noise. First as in the two dimensional case we write the nonlinear term $u\cdot\nabla u=\frac{1}{2}div(u\otimes u)$ and construct the associated regularity structure (Theorem 2.7). As in [Hai14] we construct different admissible models to denote different realizations of the equations corresponding to different noises. Then for any suitable models, we obtain local existence and uniqueness of solutions by fixed point argument. Finally, we renormalized models of approximation such that the solutions to the equations associated with these renormalized models converge to the solution of the 3D Navier-Stokes equation driven by space-time white noise in probability, locally in time (Proposition 2.12 and Theorem 2.16).

The theory of paracontrolled distribution combines the idea of Gubinelli's controlled rough path [Gub04] and Bony's paraproduct [Bon84], which is defined by the following: Let $\Delta_j f$ be the jth Littlewood-Paley block of a distribution f, define

$$\pi_<(f,g)=\pi_>(g,f)=\sum_{j\geq-1}\sum_{i<j-1}\Delta_i f\Delta_j g,\quad \pi_0(f,g)=\sum_{|i-j|\leq1}\Delta_i f\Delta_j g.$$

Formally $fg=\pi_<(f,g)+\pi_0(f,g)+\pi_>(f,g)$. Observing that if f is regular $\pi_<(f,g)$ behaves like g and is the only term in the Bony's paraproduct not raising the regularities, the authors in [GIP13] consider paracontrolled ansatz of the type

$$u=\pi_<(u',g)+u^\sharp,$$

where $\pi_<(u',g)$ represents the "bad-term" in the solution, g is some distribution we can handle and $u^\sharp$ is regular enough to define the multiplication required. Then to make sense of the product of uf we only need to define gf.

In the second part of this paper we apply paracontrolled distribution method to three dimensional Navier-Stokes equations driven by space-time white noise. First we split the equation into four equations and consider the approximation equations. By using paracontrolled ansatz we obtain uniform estimates for the approximation equations and moreover we also get the local Lipschitz continuity of solutions with respect to initial values and some extra terms independent of the solutions. Then we do suitable renormalisation for these terms and prove their convergence in suitable spaces. Here inspired by [Hai14] we prove Lemma 3.10 which makes the calculations of renormalisation much easier. Moreover by taking the limit of the solutions to the approximation equations we obtain local existence and uniqueness of solutions (Theorem 3.12). Indeed by choosing a suitable solution space we can also give a meaning of the original equation (see Remark 3.9).

This paper is organized as follows. In Section 2, we use regularity structure theory to obtain local existence and uniqueness of the solutions to 3D Navier-Stokes equation driven by space-time white noise. In Section 3, we apply paracontrolled distribution method to deduce local existence and uniqueness of the solutions.

2 NS equation by regularity structure theory

2.1 Preliminary on regularity structure theory

In this subsection we recall some preliminaries for the regularity structure theory from [Hai14].

Definition 2.1 A regularity structure $\mathfrak{T} = (A, T, G)$ consists of the following elements:

(i) An index set $A \subset \mathbb{R}$ such that $0 \in A$, A is bounded from below and locally finite.

(ii) A model space T, which is a graded vector space $T = \oplus_{\alpha\in A} T_\alpha$, with each T_α a Banach space. Furthermore, T_0 is one-dimensional and has a basis vector $\mathbf{1}$. Given $\tau \in T$ we write $\|\tau\|_\alpha$ for the norm of its component in T_α.

(iii) A structure group G of (continuous) linear operators acting on T such that for every $\Gamma \in G$, every $\alpha \in A$ and every $\tau_\alpha \in T_\alpha$ one has

$$\Gamma\tau_\alpha - \tau_\alpha \in T_{<\alpha} := \bigoplus_{\beta<\alpha} T_\beta.$$

Furthermore, $\Gamma\mathbf{1} = \mathbf{1}$ for every $\Gamma \in G$.

Now we have the regularity structure $\bar{T}$ given by all polynomials in $d+1$ indeterminates, let us call them $X_0, ..., X_d$, which denote the time and space directions respectively. Denote $X^k = X_0^{k_0} \cdots X_d^{k_d}$ with k a multi-index. The structure group can be defined by $\Gamma_h X^k = (X-h)^k$, $h \in \mathbb{R}^{d+1}$.

Given a scaling $\mathfrak{s} = (\mathfrak{s}_0, \mathfrak{s}_1, ..., \mathfrak{s}_d)$ of $\mathbb{R}^{d+1}$. We can associate the metric on $\mathbb{R}^{d+1}$ given by

$$\|z - z'\|_\mathfrak{s} := d_\mathfrak{s}(z, z') := \sum_{i=0}^{d} |z_i - z_i'|^{1/\mathfrak{s}_i}.$$

For $k = (k_0, ..., k_d)$ we define $|k|_\mathfrak{s} = \sum_{i=0}^d \mathfrak{s}_i k_i$.

Given a smooth compactly supported test function φ and a space-time coordinate $z = (t, x_1, ..., x_d) \in \mathbb{R}^{d+1}$, we denote by φ_z^λ the test function

$$\varphi_z^\lambda(s, y_1, ..., y_d) = \lambda^{-|\mathfrak{s}|}\varphi(\frac{s-t}{\lambda^{\mathfrak{s}_0}}, \frac{y_1 - x_1}{\lambda^{\mathfrak{s}_1}}, ..., \frac{y_d - x_d}{\lambda^{\mathfrak{s}_d}}).$$

Denoting by $\mathcal{B}_\alpha$ the set of smooth test function $\varphi : \mathbb{R}^{d+1} \mapsto \mathbb{R}$ that are supported in the centred ball of of radius 1 and such that their derivative of order up to $1 + |\alpha|$ are uniformly bounded by 1. We denote by $\mathcal{S}'$ the space of all distributions on $\mathbb{R}^{d+1}$ and denote by $L(E, F)$ the set of all continuous linear maps between the topological vector spaces E and F.

Definition 2.2 Given a regularity structure $\mathfrak{T}$, a model for $\mathfrak{T}$ consists of maps

$$\mathbb{R}^{d+1} \ni z \mapsto \Pi_z \in L(T, \mathcal{S}'), \quad \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \ni (z, z') \mapsto \Gamma_{zz'} \in G,$$

satisfying the algebraic compatibility conditions

$$\Pi_z \Gamma_{zz'} = \Pi_{z'}, \quad \Gamma_{zz'} \circ \Gamma_{z'z''} = \Gamma_{zz''},$$

as well as the analytical bounds

$$|\langle \Pi_z \tau, \varphi_z^\lambda \rangle| \lesssim \lambda^\alpha \|\tau\|, \quad \|\Gamma_{zz'} \tau\|_\beta \lesssim \|z - z'\|_{\mathfrak{s}}^{\alpha - \beta} \|\tau\|.$$

Here, the bounds are imposed uniformly over all $\tau \in T_\alpha$, all $\beta < \alpha \in A$ with $\alpha < \gamma$, $\gamma > 0$, and all text function $\varphi \in \mathcal{B}_r$ with $r = \inf A$. They are imposed locally uniformly in z and z'.

Then for every compact set $\mathfrak{R} \subset \mathbb{R}^{d+1}$ and any two models $Z = (\Pi, \Gamma)$ and $\bar{Z} = (\bar{\Pi}, \bar{\Gamma})$ we define

$$|||Z; \bar{Z}|||_{\gamma;\mathfrak{R}} := \sup_{z \in \mathfrak{R}} [\sup_{\varphi, \lambda, \alpha, \tau} \lambda^{-\alpha} |\langle \Pi_z \tau - \bar{\Pi}_z \tau, \varphi_z^\lambda \rangle| + \sup_{\|z - z'\|_{\mathfrak{s}} \leq 1} \sup_{\alpha, \beta, \tau} \|z - z'\|_{\mathfrak{s}}^{\beta - \alpha} \|\Gamma_{zz'} \tau - \bar{\Gamma}_{zz'} \tau\|_\beta],$$

where the suprema run over the same sets as before, but with $\|\tau\| = 1$.

On the regularity structure one can define multiplication $\star$, differentiation $\mathfrak{D}$ as in [Hai14]. Now we have the following definition for the spaces of distributions $\mathcal{C}_{\mathfrak{s}}^\alpha$, $\alpha < 0$, which is an extension of Hölder space to include $\alpha < 0$.

Definition 2.3 Let $\eta \in \mathcal{S}'$ and $\alpha < 0$. We say that $\eta \in \mathcal{C}_{\mathfrak{s}}^\alpha$ if the bound

$$|\eta(\varphi_z^\lambda)| \lesssim \lambda^\alpha,$$

holds uniformly over all $\lambda \in (0, 1]$, all $\varphi \in \mathcal{B}_\alpha$ and locally uniformly over $z \in \mathbb{R}^{d+1}$.

For every compact set $\mathfrak{R} \subset \mathbb{R}^{d+1}$, we will denote by $\|\eta\|_{\alpha;\mathfrak{R}}$ the seminorm given by

$$\|\eta\|_{\alpha;\mathfrak{R}} := \sup_{z \in \mathfrak{R}} \sup_{\varphi \in \mathcal{B}_\alpha} \sup_{\lambda \leq 1} \lambda^{-\alpha} |\eta(\varphi_z^\lambda)|.$$

We also write $\| \cdot \|_\alpha$ for the same expression with $\mathfrak{R} = \mathbb{R}^{d+1}$.

We also have Hölder spaces on the regularity structure. Consider $\mathfrak{P} = \{(t, x) : t = 0\}$. Given a subset $\mathfrak{R} \subset \mathbb{R}^{d+1}$ we also denote by $\mathfrak{R}_{\mathfrak{P}}$ the set

$$\mathfrak{R}_{\mathfrak{P}} = \{(z, \bar{z}) \in (\mathfrak{R} \setminus \mathfrak{P})^2 : z \neq \bar{z} \text{ and } \|z - \bar{z}\|_{\mathfrak{s}} \leq |t|^{\frac{1}{2}} \wedge |\bar{t}|^{\frac{1}{2}} \wedge 1\},$$

where $z = (t, x), \bar{z} = (\bar{t}, \bar{x})$.

Definition 2.4 Fix a regularity structure $\mathfrak{T}$ and a model (Π, Γ) and $\mathfrak{P}$ as above. Then for any $\gamma > 0$ and $\eta \in \mathbb{R}$, we set for $z = (t, x), \bar{z} = (\bar{t}, \bar{x})$ and every compact set $\mathfrak{R} \subset \mathbb{R}^{d+1}$,

$$\|f\|_{\gamma,\eta;\mathfrak{R}} := \sup_{z \in \mathfrak{R}\setminus\mathfrak{P}} \sup_{l<\gamma} \frac{\|f(z)\|_l}{|t|^{\frac{\eta-l}{2}\wedge 0}}.$$

The space $\mathcal{D}^{\gamma,\eta}$ then consists of all functions $f : \mathbb{R}^{d+1} \setminus \mathfrak{P} \to T_{<\gamma}$ such that for every compact set $\mathfrak{R} \subset \mathbb{R}^{d+1}$ one has

$$|||f|||_{\gamma,\eta;\mathfrak{R}} := \|f\|_{\gamma,\eta;\mathfrak{R}} + \sup_{(z,\bar{z})\in\mathfrak{R}_{\mathfrak{P}}} \sup_{l<\gamma} \frac{\|f(z) - \Gamma_{z\bar{z}} f(\bar{z})\|_l}{\|z - \bar{z}\|_{\mathfrak{s}}^{\gamma-l} (|t| \wedge |\bar{t}|)^{\frac{\eta-\gamma}{2}}} < \infty.$$

We also set

$$|||f;\bar{f}|||_{\gamma,\eta;\mathfrak{R}} := \|f - \bar{f}\|_{\gamma,\eta;\mathfrak{R}} + \sup_{(z,\bar{z})\in\mathfrak{R}_{\mathfrak{P}}} \sup_{l<\gamma} \frac{\|f(z) - \bar{f}(\bar{z}) - \Gamma_{z\bar{z}} f(\bar{z}) + \bar{\Gamma}_{z\bar{z}} \bar{f}(\bar{z})\|_l}{\|z - \bar{z}\|_{\mathfrak{s}}^{\gamma-l} (|t| \wedge |\bar{t})^{\frac{\eta-\gamma}{2}}} < \infty.$$

Given a regularity structure, we say that a subspace $V \subset T$ is a sector of regularity α if it is invariant under the action of the structure group G and it can be written as $V = \oplus_{\beta\in A} V_\beta$ with $V_\beta \subset T_\beta$, and $V_\beta = \{0\}$ for $\beta < \alpha$. We will use $\mathcal{D}^{\gamma,\eta}(V)$ to denote all functions in $\mathcal{D}^{\gamma,\eta}$ taking values in V.

Theorem 2.5 (cf. [Hai14, Proposition 6.9]) Given a regularity structure and a model (Π, Γ). Let $f \in \mathcal{D}^{\gamma,\eta}(V)$ for some sector V of regularity $\alpha \leq 0$, some $\gamma > 0$, and some $\eta \leq \gamma$. Then provided that $\alpha \wedge \eta > -2$, there exists a unique distribution $\mathcal{R}f \in \mathcal{C}_{\mathfrak{s}}^{\eta\wedge\alpha}$ such that

$$|(\mathcal{R}f - \Pi_z f(z))(\varphi_z^\lambda)| \lesssim \lambda^\gamma,$$

holds uniformly over $\lambda \in (0, 1]$ and $\varphi \in \mathcal{B}_r$ with φ_z^λ compactly supported away from $\mathfrak{P}$ and locally uniformly over $z \in \mathbb{R}^{d+1}$. Moreover, $(\Pi, \Gamma, f) \to \mathcal{R}f$ is jointly (locally) Lipschitz continuous with respect to the metric for (Π, Γ) and f defined in Definitions 2.2 and 2.4.

In order to define the integration against singular kernel K, Martin Hairer in [Hai14] introduced an abstract integration map $\mathcal{I} : T \to T$ to provide an "abstract" representation of $\mathcal{K}$ operating at the level of the regularity structure. In the regularity structure theory $\mathcal{I}$ is a linear map from T to T such that $\mathcal{I}T_\alpha \subset T_{\alpha+\beta}$ and $\mathcal{I}\bar{T} = 0$ and for every $\Gamma \in G, \tau \in T$ one has $\Gamma\mathcal{I}\tau - \mathcal{I}\Gamma\tau \in \bar{T}$.

Furthermore, we say that K is a β-regularising kernel if one can write $K = \sum_{n\geq 0} K_n$ where each of $K_n : \mathbb{R}^{d+1} \to \mathbb{R}$ is smooth and compactly supported in a ball of radius 2^{-n} around the origin. Furthermore, we assume that for every multi-index k, one has a constant C such that

$$\sup_x |D^k K_n(x)| \leq C 2^{n(d+1-\beta+|k|_{\mathfrak{s}})},$$

holds uniformly in n. Finally, we assume that $\int K_n(x)E(x)dx = 0$ for every polynomial E of degree at most N for some sufficiently large value of N.

Then we have the following results from [Hai14, Proposition 6.16].

Theorem 2.6 Let $\mathfrak{T} = (A, T, G)$ be a regularity structure and (Π, Γ) be a model for $\mathfrak{T}$. Let K be a β-regularising kernel for some $\beta > 0$, let $\mathcal{I}$ be an abstract integration map acting on some sector V of regularity $\alpha \leq 0$, and let Π be a model realising K for $\mathcal{I}$. Let $\gamma > 0$, $\eta \leq \gamma$. Then provided that $\alpha \wedge \eta > -2$, $\gamma + \beta, \eta + \beta$ not in $\mathbb{N}$, there exists a continuous linear operator $\mathcal{K}_\gamma : \mathcal{D}^{\gamma,\eta}(V) \to \mathcal{D}^{\bar{\gamma},\bar{\eta}}$ with $\bar{\gamma} = \gamma + \beta$ and $\bar{\eta} = (\eta \wedge \alpha) + \beta$, such that

$$\mathcal{R}\mathcal{K}_\gamma f = K * \mathcal{R}f,$$

holds for $f \in \mathcal{D}^{\gamma,\eta}(V)$.

In the following we use the notations $O_T = (-\infty, T] \times \mathbb{R}^d$ and use the shorthands $|||\cdot|||_{\gamma,\eta;T}$ as a short hand for $|||\cdot|||_{\gamma,\eta;O_T}$, and similarly for $|||\cdot|||_{\gamma,\eta;T}$. Moreover, we have for some $\theta > 0$

$$|||\mathcal{K}_\gamma 1_{t>0} f|||_{\bar{\gamma},\bar{\eta};T} \lesssim T^\theta |||f|||_{\gamma,\eta;T}.$$

2.2 NS equation

In this subsection we apply the regularity structure theory to 3D Navier-Stokes equations driven by space-time white noise. In this case the scaling $\mathfrak{s} = (2, 1, 1, 1)$, so that the scaling dimension of space-time is 5. Since the kernel $G^{ij}, i, j = 1, 2, 3$, given by the heat kernel composed with the Leray projection P has the scaling property $G^{ij}(\frac{t}{\delta^2}, \frac{x}{\delta}) = \delta^3 G^{ij}(t, x)$ for $\delta > 0$, by [Hai14, Lemma 5.5] it can be decomposed into $K^{ij} + R^{ij}, i, j = 1, 2, 3$, with K^{ij} is a 2-regularising kernel and $R^{ij} \in \mathcal{C}^\infty$. By [Hai14] we could choose K^{ij} is compactly supported and smooth away from the origin and such that it annihilates all polynomials up to some degree $r > 2$. Moreover, by [KT01] we could choose K^{ij} is of order -3, i.e. $|D^k K(z)| \leq C\|z\|_{\mathfrak{s}}^{-3-|k|_{\mathfrak{s}}}$ for every z with $\|z\|_{\mathfrak{s}} \leq 1$ and every multi-index k. We also use $D_j K$, $j = 1, 2, 3$, to represent the derivative of K with respect to the j-th space variable and $D_j K$ is also a 1-regularising kernel and of order -4 and $D_j R \in \mathcal{C}^\infty$.

Consider the regularity structure generated by SNS equation with $\beta = 2, -\frac{13}{5} < \alpha < -\frac{5}{2}$. In the regularity structure we use symbol Ξ^i to replace driving noise ξ^i. For $i, i_1 = 1, 2, 3$, we introduce the integration map $\mathcal{I}^{ii_1}$ associating with K^{ii_1} and the integration map $\mathcal{I}_k^{ii_1}$ for a multiindex k, which represents integration against $D^k K^{ii_1}$. We recall the following notations from [Hai14]: defining a set $\mathcal{F}$ by postulating that $\{\mathbf{1}, \Xi^i, X_j\} \subset \mathcal{F}$ and whenever $\tau, \bar{\tau} \in \mathcal{F}$, we have $\tau\bar{\tau} \in \mathcal{F}$ and $\mathcal{I}_k^{ij}(\tau) \in \mathcal{F}$; defining $\mathcal{F}^+$ as the set of all elements $\tau \in \mathcal{F}$ such that either $\tau = \mathbf{1}$ or $|\tau|_{\mathfrak{s}} > 0$ and such that, whenever τ can be written as $\tau = \tau_1\tau_2$ we have either $\tau_i = \mathbf{1}$ or $|\tau_i|_{\mathfrak{s}} > 0$; $\mathcal{H}, \mathcal{H}^+$ denote the set of finite linear combinations of all elements in $\mathcal{F}, \mathcal{F}^+$, respectively. Here for each $\tau \in \mathcal{F}$ a weight $|\tau|_{\mathfrak{s}}$ which is obtained by setting $|\mathbf{1}|_{\mathfrak{s}} = 0$,

$$|\tau\bar{\tau}|_{\mathfrak{s}} = |\tau|_{\mathfrak{s}} + |\bar{\tau}|_{\mathfrak{s}},$$

for any two formal expressions τ and $\bar{\tau}$ in $\mathcal{F}$, and such that

$$|\Xi^i|_{\mathfrak{s}} = \alpha, \quad |X_i|_{\mathfrak{s}} = \mathfrak{s}_i, \quad |\mathcal{I}_k^{ii_1}(\tau)|_{\mathfrak{s}} = |\tau|_{\mathfrak{s}} + 2 - |k|_{\mathfrak{s}}.$$

Define a linear projection operator $P_+ : \mathcal{H} \to \mathcal{H}_+$ by imposing that

$$P_+\tau = \tau, \quad \tau \in \mathcal{F}_+, \quad P_+\tau = 0, \quad \tau \in \mathcal{F} \setminus \mathcal{F}_+,$$

and two linear maps $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}_+$ and $\Delta^+ : \mathcal{H}_+ \to \mathcal{H}_+ \otimes \mathcal{H}_+$ by

$$\Delta\mathbf{1} = \mathbf{1} \otimes \mathbf{1}, \quad \Delta^+\mathbf{1} = \mathbf{1} \otimes \mathbf{1},$$

$$\Delta X_i = X_i \otimes \mathbf{1} + \mathbf{1} \otimes X_i, \quad \Delta^+ X_i = X_i \otimes \mathbf{1} + \mathbf{1} \otimes X_i,$$
$$\Delta \Xi^i = \Xi^i \otimes \mathbf{1},$$

and recursively by

$$\Delta(\tau\bar{\tau}) = (\Delta\tau)(\Delta\bar{\tau})$$
$$\Delta(\mathcal{I}_k^{ij}\tau) = (\mathcal{I}_k^{ij} \otimes I)\Delta\tau + \sum_{l,m} \frac{X^l}{l!} \otimes \frac{X^m}{m!}(P_+\mathcal{I}_{k+l+m}^{ij}\tau),$$
$$\Delta^+(\tau\bar{\tau}) = (\Delta^+\tau)(\Delta^+\bar{\tau})$$
$$\Delta^+(\mathcal{I}_k^{ij}\tau) = (I \otimes \mathcal{I}_k^{ij}\tau) + \sum_l (P_+\mathcal{I}_{k+l}^{ij} \otimes \frac{(-X)^l}{l!})\Delta\tau.$$

To apply the regularity structure theory we write the equation as follows: for $i = 1, 2, 3$

$$\begin{aligned} \partial_t v_1^i =& \nu \sum_{i_1=1}^3 P^{ii_1} \Delta v_1^{i_1} + \sum_{i_1=1}^3 P^{ii_1} \xi^{i_1}, \quad div v_1 = 0, \\ \partial_t v^i =& \nu \sum_{i_1=1}^3 P^{ii_1} \Delta v^{i_1} - \sum_{i_1,j=1}^3 P^{ii_1} \frac{1}{2} D_j[(v^{i_1} + v_1^{i_1})(v^j + v_1^j)], \quad div v = 0. \end{aligned} \tag{2.1}$$

Then $v_1 + v$ is the solution to the 3D Navier-Stokes equations driven by space-time white noise.

Now we consider the second equation in (2.1). Define for $i, j = 1, 2, 3$,

$$\mathfrak{M}_F^{ij} = \{1, \mathcal{I}^{ii_1}(\Xi_{i_1}), \mathcal{I}^{jj_1}(\Xi_{j_1}), \mathcal{I}^{ii_1}(\Xi_{i_1})\mathcal{I}^{jj_1}(\Xi_{j_1}), U_i, U_j, U_iU_j, \mathcal{I}^{ii_1}(\Xi_{i_1})U_j, U_i\mathcal{I}^{jj_1}(\Xi_{j_1}), i_1, j_1 = 1, 2, 3\}.$$

Then we build subsets $\{\mathcal{P}_n^i\}_{n\geq 0}$ and $\{\mathcal{W}_n\}_{n\geq 0}$ by the following algorithm. Set $\mathcal{W}_0^{ij} = \mathcal{P}_0^i = \varnothing$ and

$$\mathcal{W}_n^{ij} = \mathcal{W}_{n-1}^{ij} \cup \bigcup_{\mathcal{Q}\in\mathfrak{M}_F^{ij}} \mathcal{Q}(\mathcal{P}_{n-1}^i, \mathcal{P}_{n-1}^j),$$
$$\mathcal{P}_n^i = \{X^k\} \cup \{\mathcal{I}_{i_2}^{ii_1}(\tau) : \tau \in \mathcal{W}_{n-1}^{i_1 i_2}, i_1, i_2 = 1, 2, 3\},$$

and

$$\mathcal{F}_F := \bigcup_{n\geq 0} \bigcup_{i,j=1}^3 \mathcal{W}_n^{ij}, \quad \mathcal{F}_F^{ij} := \bigcup_{n\geq 0} \mathcal{W}_n^{ij}, i, j = 1, 2, 3.$$

Then $\mathcal{F}_F$ contains the elements required to describe both the solution and the terms in the equation (2.1). We denote by $\mathcal{H}_F, \mathcal{H}_F^{ij}, i, j = 1, 2, 3$, the set of finite linear combinations of elements in $\mathcal{F}_F$, $\mathcal{F}_F^{ij}$, respectively. Now by using the theory of regularity structure (see [Hai14, Section 8]) we can also define a structure group G_F of linear operators acting on $\mathcal{H}_F$ satisfying Definition 2.1 as follows: For group-like elements $g \in \mathcal{H}_+^*$, the dual of $\mathcal{H}^+$, $\Gamma_g : \mathcal{H} \to \mathcal{H}, \Gamma_g\tau = (I \otimes g)\Delta\tau$. By [Hai14, Theorem 8.24] we construct the following regularity structure.

Theorem 2.7 Let $T = \mathcal{H}_F$ with $T_\gamma = \langle\{\tau \in \mathcal{F}_F : |\tau|_\mathfrak{s} = \gamma\}\rangle$, $A = \{|\tau|_\mathfrak{s} : \tau \in \mathcal{F}_F\}$ and G_F be obtained above. Then $\mathfrak{T}_F = (A, \mathcal{H}_F, G_F)$ defines a regularity structure $\mathfrak{T}$.

Proof In our case, the nonlinearity is locally subcritical. (i) (ii) in Definition 2.1 can be checked easily. (iii) in Definition 2.1 follows from the definition of Δ and Γ_g. □

Now we come to construct suitable models associated with the regularity structure above. Given any continuous approximation ξ_ε to the driving noise ξ, we set for $x, y \in \mathbb{R}^4$

$$(\Pi_x^{(\varepsilon)}\Xi_i)(y) = \xi_\varepsilon^i(y), \quad (\Pi_x^{(\varepsilon)}X^k)(y) = (y-x)^k,$$

and recursively define

$$(\Pi_x^{(\varepsilon)}\tau\bar{\tau})(y) = (\Pi_x^{(\varepsilon)}\tau)(y)(\Pi_x^{(\varepsilon)}\bar{\tau})(y),$$

and

$$(\Pi_x^{(\varepsilon)}\mathcal{I}_k^{ij}\tau)(y) = \int D_1^k K^{ij}(y-z)(\Pi_x^{(\varepsilon)}\tau)(z)dz + \sum_l \frac{(y-x)^l}{l!} f_x^{(\varepsilon)}(P_+\mathcal{I}_{k+l}^{ij}\tau). \tag{2.2}$$

Here $f_x^{(\varepsilon)}(\mathcal{I}_l^{ij}\tau)$ are defined by

$$f_x^{(\varepsilon)}(\mathcal{I}_l^{ij}\tau) = -\int D_1^l K^{ij}(x,z)(\Pi_x^{(\varepsilon)}\tau)(z)dz. \tag{2.3}$$

Furthermore we impose $f_x^{(\varepsilon)}(X_i) = -x_i$, $f_x^{(\varepsilon)}(\tau\bar{\tau}) = f_x^{(\varepsilon)}(\tau)f_x^{(\varepsilon)}(\bar{\tau})$ and extend this to all of $\mathcal{H}^+$ by linearity. Then define

$$\Gamma_{xy}^{(\varepsilon)} = \Gamma_{f_x^{(\varepsilon)}} \circ (\Gamma_{f_y^{(\varepsilon)}})^{-1}, \tag{2.4}$$

where $\Gamma_{f_x^{(\varepsilon)}}\tau := (I \otimes f_x^{(\varepsilon)})\Delta\tau$ for $\tau \in \mathcal{H}$.

Now by [Hai14, Proposition 8.27] we have

Proposition 2.8 $(\Pi^{(\varepsilon)}, \Gamma^{(\varepsilon)})$ is a model for $\mathfrak{T}_F$ constructed in Theorem 2.7.

Definition 2.9 A model (Π, Γ) for $\mathfrak{T}$ is admissible if it satisfies $(\Pi_x X^k)(y) = (y-x)^k$ as well as (2.2), (2.3) and (2.4). We denote by $\mathcal{M}_F$ the set of admissible models.

Set

$$\begin{aligned}\mathcal{F}_0 =\{&\mathbf{1}, \Xi_i, \mathcal{I}^{ii_1}(\Xi_{i_1}), \mathcal{I}^{ii_1}(\Xi_{i_1})\mathcal{I}^{jj_1}(\Xi_{j_1}), \mathcal{I}_j^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})), \mathcal{I}_j^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{jj_1}(\Xi_{j_1})), \mathcal{I}_j^{ii_1}(\mathcal{I}^{jj_1}(\Xi_{j_1})),\\ &\mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2}))\mathcal{I}^{jj_1}(\Xi_{j_1}), \mathcal{I}_k^{ii_1}(\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{jj_1}(\Xi_{j_1}), \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{jj_1}(\Xi_{j_1}),\\ &\mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}_l^{jj_1}(\mathcal{I}^{j_1j_2}(\Xi_{j_2})\mathcal{I}^{ll_1}(\Xi_{l_1})), \mathcal{I}_l^{ii_1}(\mathcal{I}_k^{i_1i_2}(\mathcal{I}^{i_2i_3}(\Xi_{i_3})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{ll_1}(\Xi_{l_1}))\mathcal{I}^{jj_1}(\Xi_{j_1}),\\ &\mathcal{I}_l^{ii_1}(\mathcal{I}_k^{ll_1}(\mathcal{I}^{l_1l_2}(\Xi_{l_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{i_1i_2}(\Xi_{i_2}))\mathcal{I}^{jj_1}(\Xi_{j_1}), i,j,k,l,i_1,i_2,i_3,j_1,j_2,k_1,l_1,l_2 = 1,2,3\}\end{aligned}$$

and

$$\mathcal{F}_* = \{\mathcal{I}^{ik}(\Xi_k), \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{jj_1}(\Xi_{j_1}), i,k,i_1,i_2,j_1,k_1 = 1,2,3\}.$$

Then $\mathcal{F}_0 \subset \mathcal{F}_F$ contains every $\tau \in \mathcal{F}_F$ with $|\tau|_\mathfrak{s} \leq 0$ and for every $\tau \in \mathcal{F}_0$, $\Delta\tau \in \langle\mathcal{F}_0\rangle \otimes \langle\mathrm{Alg}(\mathcal{F}_*)\rangle$. Here $\langle\mathcal{F}_0\rangle$ denotes the linear span of $\mathcal{F}_0$ and $\mathrm{Alg}(\mathcal{F}_*)$ denotes the set of all elements in $\mathcal{F}_+$ of the form $X^k\prod_{i,i_1,i_2}\mathcal{I}_{l_i}^{i_1i_2}\tau_i$ for some multiindices k and l_i such that $|\mathcal{I}_{l_i}^{ii_1}\tau_i|_\mathfrak{s} > 0$ and $\tau_i \in \mathcal{F}_*$.

Then for any constants $C^1_{ii_1jj_1}, C^2_{ii_1i_2jj_1j_2kk_1ll_1}, C^3_{ii_1i_2i_3kk_1ll_1jj_1}, C^4_{ii_1i_2kk_1ll_1l_2jj_1}, i, j, k, l, i_1, i_2, i_3, j_1, k_1, l_1, l_2 = 1, 2, 3$, we define a linear map M on $\langle \mathcal{F}_0 \rangle$ by

$$\begin{aligned}
&M(\mathcal{I}^{ii_1}(\Xi_{i_1})\mathcal{I}^{jj_1}(\Xi_{j_1})) = \mathcal{I}^{ii_1}(\Xi_{i_1})\mathcal{I}^{jj_1}(\Xi_{j_1}) - C^1_{ii_1jj_1}\mathbf{1},\\
&M(\mathcal{I}^{ii_1}_k(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{jj_1}_l(\mathcal{I}^{j_1j_2}(\Xi_{j_2})\mathcal{I}^{ll_1}(\Xi_{l_1})))\\
=&\mathcal{I}^{ii_1}_k(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{jj_1}_l(\mathcal{I}^{j_1j_2}(\Xi_{j_2})\mathcal{I}^{ll_1}(\Xi_{l_1})) - C^2_{ii_1i_2jj_1j_2kk_1ll_1}\mathbf{1},\\
&M(\mathcal{I}^{ii_1}_l(\mathcal{I}^{i_1i_2}_k(\mathcal{I}^{i_2i_3}(\Xi_{i_3})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{ll_1}(\Xi_{l_1}))\mathcal{I}^{jj_1}(\Xi_{j_1}))\\
=&\mathcal{I}^{ii_1}_l(\mathcal{I}^{i_1i_2}_k(\mathcal{I}^{i_2i_3}(\Xi_{i_3})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{ll_1}(\Xi_{l_1}))\mathcal{I}^{jj_1}(\Xi_{j_1}) - C^3_{ii_1i_2i_3kk_1ll_1jj_1}\mathbf{1},\\
&M(\mathcal{I}^{ii_1}_l(\mathcal{I}^{ll_1}_k(\mathcal{I}^{l_1l_2}(\Xi_{l_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{i_1i_2}(\Xi_{i_2}))\mathcal{I}^{jj_1}(\Xi_{j_1}))\\
=&\mathcal{I}^{ii_1}_l(\mathcal{I}^{ll_1}_k(\mathcal{I}^{l_1l_2}(\Xi_{l_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{i_1i_2}(\Xi_{i_2}))\mathcal{I}^{jj_1}(\Xi_{j_1}) - C^4_{ii_1i_2kk_1ll_1l_2jj_1}\mathbf{1},
\end{aligned} \tag{2.5}$$

as well as $M(\tau) = \tau$ for the remaining basis vectors in $\mathcal{F}_0$. We claim that for any $\tau \in \mathcal{F}_0$,

$$\Delta^M \tau = (M\tau) \otimes 1. \tag{2.6}$$

Since τ satisfies $M\tau = \tau - C\mathbf{1}$ for any $\tau \in \mathcal{F}_0$, it is easy to check that (2.6) holds.

For $\tau = \mathcal{I}^{ii_1}_j(\mathcal{I}^{i_1i_2}(\Xi_{i_2}))$, $i, i_1, i_2, j = 1, 2, 3$, we have

$$\Delta^+\mathcal{I}^{ii_1}_j(\mathcal{I}^{i_1i_2}(\Xi_{i_2})) = \mathcal{I}^{ii_1}_j(\mathcal{I}^{i_1i_2}(\Xi_{i_2})) \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{I}^{ii_1}_j(\mathcal{I}^{i_1i_2}(\Xi_{i_2})).$$

$$(\mathcal{A}\hat{M}\mathcal{A} \otimes \hat{M})\Delta^+\mathcal{I}^{ii_1}_j(\mathcal{I}^{i_1i_2}(\Xi_{i_2})) = \mathcal{I}^{ii_1}_j(\mathcal{I}^{i_1i_2}(\Xi_{i_2})) \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{I}^{ii_1}_j(\mathcal{I}^{i_1i_2}(\Xi_{i_2})),$$

It follows that

$$\hat{\Delta}^M\mathcal{I}^{ii_1}_j(\mathcal{I}^{i_1i_2}(\Xi_{i_2})) = \mathcal{I}^{ii_1}_j(\mathcal{I}^{i_1i_2}(\Xi_{i_2})) \otimes \mathbf{1}.$$

For $\tau = \mathcal{I}^{ii_1}_l(\tau_1)$, where $\tau_1 = \mathcal{I}^{i_1i_2}_k(\mathcal{I}^{i_2i_3}(\Xi_{i_3})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{ll_1}(\Xi_{l_1})$, $i, i_1, i_2, i_3, k, k_1, l, l_1 = 1, 2, 3$, we have

$$\Delta^+\mathcal{I}^{ii_1}_l(\tau_1) = \mathcal{I}^{ii_1}_l(\tau_1) \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{I}^{ii_1}_l(\tau_1).$$

$$(\mathcal{A}\hat{M}\mathcal{A} \otimes \hat{M})\Delta^+\mathcal{I}^{ii_1}_l(\tau_1) = \mathcal{I}^{ii_1}_l(\tau_1) \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{I}^{ii_1}_l(\tau_1),$$

which implies that

$$\hat{\Delta}^M\mathcal{I}^{ii_1}_l(\tau_1) = \mathcal{I}^{ii_1}_l(\tau_1) \otimes \mathbf{1}.$$

Similarly, we obtain

$$\hat{\Delta}^M\mathcal{I}^{il}_{i_1}(\tau_1) = \mathcal{I}^{il}_{i_1}(\tau_1) \otimes \mathbf{1}.$$

As a consequence of the expression, we have M belongs to the renormalisation group $\mathfrak{R}_0$ defined in [Hai14, Definition 8.43]. Then by [Hai14, Theorem 8.46] we can define (Π^M, Γ^M) and it is an admissible model for $\mathfrak{T}_F$ on $\langle \mathcal{F}_0 \rangle$. Furthermore, it extends uniquely to an admissible model for all of $\mathfrak{T}_F$.

By (2.6) we also have

$$\Pi^M_x \tau = \Pi_x M\tau.$$

Now we come to the equation. First we define for any $\alpha_0 < 0$ and compact set $\mathfrak{K}$ the norm

$$|\xi|_{\alpha_0;\mathfrak{K}} = \sup_{s\in\mathbb{R}} \|\xi 1_{t\geq s}\|_{\alpha_0;\mathfrak{K}},$$

and we denote by $\bar{\mathcal{C}}^{\alpha_0}_{\mathfrak{s}}$ the intersections of the completions of smooth functions under $|\cdot|_{\alpha_0;\mathfrak{K}}$ for all compact sets $\mathfrak{K}$.

Since $\alpha < -\frac{5}{2}$, Theorem 2.5 does not apply to $\mathbf{R}^+\Xi^i$, where $\mathbf{R}^+ : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ is given by $\mathbf{R}^+(t,x) = 1$ for $t > 0$ and $\mathbf{R}^+(t,x) = 0$ otherwise. To define the reconstruction operator for $\mathbf{R}^+\Xi^i$ by hand, we need the following results, which has been proved by [Hai14, Proposition 9.5].

Proposition 2.10 Let $\xi = (\xi^1,\xi^2,\xi^3)$, with $\xi^i, i = 1,2,3$ being independent white noise on $\mathbb{R}\times\mathbb{T}^3$, which we extend periodically to $\mathbb{R}^4$. Let $\rho : \mathbb{R}^4 \to \mathbb{R}$ be a smooth compactly supported function integrating to 1, set $\rho_\varepsilon(t,x) = \varepsilon^{-5}\rho(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon})$ and define $\xi^i_\varepsilon = \rho_\varepsilon * \xi^i$. Then for every $i, i_1 = 1,2,3$, $K^{ii_1} * \xi^{i_1} \in C(\mathbb{R},\mathcal{C}^{\alpha+2}(\mathbb{R}^3))$ almost surely. Moreover, for every compact set $\mathfrak{K}\subset\mathbb{R}^4$ and every $0 < \theta < -\alpha - \frac{5}{2}$ we have

$$E|\xi^i - \xi^i_\varepsilon|_{\alpha;\mathfrak{K}} \lesssim \varepsilon^\theta.$$

Finally for every $0 < \kappa < -\alpha - \frac{5}{2}$, we have the bound

$$E\sup_{t\in[0,1]}\|K^{ii_1} * \xi^{i_1}(t,\cdot) - K^{ii_1} * \xi^{i_1}_\varepsilon)(t,\cdot)\|_{\alpha+2} \lesssim \varepsilon^\kappa.$$

Now we reformulate the fixed point map as

$$\begin{aligned} v^i_1 =& \sum_{i_1=1}^3 (\mathcal{K}^{ii_1}_{\bar\gamma} + R^{ii_1}_\gamma\mathcal{R})\boldsymbol{R}^+\Xi^{i_1}, \\ u^i =& -\frac{1}{2}\sum_{i_1,j=1}^3 ((\mathcal{D}_j\mathcal{K}^{ii_1})_{\bar\gamma} + (D_jR^{ii_1})_\gamma\mathcal{R})\boldsymbol{R}^+(u^{i_1}\star u^j) + v^i_1 + \sum_{i_1=1}^3 G^{ii_1}u^{i_1}_0. \end{aligned} \tag{2.7}$$

Here for $i, i_1, j = 1,2,3$, $\mathcal{K}^{ii_1}_{\bar\gamma}$ and $(\mathcal{D}_j\mathcal{K}^{ii_1})_{\bar\gamma}$ are the continuous linear operators obtained by Theorem 2.6 associated with the kernel K^{ii_1} and $D_jK^{ii_1}$ respectively,

$$R^{ii_1}_\gamma : \mathcal{C}^\alpha_{\mathfrak{s}} \to \mathcal{D}^{\gamma,\eta}, (R^{ii_1}_\gamma f)(z) = \sum_{|k|_{\mathfrak{s}}<\gamma}\frac{X^k}{k!}\int D^k_1 R^{ii_1}(z,\bar z)f(\bar z)d\bar z,$$

$$(D_jR^{ii_1})_\gamma : \mathcal{C}^\alpha_{\mathfrak{s}} \to \mathcal{D}^{\gamma,\eta}, (R^{ii_1}_\gamma f)(z) = \sum_{|k|_{\mathfrak{s}}<\gamma}\frac{X^k}{k!}\int D^k_1 (D_jR^{ii_1})(z,\bar z)f(\bar z)d\bar z,$$

and $\gamma,\bar\gamma$ will be chosen below and we define $\mathcal{R}\boldsymbol{R}^+\Xi$ as the distribution $\xi\mathbf{1}_{t\geq 0}$.

For the second equaiton of (2.7), define

$$V^i := \oplus^3_{i_1,j=1}\mathcal{I}^{ii_1}_j(\mathcal{H}^{i_1j}_F) \oplus \mathrm{span}\{\mathcal{I}^{ii_1}(\Xi_{i_1})\} \oplus \bar T.$$

$$V = V^1\times V^2\times V^3.$$

We define the local map $F^i_j : V \to T$ by for $\tau = (\tau^1,\tau^2,\tau^3)$ with $\tau^i\in V^i$,

$$F^i_j(\tau) := \tau^i \star \tau^j.$$

For $\gamma > 0, \eta \in \mathbb{R}$ we define

$$\mathcal{D}^{\gamma,\eta}(V) := \mathcal{D}^{\gamma,\eta}(V_1) \times \mathcal{D}^{\gamma,\eta}(V_2) \times \mathcal{D}^{\gamma,\eta}(V_3).$$

$$(\mathcal{D}^{\gamma,\eta})^3 := \mathcal{D}^{\gamma,\eta} \times \mathcal{D}^{\gamma,\eta} \times \mathcal{D}^{\gamma,\eta}.$$

Lemma 2.11 For $\gamma > |\alpha+2|$ and $-1 < \eta \leq \alpha+2$, the map $u \mapsto F_j^i(u)$ is locally Lipschitz continuous from $\mathcal{D}^{\gamma,\eta}(V)$ into $\mathcal{D}^{\gamma+\alpha+2,2\eta}$.

Proof This is a consequence of [Hai14, Proposition 6.12, Proposition 6.15]. □

Now for γ, η as in Lemma 2.11 and $u_0^{i_1} \in \mathcal{C}^\eta(\mathbb{R}^3), i_1 = 1,2,3$, periodic, we have $P^{ii_1}u_0^{i_1} \in \mathcal{C}^\eta(\mathbb{R}^3), i, i_1 = 1,2,3$, which by [Hai14, Lemma 7.5] implies that $G^{ii_1}u_0^{i_1} \in \mathcal{D}^{\gamma,\eta}, i, i_1 = 1,2,3$. By Proposition 2.10 and [Hai14, Remark 6.17] we also have for $i = 1,2,3$, $v_1^i \in \mathcal{D}^{\gamma,\eta}$. Now we can apply fixed point argument in $(\mathcal{D}^{\gamma,\eta})^3$ to obtain existence and uniqueness of local solutions.

Proposition 2.12 Let $\mathfrak{T}_F$ be the regularity structure as above associated to NS equation with $\alpha \in (-\frac{13}{5}, -\frac{5}{2})$. Let $\eta \in (-1, \alpha+2]$ and let $Z = (\Pi, \Gamma) \in \mathcal{M}_F$ be an admissible model for $\mathfrak{T}_F$ with the additional properties that for $i, i_1 = 1,2,3$, $\xi^i := \mathcal{R}\Xi^i$ belongs to $\bar{\mathcal{C}}^\alpha_\mathfrak{s}$ and that $K^{ii_1} * \xi^{i_1} \in C(\mathbb{R}, \mathcal{C}^\eta)$. Then there exists a maximal solution $\mathcal{S}^L \in (\mathcal{D}^{\gamma,\eta})^3$ for the equation (2.7).

Proof Consider the second equation in (2.7) and we have that u takes values in a sector of regularity $\zeta = \alpha+2$ and $F_j^i, i, j = 1,2,3$, takes value in a sector of regularity $\bar{\zeta} = 2\alpha+4$ satisfying $\zeta < \bar{\zeta}+1$. For η and γ as in Lemma 2.11 we have $\bar{\eta} = 2\eta$ and $\gamma > \bar{\gamma} = \gamma+\alpha+2 > 0$ and $\bar{\gamma} > \gamma+1$. By Lemma 2.11 for $i, j = 1,2,3$, F_j^i is locally Lipschitz continuous from $\mathcal{D}^{\gamma,\eta}$ to $\mathcal{D}^{\bar{\gamma},\bar{\eta}}$. Then $\eta < (\bar{\eta} \wedge \bar{\zeta}) + 1$ and $(\bar{\eta} \wedge \bar{\zeta}) + 2 > 0$ are satisfied by assumption. Denote by $M_F^i(u)$ the right hand side of the second equation in (2.7). By [Hai14, Theorem 7.1, Lemma 7.3] and local Lipschitz continuity of F_j^i we obtain that there exist $\kappa > 0$ such that

$$\begin{aligned}\sum_{i=1}^3 |||M_F^i(u) - M_F^i(\bar{u})|||_{\gamma,\eta;T} &\lesssim T^\kappa \sum_{i,j=1}^3 |||F_j^i(u) - F_j^i(\bar{u})|||_{\bar{\gamma},\bar{\eta};T} \\ &\lesssim T^\kappa \sum_{i=1}^3 |||u^i - \bar{u}^i|||_{\gamma,\eta;T}.\end{aligned}$$

Then we obtain local existence and uniqueness of the solutions by similar arguments as in the proof of [Hai14, Theorem 7.8]. Here we consider the solution is vector valued and the corresponding norm is the sum of the norm for each component. To extend this local map up to the first time where $\sum_{i=1}^3 \|(\mathcal{R}u^i)(t,\cdot)\|_\eta$ blows up, we write $u = v_1 + v_2 + v_3$ with v_1 in (2.7) and

$$\begin{aligned}v_2^i &= -\frac{1}{2}\sum_{i_1,j=1}^3 ((\mathcal{D}_j\mathcal{K}^{ii_1})_{\bar{\gamma}} + (D_jR^{ii_1})_\gamma\mathcal{R})\boldsymbol{R}^+(v_1^{i_1} \star v_1^j), \\ v_3^i &= -\frac{1}{2}\sum_{i_1=1}^3((\mathcal{D}_j\mathcal{K}^{ii_1})_{\bar{\gamma}} + (D_jR^{ii_1})_\gamma\mathcal{R})\boldsymbol{R}^+[((v_3^{i_1} + v_2^{i_1}) \star (v_3^j + v_2^j)) \\ &\quad + ((v_3^{i_1} + v_2^{i_1}) \star v_1^j) + (v_1^{i_1} \star (v_3^j + v_2^j))] + \sum_{i_1=1}^3 G^{ii_1}u_0^{i_1},\end{aligned}$$

In this case v_3^i takes values in a function-like sector with $\zeta = 3\alpha + 8$ and we can use similar arguments as in the proof of [Hai14, Proposition 7.11] to conclude results. $\square$

Remark 2.13 Here the lower bound for η is -1, which seems to be optimal by the regularity structure theory. The reason for this is as follows: the nonlinear term always contains $v \star v$ and thus $\bar{\eta} \leq 2\eta$ which should be larger than -2 required by [Hai14, Theorem 7.8]. As a result, $\eta > -1$.

Denote $O := [-1, 2] \times \mathbb{R}^3$. Given a model $Z = (\Pi, \Gamma)$ for $\mathfrak{T}_F$, a periodic initial condition $u_0 \in (\mathcal{C}_{\mathfrak{s}}^{\eta})^3$, and some cut-off value $L > 0$, we denote by $u = \mathcal{S}^L(u_0, Z) \in (\mathcal{D}^{\gamma,\eta})^3$ and $T = T^L(u_0, Z) \in \mathbb{R}_+ \cup \{+\infty\}$ the (unique) modelled distribution and time such that (2.7) holds on $[0, T]$, such that $\|(\mathcal{R}u)(t, \cdot)\|_\eta < L$ for $t < T$, and such that $\|(\mathcal{R}u)(t, \cdot)\|_\eta \geq L$ for $t \geq T$. Then by [Hai14, Corollary 7.12] we obtain the following results.

Proposition 2.14 Let $L > 0$ be fixed. In the setting of Proposition 2.12, for every $\varepsilon > 0$ and $C > 0$ there exists $\delta > 0$ such that setting $T = 1 \wedge T^L(u_0, Z) \wedge T^L(\bar{u}_0, \bar{Z})$ we have

$$\|\mathcal{S}^L(u_0, Z) - \mathcal{S}^L(\bar{u}_0, \bar{Z})\|_{\gamma,\eta;T} \leq \varepsilon,$$

for all $u_0, \bar{u}_0, Z, \bar{Z}$ such that $|||Z|||_{\gamma;O} \leq C, |||\bar{Z}|||_{\gamma;O} \leq C, \|u_0\|_\eta \leq L/2, \|\bar{u}_0\|_\eta \leq L/2, \|u_0 - \bar{u}_0\|_\eta \leq \delta$, and $|||Z; \bar{Z}|||_{\gamma;O} \leq \delta$ and

$$|\xi|_{\alpha;O} + |\bar{\xi}|_{\alpha;O} \leq C,$$

$$\sum_{i,i_1=1}^{3} \sup_{t\in[0,1]} (\|(K^{ii_1} * \xi^{i_1})(t, \cdot)\|_\eta + \|(K^{ii_1} * \bar{\xi}^{i_1})(t, \cdot)\|_\eta) \leq C,$$

as well as

$$|\xi - \bar{\xi}|_{\alpha;O} \leq \delta,$$

$$\sum_{i,i_1=1}^{3} \sup_{t\in[0,1]} (\|(K^{ii_1} * \xi^{i_1})(t, \cdot) - (K^{ii_1} * \bar{\xi}^{i_1})(t, \cdot)\|_\eta) \leq \delta,$$

where $\bar{\xi}^i = \bar{\mathcal{R}}\Xi^i$ and $\bar{\mathcal{R}}$ is the reconstruction operator associated to $\bar{Z}$.

Proposition 2.15 Given a continuous periodic vector $\xi_\varepsilon = (\xi_\varepsilon^1, \xi_\varepsilon^2, \xi_\varepsilon^3)$, denote by $Z_\varepsilon = (\Pi^{(\varepsilon)}, \Gamma^{(\varepsilon)})$ the associated canonical model realising $\mathfrak{T}_F$ given in Proposition 2.8. Let M be the renormalisation map defined in (2.5). Then for every $L > 0$ and periodic $u_0 \in C^\eta(\mathbb{R}^3; \mathbb{R}^3)$, $u_\varepsilon = \mathcal{R}S^L(u_0, Z_\varepsilon)$ satisfies the following equation on $[0, T^L(u_0, Z_\varepsilon)]$ in the mild sense:

$$\partial_t u_\varepsilon = \Delta u_\varepsilon - P(u_\varepsilon \cdot \nabla u_\varepsilon) + P\xi_\varepsilon, \quad divu_\varepsilon = 0, \quad u_\varepsilon(0, x) = Pu_0.$$

Furthermore, $u_\varepsilon^M = \mathcal{R}S^L(u_0, MZ_\varepsilon)$ satisfies the following equation on $[0, T^L(u_0, MZ_\varepsilon)]$ in the mild sense:

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + P\xi_\varepsilon - \frac{1}{2}P\sum_{j=1}^{3} D_j(u_\varepsilon u_\varepsilon^j),$$
$$divu_\varepsilon = 0, \quad u_\varepsilon(0, x) = Pu_0.$$

Proof The first result follows from the fact that ξ_ε is a continuous function and a similar argument as in the proof of [Hai14, Proposition 9.4].

Consider for $i = 1, 2, 3$, u^i is the solution to the abstract fixed point map that can be expanded as

$$\begin{aligned} u^i =& \sum_{i_1=1}^{3} \mathcal{I}^{ii_1}(\Xi_{i_1}) - \frac{1}{2} \sum_{j,i_1,i_2,j_1=1}^{3} \mathcal{I}_j^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{jj_1}(\Xi_{j_1})) + \varphi^i \mathbf{1} - \frac{1}{2} \sum_{j,i_1,j_1=1}^{3} \mathcal{I}_j^{ii_1}(\mathcal{I}^{jj_1}(\Xi_{j_1}))\varphi^{i_1} \\ & - \frac{1}{2} \sum_{j,i_1,i_2=1}^{3} \mathcal{I}_j^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2}))\varphi^j + \frac{1}{4} \sum_{i_1,i_2,i_3,j,j_1,k,k_1=1}^{3} \mathcal{I}_k^{ii_1}(\mathcal{I}_j^{i_1i_2}(\mathcal{I}^{i_2i_3}(\Xi_{i_3})\mathcal{I}^{jj_1}(\Xi_{j_1}))\mathcal{I}^{kk_1}(\Xi_{k_1})) \\ & + \frac{1}{4} \sum_{i_1,i_2,j,j_1,k,k_1,k_2=1}^{3} \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}_j^{kk_1}(\mathcal{I}^{k_1k_2}(\Xi_{k_2})\mathcal{I}^{jj_1}(\Xi_{j_1}))) + \rho_u. \end{aligned}$$

Here every component of ρ_u has homogeneity strictly greater than $3\alpha + 8$. Then for

$$F_j^i(u) := u^i u^j,$$

we have

$$\begin{aligned} F_j^i(u) =& \frac{1}{4} \sum_{i_1,i_2,j_1,j_2,k,k_1,l,l_1=1}^{3} \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}_l^{jj_1}(\mathcal{I}^{j_1j_2}(\Xi_{j_2})\mathcal{I}^{ll_1}(\Xi_{l_1})) \\ & - \frac{1}{2} \sum_{i_1,i_2,k,k_1=1}^{3} \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\varphi^j - \frac{1}{2}\varphi^i \sum_{j_1,j_2,k,k_1=1}^{3} \mathcal{I}_k^{jj_1}(\mathcal{I}^{j_1j_2}(\Xi_{j_2})\mathcal{I}^{kk_1}(\Xi_{k_1})) \\ & + \varphi^i\varphi^j - \frac{1}{2} \sum_{i_1,i_2,j_1,k,k_1=1}^{3} \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{jj_1}(\Xi_{j_1}) + \varphi^i \sum_{j_1=1}^{3} \mathcal{I}^{jj_1}(\Xi_{j_1}) \\ & - \frac{1}{2} \sum_{i_1,j_1,k,k_1=1}^{3} \mathcal{I}_k^{ii_1}(\mathcal{I}^{kk_1}(\Xi_{k_1}))\varphi^{i_1}\mathcal{I}^{jj_1}(\Xi_{j_1}) - \frac{1}{2} \sum_{i_1,i_2,j_1,k=1}^{3} \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2}))\varphi^k\mathcal{I}^{jj_1}(\Xi_{j_1}) \\ & + \frac{1}{4} \sum_{i_1,i_2,i_3,l,l_1,k,k_1,j_1=1}^{3} \mathcal{I}_k^{ii_1}(\mathcal{I}_l^{i_1i_2}(\mathcal{I}^{i_2i_3}(\Xi_{i_3})\mathcal{I}^{ll_1}(\Xi_{l_1}))\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{jj_1}(\Xi_{j_1}) \\ & + \frac{1}{4} \sum_{i_1,i_2,k,k_1,k_2,l,l_1,j_1=1}^{3} \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}_l^{kk_1}(\mathcal{I}^{k_1k_2}(\Xi_{k_2})\mathcal{I}^{ll_1}(\Xi_{l_1})))\mathcal{I}^{jj_1}(\Xi_{j_1}) \\ & - \frac{1}{2} \sum_{i_1,j_1,j_2,k,k_1=1}^{3} \mathcal{I}_k^{jj_1}(\mathcal{I}^{j_1j_2}(\Xi_{j_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{ii_1}(\Xi_{i_1}) + \sum_{i_1=1}^{3} \varphi^j\mathcal{I}^{ii_1}(\Xi_{i_1}) \\ & - \frac{1}{2} \sum_{i_1,j_1,k,k_1=1}^{3} \mathcal{I}_k^{jj_1}(\mathcal{I}^{kk_1}(\Xi_{k_1}))\varphi^{j_1}\mathcal{I}^{ii_1}(\Xi_{i_1}) - \frac{1}{2} \sum_{i_1,j_1,j_2,k=1}^{3} \mathcal{I}_k^{jj_1}(\mathcal{I}^{j_1j_2}(\Xi_{j_2}))\varphi^k\mathcal{I}^{ii_1}(\Xi_{i_1}) \\ & + \frac{1}{4} \sum_{i_1,j_1,j_2,j_3,l,l_1,k,k_1=1}^{3} \mathcal{I}_k^{jj_1}(\mathcal{I}_l^{j_1j_2}(\mathcal{I}^{j_2j_3}(\Xi_{j_3})\mathcal{I}^{ll_1}(\Xi_{l_1}))\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{ii_1}(\Xi_{i_1}) \end{aligned}$$

$$+\frac{1}{4}\sum_{i_1,j_1,j_2,l,l_1,k,k_1,k_2=1}^{3}\mathcal{I}_k^{jj_1}(\mathcal{I}^{j_1j_2}(\Xi_{j_2})\mathcal{I}_l^{kk_1}(\mathcal{I}^{k_1k_2}(\Xi_{k_2})\mathcal{I}^{ll_1}(\Xi_{l_1})))\mathcal{I}^{ii_1}(\Xi_{i_1})$$
$$+\sum_{i_1,j_1=1}^{3}\mathcal{I}^{ii_1}(\Xi_{i_1})\mathcal{I}^{jj_1}(\Xi_{j_1})+\rho_F,$$

where ρ_F has strictly positive homogeneity. Moreover we have

$$\mathcal{R}u^i=-\frac{1}{2}\sum_{i_1,i_2,j,j_1=1}^{3}D_jK^{ii_1}*(K^{i_1i_2}*\xi_\varepsilon^{i_2}\cdot K^{jj_1}*\xi_\varepsilon^{j_1})+\varphi^i+\sum_{i_1=1}^{3}K^{ii_1}*\xi_\varepsilon^{i_1}.$$

Since $\Delta^M\tau=M\tau\otimes 1$, we have the identity $(\Pi_z^{M,(\varepsilon)}\tau)(z)=(\Pi_z^{(\varepsilon)}M\tau)(z)$. It follows that for the reconstruction operator $\mathcal{R}^M$ associated with MZ_ε

$$\begin{aligned}\mathcal{R}^MF_j^i(u)=&\mathcal{R}u^i\mathcal{R}u^j-\frac{1}{4}\sum_{i_1,i_2,j_1,j_2,k,k_1,l,l_1=1}^{3}C^2_{ii_1i_2jj_1j_2kk_1ll_1}-\sum_{i_1,j_1=1}^{3}C^1_{ii_1jj_1}\\&-\frac{1}{4}\sum_{i_1,i_2,i_3,k,k_1,l,l_1,j_1=1}^{3}C^3_{ii_1i_2i_3ll_1kk_1jj_1}-\frac{1}{4}\sum_{i_1,i_2,k,k_1,k_2,l,l_1,j_1=1}^{3}C^4_{ii_1i_2ll_1kk_1k_2jj_1}\\&-\frac{1}{4}\sum_{i_1,k,k_1,l,l_1,j_1,j_2,j_3=1}^{3}C^3_{jj_1j_2j_3ll_1kk_1ii_1}-\frac{1}{4}\sum_{i_1,k,k_1,k_2,l,l_1,j_1,j_2=1}^{3}C^4_{jj_1j_2ll_1kk_1k_2ii_1},\end{aligned}$$

which combining with the fact that $\int_0^t\int D_jG^{ii_1}(t-s,x-y)dyds=0$ implies the results. □

Theorem 2.16 Let $\mathfrak{T}_F$ be the regularity structure associated to the dynamical SNS model for $\beta=2,\alpha\in(-\frac{13}{5},-\frac{5}{2})$, let $\xi_\varepsilon=\rho_\varepsilon*\xi$ and let Z_ε be the associated canonical model and M_ε be a sequence of renormalisation linear map defined in (2.5) corresponding to $C^{1,\varepsilon},C^{2,\varepsilon},C^{3,\varepsilon},C^{4,\varepsilon}$, which will be defined in the proof. $\hat{Z}_\varepsilon=M_\varepsilon Z_\varepsilon$. Then, there exists a random model $\hat{Z}$ independent of the choice of mollifier ρ and $M_\varepsilon\in\mathfrak{R}_0$ such that $M_\varepsilon Z_\varepsilon\to\hat{Z}$ in probability.

More precisely, for any $\theta<-\frac{5}{2}-\alpha$, any compact set $\mathfrak{K}$ and any $\gamma<r$ we have

$$E|||M_\varepsilon Z_\varepsilon;\hat{Z}|||_{\gamma;\mathfrak{K}}\lesssim\varepsilon^\theta,$$

uniformly over $\varepsilon\in(0,1]$.

Proof By [Hai14, Theorem 10.7] it is sufficient to prove that for $\tau\in\mathcal{F}$ with $|\tau|_\mathfrak{s}<0$, any test function $\varphi\in\mathcal{B}_r$, every $x\in\mathbb{R}^4$, there exist random variables $\hat{\Pi}_x\tau(\varphi)$ such that for κ small enough

$$E|(\hat{\Pi}_x\tau)(\varphi_x^\lambda)|^2\lesssim\lambda^{2|\tau|_\mathfrak{s}+\kappa},\tag{2.8}$$

and such that for some $0<\theta<-\frac{5}{2}-\alpha$,

$$E|(\hat{\Pi}_x\tau-\hat{\Pi}_x^{(\varepsilon)}\tau)(\varphi_x^\lambda)|^2\lesssim\varepsilon^{2\theta}\lambda^{2|\tau|_\mathfrak{s}+\kappa}.\tag{2.9}$$

For $\tau=\Xi_i,\mathcal{I}^{ii_1}(\Xi_{i_1}),i,i_1=1,2,3$, it is easy to conclude (2.8), (2.9) hold in this case. For $\tau=\mathcal{I}^{ii_1}(\Xi_{i_1})\mathcal{I}^{jj_1}(\Xi_{j_1})$, $i,i_1,j,j_1=1,2,3$, we have

$$\hat{\Pi}_x^{(\varepsilon)}\tau(y)=\int K^{ii_1}(y-z)\xi_\varepsilon^{i_1}(z)dz\int K^{jj_1}(y-z)\xi_\varepsilon^{j_1}(z)dz-C^{1,\varepsilon}_{ii_1jj_1}.$$

If we choose $C^{1,\varepsilon}_{ii_1jj_1} := \langle K^{ii_1}_\varepsilon, K^{jj_1}_\varepsilon\rangle$, where $K_\varepsilon = \rho_\varepsilon * K$ we have

$$\hat{\Pi}^{(\varepsilon)}_x\tau(y) = \int K^{ii_1}(y-z_1)K^{jj_1}(y-z_2)\xi^{i_1}_\varepsilon(z_1)\diamond\xi^{j_1}_\varepsilon(z_2)dz_1dz_2,$$

so that $\hat{\Pi}^{(\varepsilon)}_x\tau(y)$ belongs to the homogeneous chaos of order 2 with

$$(\hat{\mathcal{W}}^{(\varepsilon;2)}(\tau)(y;z_1,z_2) = K^{ii_1}_\varepsilon(y-z_1)K^{jj_1}_\varepsilon(y-z_2).$$

Then applying [Hai14, Lemma 10.14] we deduce that

$$|\langle(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(y),(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(\bar{y})\rangle| \lesssim \|y-\bar{y}\|_{\mathfrak{s}}^{-2},$$

holds uniformly over $\varepsilon\in(0,1]$, which implies the bound for $4\alpha+10+\kappa<0$

$$\begin{aligned}|\int\int\psi^\lambda(y)\psi^\lambda(\bar{y})\langle(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(y),(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(\bar{y})\rangle dyd\bar{y}| &\lesssim \lambda^{-10}\int_{\|y\|_{\mathfrak{s}}\leq\lambda,\|\bar{y}\|_{\mathfrak{s}}\leq\lambda}\|y-\bar{y}\|_{\mathfrak{s}}^{-2}dyd\bar{y}\\ \lesssim&\lambda^{-5}\int_{\|y\|_{\mathfrak{s}}\leq2\lambda}\|y\|_{\mathfrak{s}}^{-2}dy\lesssim\lambda^{-2}\lesssim\lambda^{\kappa+2(2\alpha+4)}.\end{aligned}$$

Hence we could choose

$$(\hat{\mathcal{W}}^{(2)}\tau)(y;z_1,z_2) = K^{ii_1}(y-z_1)K^{jj_1}(y-z_2).$$

In the same way, it is straightforward to obtain an analogous bound on $(\hat{\mathcal{W}}^{(2)})(\tau)$, which implies (2.8) holds in this case, so it remains to find similar bounds on $(\delta\hat{\mathcal{W}}^{(\varepsilon;2)}\tau) = (\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)-(\hat{\mathcal{W}}^{(2)}\tau)$. Similarly by [Hai14, Lemma 10.17] we have for $0<\kappa+\theta<-2(2\alpha+5)$

$$|\langle(\delta\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(y),(\delta\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(\bar{y})\rangle| \lesssim \varepsilon^\theta\|y-\bar{y}\|_{\mathfrak{s}}^{-2-\theta},$$

holds uniformly over $\varepsilon\in(0,1]$. Then we have the bound

$$|\int\int\psi^\lambda(y)\psi^\lambda(\bar{y})\langle(\delta\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(y),(\delta\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(\bar{y})\rangle dyd\bar{y}| \lesssim \varepsilon^\theta\lambda^{\kappa+2(2\alpha+4)},$$

which implies (2.9) holds in this case.

For $\tau = \mathcal{I}^{ii_1}_j(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{jj_1}(\Xi_{j_1}))$, $i,i_1,i_2,j,j_1=1,2,3$, we have the following identity

$$\begin{aligned}\hat{\Pi}^{(\varepsilon)}_x\tau(y) &= \int D_jK^{ii_1}(y-y_1)\int K^{i_1i_2}(y_1-z)\xi^{i_2}_\varepsilon(z)dz\int K^{jj_1}(y_1-z)\xi^{j_1}_\varepsilon(z)dzdy_1\\ &= \int D_jK^{ii_1}(y-y_1)\int\int K^{i_1i_2}(y_1-z_1)K^{jj_1}(y_1-z_2)\xi^{i_2}_\varepsilon(z_1)\diamond\xi^{j_1}_\varepsilon(z_2)dz_1dz_2dy_1,\end{aligned}$$

so that $\hat{\Pi}^{(\varepsilon)}_x\tau(y)$ belongs to the homogeneous chaos of order 2 with

$$(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(y;z_1,z_2) = \int D_jK^{ii_1}(y-y_1)K^{i_1i_2}_\varepsilon(y_1-z_1)K^{jj_1}_\varepsilon(y_1-z_2)dy_1.$$

Then by [Hai14, Lemma 10.14] we have for any $\delta>0$

$$|\langle(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(y),(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(\bar{y})\rangle| \lesssim \|y-\bar{y}\|_{\mathfrak{s}}^{-\delta},$$

holds uniformly over $\varepsilon \in (0,1]$, which implies the bound

$$|\int\int \psi^\lambda(y)\psi^\lambda(\bar y)\langle(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(y),(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(\bar y)\rangle dyd\bar y| \lesssim \lambda^{-10}\int_{\|y\|_\mathfrak{s}\leq\lambda,\|\bar y\|_\mathfrak{s}\leq\lambda}\|y-\bar y\|_\mathfrak{s}^{-\delta}dyd\bar y$$
$$\lesssim\lambda^{-5}\int_{\|y\|_\mathfrak{s}\leq 2\lambda}\|y\|_\mathfrak{s}^{-\delta}dy \lesssim \lambda^{-\delta}\lesssim\lambda^{\kappa+2(2\alpha+5)},$$

for $0<\kappa+\delta<-2(2\alpha+5)$. Hence we could choose

$$(\hat{\mathcal{W}}^{(2)}\tau)(y;z_1,z_2)=\int D_jK^{ii_1}(y-y_1)K^{i_1i_2}(y_1-z_1)K^{jj_1}(y_1-z_2)dy_1,$$

and deduce easily that (2.9) holds for $\tau=\mathcal{I}^{ii_1}_j(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{jj_1}(\Xi_{j_1}))$. Similarly we have the bound for $0<\kappa+\delta+\theta<-2(2\alpha+5)$

$$|\int\int \psi^\lambda(y)\psi^\lambda(\bar y)\langle(\delta\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(y),(\delta\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(\bar y)\rangle dyd\bar y|\lesssim\varepsilon^\theta\lambda^{\kappa+2(2\alpha+5)},$$

holds uniformly over $\varepsilon\in(0,1]$, which also implies that (2.10) holds for $\tau=\mathcal{I}^{ii_1}_j(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{jj_1}(\Xi_{j_1}))$.

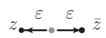
In the following we use to represent a factor K and to represent DK, where for simplicity we write $K^{ii_1}=K, D_jK^{ii_1}=DK$ and we do not make the difference of the graphs associated with different K^{ii_1} since they have the same order. In the graphs below we also omit the dependence on ε if there's no confusion. We also use the convention that if a vertex is drawn in grey, then the corresponding variable is integrated out.

For $\tau=\mathcal{I}^{ii_1}_k(\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{jj_1}(\Xi_{j_1})$, $i,i_1,k,k_1,j,j_1=1,2,3$ we have

$(\mathcal{W}^{(\varepsilon;2)}\tau)(z)=$ 

.

Defining kernels Q^0_ε, P^0_ε by

$P^0_\varepsilon(z-\bar z)=$ 

, $Q^0_\varepsilon(z-\bar z)=$ 

,

we have

$$\langle\mathcal{W}^{(\varepsilon;2)}\tau(z),\mathcal{W}^{(\varepsilon;2)}\tau(\bar z)\rangle=P^0_\varepsilon(z-\bar z)\delta^{(2)}Q^0_\varepsilon(z,\bar z),$$

where, for any function Q of two variables we have set

$$\delta^{(2)}Q(z,\bar z)=Q(z,\bar z)-Q(z,0)-Q(0,\bar z)+Q(0,0).$$

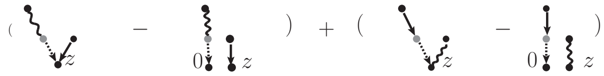
It follows from [Hai14, Lemma 10.14, Lemma 10.17] that for every $\delta>0$ we have

$$|Q^0_\varepsilon(z)-Q^0_\varepsilon(0)|\lesssim\|z\|_\mathfrak{s}^{1-\delta},\quad |P^0_\varepsilon(z)|\lesssim\|z\|_\mathfrak{s}^{-1}.$$

As a consequence we have the desired a priori bounds for $\mathcal{W}^{(\varepsilon;2)}\tau$, namely for every $\delta>0$

$$\langle(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(z),(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(\bar z)\rangle\lesssim\|z-\bar z\|_\mathfrak{s}^{-1}(\|z-\bar z\|_\mathfrak{s}^{1-\delta}+\|z\|_\mathfrak{s}^{1-\delta}+\|\bar z\|_\mathfrak{s}^{1-\delta}),$$

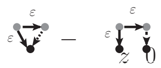
holds uniformly over $\varepsilon \in (0,1]$. Defining as previously $\hat{W}^{(2)}\tau$ like $\hat{\mathcal{W}}^{(\varepsilon;2)}\tau$ but with each instance of K_ε replaced by K. Moreover, we use $\rightsquigarrow$ to represent the kernel $K - K_\varepsilon$ and we have

$$(\delta\mathcal{W}^{(\varepsilon;2)}\tau)(z) =$$  .

By a similar calculation as above we obtain the following bounds

$$\begin{aligned}\langle(\delta\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(z),(\delta\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(\bar{z})\rangle \lesssim & \varepsilon^{2\theta}\|z-\bar{z}\|_{\mathfrak{s}}^{-1}(\|z-\bar{z}\|_{\mathfrak{s}}^{1-2\theta-\delta}+\|z\|_{\mathfrak{s}}^{1-2\theta-\delta}+\|\bar{z}\|_{\mathfrak{s}}^{1-2\theta-\delta}) \\ & + \varepsilon^{2\theta}\|z-\bar{z}\|_{\mathfrak{s}}^{-1-2\theta}(\|z-\bar{z}\|_{\mathfrak{s}}^{1-\delta}+\|z\|_{\mathfrak{s}}^{1-\delta}+\|\bar{z}\|_{\mathfrak{s}}^{1-\delta}),\end{aligned}$$

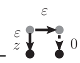
which is valid uniformly over $\varepsilon \in (0,1]$, provided that $\theta < 1, \delta > 0$. Here we used [Hai14, Lemma 10.17]. We come to $\hat{\mathcal{W}}^{(\varepsilon;0)}\tau$ and have

$$(\hat{\mathcal{W}}^{(\varepsilon;0)}\tau)(z) =$$  .

Since K is symmetric and DK is anti-symmetric with respect to space variable, we conclude that

 $= 0,$

which deduces the following

$$(\hat{\mathcal{W}}^{(\varepsilon;0)}\tau)(z) = -$$  .

Then by [Hai14, Lemma 10.14, Lemma 10.17] we have for every $\delta > 0$

$$|(\hat{\mathcal{W}}^{(\varepsilon;0)}\tau)(z)| \lesssim \|z\|_{\mathfrak{s}}^{-\delta},$$

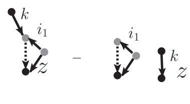
holds uniformly over $\varepsilon \in (0,1]$. Similarly bounds also hold for $(\delta\hat{\mathcal{W}}^{(\varepsilon;0)}\tau)$. Then we can easily conclude that (2.8) (2.9) hold for $\tau = \mathcal{I}_k^{ii_1}(\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{jj_1}(\Xi_{j_1})$.

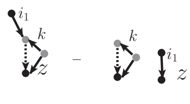
For $\tau = \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2}))\mathcal{I}^{jj_1}(\Xi_{j_1})$, $i,i_1,i_2,k,j,j_1 = 1,2,3$, we could prove similar bounds as above since in this case we also have

 $= 0.$

For $\tau = \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{jj_1}(\Xi_{j_1})$, $i,i_1,i_2,k,k_1,j,j_1 = 1,2,3$, we have the following identities

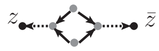
$$(\hat{\mathcal{W}}^{(\varepsilon;3)}\tau)(z) =$$  ,

$$(\hat{\mathcal{W}}_1^{(\varepsilon;1)}\tau)(z) =$$  ,

$$(\hat{\mathcal{W}}_2^{(\varepsilon;1)}\tau)(z) =$$  ,

$$\langle\hat{\mathcal{W}}^{(\varepsilon;3)}\tau(z),\hat{\mathcal{W}}^{(\varepsilon;3)}\tau(\bar{z})\rangle = P_\varepsilon^0(z-\bar{z})Q_\varepsilon(z-\bar{z}),$$

where

$$Q_\varepsilon(z-\bar z) = $$  ,  $$= 0.$$

By [Hai14, Lemma 10.14, Lemma 10.17] for every $\delta > 0$ we have the bound

$$|Q_\varepsilon(z-\bar z)| \lesssim \|z-\bar z\|_{\mathfrak{s}}^{-\delta},$$

which implies that

$$|\langle \hat{\mathcal{W}}^{(\varepsilon;3)}\tau(z), \hat{\mathcal{W}}^{(\varepsilon;3)}\tau(\bar z)\rangle| \lesssim \|z-\bar z\|_{\mathfrak{s}}^{-1-\delta},$$

holds uniformly over $\varepsilon \in (0,1]$. Defining as previously $\hat{W}^{(3)}\tau$ like $\hat{\mathcal{W}}^{(\varepsilon;3)}\tau$ but with each instance of K_ε replaced by K. Then $\delta\hat{\mathcal{W}}^{(\varepsilon;3)}\tau$ can be bounded in a manner similar to before. Now for $\hat{\mathcal{W}}^{(\varepsilon;1)}\tau$, we have

$$(\hat{\mathcal{W}}_1^{(\varepsilon;1)}\tau)(z) = ((\mathcal{R}_1 L_\varepsilon) * K_\varepsilon^{kk_1})(z),$$

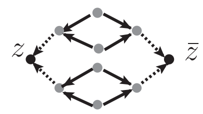
where $L_\varepsilon(z) =$  and $(\mathcal{R}_1 L_\varepsilon)(\psi) = \int L_\varepsilon(x)(\psi(x)-\psi(0))dx$ for ψ smooth and compactly support. It follows from [Hai14, Lemma 10.16] that, the bounds

$$|\langle (\hat{\mathcal{W}}_1^{(\varepsilon;1)}\tau)(z), (\hat{\mathcal{W}}_1^{(\varepsilon;1)}\tau)(\bar z)\rangle| \lesssim \|z-\bar z\|_{\mathfrak{s}}^{-1},$$

holds uniformly for $\varepsilon \in (0,1]$. Similarly, this bounds also holds for $(\hat{\mathcal{W}}_2^{(\varepsilon;1)}\tau)(z)$. Again, $\delta\hat{\mathcal{W}}_i^{(\varepsilon;1)}\tau, i = 1,2$ can be bounded in a manner similar to before. Then we can easily conclude that (2.8), (2.9) holds for $\tau = \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{jj_1}(\Xi_{j_1})$.

For $\tau = \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}_l^{jj_1}(\mathcal{I}^{j_1j_2}(\Xi_{j_2})\mathcal{I}^{ll_1}(\Xi_{l_1})), i, i_1, i_2, k, k_1, j, j_1, j_2, l, l_1 = 1,2,3$, we have the identities

$$(\hat{\mathcal{W}}^{(\varepsilon;4)}\tau)(z) = $$  ,

$$\langle (\hat{\mathcal{W}}^{(\varepsilon;4)}\tau)(z), (\hat{\mathcal{W}}^{(\varepsilon;4)}\tau)(\bar z)\rangle = $$  .

Then we obtain the bound for every $\delta > 0$

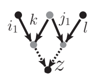
$$|\langle (\hat{\mathcal{W}}^{(\varepsilon;4)}\tau)(z), (\hat{\mathcal{W}}^{(\varepsilon;4)}\tau)(\bar z)\rangle| \lesssim \|z-\bar z\|_{\mathfrak{s}}^{-\delta}.$$

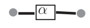
Similarly, we obtain

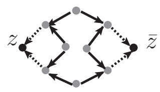
$$|\langle (\delta\hat{\mathcal{W}}^{(\varepsilon;4)}\tau)(z), (\delta\hat{\mathcal{W}}^{(\varepsilon;4)}\tau)(\bar z)\rangle| \lesssim \varepsilon^{2\theta}\|z-\bar z\|_{\mathfrak{s}}^{-2\theta},$$

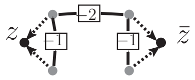
holds uniformly for $\varepsilon \in (0,1]$, provided $\theta < 1$.

For $(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(z)$, we have the identity

$$(\hat{\mathcal{W}}_1^{(\varepsilon;2)}\tau)(z) = $$  .

Other terms can be obtained by changing the position for i_1, k or j_1, l. Since the estimates are similar, we omit them here. We also use the notation  for $\|z-\bar{z}\|_{\mathfrak{s}}^{\alpha} 1_{\|z-\bar{z}\|_{\mathfrak{s}} \leq C}$ for a constant C. We obtain for $\delta > 0$,

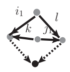
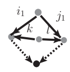
$$\langle (\hat{\mathcal{W}}_1^{(\varepsilon;2)}\tau)(z), (\hat{\mathcal{W}}_1^{(\varepsilon;2)}\tau)(\bar{z}) \rangle =$$ 

$$\lesssim$$  $$\lesssim \|z-\bar{z}\|_{\mathfrak{s}}^{-\delta},$$

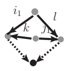
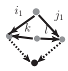
holds uniformly for $\varepsilon \in (0,1]$, where we used Young's inequality in the first inequality. Similarly, we have

$$\langle (\delta\hat{\mathcal{W}}_1^{(\varepsilon;2)}\tau)(z), (\delta\hat{\mathcal{W}}_1^{(\varepsilon;2)}\tau)(\bar{z}) \rangle \lesssim \varepsilon^{2\theta} \|z-\bar{z}\|_{\mathfrak{s}}^{-2\theta},$$

provided $\theta < 1$. Now for $\hat{\mathcal{W}}^{(\varepsilon;0)}\tau$ we have

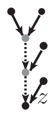
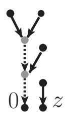
$$(\hat{\mathcal{W}}^{(\varepsilon;0)}\tau)(z) =$$  $+$  $- C^{2,\varepsilon}_{ii_1i_2jj_1j_2kk_1ll_1}.$

Hence we will choose

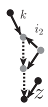
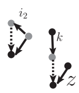
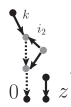
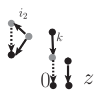
$$C^{2,\varepsilon}_{ii_1i_2jj_1j_2kk_1ll_1} =$$  $+$  .

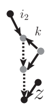
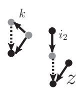
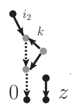
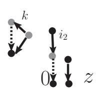
Now in this case (2.8), (2.9) follow.

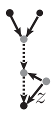
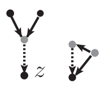
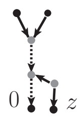
For $\tau = \mathcal{I}_l^{ii_1}(\mathcal{I}_k^{i_1i_2}(\mathcal{I}^{i_2i_3}(\Xi_{i_3})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{ll_1}(\Xi_{l_1}))\mathcal{I}^{jj_1}(\Xi_{j_1})$, $i, i_1, i_2, i_3, j, j_1, k, k_1, l, l_1 = 1, 2, 3$, we have the following identities:

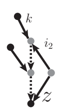
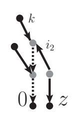
$$(\hat{\mathcal{W}}^{(\varepsilon;4)}\tau)(z) =$$  $-$  .

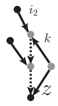
$$(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(z) = \sum_{i=1}^{5} (\hat{\mathcal{W}}_i^{(\varepsilon;2)}\tau)(z) = \sum_{i=1}^{5} [(\hat{\mathcal{W}}_{i1}^{(\varepsilon;2)}\tau)(z) - (\hat{\mathcal{W}}_{i2}^{(\varepsilon;2)}\tau)(z)].$$

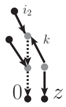
$(\hat{\mathcal{W}}_{11}^{(\varepsilon;2)}\tau)(z) =$  $-$  , $(\hat{\mathcal{W}}_{12}^{(\varepsilon;2)}\tau)(z) =$  $-$  ,

$(\hat{\mathcal{W}}_{21}^{(\varepsilon;2)}\tau)(z) =$  $-$  , $(\hat{\mathcal{W}}_{22}^{(\varepsilon;2)}\tau)(z) =$  $-$  ,

$(\hat{\mathcal{W}}_{31}^{(\varepsilon;2)}\tau)(z) =$  $-$  , $(\hat{\mathcal{W}}_{32}^{(\varepsilon;2)}\tau)(z) =$  ,

$(\hat{\mathcal{W}}_{4}^{(\varepsilon;2)}\tau)(z) = (\hat{\mathcal{W}}_{41}^{(\varepsilon;2)}\tau)(z) - (\hat{\mathcal{W}}_{42}^{(\varepsilon;2)}\tau)(z) =$  $-$  ,

$$(\hat{\mathcal{W}}_5^{(\varepsilon;2)}\tau)(z) = (\hat{\mathcal{W}}_{51}^{(\varepsilon;2)}\tau)(z) - (\hat{\mathcal{W}}_{52}^{(\varepsilon;2)}\tau)(z) =$$ 

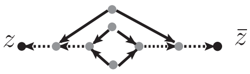
$-$ 

.

Now for $\hat{\mathcal{W}}^{(\varepsilon;4)}\tau$ we have

$$\langle \hat{\mathcal{W}}^{(\varepsilon;4)}\tau(z), \hat{\mathcal{W}}^{(\varepsilon;4)}\tau(\bar{z})\rangle = P_\varepsilon^0(z-\bar{z})\delta^{(2)}Q_\varepsilon^2(z,\bar{z}),$$

where

$$Q_\varepsilon^2(z,\bar{z}) =$$ 

,  $= 0.$

By [Hai14, Lemmas 10.14, 10.16 and 10.17] for every $\delta > 0$ we have the bound

$$|\langle \hat{\mathcal{W}}^{(\varepsilon;4)}\tau(z), \hat{\mathcal{W}}^{(\varepsilon;4)}\tau(\bar{z})\rangle| \lesssim \|z-\bar{z}\|_{\mathfrak{s}}^{-1}(\|z-\bar{z}\|_{\mathfrak{s}}^{1-\delta} + \|z\|_{\mathfrak{s}}^{1-\delta} + \|\bar{z}\|_{\mathfrak{s}}^{1-\delta}),$$

holds uniformly for $\varepsilon \in (0,1]$, and

$$\begin{aligned}&|\langle \hat{\mathcal{W}}_{11}^{(\varepsilon;2)}\tau(z) - \hat{\mathcal{W}}_{12}^{(\varepsilon;2)}\tau(z), \hat{\mathcal{W}}_{11}^{(\varepsilon;2)}\tau(\bar{z}) - \hat{\mathcal{W}}_{12}^{(\varepsilon;2)}\tau(z)\rangle|\\ \lesssim&\|z-\bar{z}\|_{\mathfrak{s}}^{-1}|\langle K * \mathcal{R}_1 L_\varepsilon^1 * DK(z-\cdot) - K * \mathcal{R}_1 L_\varepsilon^1 * DK(-\cdot),\\ &K * \mathcal{R}_1 L_\varepsilon^1 * DK(\bar{z}-\cdot) - K * \mathcal{R}_1 L_\varepsilon^1 * DK(-\cdot)\rangle|\\ \lesssim&\|z-\bar{z}\|_{\mathfrak{s}}^{-1}(\|z-\bar{z}\|_{\mathfrak{s}}^{1-\delta} + \|z\|_{\mathfrak{s}}^{1-\delta} + \|\bar{z}\|_{\mathfrak{s}}^{1-\delta}),\end{aligned}$$

holds uniformly for $\varepsilon \in (0,1]$, where $L_\varepsilon^1(z) =$ . Similarly, this bounds also holds for $(\hat{\mathcal{W}}_2^{(\varepsilon;2)}\tau)(z)$. Again, $\delta\hat{\mathcal{W}}^{(\varepsilon;4)}\tau$, $\delta\hat{\mathcal{W}}_i^{(\varepsilon;2)}\tau, i = 1,2$ can be bounded in a manner similar to before. For $\hat{\mathcal{W}}_3^{(\varepsilon;2)}\tau$ we have

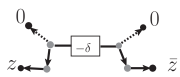
$$(\hat{\mathcal{W}}_{31}^{(\varepsilon;2)}\tau)(z) = ((\mathcal{R}_1 L_\varepsilon^1) * L_\varepsilon^2)(z),$$

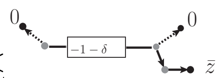
where $L_\varepsilon^1(z) =$ , $L_\varepsilon^2(z) =$ 

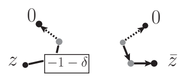
. It follows from [Hai14, Lemma 10.16] that for every $\delta > 0$, the bounds

$$|\langle(\hat{\mathcal{W}}_{31}^{(\varepsilon;2)}\tau)(z), (\hat{\mathcal{W}}_{31}^{(\varepsilon;2)}\tau)(\bar{z})\rangle| \lesssim \|z-\bar{z}\|_{\mathfrak{s}}^{-\delta},$$

holds uniformly for $\varepsilon \in (0,1]$. Moreover for $\hat{\mathcal{W}}_{32}^{(\varepsilon;2)}\tau$ we have for every $\delta \in (0,1)$

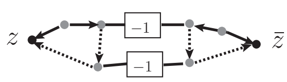
$$|\langle(\hat{\mathcal{W}}_{32}^{(\varepsilon;2)}\tau)(z), (\hat{\mathcal{W}}_{32}^{(\varepsilon;2)}\tau)(\bar{z})\rangle| =$$ 

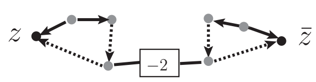
$\lesssim$ 

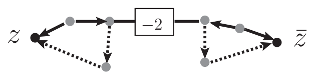
$+$ 

$\lesssim \|z\|_{\mathfrak{s}}^{-\delta}\|\bar{z}\|_{\mathfrak{s}}^{-\delta} + \|\bar{z}\|_{\mathfrak{s}}^{-\delta},$

where we used Young's inequality. Again, $\delta\hat{\mathcal{W}}_3^{(\varepsilon;2)}\tau$, can be bounded in a manner similar to before. For $\hat{\mathcal{W}}_{41}^{(\varepsilon;2)}\tau$ we have for $\delta > 0$

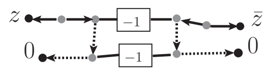
$$|\langle(\hat{\mathcal{W}}_{41}^{(\varepsilon;2)}\tau)(z), (\hat{\mathcal{W}}_{41}^{(\varepsilon;2)}\tau)(\bar{z})\rangle| =$$ 

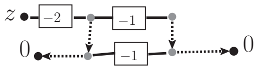
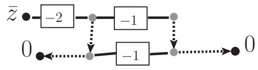
$\lesssim$ 

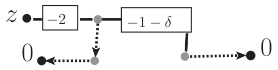
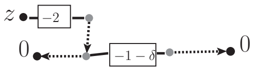
$+$ 

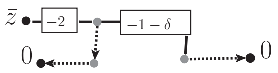
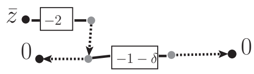
$\lesssim \|z-\bar{z}\|_{\mathfrak{s}}^{-\delta},$

holds uniformly for $\varepsilon \in (0,1]$, where we used Young's inequality. For $\delta \in (0,1)$ we have

$$|\langle (\hat{\mathcal{W}}_{42}^{(\varepsilon;2)}\tau)(z), (\hat{\mathcal{W}}_{42}^{(\varepsilon;2)}\tau)(\bar{z})\rangle| =$$ 

$$\lesssim$$  $+$ 

$$\lesssim$$  $+$ 

$+$  $+$ 

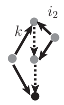
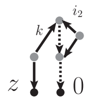
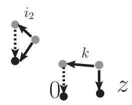
$$\lesssim \|z\|_{\mathfrak{s}}^{-\delta} + \|\bar{z}\|_{\mathfrak{s}}^{-\delta},$$

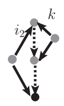
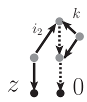
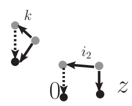
holds uniformly for $\varepsilon \in (0,1]$, where we used Young's inequality in the inequalities. Similarly, these bounds also holds for $(\hat{\mathcal{W}}_5^{(\varepsilon;2)}\tau)(z)$. Again, $\delta\hat{\mathcal{W}}_i^{(\varepsilon;2)}\tau, i = 4,5$ can be bounded in a manner similar to before.

We now turn to $\hat{\mathcal{W}}^{(\varepsilon;0)}\tau$:

$$(\hat{\mathcal{W}}^{(\varepsilon;0)}\tau)(z) = \sum_{i=1}^{2}(\hat{\mathcal{W}}_i^{(\varepsilon;0)}\tau)(z) = \sum_{i=1}^{2}[(\hat{\mathcal{W}}_{i1}^{(\varepsilon;2)}\tau)(z) - (\hat{\mathcal{W}}_{i2}^{(\varepsilon;2)}\tau)(z)] - C^{3,\varepsilon}_{ii_1i_2i_3kk_1ll_1jj_1},$$

where

$$(\hat{\mathcal{W}}_{11}^{(\varepsilon;0)}\tau)(z) =$$  $, (\hat{\mathcal{W}}_{12}^{(\varepsilon;0)}\tau)(z) =$  $-$  ,

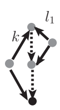
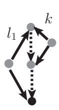
$$(\hat{\mathcal{W}}_{21}^{(\varepsilon;0)}\tau)(z) =$$  $, (\hat{\mathcal{W}}_{22}^{(\varepsilon;0)}\tau)(z) =$  $-$  ,

we choose $C^{3,\varepsilon}_{ii_1i_2i_3kk_1ll_1jj_1} = (\hat{\mathcal{W}}_{11}^{(\varepsilon;0)}\tau)(z) + (\hat{\mathcal{W}}_{21}^{(\varepsilon;0)}\tau)(z)$. By [Hai14, Lemma 10.16] we have that for every $\delta > 0$, $i = 1,2$,

$$|(\hat{\mathcal{W}}_{i2}^{(\varepsilon;0)}\tau)(z)| \lesssim \|z\|_{\mathfrak{s}}^{-\delta}$$

holds uniformly for $\varepsilon \in (0,1]$. Similarly as before, we obtain the bounds for $\delta\hat{\mathcal{W}}_{i2}^{(\varepsilon;0)}\tau$. Then (2.8), (2.9) follow in this case.

For $\tau = \mathcal{I}_l^{ii_1}(\mathcal{I}_k^{ll_1}(\mathcal{I}^{l_1l_2}(\Xi_{l_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{i_1i_2}(\Xi_{i_2}))\mathcal{I}^{jj_1}(\Xi_{j_1})$, $i, i_1, i_2, l, l_1, l_2, k, k_1, j, j_1 = 1,2,3$, we have similar bounds as above with

$$C^4_{ii_1i_2kk_1ll_1l_2jj_1} =$$  $+$  .

□

3 NS equation by paracontrolled distributions

3.1 Besov spaces and paraproduct

In the following we recall the definitions and some properties of Besov spaces and paraproducts. For a general introduction to these theories we refer to [BCD11, GIP13]. Here the notations are differen from the previous section.

First we introduce the following notations. The space of real valued infinitely differentiable functions of compact support is denoted by $\mathcal{D}(\mathbb{R}^d)$ or $\mathcal{D}$. The space of Schwartz functions is denoted by $\mathcal{S}(\mathbb{R}^d)$. Its dual, the space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^d)$. If u is a vector of n tempered distributions on $\mathbb{R}^d$, then we write $u \in \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^n)$. The Fourier transform and the inverse Fourier transform are denoted by $\mathcal{F}u$ and $\mathcal{F}^{-1}u$.

Let $\chi, \theta \in \mathcal{D}$ be nonnegative radial functions on $\mathbb{R}^d$, such that

i. the support of χ is contained in a ball and the support of θ is contained in an annulus;

ii. $\chi(z) + \sum_{j\geq 0} \theta(2^{-j}z) = 1$ for all $z \in \mathbb{R}^d$.

iii. $\mathrm{supp}(\chi) \cap \mathrm{supp}(\theta(2^{-j}\cdot)) = \emptyset$ for $j \geq 1$ and $\mathrm{supp}(\theta(2^{-i}\cdot) \cap \mathrm{supp}\theta(2^{-j}\cdot) = \emptyset$ for $|i-j| > 1$.

We call such (χ, θ) dyadic partition of unity, and for the existence of dyadic partitions of unity see [BCD11, Proposition 2.10]. The Littlewood-Paley blocks are now defined as

$$\Delta_{-1}u = \mathcal{F}^{-1}(\chi\mathcal{F}u) \quad \Delta_j u = \mathcal{F}^{-1}(\theta(2^{-j}\cdot)\mathcal{F}u).$$

For $\alpha \in \mathbb{R}$, the Hölder-Besov space $\mathcal{C}^\alpha$ is given by $\mathcal{C}^\alpha = B^\alpha_{\infty,\infty}(\mathbb{R}^d, \mathbb{R}^n)$, where for $p, q \in [1, \infty]$ we define

$$B^\alpha_{p,q}(\mathbb{R}^d, \mathbb{R}^n) = \{u = (u^1, ..., u^n) \in \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^n) : \|u\|_{B^\alpha_{p,q}} = \sum_{i=1}^n (\sum_{j\geq -1} (2^{j\alpha}\|\Delta_j u^i\|_{L^p})^q)^{1/q} < \infty\},$$

with the usual interpretation as l^∞ norm in case $q = \infty$. We write $\|\cdot\|_\alpha$ instead of $\|\cdot\|_{B^\alpha_{\infty,\infty}}$.

We point out that everything above and everything that follows can be applied to distributions on the torus. More precisely, let $\mathcal{D}'(\mathbb{T}^d)$ be the space of distributions on $\mathbb{T}^d$. Therefore, Besov spaces on the torus with general indices $p, q \in [1, \infty]$ are defined as

$$B^\alpha_{p,q}(\mathbb{T}^d, \mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{T}^d, \mathbb{R}^n) : \|u\|_{B^\alpha_{p,q}} = \sum_{i=1}^n (\sum_{j\geq -1} (2^{j\alpha}\|\Delta_j u^i\|_{L^p(\mathbb{T}^d)})^q)^{1/q} < \infty\}.$$

We will need the following Besov embedding theorem on the torus (c.f. [GIP13, Lemma 41]):

Lemma 3.1 Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$, and let $\alpha \in \mathbb{R}$. Then $B^\alpha_{p_1,q_1}(\mathbb{T}^d)$ is continuously embedded in $B^{\alpha-d(1/p_1-1/p_2)}_{p_2,q_2}(\mathbb{T}^d)$.

Now we recall the following paraproduct introduced by Bony (see [Bon81]). In general, the product fg of two distributions $f \in \mathcal{C}^\alpha, g \in \mathcal{C}^\beta$ is well defined if and only if $\alpha + \beta > 0$. In terms of Littlewood-Paley blocks, the product fg can be formally decomposed as

$$fg = \sum_{j\geq -1}\sum_{i\geq -1} \Delta_i f \Delta_j g = \pi_<(f, g) + \pi_0(f, g) + \pi_>(f, g),$$

with

$$\pi_<(f,g) = \pi_>(g,f) = \sum_{j\geq -1}\sum_{i<j-1} \Delta_i f \Delta_j g, \quad \pi_0(f,g) = \sum_{|i-j|\leq 1} \Delta_i f \Delta_j g.$$

We also use the notation

$$S_j f = \sum_{i\leq j-1} \Delta_i f.$$

We will use without comment that $\|\cdot\|_\alpha \leq \|\cdot\|_\beta$ for $\alpha \leq \beta$, that $\|\cdot\|_{L^\infty} \lesssim \|\cdot\|_\alpha$ for $\alpha > 0$, and that $\|\cdot\|_\alpha \lesssim \|\cdot\|_{L^\infty}$ for $\alpha \leq 0$. We will also use that $\|S_j u\|_{L^\infty} \lesssim 2^{-j\alpha}\|u\|_\alpha$ for $\alpha < 0$ and $u \in \mathcal{C}^\alpha$.

The basic result about these bilinear operations is given by the following estimates:

Lemma 3.2 (Paraproduct estimates, [Bon 81, GIP13, Lemma 2]) For any $\beta \in \mathbb{R}$ we have

$$\|\pi_<(f,g)\|_\beta \lesssim \|f\|_{L^\infty}\|g\|_\beta \quad f \in L^\infty, g \in \mathcal{C}^\beta,$$

and for $\alpha < 0$ furthermore

$$\|\pi_<(f,g)\|_{\alpha+\beta} \lesssim \|f\|_\alpha \|g\|_\beta \quad f \in \mathcal{C}^\alpha, g \in \mathcal{C}^\beta.$$

For $\alpha + \beta > 0$ we have

$$\|\pi_0(f,g)\|_{\alpha+\beta} \lesssim \|f\|_\alpha \|g\|_\beta \quad f \in \mathcal{C}^\alpha, g \in \mathcal{C}^\beta.$$

The following basic commutator lemma is important for our use:

Lemma 3.3 ([GIP13, Lemma 5]) Assume that $\alpha \in (0,1)$ and $\beta, \gamma \in \mathbb{R}$ are such that $\alpha+\beta+\gamma > 0$ and $\beta + \gamma < 0$. Then for smooth f, g, h, the trilinear operator

$$C(f,g,h) = \pi_0(\pi_<(f,g),h) - f\pi_0(g,h)$$

allows for the bound

$$\|C(f,g,h)\|_{\alpha+\beta+\gamma} \lesssim \|f\|_\alpha \|g\|_\beta \|h\|_\gamma.$$

Thus, C can be uniquely extended to a bounded trilinear operator in $\mathcal{L}^3(\mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma, \mathcal{C}^{\alpha+\beta+\gamma})$.

Now we prove the following commutator estimate.

Lemma 3.4 Let $u \in \mathcal{C}^\alpha$ for some $\alpha < 1$ and $v \in \mathcal{C}^\beta$ for some $\beta \in \mathbb{R}$. Then for every $k, l = 1, 2, 3$ we have

$$\|P^{kl}\pi_<(u,v) - \pi_<(u, P^{kl}v)\|_{\alpha+\beta} \lesssim \|u\|_\alpha \|v\|_\beta,$$

where P is the Leray projection.

Proof By the same argument as the proof of [CC13, Lemma A.1] we have for $j \geq 0$

$$\|[(\psi(2^{-j}\cdot)\hat{P}^{kl})(D), S_{j-1}u]\Delta_j v\|_{L^\infty} \lesssim \sum_{\eta\in\mathbb{N}^d, |\eta|=1} \|x^\eta \mathcal{F}^{-1}(\psi(2^{-j}\cdot)\hat{P}^{kl})\|_{L^1} \|\partial^\eta S_{j-1}u\|_{L^\infty} \|\Delta_j v\|_{L^\infty}.$$

Here $\hat{P}^{kl}(x) = \delta_{kl} - \frac{x_k x_l}{|x|^2}$, $(\psi(2^{-j}\cdot)\hat{P}^{kl})(D)u = \mathcal{F}^{-1}(\psi(2^{-j}\cdot)\hat{P}^{kl}\mathcal{F}u)$, $[(\psi(2^{-j}\cdot)\hat{P}^{kl})(D), S_{j-1}u]$ denotes the commutator and $\psi \in \mathcal{D}$ with support in an annulus and satisfies $[(\psi(2^{-j}\cdot)\hat{P}^{kl})(D), S_{j-1}u]\Delta_j v = [\hat{P}^{kl}(D), S_{j-1}u]\Delta_j v$.

Now we have

$$\begin{aligned}
&\|x^\eta \mathcal{F}^{-1}(\psi(2^{-j}\cdot)\hat{P}^{kl})\|_{L^1}\\
\leq&2^{-j}\|\mathcal{F}^{-1}(\partial^\eta\psi)(2^{-j}\cdot)\hat{P}^{kl})\|_{L^1}+\|\mathcal{F}^{-1}(\psi(2^{-j}\cdot)\partial^\eta\hat{P}^{kl})\|_{L^1}\\
=&2^{-j}\|\mathcal{F}^{-1}(\partial^\eta\psi(\cdot)\hat{P}^{kl}(2^j\cdot))\|_{L^1}+\|\mathcal{F}^{-1}(\psi(\cdot)\partial^\eta\hat{P}^{kl}(2^j\cdot))\|_{L^1}\\
\lesssim&2^{-j}\|(1+|\cdot|^2)^d\mathcal{F}^{-1}(\partial^\eta\psi(\cdot)\hat{P}^{kl}(2^j\cdot))\|_{L^\infty}+\|(1+|\cdot|^2)^d\mathcal{F}^{-1}(\psi(\cdot)\partial^\eta\hat{P}^{kl}(2^j\cdot))\|_{L^\infty}\\
=&2^{-j}\|\mathcal{F}^{-1}((1-\Delta)^d(\partial^\eta\psi(\cdot)\hat{P}^{kl}(2^j\cdot)))\|_{L^\infty}+\|\mathcal{F}^{-1}((1-\Delta)^d(\psi(\cdot)\partial^\eta\hat{P}^{kl}(2^j\cdot)))\|_{L^\infty}\\
\lesssim&2^{-j}\|(1-\Delta)^d(\partial^\eta\psi(\cdot)\hat{P}^{kl}(2^j\cdot))\|_{L^1}+\|(1-\Delta)^d(\psi(\cdot)\partial^\eta\hat{P}^{kl}(2^j\cdot))\|_{L^1}\\
\lesssim&2^{-j}\sum_{0\leq|m|\leq 2d}(2^j)^{|m|}\frac{1}{(2^j)^{|m|}}+\sum_{|m|\leq 2d}(2^j)^{|m|}\frac{1}{(2^j)^{|m|+1}}\\
\lesssim&2^{-j},
\end{aligned}$$

where in the last second inequality we used $|D^m\hat{P}^{kl}(x)|\lesssim|x|^{-|m|}$ for any multiindices m. Thus we get that

$$\|[\psi(2^{-j}\cdot)\hat{P}^{kl}(D),S_{j-1}u]\Delta_j v\|_{L^\infty}\lesssim 2^{-j(\alpha+\beta)}\|u\|_\alpha\|v\|_\beta,$$

which implies the result by the same argument as in the proof of [CC13, Lemma A.1]. □

Now we recall the following lemma which is important for our estimate.

Lemma 3.5 ([GIP13, Lemma 47]) Let $u\in\mathcal{C}^\alpha$ for some $\alpha\in\mathbb{R}$. Then we have for every $\delta\geq 0$

$$\|P_t u\|_{\alpha+\delta}\lesssim t^{-\delta/2}\|u\|_\alpha,$$

where P_t is the heat semigroup.

By the same argument as Lemma 3.5 we also have the following result on $\mathbb{T}^d$:

Lemma 3.6 Let $u\in\mathcal{C}^\alpha$ for some $\alpha\in\mathbb{R}$. Then we have for every $k,l=1,2,3$

$$\|P^{kl}u\|_\alpha\lesssim\|u\|_\alpha,$$

where P is the Leray projection.

Proof We have for $j\geq 0$

$$\|\Delta_j P^{kl}u\|_{L^\infty}\lesssim\|\mathcal{F}^{-1}(\hat{P}^{kl}(\cdot)\theta(2^{-j}\cdot))\|_{L^1}2^{-j\alpha}\|u\|_\alpha=\|\mathcal{F}^{-1}(\hat{P}^{kl}(2^j\cdot)\theta)\|_{L^1}2^{-j\alpha}\|u\|_\alpha.$$

Here $\hat{P}^{kl}(x)=\delta_{kl}-\frac{x^kx^l}{|x|^2}$. By the same argument as in the proof of Lemma 3.4 we get that

$$\|\mathcal{F}^{-1}(\hat{P}^{kl}(2^j\cdot)\theta)\|_{L^1}\lesssim\|(1-\Delta)^d(\hat{P}^{kl}(2^j\cdot)\theta)\|_{L^1}\lesssim\sum_{0\leq|m|\leq 2d}(2^j)^{|m|}\frac{1}{(2^j)^{|m|}}\lesssim C.$$

The above calculation is satisfied on $\mathbb{R}^d$ and $\mathbb{T}^d$. Moreover, on $\mathbb{T}^d$ for $1<p<\infty$

$$\|\Delta_{-1}P^{kl}u\|_{L^\infty(\mathbb{T}^d)}=\|\mathcal{F}^{-1}\hat{P}^{kl}\chi\mathcal{F}u\|_{L^\infty(\mathbb{T}^d)}\lesssim\|\mathcal{F}^{-1}\hat{P}^{kl}\chi\mathcal{F}u\|_{L^p(\mathbb{T}^d)}\lesssim\|\Delta_{-1}u\|_{L^p(\mathbb{T}^d)}\lesssim\|\Delta_{-1}u\|_{L^\infty(\mathbb{T}^d)},$$

where in the first inequality we used that $\text{supp}(\chi\hat{P}\mathcal{F}u)$ is contained in a ball and in the second inequality we used Mihlin's multiplier theorem. Thus the result follows. $\square$

Now we consider the scaling of the spatial variable:

Lemma 3.7 ([GIP13, Lemma 44]) For all $\lambda > 0$ and $u \in \mathcal{S}'$ define the scaling transformation $\Lambda_\lambda u(\cdot) = u(\lambda\cdot)$. Then we have

$$\|\Lambda_\lambda u\|_\alpha \lesssim (1+\lambda^\alpha)\|u\|_\alpha$$

for all $\alpha \in \mathbb{R}\backslash\{0\}$ and all $u \in \mathcal{C}^\alpha$.

3.2 Navier-Stokes equations

Let us focus on the equation on the $\mathbb{T}^3$:

$$Lu^i = \sum_{i_1=1}^3 P^{ii_1}\xi^{i_1} - \frac{1}{2}\sum_{i_1=1}^3 P^{ii_1}(\sum_{j=1}^3 D_j(uu^j)), \tag{3.1}$$

$$u(0) = Pu_0 \in \mathcal{C}^{-z},$$

where $\xi = (\xi^1, \xi^2, \xi^3)$, ξ^i is the periodic independent space time white noise, $L = \partial_t - \Delta$ and $z \in (1/2, 1/2 + \delta_0)$ with $0 < \delta_0 < 1/2$. As we mentioned in the introduction the nonlinear term of this equation is not well defined since the singularity of ξ. Now we follow the idea of [GIP13] to give the definition of the solution of the equation as limit of solutions u^ε to the following equation:

$$Lu^{\varepsilon,i} = \sum_{i_1=1}^3 P^{ii_1}\xi^{\varepsilon,i_1} - \frac{1}{2}\sum_{i_1=1}^3 P^{ii_1}(\sum_{j=1}^3 D_j(u^\varepsilon u^{\varepsilon,j})),$$

$$u(0) = Pu_0 \in \mathcal{C}^{-z},$$

for a family of smooth approximations (ξ^ε) of ξ such that $\xi^\varepsilon \to \xi$ as $\varepsilon \to 0$. Now we want to prove a priori estimate for u^ε.

In the following to avoid notations we omit the dependence on ε and consider (3.1) for smooth ξ and we use $\diamond$ to replace the product of some terms and we will give the meaning later. Consider

$$Lu_1^i = \sum_{i_1=1}^3 P^{ii_1}\xi^{i_1},$$

$$Lu_2^i = -\frac{1}{2}\sum_{i_1=1}^3 P^{ii_1}(\sum_{j=1}^3 D_j(u_1^{i_1} \diamond u_1^j)) \quad u_2(0) = 0,$$

$$Lu_3^i = -\frac{1}{2}\sum_{i_1=1}^3 P^{ii_1}(\sum_{j=1}^3 D_j(u_1^{i_1} \diamond u_2^j + u_2^{i_1} \diamond u_1^j)) \quad u_3(0) = 0,$$

$$LK^i = u_1^i \quad K^i(0) = 0.$$

Here for $i=1,2,3$, $u_1^i=\int_{-\infty}^t\sum_{i_1=1}^3 P^{ii_1}P_{t-s}\xi^{\varepsilon,i_1}ds$. Then we get that for any $\delta>0$ small enough, $u_1^i\in C([0,T];\mathcal{C}^{-\frac{1}{2}-\frac{\delta}{2}})$ and $K^i\in C([0,T];\mathcal{C}^{\frac{3}{2}-\delta})$ and by Lemma 3.5

$$\sup_{t\in[0,T]}\|K^i\|_{\frac{3}{2}-\delta}\lesssim\sup_{t\in[0,T]}\|u_1^i\|_{-1/2-\delta/2}$$

If we assume that for $i,j,i_1,j_1=1,2,3$, $u_1^i\diamond u_1^j\in C([0,T];\mathcal{C}^{-1-\delta/2})$, $u_1^i\diamond u_2^j=u_2^j\diamond u_1^i\in C([0,T];\mathcal{C}^{-1/2-\delta/2})$, $u_2^i\diamond u_2^j\in C([0,T];\mathcal{C}^{-\delta})$, $\pi_{0,\diamond}(u_3^i,u_1^j)\in C([0,T];\mathcal{C}^{-\delta})$ and $\pi_{0,\diamond}(P^{ii_1}D_jK^j,u_1^{j_1})$, $\pi_{0,\diamond}(P^{ii_1}D_jK^{i_1},u_1^{j_1})\in C([0,T];\mathcal{C}^{-\delta})$ and

$$\begin{aligned}C_\xi:=\sup_{t\in[0,T]}(&\sum_{i=1}^3\|u_1^i\|_{-1/2-\delta/2}+\sum_{i,j=1}^3\|u_1^i\diamond u_1^j\|_{-1-\delta/2}+\sum_{i,j=1}^3\|u_1^i\diamond u_2^j\|_{-1/2-\delta/2}+\sum_{i,j=1}^3\|u_2^i\diamond u_2^j\|_{-\delta}\\&+\sum_{i,j=1}^3\|\pi_{0,\diamond}(u_3^i,u_1^j)\|_{-\delta}+\sum_{i,i_1,j,j_1=1}^3\|\pi_{0,\diamond}(P^{ii_1}D_jK^j,u_1^{j_1})\|_{-\delta}+\sum_{i,i_1,j,j_1=1}^3\|\pi_{0,\diamond}(P^{ii_1}D_jK^{i_1},u_1^{j_1})\|_{-\delta})\\&<\infty.\end{aligned}\tag{3.2}$$

Moreover by Lemmas 3.5 and 3.6 we get for $i=1,2,3$, $u_2^i\in C([0,T];\mathcal{C}^{-\delta})$, $u_3^i\in C([0,T];\mathcal{C}^{1/2-\delta})$ and

$$\sup_{t\in[0,T]}(\sum_{i=1}^3\|u_2^i\|_{-\delta}+\sum_{i=1}^3\|u_3^i\|_{1/2-\delta})\lesssim C_\xi.\tag{3.3}$$

Here the meaning of $\diamond,\pi_{0,\diamond}$ will be given later.

Then $u=u_1+u_2+u_3+u_4$ solves (3.1) if and only if u_4 solves

$$\begin{aligned}Lu_4^i=-\frac{1}{2}\sum_{i_1,j=1}^3P^{ii_1}D_j(&u_1^{i_1}\diamond(u_3^j+u_4^j)+(u_3^{i_1}+u_4^{i_1})\diamond u_1^j+u_2^{i_1}\diamond u_2^j+u_2^{i_1}(u_3^j+u_4^j)+u_2^j(u_3^{i_1}+u_4^{i_1})+(u_3^{i_1}+u_4^{i_1})(u_3^j+u_4^j)).\\u_4(0)&=Pu_0-u_1(0).\end{aligned}\tag{3.4}$$

By a fixed point argument it is easy to obtain local existence and uniqueness of solutions of equation (3.1): More precisely, for each $\varepsilon\in(0,1)$ there exists a maximal time T_ε and u_4 satisfying equation (3.4) before T_ε such that $u_4\in C((0,T_\varepsilon);\mathcal{C}^{1/2-\delta_0})$ with respect to the norm $\sup_{t\in[0,T]}t^{\frac{1/2-\delta_0+z}{2}}\|u_4(t)\|_{1/2-\delta_0}$ and satisfying

$$\sup_{t\in[0,T_\varepsilon)}t^{\frac{1/2-\delta_0+z}{2}}\|u_4(t)\|_{1/2-\delta_0}=\infty.$$

Indeed since ξ is smooth by (3.4) and Lemmas 3.5 and 3.6 we have the following estimate

$$\sup_{t\in[0,T]}t^{\frac{1/2-\delta_0+z}{2}}\|u_4(t)\|_{1/2-\delta_0}\lesssim C_\varepsilon(\|u_0\|_{-z},u_1,u_2,u_3)+T^{\frac{1/2+\delta_0-z}{2}}(\sup_{t\in[0,T]}t^{\frac{1/2-\delta_0+z}{2}}\|u_4(t)\|_{1/2-\delta_0})^2,$$

where $C_\varepsilon(\|u_0\|_{-z},u_1,u_2,u_3)$ is a constant depending on ε and we used $z<1/2+\delta_0$.

Now consider the paracontrolled ansatz for $i=1,2,3$,

$$u_4^i=-\frac{1}{2}\sum_{i_1=1}^3P^{ii_1}(\sum_{j=1}^3D_j[\pi_<(u_3^{i_1}+u_4^{i_1},K^j)+\pi_<(u_3^j+u_4^j,K^{i_1})])+u^{\sharp,i}$$

with $u^{\sharp,i}(t)\in\mathcal{C}^{1/2+\beta}$ for some $\delta/2<\beta<(z+2\delta-1/2)\wedge(1/2-2\delta)$ and $t\in(0,T_\varepsilon)$ (which can be done for fix $\varepsilon>0$ since ξ_ε is smooth and we have

$$t^{\frac{1/2+\beta+z}{2}}\|u_4(t)\|_{1/2+\beta}\lesssim C_\varepsilon(\|u_0\|_{-z},u_1,u_2,u_3)+t^{\frac{1/2+\delta_0-z}{2}}(\sup_{s\in[0,t]}s^{\frac{1/2-\delta_0+z}{2}}\|u_4(s)\|_{1/2-\delta_0})^2).$$

By paracontrolled ansatz and Lemma 3.2 we also have the following estimate:

$$\|u_4^i\|_{1/2-\delta}\lesssim\sum_{i_1,j=1}^3\|u_3^{i_1}+u_4^{i_1}\|_{1/2-\delta_0}\|K^j\|_{3/2-\delta}+\|u^{\sharp,i}\|_{1/2+\beta}. \tag{3.5}$$

Then $u=u_1+u_2+u_3+u_4$ solves (3.1) if and only if $u^\sharp$ solves the following equation:

$$\begin{aligned}Lu^{\sharp,i}=&-\frac{1}{2}\sum_{i_1,j=1}^3P^{ii_1}D_j(u_2^{i_1}\diamond u_2^j+u_2^{i_1}(u_3^j+u_4^j)+u_2^j(u_3^{i_1}+u_4^{i_1})+(u_3^{i_1}+u_4^{i_1})(u_3^j+u_4^j)\\&-\pi_<(L(u_3^{i_1}+u_4^{i_1}),K^j)+2\sum_{l=1}^3\pi_<(D_l(u_3^{i_1}+u_4^{i_1}),D_lK^j)+\pi_>(u_3^{i_1}+u_4^{i_1},u_1^j)+\pi_{0,\diamond}(u_3^{i_1},u_1^j)+\pi_{0,\diamond}(u_4^{i_1},u_1^j)\\&-\pi_<(L(u_3^j+u_4^j),K^{i_1})+2\sum_{l=1}^3\pi_<(D_l(u_3^j+u_4^j),D_lK^{i_1})+\pi_>(u_3^j+u_4^j,u_1^{i_1})+\pi_{0,\diamond}(u_3^j,u_1^{i_1})+\pi_{0,\diamond}(u_4^j,u_1^{i_1}))\\&:=\phi^{\sharp,i}.\end{aligned}\tag{3.6}$$

First we consider $\pi_{0,\diamond}(u_4^i,u_1^j)$: by the paracontrolled ansatz we have for $i,j=1,2,3$,

$$\begin{aligned}\pi_{0,\diamond}(u_4^i,u_1^j)=&-\frac{1}{2}(\pi_{0,\diamond}(\sum_{i_1,j_1=1}^3P^{ii_1}\pi_<(u_3^{i_1}+u_4^{i_1},D_{j_1}K^{j_1}),u_1^j)+\pi_{0,\diamond}(\sum_{i_1,j_1=1}^3P^{ii_1}\pi_<(u_3^{j_1}+u_4^{j_1},D_{j_1}K^{i_1}),u_1^j)\\&+\sum_{i_1,j_1=1}^3\pi_0(P^{ii_1}\pi_<(D_{j_1}(u_3^{i_1}+u_4^{i_1}),K^{j_1}),u_1^j))+\sum_{i_1,j_1=1}^3\pi_0(P^{ii_1}\pi_<(D_{j_1}(u_3^{j_1}+u_4^{j_1}),K^{i_1}),u_1^j))\\&+\pi_0(u^{\sharp,i},u_1^j).\end{aligned}$$

The bound for the last three terms can be easily obtained by Lemma 3.2, and we only need to consider the first two terms: for $i,i_1,j,j_1=1,2,3$, we have

$$\begin{aligned}&\pi_{0,\diamond}(P^{ii_1}\pi_<(u_3^{i_1}+u_4^{i_1},D_{j_1}K^{j_1}),u_1^j)\\=&\pi_0(P^{ii_1}\pi_<(u_3^{i_1}+u_4^{i_1},D_{j_1}K^{j_1}),u_1^j)-\pi_0(\pi_<(u_3^{i_1}+u_4^{i_1},P^{ii_1}D_{j_1}K^{j_1}),u_1^j)\\&+\pi_0(\pi_<(u_3^{i_1}+u_4^{i_1},P^{ii_1}D_{j_1}K^{j_1}),u_1^j)-(u_3^{i_1}+u_4^{i_1})\pi_0(P^{ii_1}D_{j_1}K^{j_1},u_1^j)\\&+(u_3^{i_1}+u_4^{i_1})\pi_{0,\diamond}(P^{ii_1}D_{j_1}K^{j_1},u_1^j).\end{aligned}$$

Thus by Lemmas 3.2 and 3.3 we have for $\delta\le\delta_0<1/2-3\delta/2$

$$\begin{aligned}&\|\pi_{0,\diamond}(P^{ii_1}\pi_<(u_3^{i_1}+u_4^{i_1},D_{j_1}K^{j_1}),u_1^j)\|_{-\delta}\\\lesssim&\|P^{ii_1}\pi_<(u_3^{i_1}+u_4^{i_1},D_{j_1}K^{j_1})-\pi_<(u_3^{i_1}+u_4^{i_1},P^{ii_1}D_{j_1}K^{j_1})\|_{1-\delta-\delta_0}\|u_1^j\|_{-1/2-\delta/2}\\&+\|u_3^{i_1}+u_4^{i_1}\|_{1/2-\delta_0}\|P^{ii_1}D_{j_1}K^{j_1}\|_{1/2-\delta}\|u_1^j\|_{-1/2-\delta/2}\\&+\|u_3^{i_1}+u_4^{i_1}\|_{1/2-\delta_0}\|\pi_{0,\diamond}(P^{ii_1}D_{j_1}K^{j_1},u_1^j)\|_{-\delta}\\\lesssim&\|u_3^{i_1}+u_4^{i_1}\|_{1/2-\delta_0}\|K^{j_1}\|_{3/2-\delta}\|u_1^j\|_{-1/2-\delta/2}+\|u_3^{i_1}+u_4^{i_1}\|_{1/2-\delta_0}\|\pi_{0,\diamond}(P^{ii_1}D_{j_1}K^{j_1},u_1^j)\|_{-\delta}.\end{aligned}$$

Here in the last inequality we used Lemmas 3.4 and 3.6. We also obtain similar estimates for $\pi_{0,\diamond}(\sum_{i_1,j_1=1}^3 P^{ii_1}\pi_<(u_3^{j_1}+u_4^{j_1},D_{j_1}K^{i_1}),u_1^j)$.

Hence we obtain for $i,j=1,2,3$,

$$\begin{aligned}\|\pi_{0,\diamond}(u_4^i,u_1^j)\|_{-\delta}\lesssim&\sum_{i_1=1}^3\|u_3^{i_1}+u_4^{i_1}\|_{1/2-\delta_0}\sum_{j_1=1}^3\|K^{j_1}\|_{3/2-\delta}\|u_1^j\|_{-1/2-\delta/2}\\&+\sum_{i_1,j_1=1}^3\|u_3^{i_1}+u_4^{i_1}\|_{1/2-\delta_0}\|\pi_{0,\diamond}(P^{ii_1}D_{j_1}K^{j_1},u_1^j)\|_{-\delta}\\&+\sum_{i_1,j_1=1}^3\|u_3^{j_1}+u_4^{j_1}\|_{1/2-\delta_0}\|\pi_{0,\diamond}(P^{ii_1}D_{j_1}K^{i_1},u_1^j)\|_{-\delta}\\&+\|u^{\sharp,i}\|_{1/2+\beta}\|u_1^j\|_{-1/2-\delta/2}.\end{aligned}$$

Now we consider $\pi_<(L(u_3^i+u_4^i),K^j),i,j=1,2,3$, in (3.6): Indeed by (3.1) and (3.4) we have for $i=1,2,3$,

$$\begin{aligned}L(u_3^i+u_4^i)=&-\frac{1}{2}\sum_{i_1,j=1}^3P^{ii_1}D_j(u_1^{i_1}\diamond u_2^j+u_1^j\diamond u_2^{i_1}+u_1^{i_1}\diamond(u_3^j+u_4^j)+u_1^j\diamond(u_3^{i_1}+u_4^{i_1})\\&+u_2^{i_1}\diamond u_2^j+u_2^{i_1}(u_3^j+u_4^j)+u_2^j(u_3^{i_1}+u_4^{i_1})+(u_3^{i_1}+u_4^{i_1})(u_3^j+u_4^j)),\end{aligned}$$

where for $i,j=1,2,3$,

$$u_1^i\diamond(u_3^j+u_4^j)=\pi_<(u_3^j+u_4^j,u_1^i)+\pi_{0,\diamond}(u_3^j,u_1^i)+\pi_>(u_3^j+u_4^j,u_1^i)+\pi_{0,\diamond}(u_4^j,u_1^i).$$

Thus by Lemmas 3.6 and 3.2 we obtain for $i=1,2,3$,

$$\begin{aligned}\|L(u_3^i+u_4^i)\|_{-3/2-\delta/2}\lesssim&\sum_{i_1,j_1=1}^3[\|u_1^{i_1}\diamond u_2^{j_1}\|_{-1/2-\delta/2}+\|u_2^{i_1}\diamond u_2^{j_1}\|_{-\delta}+\|u_1^{i_1}\|_{-1/2-\delta/2}\|u_3^{j_1}+u_4^{j_1}\|_{1/2-\delta_0}\\&+\|\pi_{0,\diamond}(u_3^{i_1},u_1^{j_1})\|_{-\delta}+\|u_2^{i_1}\|_{-\delta}\|u_3^{j_1}+u_4^{j_1}\|_{1/2-\delta_0}\\&+\|u_3^{i_1}+u_4^{i_1}\|_{\delta}\|u_3^{j_1}+u_4^{j_1}\|_{\delta}+\|u_3^{i_1}+u_4^{i_1}\|_{1/2-\delta_0}\|K^{j_1}\|_{3/2-\delta}\sum_{i_2=1}^3\|u_1^{i_2}\|_{-1/2-\delta/2}\\&+\sum_{j,i_2=1}^3\|u_3^{i_1}+u_4^{i_1}\|_{1/2-\delta_0}\|\pi_{0,\diamond}(P^{i_2i_1}D_{j_1}K^{j_1},u_1^j)\|_{-\delta}\\&+\sum_{j,i_2=1}^3\|u_3^{j_1}+u_4^{j_1}\|_{1/2-\delta_0}\|\pi_{0,\diamond}(P^{i_2i_1}D_{j_1}K^{i_1},u_1^j)\|_{-\delta}\\&+\|u^{\sharp,i_1}\|_{1/2+\beta}\|u_1^{j_1}\|_{-1/2-\delta/2}],\end{aligned}$$

which by Lemma 3.2 implies that

$$
\begin{aligned}
&\|\pi_<(L(u_3^i+u_4^i),K^j)\|_{-3\delta/2}\\
\lesssim&\|K^j\|_{3/2-\delta}\sum_{i_1,j_1=1}^3[\|u_1^{i_1}\diamond u_2^{j_1}\|_{-1/2-\delta/2}+\|u_2^{i_1}\diamond u_2^{j_1}\|_{-\delta}+\|u_1^{i_1}\|_{-1/2-\delta/2}\|u_3^{j_1}+u_4^{j_1}\|_{1/2-\delta_0}\\
&+\|\pi_{0,\diamond}(u_3^{i_1},u_1^{j_1})\|_{-\delta}+\|u_2^{i_1}\|_{-\delta}\|u_3^{j_1}+u_4^{j_1}\|_{1/2-\delta_0}\\
&+\|u_3^{i_1}+u_4^{i_1}\|_{\delta}\|u_3^{j_1}+u_4^{j_1}\|_{\delta}+\|u_3^{i_1}+u_4^{i_1}\|_{1/2-\delta_0}\|K^{j_1}\|_{3/2-\delta}\sum_{i_2=1}^3\|u_1^{i_2}\|_{-1/2-\delta/2}\\
&+\sum_{i_2,j_2=1}^3\|u_3^{i_1}+u_4^{i_1}\|_{1/2-\delta_0}\|\pi_{0,\diamond}(P^{i_2i_1}D_{j_1}K^{j_1},u_1^{j_2})\|_{-\delta}\\
&+\sum_{i_2,j_2=1}^3\|u_3^{j_1}+u_4^{j_1}\|_{1/2-\delta_0}\|\pi_{0,\diamond}(P^{i_2i_1}D_{j_1}K^{i_1},u_1^{j_2})\|_{-\delta}+\|u^{\sharp,i_1}\|_{1/2+\beta}\|u_1^{j_1}\|_{-1/2-\delta/2}].
\end{aligned}
$$

Now we consider $\pi_<(D_l(u_3^{i_1}+u_4^{i_1}),D_lK^j)+\pi_>(u_3^{i_1}+u_4^{i_1},u_1^j)$ for $i_1,l,j=1,2,3$ in (3.6): Indeed by Lemma 3.2 we have

$$
\begin{aligned}
&\|\pi_<(D_l(u_3^{i_1}+u_4^{i_1}),D_lK^j)+\pi_>(u_3^{i_1}+u_4^{i_1},u_1^j)\|_{-2\delta}\\
\lesssim&(\|u_3^{i_1}\|_{1/2-\delta}+\|u_4^{i_1}\|_{1/2-\delta})(\|K^j\|_{3/2-\delta}+\|u_1^j\|_{-1/2-\delta/2})\\
\lesssim&(\|u_3^{i_1}\|_{1/2-\delta}+\sum_{i_2,j_1=1}^3\|u_3^{i_2}+u_4^{i_2}\|_{1/2-\delta_0}\|K^{j_1}\|_{3/2-\delta}+\|u^{\sharp,i_1}\|_{1/2+\beta})(\|K^j\|_{3/2-\delta}+\|u_1^j\|_{-1/2-\delta/2}),
\end{aligned}
$$

where in the last inequality we used (3.5).

Hence by (3.6) we get that

$$
\begin{aligned}
&\|\phi^{\sharp,i}\|_{-1-2\delta}\\
\lesssim&\sum_{j=1}^3(\|K^j\|_{3/2-\delta}+1)\sum_{i_1,j_1=1}^3[\|u_1^{i_1}\diamond u_2^{j_1}\|_{-1/2-\delta/2}+\|u_2^{i_1}\diamond u_2^{j_1}\|_{-\delta}+\|u_1^{i_1}\|_{-1/2-\delta/2}\|u_3^{j_1}+u_4^{j_1}\|_{1/2-\delta_0}\\
&+\|\pi_{0,\diamond}(u_3^{i_1},u_1^{j_1})\|_{-\delta}+\|u_2^{i_1}\|_{-\delta}\|u_3^{j_1}+u_4^{j_1}\|_{1/2-\delta_0}\\
&+\|u_3^{i_1}+u_4^{i_1}\|_{\delta}\|u_3^{j_1}+u_4^{j_1}\|_{\delta}+\|u_3^{i_1}+u_4^{i_1}\|_{1/2-\delta_0}\|K^{j_1}\|_{3/2-\delta}\sum_{i_2=1}^3\|u_1^{i_2}\|_{-1/2-\delta/2}\\
&+\sum_{i_2,j_2=1}^3\|u_3^{i_1}+u_4^{i_1}\|_{1/2-\delta_0}\|\pi_{0,\diamond}(P^{i_2i_1}D_{j_1}K^{j_1},u_1^{j_2})\|_{-\delta}\\
&+\sum_{i_2,j_2=1}^3\|u_3^{j_1}+u_4^{j_1}\|_{1/2-\delta_0}\|\pi_{0,\diamond}(P^{i_2i_1}D_{j_1}K^{i_1},u_1^{j_2})\|_{-\delta}+\|u^{\sharp,i_1}\|_{1/2+\beta}\|u_1^{j_1}\|_{-1/2-\delta/2}]\\
&+\sum_{i_1,j_1,l=1}^3(\|u_3^{i_1}\|_{1/2-\delta}+\sum_{i_2,j_1=1}^3\|u_3^{i_2}+u_4^{i_2}\|_{1/2-\delta_0}\|K^{j_1}\|_{3/2-\delta}+\|u^{\sharp,i_1}\|_{1/2+\beta})(\|K^j\|_{3/2-\delta}+\|u_1^j\|_{-1/2-\delta/2})
\end{aligned}
$$

$$\lesssim(1+C_\xi^3)(1+\sum_{i_1=1}^3\|u^{\sharp,i_1}\|_{1/2+\beta}+\sum_{i_1=1}^3\|u_4^{i_1}\|_{1/2-\delta_0}+(\sum_{i_1=1}^3\|u_4^{i_1}\|_\delta)^2), \tag{3.7}$$

where we used (3.2) (3.3) and $\delta\leq\delta_0$ in the last inequality.

In order to use this estimate to bound u_4, we apply the scaling argument as [GIP13]. More precisely, for $\lambda\in(0,1)$ we set $\Lambda_\lambda u(t,x)=u(\lambda^2t,\lambda x)$, so that $L\Lambda_\lambda=\lambda^2\Lambda_\lambda L$. Now let $u_1^\lambda=\lambda^{1/2+\delta/2}\Lambda_\lambda u_1, u_2^\lambda=\lambda^\delta\Lambda_\lambda u_2, u_3^\lambda=\lambda^z\Lambda_\lambda u_3, u_4^\lambda=\lambda^z\Lambda_\lambda u_4, LK^\lambda=u_1^\lambda$. Note that $u_i^\lambda:[0,T/\lambda^2]\times\mathbb{T}_\lambda^3\to\mathbb{R}$, $i=1,2,3,4$, where $\mathbb{T}_\lambda^3=(\mathbb{R}/(2\pi\lambda^{-1}\mathbb{Z}))^3$ is a rescaled torus, and that u_4^λ solves the equation:

$$\begin{aligned}Lu_4^{\lambda,i}=&-\frac{1}{2}\sum_{i_1,j=1}^3P^{ii_1}D_j(\lambda^{1/2-\delta/2}u_1^{\lambda,i_1}\diamond(u_3^{\lambda,j}+u_4^{\lambda,j})+\lambda^{1/2-\delta/2}u_1^{\lambda,j}\diamond(u_3^{\lambda,i_1}+u_4^{\lambda,i_1})\\&+\lambda^{1-2\delta+z}u_2^{\lambda,i_1}\diamond u_2^{\lambda,j}+\lambda^{1-\delta}u_2^{\lambda,i_1}(u_3^{\lambda,j}+u_4^{\lambda,j})+\lambda^{1-\delta}u_2^{\lambda,j}(u_3^{\lambda,i_1}+u_4^{\lambda,i_1})\\&+\lambda^{1-z}(u_3^{\lambda,i_1}+u_4^{\lambda,i_1})(u_3^{\lambda,j}+u_4^{\lambda,j})),\end{aligned}$$

$$u_4^\lambda(0)=\lambda^z\Lambda_\lambda(u_0-u_1(0)).$$

The scaling is chosen in such a way that $C_\xi^\lambda\lesssim C_\xi$,

$$\sup_{t\in[0,T]}\|u_2^\lambda\|_{-\delta}+\|u_3^\lambda\|_{1/2-\delta})\lesssim C_\xi$$

and $\|\lambda^z\Lambda_\lambda(u_0-u_1(0))\|_{-z}\lesssim\|u_0-u_1(0)\|_{-z}$ uniformly over $\lambda\in(0,1)$ by Lemma 3.7, where for $i,i_1,j,j_2=1,2,3$, we have for $j_1=i_1$ or $j_1=j$

$$\|\pi_{0,\diamond}(P^{ii_1}D_jK^{\lambda,j_1},u_1^{\lambda,j_2})\|_{-\delta}=\|\lambda^\delta\Lambda_\lambda\pi_{0,\diamond}(P^{ii_1}D_jK^{j_1},u_1^{j_2})\|_{-\delta}\lesssim\|\pi_{0,\diamond}(P^{ii_1}D_jK^{j_1},u_1^{j_2})\|_{-\delta},$$

holds uniformly over $\lambda\in(0,1)$.

Moreover, we obtain

$$Lu_3^{\lambda,i}=\lambda^{z+2}\Lambda_\lambda Lu_3^i=-\frac{\lambda^{1/2+z-3\delta/2}}{2}\sum_{i_1=1}^3P^{ii_1}(\sum_{j=1}^3D_j(u_1^{\lambda,i_1}\diamond u_2^{\lambda,j}+u_2^{\lambda,i_1}\diamond u_1^{\lambda,j})).$$

Then by the same argument as above we define $u^{\sharp,\lambda},\phi^{\sharp,\lambda}$ in the same way as $u^\sharp,\phi^\sharp$:

$$u_4^{\lambda,i}=-\frac{\lambda^{1/2-\delta/2}}{2}\sum_{i_1=1}^3P^{ii_1}(\sum_{j=1}^3D_j[\pi_<(u_3^{\lambda,i_1}+u_4^{\lambda,i_1},K^{\lambda,j})+\pi_<(u_3^{\lambda,j}+u_4^{\lambda,j},K^{\lambda,i_1})])+u^{\sharp,\lambda,i}$$

and by Lemma 3.2 we get

$$\|u_4^{\lambda,i}\|_{1/2-\delta_0}\lesssim\lambda^{1/2-\delta/2}\sum_{i_1,j=1}^3\|u_3^{\lambda,i_1}+u_4^{\lambda,i_1}\|_{1/2-\delta_0}\|K^{\lambda,j}\|_{3/2-\delta}+\|u^{\sharp,\lambda,i}\|_{1/2-\delta_0},$$

which shows that for λ small enough (only depend on C_ξ)

$$\sum_{i=1}^3\|u_4^{\lambda,i}\|_{1/2-\delta_0}\lesssim\lambda^{1/2-\delta/2}C_\xi^2+\sum_{i=1}^3\|u^{\sharp,\lambda,i}\|_{1/2-\delta_0}. \tag{3.8}$$

Similarly, we have for λ small enough (only depend on C_ξ)

$$\sum_{i=1}^{3}\|u_4^{\lambda,i}\|_\delta \lesssim \lambda^{1/2-\delta/2}C_\xi^2+\sum_{i=1}^{3}\|u^{\sharp,\lambda,i}\|_\delta. \tag{3.9}$$

Moreover we have a similar estimate as (3.7) and obtain

$$\|\phi^{\sharp,\lambda}\|_{-1-2\delta}\lesssim \lambda^{1-z}(1+C_\xi^3)(1+\|u^{\sharp,\lambda}\|_{1/2+\beta}+\|u_4^\lambda\|_{1/2-\delta_0}+\|u_4^\lambda\|_\delta^2), \tag{3.10}$$

where we used $1-z\leq(1-\delta)/2$. Then by Lemma 3.5 we get that for $\delta+z<1$

$$\begin{aligned}&t^{\delta+z}\|u^{\sharp,\lambda}(t)\|_{1/2+\beta}\\ \lesssim&\|Pu_0-u_1(0)\|_{-z}+t^{\delta+z}\int_0^t(t-s)^{-3/4-\delta-\beta/2}s^{-(\delta+z)}s^{\delta+z}\|\phi^{\sharp,\lambda}(s)\|_{-1-2\delta}ds,\end{aligned} \tag{3.11}$$

where we used the condition on β to deduce $\beta+2\delta<1/2$ and $\frac{1/2+\beta+z}{2}\leq\delta+z$. Also we have

$$\begin{aligned}t^{\delta+z}\|u^{\sharp,\lambda}(t)\|_\delta^2\lesssim&\|Pu_0-u_1(0)\|_{-z}^2+t^{\delta+z}(\int_0^t(t-s)^{-\frac{1+3\delta}{2}}s^{-(\delta+z)}s^{\delta+z}\|\phi^{\sharp,\lambda}(s)\|_{-1-2\delta}ds)^2\\ \lesssim&\|Pu_0-u_1(0)\|_{-z}^2+t^{(1-3\delta)/2}\int_0^t(t-s)^{-\frac{1+3\delta}{2}}s^{-(\delta+z)}(s^{\delta+z}\|\phi^{\sharp,\lambda}(s)\|_{-1-2\delta})^2ds.\end{aligned} \tag{3.12}$$

Here in the last inequality we used Hölder inequality. Thus by (3.8-3.12) we get that

$$\begin{aligned}t^{\delta+z}\|\phi^{\sharp,\lambda}\|_{-1-2\delta}\lesssim&\lambda^{1-z}(1+C_\xi^3)(\|Pu_0-u_1(0)\|_{-z}^2+\lambda^{1-\delta}C_\xi^4+1\\ &+\int_0^t t^{\delta+z}(t-s)^{-3/4-\delta-\beta/2}s^{-(\delta+z)}(s^{\delta+z}\|\phi^{\sharp,\lambda}(s)\|_{-1-2\delta})\\ &+t^{(1-3\delta)/2}(t-s)^{-\frac{1+3\delta}{2}}s^{-(\delta+z)}(s^{\delta+z}\|\phi^{\sharp,\lambda}(s)\|_{-1-2\delta})^2ds).\end{aligned}$$

Then Bihari's inequality implies that for $z<1-4\delta$ there exists some T_0 such that

$$\sup_{t\in[0,T_0]}t^{\delta+z}\|\phi^{\sharp,\lambda}\|_{-1-2\delta}\lesssim C(T_0,C_\xi,\|u_0\|_{-z}),$$

where $C(T_0,C_\xi)$ is a locally Lipschitz function on $T_0,\|u_0\|_{-z}$ and C_ξ. Here T_0 can be chosen such that the result is satisfied for all $\varepsilon\in(0,1)$ if C_ξ^ε and $\|u_0\|_{-z}$ is uniformly bounded over $\varepsilon\in(0,1)$. Similarly as (3.11) we have

$$\begin{aligned}&t^{(1/2-\delta_0+z)/2}\|u^{\sharp,\lambda}(t)\|_{1/2-\delta_0}\\ \lesssim&\|Pu_0-u_1(0)\|_{-z}+t^{(1/2-\delta_0+z)/2}\int_0^t(t-s)^{-3/4-\delta+\delta_0/2}s^{-(\delta+z)}s^{\delta+z}\|\phi^{\sharp,\lambda}(s)\|_{-1-2\delta}ds\\ \lesssim&\|Pu_0-u_1(0)\|_{-z}+t^{(1-4\delta-z)/2}\sup_{s\in[0,t]}s^{\delta+z}\|\phi^{\sharp,\lambda}(s)\|_{-1-2\delta}.\end{aligned} \tag{3.13}$$

Thus by (3.8) (3.13) we obtain that

$$\sup_{t\in[0,T_0]}t^{\frac{1/2-\delta_0+z}{2}}\|u_4^\lambda(t)\|_{1/2-\delta_0}\lesssim C_\xi^2+\|u_0\|_{-z}+C(T_0,C_\xi,\|u_0\|_{-z}),$$

which implies that $T_\varepsilon \geq T_0$. Here we used $z \geq 1/2 + \delta/2$. Moreover by paracontrolled ansatz we also obtain

$$\|u_4^{\lambda,i}\|_{-z} \lesssim \lambda^{1/2-\delta/2} \sum_{i_1,j=1}^{3} \|u_3^{\lambda,i_1} + u_4^{\lambda,i_1}\|_{-z} \|K^{\lambda,j}\|_{3/2-\delta} + \|u^{\sharp,\lambda,i}\|_{-z},$$

which by Lemma 3.5 implies that for λ small enough (only depend on C_ξ) and $t \in [0, T_0]$

$$\begin{aligned}\|u_4^\lambda(t)\|_{-z} &\lesssim C_\xi^2 + \|u^\sharp\|_{-z} \\ &\lesssim C_\xi^2 + \|u_0\|_{-z} + \int_0^t (t-s)^{\frac{-1-2\delta+z}{2}} s^{-(\delta+z)} s^{\delta+z} \|\phi^{\sharp,\lambda}\|_{-1-2\delta} ds,\end{aligned}$$

where we used $z < 1 - 4\delta$. Thus we obtain

$$\sup_{t\in[0,T_0]} \|u_4^\lambda(t)\|_{-z} \lesssim C_\xi^2 + \|u_0\|_{-z} + C(T_0, C_\xi, \|u_0\|_{-z}).$$

Similar arguments show that for every $a > 0$ there exists a sufficiently small $\lambda > 0$ such that the map $(u_0, u_1, u_1 \diamond u_1, u_1 \diamond u_2, u_2 \diamond u_2, \pi_{0,\diamond}(u_3, u_1), \pi_{0,\diamond}(PDK, u_1)) \mapsto u_4^\lambda$ is Lipschitz continuous on the set

$$\max\{\|u_0\|_{-z}, C_\xi\} \leq a.$$

Here we consider u_4^λ with respect to the norm of

$$\sup_{t\in[0,T_0]} \|u_4^\lambda(t)\|_{-z}.$$

Since $u_4 = \lambda^{-z}\Lambda_{\lambda^{-1}} u_4^\lambda$, we also obtain that u_4 restricted to $[0, \lambda^2 T]$ depends in a locally Lipschitz continuous way on the data $(u_0, , u_1, u_1 \diamond u_1, u_1 \diamond u_2, u_2 \diamond u_2, \pi_{0,\diamond}(u_3, u_1), \pi_{0,\diamond}(PDK, u_1))$. Hence we obtain for given $(u_0, u_1, u_1 \diamond u_1, u_1 \diamond u_2, u_2 \diamond u_2, \pi_{0,\diamond}(u_3, u_1), \pi_{0,\diamond}(PDK, u_1))$ there exists a unique local solution u to (3.1) with initial condition u_0, which is the limit of the solutions $u^\varepsilon, \varepsilon > 0$, to the following equation

$$Lu^{\varepsilon,i} = \sum_{i_1=1}^{3} P^{ii_1}\xi^{\varepsilon,i_1} - \frac{1}{2}\sum_{i_1=1}^{3} P^{ii_1}(\sum_{j=1}^{3} D_j(u^{\varepsilon,i_1}u^{\varepsilon,j})) \quad u^\varepsilon(0) = u_0,$$

provided that for any $\delta > 0$ and $i, i_1, j, j_2 = 1, 2, 3$, $u_1^{\varepsilon,i} \to u_1^i$ in $C([0,T]; \mathcal{C}^{-1/2-\delta/2})$, $u_1^{\varepsilon,i} \diamond u_1^{\varepsilon,j} \to u_1^i \diamond u_1^j$ in $C([0,T]; \mathcal{C}^{-1-\delta/2})$, $u_1^{\varepsilon,i} \diamond u_2^{\varepsilon,j} \to u_1^i \diamond u_2^j$ in $C([0,T]; \mathcal{C}^{-1/2-\delta/2})$, $u_2^{\varepsilon,i} \diamond u_2^{\varepsilon,j} \to u_2^i \diamond u_2^j$ in $C([0,T]; \mathcal{C}^{-\delta})$, $\pi_{0,\diamond}(u_3^{\varepsilon,i}, u_1^{\varepsilon,j}) \to \pi_{0,\diamond}(u_3^i, u_1^j)$ in $C([0,T]; \mathcal{C}^{-\delta})$, $\pi_{0,\diamond}(P^{ii_1}D_jK^{\varepsilon,j}, u_1^{\varepsilon,j_2}) \to \pi_{0,\diamond}(P^{ii_1}D_jK^j, u_1^{j_2})$ in $C([0,T]; \mathcal{C}^{-\delta})$ and $\pi_{0,\diamond}(P^{ii_1}D_jK^{\varepsilon,i_1}, u_1^{\varepsilon,j_2}) \to \pi_{0,\diamond}(P^{ii_1}D_jK^{i_1}, u_1^{j_2})$ in $C([0,T]; \mathcal{C}^{-\delta})$. Here $u_i^\varepsilon, i = 1, 2, 3, 4$ is defined as above with ξ replaced by ξ^ε. Here

$$u_1^{\varepsilon,i} \diamond u_1^{\varepsilon,j} := u_1^{\varepsilon,i}u_1^{\varepsilon,j} - C_0^{\varepsilon,ij},$$

$$u_1^{\varepsilon,i} \diamond u_2^{\varepsilon,j} := u_1^{\varepsilon,i}u_2^{\varepsilon,j},$$

$$u_2^{\varepsilon,i} \diamond u_2^{\varepsilon,j} := u_2^{\varepsilon,i}u_2^{\varepsilon,j} - \varphi_2^{\varepsilon,ij}(t) - C_2^{\varepsilon,ij},$$

$$\pi_{0,\diamond}(u_3^{\varepsilon,i}, u_1^{\varepsilon,j}) := \pi_0(u_3^{\varepsilon,i}, u_1^{\varepsilon,j}) - \varphi_1^{\varepsilon,ij}(t) - C_1^{\varepsilon,ij},$$

$$\pi_{0,\diamond}(P^{ii_1}D_jK^{\varepsilon,j},u_1^{\varepsilon,j_2}):=\pi_0(P^{ii_1}D_jK^{\varepsilon,j},u_1^{\varepsilon,j_2}),$$
$$\pi_{0,\diamond}(P^{ii_1}D_jK^{\varepsilon,i_1},u_1^{\varepsilon,j_2}):=\pi_0(P^{ii_1}D_jK^{\varepsilon,i_1},u_1^{\varepsilon,j_2}),$$

and C_0^ε is defined in section 3.3, C_1^ε and φ_1^ε are defined in Section 3.3.2 and C_2^ε and φ_2^ε are defined in Section 3.3.4 and φ_i^ε converges to some φ_i with respect to $\|\varphi\|=\sup_{t\in[0,T]}t^\rho|\varphi(t)|$ for any $\rho>0$ and $i=1,2$. Thus we obtain the following theorem:

Theorem 3.8 Let $z\in(1/2,1/2+\delta_0)$ with $0<\delta_0<1/2$ and assume that $(\xi^\varepsilon)_{\varepsilon>0}$ is a family of smooth functions converging to ξ. Suppose that there exist $v_1^i,v_2^{ij},v_3^{ij},v_4^{ij},v_5^{ij},v_6^{ii_1jj_2},v_7^{ii_1jj_2}$ such that for any $\delta>0$ and $i,i_1,j,j_2=1,2,3$, $u_1^{\varepsilon,i}\to v_1^i$ in $C([0,T];\mathcal{C}^{-1/2-\delta/2})$, $u_1^{\varepsilon,i}\diamond u_1^{\varepsilon,j}\to v_2^{ij}$ in $C([0,T];\mathcal{C}^{-1-\delta/2})$, $u_1^{\varepsilon,i}\diamond u_2^{\varepsilon,j}\to v_3^{ij}$ in $C([0,T];\mathcal{C}^{-1/2-\delta/2})$, $u_2^{\varepsilon,i}\diamond u_2^{\varepsilon,j}\to v_4^{ij}$ in $C([0,T];\mathcal{C}^{-\delta})$, $\pi_{0,\diamond}(u_3^{\varepsilon,i},u_1^{\varepsilon,j})\to v_5^{ij}$ in $C([0,T];\mathcal{C}^{-\delta})$, $\pi_{0,\diamond}(P^{ii_1}D_jK^{\varepsilon,j},u_1^{\varepsilon,j_2})\to v_6^{ii_1jj_2}$ in $C([0,T];\mathcal{C}^{-\delta})$ and $\pi_{0,\diamond}(P^{ii_1}D_jK^{\varepsilon,i_1},u_1^{\varepsilon,j_2})\to v_7^{ii_1jj_2}$ in $C([0,T];\mathcal{C}^{-\delta})$. Let for $\varepsilon>0$ the function u^ε be the unique maximal solution to the Cauchy problem

$$Lu^{\varepsilon,i}=\sum_{i_1=1}^3P^{ii_1}\xi^{\varepsilon,i_1}-\frac{1}{2}\sum_{i_1=1}^3P^{ii_1}(\sum_{j=1}^3D_j(u^{\varepsilon,i_1}u^{\varepsilon,j}))\quad u^\varepsilon(0)=Pu_0,$$

such that u_4^ε defined as above in $C((0,T_\varepsilon);\mathcal{C}^{1/2-\delta_0})$, where $u_0\in\mathcal{C}^{-z}$. Then there exists $\tau=\tau(u_0,v_1,v_2,v_3,v_4,v_5,v_6)>0$ such that

$$\sup_{t\in[0,\tau]}\|u^\varepsilon-u\|_{-z}\to0.$$

The limit u depends only on $(u_0,v_i),i=1....,6$, and not on the approximating family.

Remark 3.9 Indeed we can define the solution space as the following: $u-u_1\in\mathcal{D}_X^L$ if

$$u-u_1=u_2+u_3-\frac{1}{2}\int_0^tP_{t-s}P\sum_{j=1}^3D_j[\pi_<(\Phi',u_1^j)+\pi_<(\Phi'^j,u_1)]ds+\Phi^\sharp$$

such that

$$\|\Phi^\sharp\|_{\star,1,L,T}:=\sup_{t\in[0,T]}t^{\frac{1-\eta+z}{2}}\|\Phi_t^\sharp\|_{1-\eta}+\sup_{t\in[0,T]}t^{\frac{\gamma+z}{2}}\|\Phi_t^\sharp\|_\gamma+\sup_{s,t\in[0,T]}s^{\frac{z+a}{2}}\frac{\|\Phi_t^\sharp-\Phi_s^\sharp\|_{a-2b}}{|t-s|^b}<\infty,$$

and

$$\|\Phi'\|_{\star,2,L,T}:=\sup_{t\in[0,T]}t^{\frac{2\gamma+z}{2}}\|\Phi_t'\|_{1/2-\kappa}+\sup_{s,t\in[0,T]}s^{\frac{z+a}{2}}\frac{\|\Phi_t'-\Phi_s'\|_{c-2d}}{|t-s|^d}<\infty.$$

Here $\eta,\gamma\in(0,1),a\geq2b,0<\kappa<1/2,c\geq2d$. By a similar argument as [CC13] if $u-u_1\in\mathcal{D}_X^L$ then the equation

$$u-u_1=P_t(u_0-u_1(0))-\frac{1}{2}\int_0^tP_{t-s}P\sum_{j=1}^3D_j(u_1\diamond u_1^j+(u-u_1)\diamond u_1^j+u_1\diamond(u-u_1)^j+(u-u_1)\diamond(u-u_1)^jds$$

can be well defined and by a fixed point argument we also obtain local existence and uniqueness of solution. The calculation for this method is more complicated and we will not go to details here.

3.3 Renormalisation

In the following we use notation X to represent u_1 in the calculation and $\hat{f}(k) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{T}^3} f(x) e^{\imath x\cdot k} dx$ for $k \in \mathbb{Z}^3$. To simplify the arguments below, we assume that $\hat{\xi}(0) = 0$ and restrict ourselves to the flow of $\int_{\mathbb{T}^3} u(x)dx = 0$. Then $X_t = \sum_{k\in\mathbb{Z}^3\backslash\{0\}} \hat{X}_t(k)e_k$ is a centered Gaussian process with covariance function given by

$$E[\hat{X}_t^i(k)\hat{X}_s^j(k')] = \delta_{k+k'=0} \sum_{i_1=1}^{3} \frac{e^{-|k|^2|t-s|}}{2|k|^2} \hat{P}^{ii_1}(k)\hat{P}^{ji_1}(k),$$

and $\hat{X}_t(0) = 0$, where $e_k(x) = (2\pi)^{-3/2} e^{\imath x\cdot k}, x \in \mathbb{T}^3$ and $\hat{P}^{ii_1}(k) = \delta_{ii_1} - \frac{k_i k_{i_1}}{|k|^2}$ for $k \in \mathbb{Z}^3\backslash\{0\}$. Let $X_t^{\varepsilon,i} = \int_{-\infty}^t \sum_{i_1=1}^3 P^{ii_1} P_{t-s}\xi^{\varepsilon,i_1} ds$, more precisely $\hat{\xi}^\varepsilon(k) = f(\varepsilon k)\hat{\xi}(k)$, where f is a smooth radial function with bounded support such that $f(0) = 1$. In this subsection we will prove that there exist $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ such that for $i, i_1, j, j_2 = 1, 2, 3$, $u_1^{\varepsilon,i} \to v_1^i$ in $C([0,T]; \mathcal{C}^{-1/2-\delta/2})$, $u_1^{\varepsilon,i} \diamond u_1^{\varepsilon,j} \to v_2^{ij}$ in $C([0,T]; \mathcal{C}^{-1-\delta})$, $u_1^{\varepsilon,i} \diamond u_2^{\varepsilon,j} \to v_3^{ij}$ in $C([0,T]; \mathcal{C}^{-1/2-\delta})$, $u_2^{\varepsilon,i} \diamond u_2^{\varepsilon,j} \to v_4^{ij}$ in $C([0,T]; \mathcal{C}^{-\delta})$, $\pi_{0,\diamond}(u_3^{\varepsilon,i}, u_1^{\varepsilon,j}) \to v_5^{ij}$ in $C([0,T]; \mathcal{C}^{-\delta})$, $\pi_{0,\diamond}(P^{ii_1} D_j K^{\varepsilon,j}, u_1^{\varepsilon,j_2}) \to v_6^{ii_1jj_2}$ in $C([0,T]; \mathcal{C}^{-\delta})$ and $\pi_{0,\diamond}(P^{ii_1} D_j K^{\varepsilon,i_1}, u_1^{\varepsilon,j_2}) \to v_7^{ii_1jj_2}$ in $C([0,T]; \mathcal{C}^{-\delta})$.

It is easy to obtain that $u_1^\varepsilon \to v_1$ in $C([0,T]; \mathcal{C}^{-1/2-\delta/2})$. Renormalisation of $u_1^{\varepsilon,i} \diamond u_1^{\varepsilon,j}, i,j = 1,2,3$ and the fact that there exists $v_2 \in C([0,T]; \mathcal{C}^{-1-\delta})$ such that $u_1^{\varepsilon,i} \diamond u_1^{\varepsilon,j} \to v_2^{ij}$ in $C([0,T]; \mathcal{C}^{-1-\delta})$ can be easily obtained by using wick product (c.f.[CC13]), where we choose

$$C_0^{\varepsilon,i,j} = \sum_{i_1=1}^{3} \sum_{k\in\mathbb{Z}^3\backslash\{0\}} \frac{f(\varepsilon k)^2}{2|k|^2} \hat{P}^{ii_1}(k)\hat{P}^{ji_1}(k).$$

It is easy to check that $C_0^{\varepsilon,i,j} \to \infty$ as $\varepsilon \to 0$. Now we introduce the following notations: $k_{1,\dots,n} = \sum_{i=1}^n k_i$. To obtain the results we first prove the following two lemmas for our later use. Inspired by [Hai14, Lemma 10.14] we prove the following theorem.

Lemma 3.10 Let $0 < l, m < d, l + m - d > 0$. Then we have

$$\sum_{k_1,k_2\in\mathbb{Z}^d\backslash\{0\}, k_1+k_2=k} \frac{1}{|k_1|^l |k_2|^m} \lesssim \frac{1}{|k|^{l+m-d}}.$$

Proof We have the following calculations:

$$\sum_{k_1,k_2\in\mathbb{Z}^d\backslash\{0\}, k_1+k_2=k} \frac{1}{|k_1|^l |k_2|^m} \lesssim \sum_{k_1,k_2\in\mathbb{Z}^d\backslash\{0\}, k_1+k_2=k, |k_1|\le\frac{|k|}{2}} \frac{1}{|k_1|^l |k_2|^m} + \sum_{k_1,k_2\in\mathbb{Z}^d\backslash\{0\}, k_1+k_2=k, |k_2|\le\frac{|k|}{2}} \frac{1}{|k_1|^l |k_2|^m}$$
$$+ \sum_{k_1,k_2\in\mathbb{Z}^d\backslash\{0\}, k_1+k_2=k, |k_1|>\frac{|k|}{2}, |k_2|>\frac{|k|}{2}} \frac{1}{|k_1|^l |k_2|^m}.$$

Since if $|k_1| \le |k|/2$, $|k_2| \ge |k| - |k_1| \ge |k|/2$, we obtain

$$\sum_{k_1,k_2\in\mathbb{Z}^d\backslash\{0\}, k_1+k_2=k, |k_1|\le\frac{|k|}{2}} \frac{1}{|k_1|^l |k_2|^m} \lesssim \sum_{k_1\in\mathbb{Z}^d\backslash\{0\}, |k_1|\le\frac{|k|}{2}} \frac{1}{|k_1|^l |k|^m} \lesssim |k|^{-l-m+d}.$$

For the second term we have a similar argument and obtain the same estimate. If $|k_1| > |k|/2, |k_2| > |k|/2$ since $|k_2| \geq |k_1| - |k|$ by the triangle inequality, one has

$$|k_2| \geq \frac{1}{4}(|k_1| - |k|) + \frac{1-1/4}{2}|k| \geq \frac{1}{4}|k|,$$

which implies that

$$\sum_{k_1,k_2 \in \mathbb{Z}^d \setminus \{0\}, k_1+k_2=k, |k_1|>\frac{|k|}{2}, |k_2|>\frac{|k|}{2}} \frac{1}{|k_1|^l |k_2|^m} \lesssim |k|^{-l-m+d}.$$

Hence the result follows. □

Lemma 3.11 For any $0 < \eta < 1$, $i, j, l = 1, 2, 3$ we have

$$|e^{-|k_{12}|^2(t-s)} k_{12}^i \hat{P}^{jl}(k_{12}) - e^{-|k_2|^2(t-s)} k_2^i \hat{P}^{jl}(k_2)| \lesssim |k_1|^\eta |t-s|^{-(1-\eta)/2}.$$

Here $\hat{P}^{ij}(x) = \delta_{ij} - \frac{x^i \otimes x^j}{|x|^2}$.
Proof First we have the following bounds:

$$|e^{-|k_{12}|^2(t-s)} k_{12} \hat{P}(k_{12}) - e^{-|k_2|^2(t-s)} k_2 \hat{P}(k_2)| \lesssim |t-s|^{-1/2}.$$

Consider function $F(x) = e^{-|x|^2(t-s)} x \hat{P}(x)$ and it is easy to check that $|DF(x)| \leq C$, which implies that

$$|e^{-|k_{12}|^2(t-s)} k_{12} \hat{P}(k_{12}) - e^{-|k_2|^2(t-s)} k_2 \hat{P}(k_2)| \lesssim |k_1|.$$

Thus the result follows by the interpolation. □

3.3.1 Renormalization for $u_1^\varepsilon u_2^\varepsilon$

In this subsection we focus on $u_1^\varepsilon u_2^\varepsilon$ and prove that $u_1^{\varepsilon,i} \diamond u_2^{\varepsilon,j} \to v_3^{ij}$ in $C([0,T]; \mathcal{C}^{-1/2-\delta})$ for $i, j = 1, 2, 3$. Now we have the following identity: for $t \in [0,T]$, $i, j = 1, 2, 3$

$$\begin{aligned}
u_1^{\varepsilon,j} u_2^{\varepsilon,i}(t) =&(2\pi)^{-3} \sum_{i_1,i_2=1}^{3} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \sum_{k_{123}=k} \int_0^t e^{-|k_{12}|^2(t-s)} \imath k_{12}^{i_2} : \hat{X}_s^{\varepsilon,i_1}(k_1) \hat{X}_s^{\varepsilon,i_2}(k_2) \hat{X}_t^{\varepsilon,j}(k_3) : ds \hat{P}^{ii_1}(k_{12}) e_k \\
&+ 2(2\pi)^{-3} \sum_{i_1,i_2,i_3=1}^{3} \sum_{k_1,k_2 \in \mathbb{Z}^3 \setminus \{0\}} \int_0^t e^{-|k_{12}|^2(t-s)} \imath k_{12}^{i_2} \hat{X}_s^{\varepsilon,i_1}(k_1) \frac{e^{-|k_2|^2(t-s)} f(\varepsilon k_2)^2}{2|k_2|^2} ds \\
&\hat{P}^{ii_1}(k_{12}) \hat{P}^{i_2 i_3}(k_2) \hat{P}^{j i_3}(k_2) e_{k_1} \\
=& I_t^1 + I_t^2.
\end{aligned}$$

Term in the first chaos: First we consider I_t^2 and we have

$$I_t^2 = I_t^2 - \tilde{I}_t^2 + \tilde{I}_t^2 - \sum_{i_1=1}^{3} X_t^{\varepsilon,i_1} C_t^{\varepsilon,i_1},$$

where

$$\tilde{I}_t^2 =(2\pi)^{-3}\sum_{i_1,i_2,i_3=1}^{3}\sum_{k_1,k_2\in\mathbb{Z}^3\backslash\{0\}}\hat{X}_t^{\varepsilon,i_1}(k_1)e_{k_1}\int_0^t e^{-|k_{12}|^2(t-s)}\imath k_{12}^{i_2}\frac{e^{-|k_2|^2(t-s)}f(\varepsilon k_2)^2}{|k_2|^2}ds$$
$$\hat{P}^{ii_1}(k_{12})\hat{P}^{i_2i_3}(k_2)\hat{P}^{ji_3}(k_2),$$

and

$$C_t^{\varepsilon,i_1} = (2\pi)^{-3}\sum_{i_2,i_3=1}^{3}\sum_{k_2\in\mathbb{Z}^3\backslash\{0\}}\int_0^t e^{-2|k_2|^2(t-s)}\imath k_2^{i_2}\frac{f(\varepsilon k_2)^2}{|k_2|^2}\hat{P}^{ii_1}(k_2)\hat{P}^{i_2i_3}(k_2)\hat{P}^{ji_3}(k_2)ds=0.$$

By a straightforward calculation we obtain

$$\begin{aligned}E[|\Delta_q(I_t^2-\tilde{I}_t^2)|^2] \lesssim& E[|\sum_{i_1,i_2,i_3=1}^{3}\int_0^t\sum_{k_1}\theta(2^{-q}k_1)e_{k_1}a_{k_1}^{i_1i_2i_3}(t-s)(\hat{X}_s^{\varepsilon,i_1}(k_1)-\hat{X}_t^{\varepsilon,i_1}(k_1))ds|^2]\\ \lesssim& \sum_{i_1,i_2,i_3=1}^{3}\sum_{i_1',i_2',i_3'=1}^{3}\int_0^t\int_0^t dsd\bar{s}\sum_{k_1,k_1'}\theta(2^{-q}k_1)\theta(2^{-q}k_1')|a_{k_1}^{i_1i_2i_3}(t-s)a_{k_1'}^{i_1'i_2'i_3'}(t-\bar{s})|\\ &E|(\hat{X}_s^{\varepsilon,i_1}(k_1)-\hat{X}_t^{\varepsilon,i_1}(k_1))\overline{(\hat{X}_{\bar{s}}^{\varepsilon,i_1'}(k_1')-\hat{X}_t^{\varepsilon,i_1'}(k_1'))}|\\ \lesssim& \sum_{k_1}\theta(2^{-q}k_1)^2\frac{f(\varepsilon k_1)^2}{|k_1|^{2(1-\eta)}}(\int_0^t|t-s|^{\eta/2}a_{k_1}^{i_1i_2i_3}(t-s)ds)^2.\end{aligned}$$

Here

$$a_{k_1}^{i_1i_2i_3}(t-s)=\sum_{k_2}e^{-|k_{12}|^2(t-s)}k_{12}^{i_2}\frac{e^{-|k_2|^2(t-s)}f(\varepsilon k_2)^2}{|k_2|^2}\hat{P}^{ii_1}(k_{12})\hat{P}^{i_2i_3}(k_{12})\hat{P}^{ji_3}(k_{12}),$$

and we used that for $\eta>0$ small enough

$$\begin{aligned}&E|(\hat{X}_s^{\varepsilon,i_1}(k_1)-\hat{X}_t^{\varepsilon,i_1}(k_1))\overline{(\hat{X}_{\bar{s}}^{\varepsilon,i_1'}(k_1')-\hat{X}_t^{\varepsilon,i_1'}(k_1'))}|\\ \leq&\delta_{k_1=k_1'}(E|(\hat{X}_s^{\varepsilon,i_1}(k_1)-\hat{X}_t^{\varepsilon,i_1}(k_1))|^2)^{1/2}(E|\overline{(\hat{X}_{\bar{s}}^{\varepsilon,i_1'}(k_1')-\hat{X}_t^{\varepsilon,i_1'}(k_1'))}|^2)^{1/2}\\ \lesssim&\delta_{k_1=k_1'}(\frac{f(\varepsilon k_1)^2}{|k_1|^2}(1-e^{-|k_1|^2(t-s)}))^{1/2}(\frac{f(\varepsilon k_1')^2}{|k_1'|^2}(1-e^{-|k_1'|^2(t-\bar{s})}))^{1/2}\\ \lesssim&\frac{f(\varepsilon k_1)^2}{|k_1|^2}|k_1|^{2\eta}|t-s|^{\eta/2}|t-\bar{s}|^{\eta/2}.\end{aligned}$$

Since for $\eta>\epsilon>0$, ϵ small enough and $|a_{k_1}^{i_1i_2i_3}(t-s)|\lesssim|t-s|^{-1-\epsilon/2}ds\sum_{k_2}\frac{1}{|k_2|^{3+\epsilon}}$, it follows that

$$\int_0^t|t-s|^{\eta/2}a_{k_1}^{i_1i_2i_3}(t-s)ds\lesssim\int_0^t|t-s|^{\eta/2-1-\epsilon/2}ds\sum_{k_2}\frac{1}{|k_2|^{3+\epsilon}}\lesssim t^{(\eta-\epsilon)/2},$$

which implies that

$$E[|\Delta_q(I_t^2-\tilde{I}_t^2)|^2]\lesssim 2^{q(1+2\eta)}t^{\eta-\epsilon}.$$

Moreover, by Lemma 3.11 we deduce that for $\epsilon > 0$ small enough

$$
\begin{aligned}
&E[|\Delta_q(\tilde{I}_t^2 - \sum_{i_1=1}^{3} X_t^{\varepsilon,i_1} C_t^{\varepsilon,i_1})|^2] \\
\lesssim &\sum_{k_1} \sum_{i_1,i_2,i_3=1}^{3} \frac{f(\varepsilon k_1)^2}{2|k_1|^2} \theta(2^{-q}k_1)^2 (\sum_{k_2} \int_0^t \frac{e^{-|k_2|^2(t-s)} f(\varepsilon k_2)^2}{|k_2|^2} \\
&(e^{-|k_{12}|^2(t-s)} k_{12}^{i_2} \hat{P}^{ii_1}(k_{12}) \hat{P}^{i_2 i_3}(k_2) \hat{P}^{j i_3}(k_2) - e^{-|k_2|^2(t-s)} k_2^{i_2} \hat{P}^{ii_1}(k_2) \hat{P}^{i_2 i_3}(k_2) \hat{P}^{j i_3}(k_2)) ds)^2 \\
\lesssim &\sum_{k_1} \frac{f(\varepsilon k_1)^2}{|k_1|^{2-2\eta}} \theta(2^{-q}k_1)^2 (\sum_{k_2} \int_0^t \frac{e^{-|k_2|^2(t-s)} f(\varepsilon k_2)^2}{|k_2|^2} (t-s)^{-(1-\eta)/2} ds)^2 \\
\lesssim &t^{\eta-\epsilon} 2^{q(1+2\eta)},
\end{aligned}
\tag{3.14}
$$

holds uniformly over $\varepsilon \in (0,1)$, which is the desired bound for I_t^2.

Term in the third chaos: Now we focus on the bounds for I_t^1. Let $b_{k_{12}}^{i_1,i_2}(t-s) = e^{-|k_{12}|^2(t-s)} k_{12}^{i_2} \hat{P}^{ii_1}(k_{12})$. We obtain the following inequalities:

$$
\begin{aligned}
&E|\Delta_q I_t^1|^2 \\
\lesssim &2 \sum_{i_1,i_2=1}^{3} \sum_{i_1',i_2'=1}^{3} \sum_k \theta(2^{-q}k) \sum_{k_{123}=k} \Pi_{i=1}^3 \frac{f(\varepsilon k_i)^2}{|k_i|^2} \int_0^t \int_0^t e^{-(|k_1|^2+|k_2|^2)|s-\bar{s}|} |b_{k_{12}}^{i_1,i_2}(t-s) b_{k_{12}}^{i_1',i_2'}(t-\bar{s})| ds d\bar{s} \\
&+ 2 \sum_k \theta(2^{-q}k) \sum_{k_{123}=k} \Pi_{i=1}^3 \frac{f(\varepsilon k_i)^2}{|k_i|^2} \int_0^t \int_0^t e^{-|k_2|^2|s-\bar{s}| - |k_1|^2(t-s) - |k_3|^2(t-\bar{s})} |b_{k_{12}}^{i_1,i_2}(t-s) b_{k_{32}}^{i_1',i_2'}(t-\bar{s})| ds d\bar{s} \\
= &J_t^1 + J_t^2.
\end{aligned}
$$

Since $|b_{k_{12}}^{i_1,i_2}(t-s)| \lesssim \frac{1}{|k_{12}|^{1-\eta}(t-s)^{1-\eta/2}}$ it follows by Lemma 3.10 that for $\eta > 0$ small enough

$$
\begin{aligned}
J_t^1 \lesssim &\sum_k \theta(2^{-q}k) \sum_{k_{123}=k} \Pi_{i=1}^3 \frac{1}{|k_i|^2} \frac{t^\eta}{|k_{12}|^{2-2\eta}} \\
\lesssim &\sum_k \theta(2^{-q}k) \sum_{k_{123}=k} \frac{t^\eta}{|k_3|^2 |k_{12}|^{3-2\eta}} \\
\lesssim &t^\eta 2^{q(1+2\eta)},
\end{aligned}
$$

and

$$
\begin{aligned}
J_t^2 \lesssim &\sum_k \theta(2^{-q}k) \sum_{k_{123}=k} \frac{t^\eta}{|k_1|^2 |k_2|^2 |k_3|^2 |k_{12}|^{1-\eta} |k_{32}|^{1-\eta}} \\
\lesssim &\sum_k \theta(2^{-q}k) (\sum_{k_{123}=k} \frac{t^\eta}{|k_1|^2 |k_2|^2 |k_3|^2 |k_{12}|^{2-2\eta}})^{1/2} (\sum_{k_{123}=k} \frac{t^\eta}{|k_1|^2 |k_2|^2 |k_3|^2 |k_{32}|^{2-2\eta}})^{1/2} \\
\lesssim &t^\eta 2^{q(1+2\eta)},
\end{aligned}
$$

which gives the desired estimate for I_t^1. By a similar calculation we also obtain that for $\eta > \epsilon > 0$ small enough,

$$
E[|\Delta_q(u_2^{\varepsilon_1,i} u_1^{\varepsilon_1,j}(t_1) - u_2^{\varepsilon_1,i} u_1^{\varepsilon_1,j}(t_2) - u_2^{\varepsilon_2,i} u_1^{\varepsilon_1,j}(t_1) + u_2^{\varepsilon_2,i} u_1^{\varepsilon_1,j}(t_2))|^2] \lesssim C(\varepsilon_1,\varepsilon_2)|t_1-t_2|^{\eta-\epsilon} 2^{q(1+2\eta)},
\tag{3.15}
$$

where $C(\varepsilon_1,\varepsilon_2)\to 0$ as $\varepsilon_1,\varepsilon_2\to 0$, which by Gaussian hypercontractivity and Lemma 3.1 implies that

$$\begin{aligned}&E[\|(u_2^{\varepsilon_1,i}u_1^{\varepsilon_1,j}(t_1)-u_2^{\varepsilon_1,i}u_1^{\varepsilon_1,j}(t_2)-u_2^{\varepsilon_2,i}u_1^{\varepsilon_1,j}(t_1)+u_2^{\varepsilon_2,i}u_1^{\varepsilon_1,j}(t_2))\|^p_{\mathcal{C}^{-1/2-\eta-\epsilon-3/p}}]\\ \lesssim&E[\|(u_2^{\varepsilon_1,i}u_1^{\varepsilon_1,j}(t_1)-u_2^{\varepsilon_1,i}u_1^{\varepsilon_1,j}(t_2)-u_2^{\varepsilon_2,i}u_1^{\varepsilon_1,j}(t_1)+u_2^{\varepsilon_2,i}u_1^{\varepsilon_1,j}(t_2))\|^p_{B^{-1/2-\eta-\epsilon}_{p,p}}]\\ \lesssim&C(\varepsilon_1,\varepsilon_2)^{p/2}|t_1-t_2|^{p(\eta-\epsilon)/2}.\end{aligned}\tag{3.16}$$

Thus for every $i,j=1,2,3$, we choose p large enough and deduce that there exists $v_3^{ij}\in C([0,T],\mathcal{C}^{-1/2-\delta/2})$ such that

$$u_2^{\varepsilon,i}\diamond u_1^{\varepsilon,j}\to v_3^{ij}\in C([0,T],\mathcal{C}^{-1/2-\delta/2}).$$

To prove (3.15) we only calculate for the term as (3.14) with ε and $0\le t_1<t_2\le T$ and other terms can be obtained similarly. It is straightforward to calculate that

$$\begin{aligned}&E[|\Delta_q(\tilde I^2_{t_1}-\sum_{i_1=1}^3X^{\varepsilon,i_1}_{t_1}C^{\varepsilon,i_1}_{t_1}-\tilde I^2_{t_2}+\sum_{i_1=1}^3X^{\varepsilon,i_1}_{t_2}C^{\varepsilon,i_1}_{t_2})|^2]\\
\lesssim&E|\sum_{i_1,i_2,i_3=1}^3\sum_{k_1}\hat X^{\varepsilon,i_1}_{t_1}(k_1)\theta(2^{-q}k_1)e_{k_1}(\sum_{k_2}\int_0^{t_1}\frac{e^{-|k_2|^2(t_1-s)}f(\varepsilon k_2)^2}{|k_2|^2}(e^{-|k_{12}|^2(t_1-s)}k_{12}^{i_2}\hat P^{ii_1}(k_{12})\hat P^{i_2i_3}(k_2)\hat P^{ji_3}(k_2)\\
&-e^{-|k_2|^2(t_1-s)}k_2^{i_2}\hat P^{ii_1}(k_2)\hat P^{i_2i_3}(k_2)\hat P^{ji_3}(k_2))ds-\sum_{k_2}\int_0^{t_2}\frac{e^{-|k_2|^2(t_2-s)}f(\varepsilon k_2)^2}{|k_2|^2}(e^{-|k_{12}|^2(t_2-s)}k_{12}^{i_2}\\
&\hat P^{ii_1}(k_{12})\hat P^{i_2i_3}(k_2)\hat P^{ji_3}(k_2)-e^{-|k_2|^2(t_2-s)}k_2^{i_2}\hat P^{ii_1}(k_2)\hat P^{i_2i_3}(k_2)\hat P^{ji_3}(k_2))ds)|^2\\
&+E|\sum_{i_1,i_2,i_3=1}^3\sum_{k_1}(\hat X^{\varepsilon,i_1}_{t_1}(k_1)-\hat X^{\varepsilon,i_1}_{t_2}(k_1))\theta(2^{-q}k_1)e_{k_1}\int_0^{t_2}\frac{e^{-|k_2|^2(t_2-s)}f(\varepsilon k_2)^2}{|k_2|^2}\\
&(e^{-|k_{12}|^2(t_2-s)}k_{12}^{i_2}\hat P^{ii_1}(k_{12})\hat P^{i_2i_3}(k_2)\hat P^{ji_3}(k_2)-e^{-|k_2|^2(t_2-s)}k_2^{i_2}\hat P^{ii_1}(k_2)\hat P^{i_2i_3}(k_2)\hat P^{ji_3}(k_2))ds|^2\\
\lesssim&\sum_{k_1}\sum_{i_1,i_2=1}^3\frac{1}{|k_1|^2}\theta(2^{-q}k_1)^2(\sum_{k_2}\int_0^{t_1}\frac{e^{-|k_2|^2(t_1-s)}(1-e^{-|k_2|^2(t_2-t_1)})f(\varepsilon k_2)^2}{|k_2|^2}\\
&(e^{-|k_{12}|^2(t_1-s)}k_{12}^{i_2}\hat P^{ii_1}(k_{12})-e^{-|k_2|^2(t_1-s)}k_2^{i_2}\hat P^{ii_1}(k_2))ds)^2\\
&+\sum_{k_1}\sum_{i_1,i_2=1}^3\frac{1}{|k_1|^2}\theta(2^{-q}k_1)^2(\sum_{k_2}\int_0^{t_1}\frac{e^{-|k_2|^2(t_2-s)}f(\varepsilon k_2)^2}{|k_2|^2}(e^{-|k_{12}|^2(t_1-s)}k_{12}^{i_2}\hat P^{ii_1}(k_{12})\\
&-e^{-|k_2|^2(t_1-s)}k_2^{i_2}\hat P^{ii_1}(k_2)-e^{-|k_{12}|^2(t_2-s)}k_{12}^{i_2}\hat P^{ii_1}(k_{12})+e^{-|k_2|^2(t_2-s)}k_2^{i_2}\hat P^{ii_1}(k_2))ds)^2\\
&+\sum_{k_1}\sum_{i_1,i_2=1}^3\frac{1}{|k_1|^2}\theta(2^{-q}k_1)^2(\sum_{k_2}\int_{t_1}^{t_2}\frac{e^{-|k_2|^2(t_2-s)}f(\varepsilon k_2)^2}{|k_2|^2}(e^{-|k_{12}|^2(t_2-s)}k_{12}^{i_2}\hat P^{ii_1}(k_{12})\\
&-e^{-|k_2|^2(t_2-s)}k_2^{i_2}\hat P^{ii_1}(k_2))ds)^2+E|\sum_{k_1}\sum_{i_1,i_2=1}^3(\hat X^{\varepsilon,i_1}_{t_1}(k_1)-\hat X^{\varepsilon,i_1}_{t_2}(k_1))\\
&\sum_{k_2}\theta(2^{-q}k_1)e_{k_1}\int_0^{t_2}\frac{e^{-|k_2|^2(t_2-s)}f(\varepsilon k_2)^2}{|k_2|^2}(e^{-|k_{12}|^2(t_2-s)}k_{12}^{i_2}\hat P^{ii_1}(k_{12})-e^{-|k_2|^2(t_2-s)}k_2^{i_2}\hat P^{ii_1}(k_2))ds|^2\\
:=&L^1_t+L^2_t+L^3_t+L^4_t\end{aligned}$$

It is easy to deduce the desired estimates for L_t^1, L_t^3, L_t^4 as (3.14) and we only need to consider L_t^2: for some $0<\beta_0<1/2, \eta>0$ small enough by Lemma 3.11 and interpolation we have

$$\begin{aligned}L_t^2 \lesssim& \sum_{k_1}\frac{1}{|k_1|^2}\theta(2^{-q}k_1)^2(\sum_{k_2}\int_0^{t_1}\frac{e^{-|k_2|^2(t_1-s)}}{|k_2|^2}[|k_1|^\eta\wedge|t_2-t_1|^{\frac{\eta}{2}}(|k_{12}|^{2\eta}+|k_2|^{2\eta})](t_1-s)^{-\frac{1-\eta}{2}}ds)^2\\ \lesssim&\sum_{k_1}\frac{1}{|k_1|^2}\theta(2^{-q}k_1)^2(\sum_{k_2}\int_0^{t_1}\frac{e^{-|k_2|^2(t_1-s)}}{|k_2|^2}|k_1|^{\eta(1-\beta_0)}|t_2-t_1|^{\frac{\eta\beta_0}{2}}(|k_{12}|^{2\eta\beta_0}+|k_2|^{2\eta\beta_0})(t_1-s)^{-\frac{1-\eta}{2}}ds)^2\\ \lesssim&|t_1-t_2|^{\eta\beta_0/2}2^{q(1+2\eta(1+\beta_0))},\end{aligned}$$

which is the required estimate for L_t^2.

3.3.2 Renormalisation for $\pi_0(u_3^{\varepsilon,i_0}, u_1^{\varepsilon,j_0})$

Now we treat $\pi_0(u_{31}^{\varepsilon,i_0}, u_1^{\varepsilon,j_0})$ and the estimates for $\pi_0(u_3^{\varepsilon,i_0}-u_{31}^{\varepsilon,i_0}, u_1^{\varepsilon,j_0})$ can be obtained similarly, where $Lu_{31}^i=-\frac{1}{2}\sum_{i_1=1}^3P^{ii_1}\sum_{j=1}^3D_j(u_2^{i_1}\diamond u_1^j)$. We have the following identity:

$$\begin{aligned}&\pi_0(u_{31}^{\varepsilon,i_0i_1}, u_1^{\varepsilon,j_0})\\ =&(2\pi)^{-9/2}[\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq1}\sum_{k_{1234}=k}\sum_{i_1,i_2,i_3,j_1=1}^3\theta(2^{-i}k_{123})\theta(2^{-j}k_4)\int_0^t dse^{-|k_{123}|^2(t-s)}\int_0^s:\hat{X}_\sigma^{\varepsilon,i_2}(k_1)\hat{X}_\sigma^{\varepsilon,i_3}(k_2)\\ &\hat{X}_s^{\varepsilon,j_1}(k_3)\hat{X}_t^{\varepsilon,j_0}(k_4):e^{-|k_{12}|^2(s-\sigma)}d\sigma\imath k_{12}^{i_3}\imath k_{123}^{j_1}\hat{P}^{i_1i_2}(k_{12})\hat{P}^{i_0i_1}(k_{123})e_k+2\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq1}\sum_{k_{23}=k,k_1}\\ &\sum_{i_1,i_2,i_3,j_1=1}^3\theta(2^{-i}k_{123})\theta(2^{-j}k_1)\int_0^t dse^{-|k_{123}|^2(t-s)}\int_0^s:\hat{X}_\sigma^{\varepsilon,i_3}(k_2)\hat{X}_s^{\varepsilon,j_1}(k_3):\frac{e^{-|k_1|^2(t-\sigma)}f(\varepsilon k_1)^2}{2|k_1|^2}\\ &\sum_{i_4=1}^3\hat{P}^{i_2i_4}(k_1)\hat{P}^{j_0i_4}(k_1)e^{-|k_{12}|^2(s-\sigma)}d\sigma\imath k_{12}^{i_3}\imath k_{123}^{j_1}\hat{P}^{i_1i_2}(k_{12})\hat{P}^{i_0i_1}(k_{123})e_k\\ &+2\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq1}\sum_{k_{12}=k,k_3}\sum_{i_1,i_2,i_3,j_1=1}^3\theta(2^{-i}k_{123})\theta(2^{-j}k_3)\int_0^t dse^{-|k_{123}|^2(t-s)}\int_0^s:\hat{X}_\sigma^{\varepsilon,i_2}(k_1)\hat{X}_\sigma^{\varepsilon,i_3}(k_2):\\ &\frac{e^{-|k_3|^2(t-s)}f(\varepsilon k_3)^2}{2|k_3|^2}\sum_{i_4=1}^3\hat{P}^{j_1i_4}(k_3)\hat{P}^{j_0i_4}(k_3)e^{-|k_{12}|^2(s-\sigma)}d\sigma\imath k_{12}^{i_3}\imath k_{123}^{j_1}\hat{P}^{i_1i_2}(k_{12})\hat{P}^{i_0i_1}(k_{123})e_k\\ &+2\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq1}\sum_{k_{14}=k,k_2}\sum_{i_1,i_2,i_3,j_1=1}^3\theta(2^{-i}k_1)\theta(2^{-j}k_4)\int_0^t dse^{-|k_1|^2(t-s)}\int_0^s:\hat{X}_\sigma^{\varepsilon,i_2}(k_1)\hat{X}_t^{\varepsilon,j_0}(k_4):\\ &\frac{e^{-|k_2|^2(s-\sigma)}f(\varepsilon k_2)^2}{2|k_2|^2}\sum_{i_4=1}^3\hat{P}^{i_3i_4}(k_2)\hat{P}^{j_1i_4}(k_2)e^{-|k_{12}|^2(s-\sigma)}d\sigma\imath k_{12}^{i_3}\imath k_1^{j_1}\hat{P}^{i_1i_2}(k_{12})\hat{P}^{i_0i_1}(k_1)e_k\\ &+2\sum_{|i-j|\leq1}\sum_{k_1,k_2}\sum_{i_1,i_2,i_3,j_1=1}^3\theta(2^{-i}k_2)\theta(2^{-j}k_2)\int_0^t dse^{-|k_2|^2(t-s)}\int_0^s\frac{f(\varepsilon k_1)^2f(\varepsilon k_2)^2}{4|k_1|^2|k_2|^2}\end{aligned}$$

$$\sum_{i_4,i_5=1}^{3}\hat P^{i_3i_4}(k_1)\hat P^{j_1i_4}(k_1)\hat P^{i_2i_5}(k_2)\hat P^{j_0i_5}(k_2)e^{-|k_{12}|^2(s-\sigma)-|k_1|^2(s-\sigma)-|k_2|^2(t-\sigma)}d\sigma\imath k_{12}^{i_3}\imath k_2^{j_1}\hat P^{i_1i_2}(k_{12})\hat P^{i_0i_1}(k_2)]$$

$$:=I_t^1+I_t^2+I_t^3+I_t^4+I_t^5$$

First we consider I_t^5: by simple calculations we have

$$\begin{aligned}I_t^5=&(2\pi)^{-\frac{9}{2}}\sum_{|i-j|\leq1}\sum_{k_1,k_2}\sum_{i_1,i_2,i_3,j_1=1}^{3}\theta(2^{-i}k_2)\theta(2^{-j}k_2)\imath k_{12}^{i_3}\imath k_2^{j_1}\hat P^{i_1i_2}(k_{12})\hat P^{i_0i_1}(k_2)\\&\frac{f(\varepsilon k_1)^2f(\varepsilon k_2)^2}{2|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|k_{12}|^2)}\sum_{i_4,i_5=1}^{3}\hat P^{i_3i_4}(k_1)\hat P^{j_1i_4}(k_1)\hat P^{i_2i_5}(k_2)\hat P^{j_0i_5}(k_2)(\frac{1-e^{-2|k_2|^2t}}{2|k_2|^2}\\&-\int_0^t dse^{-2|k_2|^2(t-s)}e^{-(|k_{12}|^2+|k_1|^2+|k_2|^2)s}).\end{aligned}$$

Let

$$\begin{aligned}&C_{11}^{\varepsilon,i_0,j_0}(t)\\=&(2\pi)^{-\frac{9}{2}}\sum_{|i-j|\leq1}\sum_{k_1,k_2}\sum_{i_1,i_2,i_3,j_1=1}^{3}\theta(2^{-i}k_2)\theta(2^{-j}k_2)\imath k_{12}^{i_3}\imath k_2^{j_1}\hat P^{i_1i_2}(k_{12})\hat P^{i_0i_1}(k_2)\frac{f(\varepsilon k_1)^2f(\varepsilon k_2)^2}{2|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|k_{12}|^2)}\\&\frac{1-e^{-2|k_2|^2t}}{2|k_2|^2}\sum_{i_4,i_5=1}^{3}\hat P^{i_3i_4}(k_1)\hat P^{j_1i_4}(k_1)\hat P^{i_2i_5}(k_2)\hat P^{j_0i_5}(k_2)\to\infty,\end{aligned}$$

as $\varepsilon\to0$. Define

$$\varphi_{11}^{\varepsilon,i_0j_0}:=I_t^5-C_{11}^{\varepsilon,i_0j_0}.$$

Then for any $\rho>0$ we deduce that

$$\begin{aligned}|\varphi_1^{\varepsilon,i_0,j_0}|\lesssim&\sum_{|i-j|\leq1}\sum_{k_1,k_2}\frac{\theta(2^{-i}k_2)\theta(2^{-j}k_2)|k_{12}^{i_3}k_2^{j_1}|}{|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|k_{12}|^2)}\int_0^t e^{-(|k_2|^2t+|k_1|^2s+|k_{12}|^2s)}ds\\\lesssim&\sum_{|i-j|\leq1}\sum_{k_1,k_2}\frac{\theta(2^{-i}k_2)\theta(2^{-j}k_2)|k_{12}^{i_3}k_2^{j_1}|}{|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|k_{12}|^2)(|k_1|^2+|k_{12}|^2)}e^{-|k_2|^2t}(1-e^{-(|k_1|^2t+|k_{12}|^2t)})\\\lesssim&t^{-\rho}\sum_{i=1}^{\infty}2^{-i\eta}\sum_{k_1,k_2}\frac{1}{|k_1|^{3+r}|k_2|^{3+2\rho-r-\eta}}\lesssim t^{-\rho},\end{aligned}$$

holds uniformly over $\varepsilon\in(0,1)$. Here $r,\eta>0$ are small enough such that $2\rho>r+\eta$. By a similar calculation we obtain some φ_{11} such that φ_{11}^ε converges to some φ_{11} with respect to $\|\varphi\|=\sup_{t\in[0,T]}t^\rho|\varphi(t)|$ for any $\rho>0$. Similarly, we can also find similar $C_{12}^\varepsilon,\varphi_{12}^\varepsilon,\varphi_{12}$ for u_3-u_{31} and satisfy similar estimates as φ_{11}^ε. Now define $C_1^\varepsilon=C_{11}^\varepsilon+C_{12}^\varepsilon$, $\varphi_1^\varepsilon=\varphi_{11}^\varepsilon+\varphi_{12}^\varepsilon$ and $\varphi_1=\varphi_{11}+\varphi_{12}$.

Terms in the second chaos: We come to I_t^2 and have the following calculations:

$$E|\Delta_q I_t^2|^2$$

$$\lesssim \sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq 1,|i'-j'|\leq 1}\sum_{k_{23}=k,k_1,k_4}\sum_{i_1,i_2,i_3,j_1=1}^{3}\sum_{i_1',i_2',i_3',j_1'=1}^{3}\theta(2^{-i}k_{123})\theta(2^{-j}k_1)\theta(2^{-i'}k_{234})\theta(2^{-j'}k_4)\theta(2^{-q}k)^2$$

$$\Pi_{i=1}^4\frac{f(\varepsilon k_i)^2}{|k_i|^2}\int_0^t\int_0^t dsd\bar{s}e^{-|k_{123}|^2(t-s)-|k_{234}|^2(t-\bar{s})}\int_0^s\int_0^{\bar{s}}d\sigma d\bar{\sigma}e^{-|k_1|^2(t-\sigma)-|k_4|^2(t-\bar{\sigma})}e^{-(|k_{12}|^2(s-\sigma)+|k_{24}|^2(s-\bar{\sigma}))}$$

$$(e^{-|k_2|^2|\sigma-\bar{\sigma}|-|k_3|^2|s-\bar{s}|}+e^{-|k_2|^2|\bar{s}-\sigma|-|k_3|^2|s-\bar{\sigma}|})|k_{12}^{i_3}k_{123}^{j_1}\hat{P}^{i_1i_2}(k_{12})\hat{P}^{i_0i_1}(k_{123})k_{24}^{i_3'}k_{234}^{j_1'}\hat{P}^{i_1'i_2'}(k_{24})\hat{P}^{i_0i_1'}(k_{234})|$$

$$\lesssim \sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq 1,|i'-j'|\leq 1}\sum_{k_{23}=k,k_1,k_4}\theta(2^{-i}k_{123})\theta(2^{-j}k_1)\theta(2^{-i'}k_{234})\theta(2^{-j'}k_4)\theta(2^{-q}k)^2$$

$$\frac{t^\eta}{|k_2|^2|k_3|^2|k_1|^{4-\eta}|k_4|^{4-\eta}}$$

$$\lesssim \sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{q\lesssim i}2^{-(1-\eta-\epsilon)i}\sum_{q\lesssim i'}2^{-(1-\eta-\epsilon)i'}\sum_{k_{23}=k}\theta(2^{-q}k)^2\frac{t^\eta}{|k_2|^2|k_3|^2}\lesssim t^\eta 2^{2q(\eta+2\epsilon)},$$

where $\eta,\epsilon>0$ are small enough, we used Lemma 3.10 in the last inequality and $q\lesssim i$ follows from $|k|\leq|k_{123}|+|k_1|\lesssim 2^i$ and $q\lesssim i'$ is similar.

Now we deal with $I_t^3=I_t^3-\tilde{I}_t^3+\tilde{I}_t^3-\sum_{i_1=1}^3u_2^{i_1}C_3^{\varepsilon,i_1}(t)$ with

$$\tilde{I}_t^3=(2\pi)^{-\frac{9}{2}}\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq 1}\sum_{k_{12}=k,k_3}\sum_{i_1,i_2,i_3,j_1=1}^{3}\theta(2^{-i}k_{123})\theta(2^{-j}k_3)\int_0^t:\hat{X}_\sigma^{\varepsilon,i_2}(k_1)\hat{X}_\sigma^{\varepsilon,i_3}(k_2):e^{-|k_{12}|^2(t-\sigma)}\imath k_{12}^{i_3}$$

$$\hat{P}^{i_1i_2}(k_{12})e_kd\sigma\int_0^t dse^{-|k_{123}|^2(t-s)}\frac{e^{-|k_3|^2(t-s)}f(\varepsilon k_3)^2}{|k_3|^2}\sum_{i_4}\hat{P}^{j_1i_4}(k_3)\hat{P}^{j_0i_4}(k_3)\imath k_{123}^{j_1}\hat{P}^{i_0i_1}(k_{123}),$$

and

$$C_3^{\varepsilon,i_1}(t)=(2\pi)^{-\frac{9}{2}}\sum_{|i-j|\leq 1}\sum_{k_3}\sum_{j_1=1}^{3}\theta(2^{-i}k_3)\theta(2^{-j}k_3)\int_0^t ds\frac{e^{-2|k_3|^2(t-s)}f(\varepsilon k_3)^2}{|k_3|^2}$$

$$\sum_{i_4}\hat{P}^{j_1i_4}(k_3)\hat{P}^{j_0i_4}(k_3)\imath k_3^{j_1}\hat{P}^{i_0i_1}(k_3)$$

$$=0.$$

Let $c_{k_{123},k_3}^{j_1}(t-s)=\sum_{i_1=1}^3e^{-|k_{123}|^2(t-s)}\frac{e^{-|k_3|^2(t-s)}f(\varepsilon k_3)^2}{|k_3|^2}|k_{123}^{j_1}\hat{P}^{i_0i_1}(k_{123})|$. Then we have for $\epsilon>0$ small enough,

$$E|\Delta_q(I_t^3-\tilde{I}_t^3)|^2$$

$$\lesssim \sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq 1,|i'-j'|\leq 1}\sum_{k_{12}=k,k_3,k_4}\theta(2^{-q}k)^2\theta(2^{-i}k_{123})\theta(2^{-j}k_3)\theta(2^{-i'}k_{124})\theta(2^{-j'}k_4)$$

$$(\int_0^t ds\int_0^t d\bar{s}\int_0^s d\sigma\int_0^{\bar{s}}d\bar{\sigma}(e^{-|k_{12}|^2(s-\sigma)}-e^{-|k_{12}|^2(t-\sigma)})(e^{-|k_{12}|^2(\bar{s}-\bar{\sigma})}-e^{-|k_{12}|^2(t-\bar{\sigma})})|k_{12}|^2$$

$$\frac{1}{|k_1|^2|k_2|^2}\sum_{j_1,j_1'=1}^{3}c_{k_{123},k_3}^{j_1}(t-s)c_{k_{124},k_4}^{j_1'}(t-\bar{s})$$

$$
\begin{aligned}
&+\int_0^t ds\int_0^t d\bar{s}\int_s^t d\sigma\int_{\bar{s}}^t d\bar{\sigma}e^{-|k_{12}|^2(t-\sigma)-|k_{12}|^2(t-\bar{\sigma})}|k_{12}|^2\\
&\frac{1}{|k_1|^2|k_2|^2}\sum_{j_1,j_1'=1}^3 c^{j_1}_{k_{123},k_3}(t-s)c^{j_1'}_{k_{124},k_4}(t-\bar{s}))\\
\lesssim&\sum_{k\in\mathbb{Z}^3\backslash\{0\}}\sum_{|i-j|\leq1,|i'-j'|\leq1}\sum_{k_{12}=k,k_3,k_4}\theta(2^{-q}k)^2\theta(2^{-i}k_{123})\theta(2^{-j}k_3)\theta(2^{-i'}k_{124})\theta(2^{-j'}k_4)\\
&\int_0^t ds\int_0^t d\bar{s}\frac{1}{|k_{12}||k_1|^2|k_2|^2}(t-s)^{1/4}(t-\bar{s})^{1/4}\sum_{j_1,j_1'=1}^3 c^{j_1}_{k_{123},k_3}(t-s)c^{j_1'}_{k_{124},k_4}(t-\bar{s})\\
\lesssim&\sum_{k\in\mathbb{Z}^3\backslash\{0\}}\sum_{|i-j|\leq1,|i'-j'|\leq1}\sum_{k_{12}=k,k_3,k_4}\theta(2^{-q}k)^2\theta(2^{-i}k_{123})\theta(2^{-j}k_3)\theta(2^{-i'}k_{124})\theta(2^{-j'}k_4)\\
&\frac{t^{2\epsilon}}{|k_{12}||k_1|^2|k_2|^2|k_3|^2|k_4|^2(|k_{123}|^2+|k_3|^2)^{3/4-\epsilon}(|k_{124}|^2+|k_4|^2)^{3/4-\epsilon}}\\
\lesssim& t^{2\epsilon}\sum_{q\lesssim i}\sum_{q\lesssim i'}2^{-(i+i')(1/2-3\epsilon)}\sum_k\sum_{k_{12}=k}\theta(2^{-q}k)\frac{1}{|k_{12}||k_1|^2|k_2|^2}\\
\lesssim& t^{2\epsilon}2^{-2q(1/2-3\epsilon)}\sum_k\sum_{k_{12}=k}\theta(2^{-q}k)\frac{1}{|k_{12}||k_1|^2|k_2|^2}\lesssim t^{2\epsilon}2^{2q(3\epsilon)}.
\end{aligned}
$$

Moreover, by Lemma 3.11 we obtain for $\eta>\epsilon>0$ small enough

$$
\begin{aligned}
&E[|\Delta_q(\tilde{I}^3_t-\sum_{i_1=1}^3 u_2^{\varepsilon,i_1}(t)C_3^{\varepsilon,i_1}(t))|^2]\\
\lesssim&\sum_k\sum_{k_{12}=k}\sum_{i_1,j_1=1}^3\frac{1}{|k_1|^2|k_2|^2|k_{12}|^2}\theta(2^{-q}k)^2(\sum_{|i-j|\leq1}\sum_{k_3}\theta(2^{-j}k_3)\int_0^t\frac{e^{-|k_3|^2(t-s)}f(\varepsilon k_3)^2}{|k_3|^2}\\
&(\theta(2^{-i}k_{123})e^{-|k_{123}|^2(t-s)}k^{j_1}_{123}\hat{P}^{i_0i_1}(k_{123})-\theta(2^{-i}k_3)e^{-|k_2|^2(t-s)}k_3^{j_1}\hat{P}^{i_0i_1}(k_3))ds)^2\\
\lesssim&\sum_k\sum_{k_{12}=k}\frac{1}{|k_1|^2|k_2|^2|k_{12}|^{2-2\eta}}\theta(2^{-q}k)^2(\sum_{j=0}^\infty\sum_{k_3}\theta(2^{-j}k_3)\int_0^t\frac{e^{-|k_3|^2(t-s)}}{|k_3|^2}(t-s)^{-(1-\eta)/2}ds)^2\\
\lesssim& t^{\eta-\epsilon}2^{q(2\eta)}.
\end{aligned}
$$

Now we consider $I^4_t=I^4_t-\tilde{I}^4_t+\tilde{I}^4_t-\bar{I}^4_t$ with

$$
\begin{aligned}
\tilde{I}^4_t=&(2\pi)^{-9/2}\sum_{k\in\mathbb{Z}^3\backslash\{0\}}\sum_{|i-j|\leq1}\sum_{k_{14}=k,k_2}\sum_{i_1,i_2,i_3,j_1=1}^3\theta(2^{-i}k_1)\theta(2^{-j}k_4)\int_0^t:\hat{X}^{\varepsilon,i_2}_s(k_1)\hat{X}^{\varepsilon,j_0}_t(k_4):e^{-|k_1|^2(t-s)}\\
&\imath k_1^{j_1}\hat{P}^{i_0i_1}(k_1)e_kd\sigma\int_0^s d\sigma e^{-|k_{12}|^2(s-\sigma)}\frac{e^{-|k_2|^2(s-\sigma)}f(\varepsilon k_2)^2}{|k_2|^2}\imath k^{i_3}_{12}\hat{P}^{i_1i_2}(k_{12})\sum_{i_4=1}^3\hat{P}^{i_3i_4}(k_2)\hat{P}^{j_1i_4}(k_2),
\end{aligned}
$$

and

$$\bar{I}^4_t =(2\pi)^{-9/2}\sum_{k\in\mathbb{Z}^3\backslash\{0\}}\sum_{|i-j|\leq 1}\sum_{k_{14}=k,k_2}\sum_{i_1,i_2,i_3,j_1=1}^{3}\theta(2^{-i}k_1)\theta(2^{-j}k_4)\int_0^t : \hat{X}^{\varepsilon,i_2}_s(k_1)\hat{X}^{\varepsilon,j_0}_t(k_4) : e^{-|k_1|^2(t-s)}$$
$$\imath k_1^{j_1}\hat{P}^{i_0i_1}(k_1)e_k d\sigma\int_0^s d\sigma e^{-|k_2|^2(s-\sigma)}\frac{e^{-|k_2|^2(s-\sigma)}f(\varepsilon k_2)^2}{|k_2|^2}\imath k_2^{i_3}\hat{P}^{i_1i_2}(k_2)\sum_{i_4=1}^3\hat{P}^{i_3i_4}(k_2)\hat{P}^{j_1i_4}(k_2)=0.$$

Let $d_{k_{12},k_2}(s-\sigma)=\sum_{i_2,i_3=1}^3 e^{-|k_{12}|^2(s-\sigma)}\frac{e^{-|k_2|^2(s-\sigma)}f(\varepsilon k_2)^2}{|k_2|^2}|k_{12}^{i_3}\hat{P}^{i_1i_2}(k_{12})|$. Since by Hölder's inequality we obtain

$$\begin{aligned}&E(: \hat{X}^{\varepsilon,i_2}_s(k_1)\hat{X}^{\varepsilon,j_0}_t(k_4) : - : \hat{X}^{\varepsilon,i_2}_\sigma(k_1)\hat{X}^{\varepsilon,j_0}_t(k_4) :)\overline{(: \hat{X}^{\varepsilon,i_2}_{\bar{s}}(k_1')\hat{X}^{\varepsilon,j_0}_t(k_4') : - : \hat{X}^{\varepsilon,i_2}_{\bar{\sigma}}(k_1')\hat{X}^{\varepsilon,j_0}_t(k_4') :)}\\
\lesssim&(\delta_{k_1=k_1'}\delta_{k_4=k_4'}+\delta_{k_1=k_4'}\delta_{k_4=k_1'})(\frac{1-e^{-|k_1|^2|s-\sigma|}}{|k_1|^2|k_4|^2})^{1/2}(\frac{1-e^{-|k_1'|^2|\bar{s}-\bar{\sigma}|}}{|k_1'|^2|k_4'|^2})^{1/2}\\
\lesssim&(\delta_{k_1=k_1'}\delta_{k_4=k_4'}+\delta_{k_1=k_4'}\delta_{k_4=k_1'})\frac{|k_1|^\eta|k_1'|^\eta}{|k_1||k_1'||k_4||k_4'|}|s-\sigma|^{\eta/2}|\bar{s}-\bar{\sigma}|^{\eta/2},\end{aligned}$$

it follows that $\eta,\epsilon>0$ small enough

$$\begin{aligned}E|\Delta_q(I^4_t-\tilde{I}^4_t)|^2\lesssim&\sum_{k\in\mathbb{Z}^3\backslash\{0\}}\sum_{|i-j|\leq1,|i'-j'|\leq1}\sum_{k_{14}=k,k_3,k_2}\theta(2^{-q}k)^2\theta(2^{-i}k_1)\theta(2^{-j}k_4)\theta(2^{-i'}k_1)\theta(2^{-j'}k_4)\\
&\int_0^t ds\int_0^t d\bar{s}\int_0^s d\sigma\int_0^{\bar{s}} d\bar{\sigma}e^{-|k_1|^2(t-s)}e^{-|k_1|^2(t-\bar{s})}|k_1|^2\frac{1}{|k_1|^{2-2\eta}|k_4|^2}\\
&|s-\sigma|^{\eta/2}|\bar{s}-\bar{\sigma}|^{\eta/2}d_{k_{12},k_2}(s-\sigma)d_{k_{13},k_3}(\bar{s}-\bar{\sigma})\\
&+\sum_{k\in\mathbb{Z}^3\backslash\{0\}}\sum_{|i-j|\leq1,|i'-j'|\leq1}\sum_{k_{14}=k,k_3,k_2}\theta(2^{-q}k)^2\theta(2^{-i}k_1)\theta(2^{-j}k_4)\theta(2^{-i'}k_4)\theta(2^{-j'}k_1)\\
&\int_0^t ds\int_0^t d\bar{s}\int_0^s d\sigma\int_0^{\bar{s}} d\bar{\sigma}e^{-|k_1|^2(t-s)}e^{-|k_4|^2(t-\bar{s})}|k_1||k_4|\frac{1}{|k_1|^{2-\eta}|k_4|^{2-\eta}}\\
&|s-\sigma|^{\eta/2}|\bar{s}-\bar{\sigma}|^{\eta/2}d_{k_{12},k_2}(s-\sigma)d_{k_{34},k_3}(\bar{s}-\bar{\sigma})\\
\lesssim&\sum_{k\in\mathbb{Z}^3\backslash\{0\}}\sum_{|i-j|\leq1,|i'-j'|\leq1}\sum_{k_{14}=k}\theta(2^{-q}k)^2\theta(2^{-i}k_1)\theta(2^{-j}k_4)\theta(2^{-i'}k_1)\theta(2^{-j'}k_4)\\
&(\frac{t^\epsilon}{|k_1|^{4-2\eta-2\epsilon}|k_4|^2}+\frac{t^\epsilon}{|k_1|^{3-\eta-\epsilon}|k_4|^{3-\eta-\epsilon}})\\
\lesssim&t^\epsilon\sum_k\sum_{k_{14}=k}\theta(2^{-q}k)\sum_{q\lesssim i}2^{-i}\frac{1}{|k_1|^{3-2\eta-2\epsilon}|k_4|^2}\\
&+t^\epsilon\sum_k\sum_{k_{14}=k}\theta(2^{-q}k)\sum_{q\lesssim j}2^{-j\epsilon}\frac{1}{|k_1|^{3-\eta-2\epsilon}|k_4|^{3-\eta-\epsilon}}\\
\lesssim&t^\epsilon2^{q(2\epsilon+2\eta)},\end{aligned}$$

where in the last inequality we used Lemma 3.10.

Moreover, it follows by Lemma 3.11 that for $\eta, \epsilon > 0$ small enough

$$E[|\Delta_q(\tilde{I}^4_t - \bar{I}^4_t)|^2] \lesssim \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \sum_{|i-j|\leq 1, |i'-j'|\leq 1} \sum_{k_{14}=k, k_3, k_2} \theta(2^{-q}k)^2\theta(2^{-i}k_1)\theta(2^{-j}k_4)\theta(2^{-i'}k_1)\theta(2^{-j'}k_4)$$
$$\int_0^t\int_0^t |k_1|^{2+2\eta} e^{-|k_1|^2(t-s+t-\bar{s}+|s-\bar{s}|)}\frac{1}{|k_1|^2|k_4|^2}\int_0^s \frac{e^{-|k_2|^2(s-\sigma)}}{|k_2|^2}(s-\sigma)^{-(1-\eta)/2}$$
$$\int_0^{\bar{s}} \frac{e^{-|k_3|^2(\bar{s}-\bar{\sigma})}}{|k_3|^2}(\bar{s}-\bar{\sigma})^{-(1-\eta)/2} ds d\bar{s} d\sigma d\bar{\sigma}$$
$$+ \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \sum_{|i-j|\leq 1, |i'-j'|\leq 1} \sum_{k_{14}=k, k_3, k_2} \theta(2^{-q}k)^2\theta(2^{-i}k_1)\theta(2^{-j}k_4)\theta(2^{-i'}k_4)\theta(2^{-j'}k_1)$$
$$\int_0^t\int_0^t |k_1|^{1+2\eta}|k_4| e^{-2|k_1|^2(t-s)-2|k_4|^2(t-\bar{s})}\frac{1}{|k_1|^2|k_4|^2}\int_0^s \frac{e^{-|k_2|^2(s-\sigma)}}{|k_2|^2}(s-\sigma)^{-(1-\eta)/2}$$
$$\int_0^{\bar{s}} \frac{e^{-|k_3|^2(\bar{s}-\bar{\sigma})}}{|k_3|^2}(\bar{s}-\bar{\sigma})^{-(1-\eta)/2} ds d\bar{s} d\sigma d\bar{\sigma}$$
$$\lesssim t^\epsilon \sum_k \sum_{k_{14}=k} \theta(2^{-q}k)\sum_{q\lesssim i} 2^{-i}\frac{1}{|k_1|^{3-2\eta-2\epsilon}|k_4|^2}$$
$$+ t^\epsilon \sum_k \sum_{k_{14}=k} \theta(2^{-q}k)\sum_{q\lesssim j} 2^{-j\epsilon}\frac{1}{|k_1|^{3-2\eta-2\epsilon}|k_4|^{3-\epsilon}}$$
$$\lesssim t^\epsilon 2^{q(2\epsilon+2\eta)},$$

where in the last inequality we used Lemma 3.10.

Terms in the fouth chaos: Now for I^1_t we have the following calculations:

$$E[|\Delta_q I^1_t|^2] \lesssim \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \sum_{|i-j|\leq 1, |i'-j'|\leq 1} \sum_{k_{1234}=k} \theta(2^{-q}k)^2\theta(2^{-i}k_{123})\theta(2^{-j}k_4)\theta(2^{-i'}k_{123})\theta(2^{-j'}k_4)$$
$$\int_0^t ds\int_0^t d\bar{s} e^{-|k_{123}|^2(t-s+t-\bar{s})}\int_0^s\int_0^{\bar{s}}\frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2}e^{-|k_{12}|^2(s-\sigma+\bar{s}-\bar{\sigma})}d\sigma d\bar{\sigma}|k_{12}k_{123}|^2$$
$$+ \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \sum_{|i-j|\leq 1, |i'-j'|\leq 1} \sum_{k_{1234}=k} \theta(2^{-q}k)^2\theta(2^{-i}k_{123})\theta(2^{-j}k_4)\theta(2^{-i'}k_{234})\theta(2^{-j'}k_1)$$
$$\int_0^t ds\int_0^t d\bar{s} e^{-|k_{123}|^2(t-s)}e^{-|k_{234}|^2(t-\bar{s})}\int_0^s\int_0^{\bar{s}}\frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2}$$
$$e^{-|k_{12}|^2(s-\sigma)-|k_{24}|^2(\bar{s}-\bar{\sigma})}d\sigma d\bar{\sigma}|k_{12}k_{24}k_{123}k_{234}|$$
$$+ \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \sum_{|i-j|\leq 1, |i'-j'|\leq 1} \sum_{k_{1234}=k} \theta(2^{-q}k)^2\theta(2^{-i}k_{123})\theta(2^{-j}k_4)\theta(2^{-i'}k_{123})\theta(2^{-j'}k_4)$$
$$\int_0^t ds\int_0^t d\bar{s} e^{-|k_{123}|^2(t-s)}e^{-|k_{123}|^2(t-\bar{s})}\int_0^s\int_0^{\bar{s}}\frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2}$$
$$e^{-|k_{12}|^2(s-\sigma)-|k_{13}|^2(\bar{s}-\bar{\sigma})}d\sigma d\bar{\sigma}|k_{12}k_{13}||k_{123}|^2$$
$$+ \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \sum_{|i-j|\leq 1, |i'-j'|\leq 1} \sum_{k_{1234}=k} \theta(2^{-q}k)^2\theta(2^{-i}k_{123})\theta(2^{-j}k_4)\theta(2^{-i'}k_{124})\theta(2^{-j'}k_3)$$

$$\int_0^t ds\int_0^t d\bar{s}e^{-(|k_{123}|^2+|k_3|^2)(t-s)}e^{-(|k_{124}|^2+|k_4|^2)(t-\bar{s})}\int_0^s\int_0^{\bar{s}}\frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2}$$
$$e^{-|k_{12}|^2(s-\sigma+\bar{s}-\bar{\sigma})}d\sigma d\bar{\sigma}|k_{12}|^2|k_{123}k_{124}|$$
$$+\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq1,|i'-j'|\leq1}\sum_{k_{1234}=k}\theta(2^{-q}k)^2\theta(2^{-i}k_{123})\theta(2^{-j}k_4)\theta(2^{-i'}k_{134})\theta(2^{-j'}k_2)$$
$$\int_0^t ds\int_0^t d\bar{s}e^{-(|k_{123}|^2+|k_2|^2)(t-s)}e^{-(|k_{134}|^2+|k_4|^2)(t-\bar{s})}\int_0^s\int_0^{\bar{s}}\frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2}$$
$$e^{-(|k_{12}|^2+|k_2|^2)(s-\sigma)-(|k_{34}|^2+|k_4|^2)(\bar{s}-\bar{\sigma})}d\sigma d\bar{\sigma}|k_{12}k_{34}||k_{134}k_{123}|$$
$$=E_t^1+E_t^2+E_t^3+E_t^4+E_t^5.$$

For $\epsilon,\eta>0$ small enough by Lemma 3.10 we have

$$E_t^1\lesssim\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq1,|i'-j'|\leq1}\sum_{k_{1234}=k}\frac{\theta(2^{-q}k)^2\theta(2^{-i}k_{123})\theta(2^{-j}k_4)\theta(2^{-i'}k_{123})\theta(2^{-j'}k_4)t^\eta}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2|k_{12}|^2|k_{123}|^{2-2\eta}}$$
$$\lesssim\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq1,|i'-j'|\leq1}\sum_{k_{1234}=k}\theta(2^{-q}k)^2\theta(2^{-i}k_{123})\theta(2^{-j}k_4)\theta(2^{-i'}k_{123})\theta(2^{-j'}k_4)\frac{t^\eta}{|k_4|^2|k_{123}|^{4-2\eta-\epsilon}}$$
$$\lesssim\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{q\lesssim i}2^{-(2-2\eta-\epsilon)i}\theta(2^{-q}k)^2\frac{t^\eta}{|k|}\lesssim 2^{q(2\eta+\epsilon)}t^\eta,$$

and

$$E_t^2\lesssim\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq1,|i'-j'|\leq1}\sum_{k_{1234}=k}\frac{\theta(2^{-q}k)^2\theta(2^{-i}k_{123})\theta(2^{-j}k_4)\theta(2^{-i'}k_{234})\theta(2^{-j'}k_1)t^\eta}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2|k_{12}||k_{24}||k_{123}|^{1-\eta}|k_{234}|^{1-\eta}}$$
$$\lesssim\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{k_{1234}=k}\frac{\theta(2^{-q}k)^2t^\eta2^{-q(2-2\eta)}}{|k_1|^{1+\eta}|k_2|^2|k_3|^2|k_4|^{1+\eta}|k_{12}||k_{24}||k_{123}|^{1-\eta}|k_{234}|^{1-\eta}}$$
$$\lesssim\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\Big(\sum_{k_{1234}=k}\frac{\theta(2^{-q}k)^2t^\eta2^{-q(2-2\eta)}}{|k_1|^{1+\eta}|k_2|^2|k_3|^2|k_4|^{1+\eta}|k_{12}|^2|k_{123}|^{2-2\eta}}\Big)^{1/2}$$
$$\Big(\sum_{k_{1234}=k}\frac{\theta(2^{-q}k)^2t^\eta2^{-q(2-2\eta)}}{|k_1|^{1+\eta}|k_2|^2|k_3|^2|k_4|^{1+\eta}|k_{24}|^2|k_{234}|^{2-2\eta}}\Big)^{1/2}$$
$$\lesssim\sum_{k\in\mathbb{Z}^3\setminus\{0\}}2^{-(2-2\eta)q}\frac{t^\eta}{|k|}\lesssim 2^{q(2\eta)}t^\eta.$$

By a similar argument we can also obtain the same bounds for E_t^3,E_t^4 and E_t^5, which implies that

$$E[|\Delta_qI_t^1|^2]\lesssim 2^{q(2\eta+\epsilon)}t^\eta.$$

By a similar calculation as above we also get that

$$\sum_{i_0,j_0=1}^3E[|\Delta_q(\pi_{0,\diamond}(u_3^{\varepsilon_1,i_0},u_1^{\varepsilon_1,j_0})(t_1)-\pi_{0,\diamond}(u_3^{\varepsilon_1,i_0},u_1^{\varepsilon_1,j_0})(t_2)-\pi_{0,\diamond}(u_3^{\varepsilon_2,i_0},u_1^{\varepsilon_2,j_0})(t_1)$$
$$+\pi_{0,\diamond}(u_3^{\varepsilon_2,i_0},u_1^{\varepsilon_2,j_0})(t_2))|^2]$$
$$\lesssim C(\varepsilon_1,\varepsilon_2)|t_1-t_2|^\eta2^{q(\epsilon+2\eta)},$$

where $C(\varepsilon_1,\varepsilon_2)\to 0$ as $\varepsilon_1,\varepsilon_2\to 0$, which by Gaussian hypercontractivity, Lemma 3.1 and similar arguments as (3.16) implies that there exists $v_5^{i_0j_0}\in C([0,T],\mathcal{C}^{-\delta})$, $i_0,j_0=1,2,3$, such that

$$\pi_{0,\diamond}(u_3^{\varepsilon,i_0},u_1^{\varepsilon,j_0})\to v_5^{i_0j_0}\in C([0,T],\mathcal{C}^{-\delta}).$$

3.3.3 Renormalization for $\pi_0(P^{i_1i_2}D_{j_0}K^{\varepsilon,j_0},u_1^{\varepsilon,j_1})$ and $\pi_0(P^{i_1i_2}D_{j_0}K^{\varepsilon,i_2},u_1^{\varepsilon,j_1})$

In this subsection we consider $\pi_0(P^{i_1i_2}D_{j_0}K^{\varepsilon,j_0},u_1^{\varepsilon,j_1})$ and $\pi_0(P^{i_1i_2}D_{j_0}K^{\varepsilon,i_2},u_1^{\varepsilon,j_1})$ and have the following identity:

$$\begin{aligned}&\pi_0(P^{i_1i_2}D_{j_0}K^{\varepsilon,j_0},u_1^{\varepsilon,j_1})\\&=\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq1}\sum_{k_{12}=k}\theta(2^{-i}k_1)\theta(2^{-j}k_2)\int_0^t e^{-(t-s)|k_1|^2}\imath k_1^{j_0}:\hat{X}_s^{\varepsilon,j_0}(k_1)\hat{X}_t^{\varepsilon,j_1}(k_2):dse_k\hat{P}^{i_1i_2}(k_1)\\&\quad+\sum_{|i-j|\leq1}\sum_{k_1}\theta(2^{-i}k_1)\theta(2^{-j}k_1)\int_0^t e^{-2(t-s)|k_1|^2}\imath k_1^{j_0}\frac{f(\varepsilon k_1)^2}{|k_1|^2}ds\hat{P}^{i_1i_2}(k_1)\sum_{j_2=1}^3\hat{P}^{j_0j_2}(k_1)\hat{P}^{j_1j_2}(k_1).\end{aligned}$$

It is easy to get that the second term in the right hand side of the above equality equals zero. It is straightforward to calculate for $\epsilon>0$ small enough:

$$\begin{aligned}&E|\Delta_q\pi_0(P^{i_1i_2}D_{j_0}K^{\varepsilon,j_2},u_1^{\varepsilon,j_1})|^2\\&\lesssim\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq1,|i'-j'|\leq1}\sum_{k_{12}=k}\theta(2^{-q}k)^2\theta(2^{-i}k_1)\theta(2^{-j}k_2)\theta(2^{-i'}k_1)\theta(2^{-j'}k_2)\\&\quad(\int_0^t\int_0^t e^{-(t-s+t-\bar{s})|k_1|^2}|k_1|^2\frac{e^{-|k_1|^2|s-\bar{s}|}}{|k_1|^2|k_2|^2}dsd\bar{s}\\&\quad+\int_0^t\int_0^t e^{-2(t-s)|k_1|^2-2(t-\bar{s})|k_2|^2}|k_1||k_2|\frac{1}{|k_1|^2|k_2|^2}dsd\bar{s})\\&\lesssim t^\epsilon\sum_k\sum_{q\lesssim i}\sum_{k_{12}=k}\theta(2^{-q}k)\theta(2^{-i}k_1)\frac{1}{|k_1|^{4-2\epsilon}|k_2|^2}\\&\quad+t^\epsilon\sum_k\sum_{q\lesssim i}\sum_{k_{12}=k}\theta(2^{-q}k)\theta(2^{-j}k_2)\frac{1}{|k_1|^{3-2\epsilon}|k_2|^3}\\&\lesssim t^\epsilon 2^{2q\epsilon},\end{aligned}$$

where in the last inequality we used Lemma 3.10. By a similar calculation we also get that for $\epsilon,\eta>0$ small enough

$$\begin{aligned}&E[|\Delta_q(\pi_{0,\diamond}(P^{i_1i_2}D_{j_0}K^{\varepsilon,j_2},u_1^{\varepsilon,j_1})(t_1)-\pi_{0,\diamond}(P^{i_1i_2}D_{j_0}K^{\varepsilon,j_2},u_1^{\varepsilon,j_1})(t_2)\\&\quad-\pi_{0,\diamond}(P^{i_1i_2}D_{j_0}K^{\varepsilon,j_2},u_1^{\varepsilon,j_1})(t_1)+\pi_{0,\diamond}(P^{i_1i_2}D_{j_0}K^{\varepsilon,j_2},u_1^{\varepsilon,j_1})(t_2))|^2]\\&\lesssim C(\varepsilon_1,\varepsilon_2)|t_1-t_2|^\eta 2^{q(\epsilon+2\eta)},\end{aligned}$$

where $C(\varepsilon_1,\varepsilon_2)\to 0$ as $\varepsilon_1,\varepsilon_2\to 0$, which by Gaussian hypercontractivity, Lemma 3.1 and similar argument as (3.16) implies that there exists $v_6^{i_1i_2j_0j_1}\in C([0,T],\mathcal{C}^{-\delta})$ for $i_1,i_2,j_0,j_1=1,2,3$ such that

$$\pi_{0,\diamond}(P^{i_1i_2}D_{j_0}K^{\varepsilon,j_2},u_1^{\varepsilon,j_1})\to v_6^{i_1i_2j_0j_1}\in C([0,T],\mathcal{C}^{-\delta}).$$

By a similar argument we also obtain that there exists $v_7^{i_1i_2j_0j_1}\in C([0,T],\mathcal{C}^{-\delta})$ for $i_1,i_2,j_0,j_1=1,2,3$ such that

$$\pi_{0,\diamond}(P^{i_1i_2}D_{j_0}K^{\varepsilon,i_2},u_1^{\varepsilon,j_1})\to v_7^{i_1i_2j_0j_1}\in C([0,T],\mathcal{C}^{-\delta}).$$

3.3.4 Renormalisation for $u_2^{\varepsilon,i}u_2^{\varepsilon,j}$

In this subsection we deal with $u_2^{\varepsilon,i}u_2^{\varepsilon,j}$ and prove that $u_2^{\varepsilon,i}\diamond u_2^{\varepsilon,j}\to u_2^i\diamond u_2^j$ in $C([0,T];\mathcal{C}^{-\delta})$. We have the following identities:

$$\begin{aligned}
&u_2^{\varepsilon,i}u_2^{\varepsilon,j}\\
=&(2\pi)^{-9/2}\Big(\sum_{i_1,i_2,j_1,j_2=1}^3\sum_{k_{1234}=k}\int_0^t\int_0^t e^{-|k_{12}|^2(t-s)-|k_{34}|^2(t-\bar{s})}:\hat{X}_s^{\varepsilon,i_1}(k_1)\hat{X}_s^{\varepsilon,i_2}(k_2)\hat{X}_{\bar{s}}^{\varepsilon,j_1}(k_3)\hat{X}_{\bar{s}}^{\varepsilon,j_2}(k_4):dsd\bar{s}e_k\\
&\hat{P}^{ii_1}(k_{12})\imath k_{12}^{i_2}\hat{P}^{jj_1}(k_{34})\imath k_{34}^{j_2}\\
&+4\sum_{i_1,i_2,j_1,j_2=1}^3\sum_{k_{24}=k,k_1}\int_0^t\int_0^t e^{-|k_{12}|^2(t-s)-|k_4-k_1|^2(t-\bar{s})}\frac{f(\varepsilon k_1)^2e^{-|k_1|^2|s-\bar{s}|}}{2|k_1|^2}:\hat{X}_s^{\varepsilon,i_2}(k_2)\hat{X}_{\bar{s}}^{\varepsilon,j_2}(k_4):dsd\bar{s}e_k\\
&\hat{P}^{ii_1}(k_{12})\imath k_{12}^{i_2}\hat{P}^{jj_1}\imath(k_4-k_1)(k_4^{j_2}-k_1^{j_2})\sum_{j_3=1}^3\hat{P}^{i_1j_3}(k_1)\hat{P}^{j_1j_3}(k_1)\\
&+2\sum_{i_1,i_2,j_1,j_2=1}^3\sum_{k_1,k_2}\int_0^t\int_0^t e^{-|k_{12}|^2(t-s+t-\bar{s})}\frac{f(\varepsilon k_1)^2f(\varepsilon k_2)^2e^{-(|k_1|^2+|k_2|^2)|s-\bar{s}|}}{4|k_1|^2|k_2|^2}dsd\bar{s}\hat{P}^{ii_1}(k_{12})\hat{P}^{jj_1}(k_{12})\\
&\imath k_{12}^{i_2}(-\imath k_{12}^{j_2})\sum_{j_3,j_4=1}^3\hat{P}^{i_1j_3}(k_1)\hat{P}^{j_1j_3}(k_1)\hat{P}^{i_2j_4}(k_2)\hat{P}^{j_2j_4}(k_2)\Big):=I_t^1+I_t^2+I_t^3.
\end{aligned}$$

By a easy computation we obtain that

$$\begin{aligned}
I_t^3=&(2\pi)^{-9/2}\sum_{i_1,i_2,j_1,j_2=1}^3\sum_{k_1,k_2}f(\varepsilon k_1)^2f(\varepsilon k_2)^2\hat{P}^{ii_1}(k_{12})\hat{P}^{jj_1}(k_{12})k_{12}^{i_2}k_{12}^{j_2}\sum_{j_3,j_4=1}^3\hat{P}^{i_1j_3}(k_1)\hat{P}^{j_1j_3}(k_1)\hat{P}^{i_2j_4}(k_2)\\
&\hat{P}^{j_2j_4}(k_2)\frac{1}{|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|k_{12}|^2)}\Big(\frac{1-e^{-2|k_{12}|^2t}}{2|k_{12}|^2}-\int_0^t e^{-2|k_{12}|^2(t-s)-(|k_1|^2+|k_2|^2+|k_{12}|^2)s}ds\Big).
\end{aligned}$$

Let

$$\begin{aligned}
C_2^{\varepsilon,ij}=&(2\pi)^{-9/2}\sum_{i_1,i_2,j_1,j_2=1}^3\sum_{k_1,k_2}f(\varepsilon k_1)^2f(\varepsilon k_2)^2\hat{P}^{ii_1}(k_{12})\hat{P}^{jj_1}(k_{12})k_{12}^{i_2}(-k_{12}^{j_2})\sum_{j_3,j_4=1}^3\hat{P}^{i_1j_3}(k_1)\hat{P}^{j_1j_3}(k_1)\\
&\hat{P}^{i_2j_4}(k_2)\hat{P}^{j_2j_4}(k_2)\frac{1}{2|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|k_{12}|^2)}\frac{1}{|k_{12}|^2}.
\end{aligned}$$

Define

$$\varphi_2^{\varepsilon,ij}=I_t^3-C_2^{\varepsilon,ij}.$$

Then for $\rho > 0$ we have

$$|\varphi_2^\varepsilon| \lesssim \sum_{k_1,k_2} |k_{12}|^2 \frac{1}{|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|k_{12}|^2)}\Big(\frac{e^{-2|k_{12}|^2 t}}{2|k_{12}|^2} + \int_0^t e^{-2|k_{12}|^2(t-s)-(|k_1|^2+|k_2|^2+|k_{12}|^2)s}ds\Big)$$
$$\lesssim t^{-\rho}\sum_{k_1,k_2}\frac{1}{|k_1|^2|k_2|^2|k_{12}|^{2+2\rho}} \lesssim t^{-\rho}.$$

Terms in the second chaos: Now we come to I_t^2: For $\epsilon > 0$ small enough we have the following inequalities

$$E|\Delta_q I_t^2|^2 \lesssim \sum_k \sum_{k_{24}=k,k_1,k_3} \theta(2^{-q}k)^2 \int_0^t\int_0^t\int_0^t\int_0^t e^{-|k_{12}|^2(t-s)-|k_4-k_1|^2(t-\bar{s})-|k_{23}|^2(t-\sigma)-|k_4-k_3|^2(t-\bar{\sigma})}$$
$$\frac{e^{-|k_1|^2|s-\bar{s}|-|k_2|^2|s-\sigma|-|k_4|^2|\bar{s}-\bar{\sigma}|-|k_3|^2|\sigma-\bar{\sigma}|}}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2} ds d\bar{s}|k_{12}(k_4-k_1)k_{23}(k_4-k_3)|$$
$$\lesssim t^\epsilon \sum_k \sum_{k_{24}=k,k_1,k_3} \frac{\theta(2^{-q}k)^2}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2|k_1-k_4|^{1-\epsilon}|k_4-k_3||k_{12}|^{1-\epsilon}|k_{23}|}$$
$$\lesssim t^\epsilon \sum_k\sum_{k_{24}=k}\frac{\theta(2^{-q}k)^2}{|k_2|^2|k_4|^2}\sum_{k_1}\frac{1}{|k_1-k_4||k_1|^2|k_{12}|}\sum_{k_3}\frac{1}{|k_3-k_4||k_3|^2|k_{23}|}$$
$$\lesssim t^\epsilon \sum_k\sum_{k_{24}=k}\frac{\theta(2^{-q}k)^2}{|k_2|^2|k_4|^2}\Big(\sum_{k_1}\frac{1}{|k_1-k_4|^{2-2\epsilon}|k_1|^2}\Big)^{1/2}\Big(\sum_{k_1}\frac{1}{|k_{12}|^{2-2\epsilon}|k_1|^2}\Big)^{1/2}$$
$$\Big(\sum_{k_3}\frac{1}{|k_3-k_4|^2|k_3|^2}\Big)^{1/2}\Big(\sum_{k_3}\frac{1}{|k_{23}|^2|k_3|^2}\Big)^{1/2}$$
$$\lesssim t^\epsilon \sum_k\sum_{k_{24}=k}\frac{\theta(2^{-q}k)^2}{|k_2|^{3-\epsilon}|k_4|^{3-\epsilon}} \lesssim t^\epsilon 2^{2q\epsilon},$$

where in the last two inequalities we used Lemma 3.10.

Terms in the fourth chaos:

Now we consider I_t^1. For $\epsilon > 0$ small enough we have the following calculations:

$$E|\Delta_q I_t^1|^2$$
$$\lesssim \sum_k\sum_{k_{1234}=k}\theta(2^{-q}k)^2\Big(\int_0^t\int_0^t\int_0^t\int_0^t e^{-|k_{12}|^2(t-s+t-\sigma)-|k_{34}|^2(t-\bar{s}+t-\bar{\sigma})}\frac{e^{-(|k_1|^2+|k_2|^2)|s-\sigma|-(|k_3|^2+|k_4|^2)|\bar{s}-\bar{\sigma}|}}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2}$$
$$ds d\bar{s} d\sigma d\bar{\sigma}|k_{12}k_{34}|^2 + \int_0^t\int_0^t\int_0^t\int_0^t e^{-|k_{12}|^2(t-s)-|k_{23}|^2(t-\sigma)-|k_{34}|^2(t-\bar{s})-|k_{14}|^2(t-\bar{\sigma})}\frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2}$$
$$ds d\bar{s} d\sigma d\bar{\sigma}|k_{12}k_{34}k_{14}k_{23}|\Big)$$
$$\lesssim t^\epsilon \sum_k\sum_{k_{1234}=k}\theta(2^{-q}k)^2\Big(\frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2|k_{12}|^{2-\epsilon}|k_{34}|^{2-\epsilon}}$$
$$+\frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2|k_{12}|^{1-\epsilon/2}|k_{34}|^{1-\epsilon/2}|k_{14}|^{1-\epsilon/2}|k_{23}|^{1-\epsilon/2}}\Big)$$

$$\lesssim t^{\epsilon}(2^{2q\epsilon}+(\sum_{k_{1234}=k}\frac{\theta(2^{-q}k)^2}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2|k_{12}|^{2-\epsilon}|k_{34}|^{2-\epsilon}})^{1/2}(\sum_{k_{1234}=k}\frac{\theta(2^{-q}k)^2}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2|k_{14}|^{2-\epsilon}|k_{23}|^{2-\epsilon}})^{1/2})$$
$$\lesssim t^{\epsilon}2^{2q\epsilon},$$

where we used Lemma 3.10 in the last inequality. By a similar calculation we also get that for $\epsilon,\eta>0$ small enough

$$E[|\Delta_q(u_2^{\varepsilon_1,i}\diamond u_2^{\varepsilon_1,j}(t_1)-u_2^{\varepsilon_1,i}\diamond u_2^{\varepsilon_1,j}(t_2)-u_2^{\varepsilon_2,i}\diamond u_2^{\varepsilon_2,j}(t_1)+u_2^{\varepsilon_2,i}\diamond u_2^{\varepsilon_2,j})(t_2))|^2]$$
$$\lesssim C(\varepsilon_1,\varepsilon_2)|t_1-t_2|^{\eta}2^{q(\epsilon+2\eta)},$$

where $C(\varepsilon_1,\varepsilon_2)\to 0$ as $\varepsilon_1,\varepsilon_2\to 0$, which by Gaussian hypercontractivity, Lemma 3.1 and similar argument as (3.16) implies that there exists $v_4^{ij}\in C([0,T],\mathcal{C}^{-\delta}),i,j=1,2,3$ and some φ_2 such that

$$u_2^{\varepsilon,i}\diamond u_2^{\varepsilon,j}\to v_4^{ij}\in C([0,T],\mathcal{C}^{-\delta}),$$

and φ_2^{ε} converges to some φ_2 with respect to $\|\varphi\|=\sup_{t\in[0,T]}t^{\rho}|\varphi(t)|$ for any $\rho>0$.

Combining all the convergence results we obtained above and Theorem 3.8 we obtain local existence and uniqueness of the solution to 3D Navier-Stokes equation driven by space-time white noise.

Theorem 3.11 Let $z\in(1/2,1/2+\delta_0)$ with $0<\delta_0<1/2$ and $u_0\in\mathcal{C}^{-z}$. Then there exists a unique local solution to

$$Lu^i=\sum_{i_1=1}^3P^{ii_1}\xi-\frac{1}{2}\sum_{i_1=1}^3P^{ii_1}(\sum_{j=1}^3D_j(u^{i_1}u^j))\quad u(0)=u_0,$$

in the following sense: For $\xi^{\varepsilon}=\sum_k f(\varepsilon k)\hat{\xi}(k)e_k$ with f a smooth radial function with compact support satisfying $f(0)=1$ and for $\varepsilon>0$ consider the maximal unique solution u^{ε} of the following equation such that $u_4^{\varepsilon}\in C((0,T^{\varepsilon});\mathcal{C}^{1/2-\delta_0})$

$$Lu^{\varepsilon,i}=\sum_{i_1=1}^3P^{ii_1}\xi^{\varepsilon}-\frac{1}{2}\sum_{i_1=1}^3P^{ii_1}(\sum_{j=1}^3D_j(u^{\varepsilon,i_1}u^{\varepsilon,j}))\quad u^{\varepsilon}(0)=Pu_0.$$

Then there exists a strictly positive, $\sigma(u_0,\xi)$ measurable random time τ such that

$$E(\sup_{t\in[0,\tau]}\|u^{\varepsilon}-u\|_{-z})^p\to 0,$$

for all $p\geq 1$.

Acknowledgement. We are very grateful to Professor Martin Hairer for giving us some hints to complete the paper and his suggestions which helped us to improve the results of this paper. We would also like to thank Professor Michael Röckner for his encouragement and suggestions for this work.

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