Stochastic nonlinear Schrödinger equations

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Abstract. This paper is devoted to the well-posedness of stochastic nonlinear Schrödinger equations in the energy space $H^1(\mathbb{R}^d)$, which is a natural continuation of our recent work [1]. We consider both focusing and defocusing nonlinearities and prove global well-posedness in $H^1(\mathbb{R}^d)$, including also the pathwise continuous dependence on initial conditions, with exponents exactly the same as in the deterministic case. In particular, this work improves earlier results in [4]. Moreover, the local existence, uniqueness and blowup alternative are also established for the energy-critical case. The approach presented here is mainly based on the rescaling approach already used in [1] to study the L^2 case and also on the Strichartz estimates established in [12] for large perturbations of the Laplacian.

Keyword: (stochastic) nonlinear Schrödinger equation, Wiener process, Sobolev space, Strichartz estimates.

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1 Introduction and main results

Let us consider the stochastic nonlinear Schrödinger equation with linear multiplicative noise

$$
\begin{aligned}
&idX(t,\xi) = \Delta X(t,\xi)dt + \lambda |X(t,\xi)|^{\alpha-1}X(t,\xi)dt \\
&\qquad\qquad - i\mu(\xi)X(t,\xi)dt + iX(t,\xi)dW(t,\xi),\ t\in(0,T),\ \xi\in\mathbb{R}^d, \quad (1.1)\\
&X(0) = x.
\end{aligned}
$$

Here $\lambda = \pm 1$, $\alpha > 1$ and W is the colored Wiener process

$$W(t,\xi) = \sum_{j=1}^{N} \mu_j e_j(\xi)\beta_j(t), t \geq 0, \xi \in \mathbb{R}^d, \tag{1.2}$$

with $\mu_j \in \mathbb{C}$, $e_j(\xi)$ real-valued functions and $\beta_j(t)$ independent real Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural filtration $(\mathcal{F}_t)_{t\geq 0}$, $1 \leq j \leq N$. In this paper for simplicity we assume $N < \infty$.

As in the physical context [2], we choose μ of the form

$$\mu(\xi) = \frac{1}{2}\sum_{j=1}^{N} |\mu_j|^2 e_j^2(\xi), \xi \in \mathbb{R}^d,$$

so that $|X(t)|_2^2$ is a martingale, from which one can define the "physical probability law"(see [2]).

In the deterministic case $\mu_j = 0$, $1 \leq j \leq N$, it is well known (see [9, 11]) that (1.1) is globally well posed in $H^1(\mathbb{R}^d)$ in the defocusing case $\lambda = -1$ with the subcritical exponents of the nonlinearity

$$1 < \alpha < 1 + \frac{4}{(d-2)_+}, \tag{1.3}$$

while in the focusing case $\lambda = 1$ with the exponents

$$1 < \alpha < 1 + \frac{4}{d}. \tag{1.4}$$

Here $1 + \frac{4}{(d-2)_+} = 1 + \frac{4}{d-2}$ (*resp.* ∞) with $d \geq 3$ (*resp.* $d = 1, 2$).

In the stochastic case, the authors in [4] (see also [3]) studied the conservative case $Re\mu_j = 0$, $1 \leq j \leq N$, i.e. W is a purely imaginary noise. They proved the local existence and uniqueness with α satisfying

$$\begin{cases} 1 < \alpha < \infty, & \text{if } d = 1 \text{ or } 2; \\ 1 < \alpha < 5, & \text{if } d = 3; \\ 2 \leq \alpha < 1 + \frac{4}{d-2}, & \text{if } d = 4, 5; \\ \alpha < 1 + \frac{2}{d-1}, & \text{if } d \geq 6; \end{cases}$$

and then the global well-posedness under the further assumptions that $\alpha < 1 + \frac{4}{d}$ or $\lambda = -1$. Hence, when $d \geq 6$, the global well-posedness is established only for the restrictive exponents $\alpha < 1 + \frac{2}{d-1}$. We also refer to [5] for stochastic nonlinear Schrödinger equation (however, only for one-dimension noise) with real-valued potentials in the conservative case.

The starting point of this article is our recent work [1], where we obtain the global well-posedness of (1.1) in L^2 space with exponent $\alpha \in (1, 1 + \frac{4}{d})$, i.e. in the same range as in the deterministic case.

The main aim of the present work is to study the global well-posedness of (1.1) in $H^1(\mathbb{R}^d)$ with general $\mu_j \in \mathbb{C}$ as in the physical context [2], including the non-conservative case. We prove the global well-posedness, including also the pathwise continuous dependence on initial conditions, with α in the ranges (1.3) and (1.4) in the defocusing and focusing cases respectively, i.e. in exactly the same ranges as in the deterministic case. In particular, these sharper results fill the gap for α in [4] mentioned above.

Moreover, the local well-posedness is also established in Section 2 for the energy-critical case $\lambda = \pm 1$, $\alpha = 1 + \frac{4}{d-2}$ with $d \geq 3$ and also for the focusing mass-(super)critical case $\lambda = 1$, $1 + \frac{4}{d} \leq \alpha < 1 + \frac{4}{(d-2)_+}$ with $d \geq 1$. The local results established in the latter case allow to study the noise effect on blowup phenomena, which will be contained in forthcoming work.

Before we show the main global well-posedness result, let us first present the spatial decay assumption on $\{e_j\}_{j=1}^N$ and the precise definitions of solutions to (1.1).

(H1). $e_j \in C_b^\infty(\mathbb{R}^d)$ such that

$$\lim_{|\xi| \to \infty} \zeta(\xi) |\partial^\gamma e_j(\xi)| = 0,$$

where γ is a multi-index such that $|\gamma| \leq 3$, $1 \leq j \leq N$ and

$$\zeta(\xi) = \begin{cases} 1+|\xi|^2, & \text{if } d \neq 2; \\ (1+|\xi|^2)(\ln(3+|\xi|^2))^2, & \text{if } d = 2. \end{cases}$$

Definition 1.1 *Let $x \in H^1$ and let α satisfy $1 < \alpha < \infty$ if $d = 1, 2$ or $1 < \alpha \leq 1 + \frac{4}{d-2}$ if $d \geq 3$. Fix $0 < T < \infty$.*

A strong solution of (1.1) *is a pair (X, τ), where $\tau(\leq T)$ is an $(\mathscr{F}_t)$-stopping time, and $X = (X(t))_{t\in[0,T]}$ is an H^1-valued continuous $(\mathscr{F}_t)$-adapted process, such that $|X|^{\alpha-1}X \in L^1(0,\tau; H^{-1})$, $\mathbb{P}-a.s$, and it satisfies $\mathbb{P}-a.s$*

$$\begin{aligned} X(t) =& x - \int_0^t (i\Delta X(s) + \mu X(s) + \lambda i |X(s)|^{\alpha-1} X(s)) ds \\ &+ \int_0^t X(s) dW(s), \quad t \in [0, \tau], \end{aligned} \tag{1.5}$$

as an equation in H^{-1}.

We say that uniqueness holds for (1.1)*, if for any two strong solutions (X_i, τ_i), $i = 1, 2$, it holds $\mathbb{P}$-a.s. that $X_1 = X_2$ on $[0, \tau_1 \wedge \tau_2]$.*

We refer to [13] for the general theory of infinite dimensional stochastic equations. It is easy to check that, $\int_0^t X(s)dW(s)$ in Definition 1.1 is an H^1-valued continuous stochastic integral.

The main global well-posedness result in this paper is as follows.

Theorem 1.2 *Assume* (*H*1)*. Let α satisfy (1.3) and (1.4) in the defocusing and focusing cases respectively. Then for each $x \in H^1$ and $0 < T < \infty$, there exists a unique strong solution (X, T) of (1.1) in the sense of Definition 1.1, such that*

$$X \in L^2(\Omega; C([0,T]; H^1)) \cap L^{\alpha+1}(\Omega; C([0,T]; L^{\alpha+1})), \tag{1.6}$$

and

$$X \in L^\gamma(0,T; W^{1,\rho}), \quad \mathbb{P}-a.s., \tag{1.7}$$

where (ρ, γ) is any Strichartz pair (see Lemma 2.7 below).

Furthermore, for $\mathbb{P}-a.e$ ω, the map $x \to X(\cdot, x, \omega)$ is continuous from H^1 to $C([0,T]; H^1) \cap L^\gamma(0,T; W^{1,\rho})$.

The key approach here (as in [1]) is based on the rescaling transformation that reduces the stochastic equation (1.1) to a random Schrödinger equation (see (2.5)), to which one can apply the sharp deterministic estimates, e.g. the Strichartz estimates established in [12] for large perturbations of the Laplacian.

This paper is structured as follows. In Section 2 we establish the local existence, uniqueness and blowup alternative of solutions to equation (1.1). Then in Section 3 we derive a priori estimates of the energy from the Hamiltonian, which lead to the global well-posedness in the subcritical case in Section 4. An important role in our proofs is played by Itô's formulae for the L^p- and H^1- norms, which can be heuristically computed very easily. The rigorous proofs are much harder and are contained in Section 5. Furthermore, some technical proofs are postponed to the Appendix, i.e. Section 6, for simplicity of exposition.

Notations. For $1 \leq p \leq \infty$, $L^p = L^p(\mathbb{R}^d)$ is the space of all p-integrable complex valued functions with the norm $|\cdot|_{L^p}$. $L^q(0,T;L^p)$ denotes the measurable functions $u : [0,T] \to L^p$ such that $t \to |u(t)|_{L^p}$ belongs to $L^q(0,T)$. $C([0,T];L^p)$ similarly denotes the continuous L^p-valued functions with the sup norm in t.

As usual, $W^{1,p} = W^{1,p}(\mathbb{R}^d)$ is the classical Sobolev space, i.e. $W^{1,p} = \{u \in L^p : \nabla u \in L^p\}$ with the norm $\|u\|_{W^{1,p}} = |u|_{L^p} + |\nabla u|_{L^p}$. Here $\nabla = (\partial_1, ..., \partial_d)$ with $\partial_k := \frac{\partial}{\partial x_k}$, $1 \leq k \leq d$. Moreover, the spaces $L^q(0,T;W^{1,p})$ and $C([0,T];W^{1,p})$ are understood similarly as above. We also use the notation $\partial^\gamma = \partial_1^{\gamma_1} \cdots \partial_d^{\gamma_d}$ for any multi-index $\gamma = (\gamma_1, ..., \gamma_d)$ with $\gamma_j \in \mathbb{N}$. The order of γ is $|\gamma| = \gamma_1 + \cdots + \gamma_d$, and if $|\gamma| = 0$, $\partial^\gamma f = f$.

In the special case $p = 2$, L^2 is the Hilbert space endowed with the scalar product

$$\langle u, v\rangle = \int_{\mathbb{R}^d} u(\xi)\overline{v}(\xi)d\xi;\ u, v \in L^2.$$

For simplicity, we set $|\cdot|_2 = |\cdot|_{L^2}$. Let $H^1 = W^{1,2}$ and H^{-1} be the dual space of H^1. Their norms are denoted by $|\cdot|_{H^k}$, $k = \pm 1$.

$C_c^\infty(\mathbb{R}^d)$ denotes the compactly supported smooth functions on $\mathbb{R}^d$. We use $\mathcal{S}$ and $\mathcal{S}'$ for the rapidly decreasing functions and the tempered distributions respectively. Then for $f \in \mathcal{S}$, $\widehat{f}$ means the Fourier transform, i.e.

$\widehat{f}(\eta) = \int f(\xi)e^{-i\xi\cdot\eta}d\xi$, and for $f \in \mathcal{S}'$, $f^\vee$ denotes the inverse Fourier transform of f, i.e. $f^\vee(\xi) = \frac{1}{(2\pi)^d}\int f(\eta)e^{i\xi\cdot\eta}d\eta$.

We use C, $\widetilde{C}$ for various constants that may change from line to line.

2 Local results

In this section, we will establish the local existence, uniqueness and blowup alternative for equation (1.1). The main result is given in Theorem 2.1 below.

Theorem 2.1 *Assume (H1). Let α satisfy $1 < \alpha < \infty$ if $d = 1, 2$, or, $1 < \alpha \leq 1 + \frac{4}{d-2}$ if $d \geq 3$. For each $x \in H^1$ and $0 < T < \infty$, there is a sequence of strong solutions (X_n, τ_n) of (1.1), $n \in \mathbb{N}$, where τ_n is a sequence of incresing stopping times, and uniqueness holds in the sense of Definition 1.1. For every $n \geq 1$, it holds $\mathbb{P}$-a.s that*

$$X_n|_{[0,\tau_n]} \in C([0, \tau_n]; H^1) \cap L^\gamma(0, \tau_n; W^{1,\rho}), \tag{2.1}$$

where (ρ, γ) is any Strichartz pair.

Moreover, defining $\tau^*(x) = \lim\limits_{n\to\infty} \tau_n$ and $X = \lim\limits_{n\to\infty} X_n \mathbb{1}_{[0,\tau^*(x))}$, we have the blowup alternative, that is, for $\mathbb{P}$-a.e ω, if $\tau_n(\omega) < \tau^*(x)(\omega)$, $\forall n \in \mathbb{N}$, then

$$\lim_{t\to\tau^*(x)(\omega)} |X(t)(\omega)|_{H^1} = \infty, \quad if\ 1 < \alpha < 1 + \frac{4}{(d-2)_+},\ d \geq 1, \tag{2.2}$$

and

$$\|X(\omega)\|_{L^{\frac{2(d+2)}{d-2}}(0,\tau^*(x)(\omega);L^{\frac{2(d+2)}{d-2}})} = \infty, \quad if\ \alpha = 1 + \frac{4}{d-2},\ d \geq 3. \tag{2.3}$$

Remark 2.2 *As seen below in the proof of Proposition 2.5 if the norm in (2.2) or (2.3) is finite $\mathbb{P}$-a.s., then $\tau^*(x) = T$, $\mathbb{P}$-a.s.*

The key tool to prove Theorem 2.1 is based on the rescaling approach as used in [1]. Namely, we apply the rescaling transformation

$$X = e^W y \tag{2.4}$$

to reduce the original stochastic equation (1.1) to the random Schrödinger equation

$$\begin{aligned}&\frac{\partial y(t,\xi)}{\partial t} = A(t)y(t,\xi) - \lambda i e^{(\alpha-1)ReW(t,\xi)}|y(t,\xi)|^{\alpha-1}y(t,\xi),\\&y(0) = x.\end{aligned} \tag{2.5}$$

Here

$$A(t)y(t,\xi) := -i(\Delta + b(t,\xi)\cdot\nabla + c(t,\xi))y(t,\xi) \tag{2.6}$$

with $b(t,\xi) = 2\nabla W(t,\xi)$, $c(t,\xi) = \sum_{j=1}^{d}(\partial_j W(t,\xi))^2 + \Delta W(t,\xi) - i(\mu(\xi)+\widetilde{\mu}(\xi))$ and $\widetilde{\mu}(\xi) = \frac{1}{2}\sum_{j=1}^{N}\mu_j^2 e_j^2(\xi)$.

Analogously to Definition 1.1, the solutions to (2.5) are defined as follows.

Definition 2.3 *Let $x \in H^1$, $0 < T < \infty$, and $\alpha \in (1,\infty)$ if $d = 1, 2$, or $\alpha \in (1, 1+\frac{4}{d-2}]$ for $d \geq 3$. The strong solution (y,τ) and uniqueness of (2.5) are defined similarly as in Definition 1.1, just with the modifications that X and (1.5) are replaced, respectively, by y and the equation*

$$y(t) = x + \int_0^t A(s)y(s)ds - \int_0^t \lambda i e^{(\alpha-1)ReW(s)}|y(s)|^{\alpha-1}y(s)ds. \tag{2.7}$$

Remark 2.4 *The equivalence between two strong solutions (X,τ) and (y,τ) of (1.1) and (2.5), respectively, can be proved similarly as in the proof of Lemma 6.1 in [1]. We also refer to [14] for more details.*

Therefore, it is equivalent to prove the local results for the random equation (2.5). We have the following

Proposition 2.5 *Assume the conditions in Theorem 2.1 to hold. For each $x \in H^1$ and $0 < T < \infty$, there is a sequence of strong solutions (y_n,τ_n) of (2.5), $n \in \mathbb{N}$, where τ_n is a sequence of incresing stopping times, and uniqueness holds in the sense of Definition 2.3. For every $n \geq 1$, it holds $\mathbb{P}$-a.s that*

$$y_n|_{[0,\tau_n]} \in C([0,\tau_n]; H^1) \cap L^\gamma(0,\tau_n; W^{1,\rho}), \tag{2.8}$$

where (ρ,γ) is any Strichartz pair.

Moreover, defining $\tau^*(x) = \lim_{n\to\infty} \tau_n$ *and* $y = \lim_{n\to\infty} y_n \mathbb{1}_{[0,\tau^*(x))}$*, we have the blowup alternative, namely, for* $\mathbb{P}$*-a.e* ω *if* $\tau_n(\omega) < \tau^*(x)(\omega)$*,* $\forall n \in \mathbb{N}$*, then*

$$\lim_{t\to\tau^*(x)(\omega)} |y(t)(\omega)|_{H^1} = \infty, \quad if\ 1 < \alpha < 1 + \frac{4}{(d-2)_+},\ d \geq 1,$$

and

$$\|y(\omega)\|_{L^{\frac{2(d+2)}{d-2}}(0,\tau^*(x)(\omega);L^{\frac{2(d+2)}{d-2}})} = \infty, \quad if\ \alpha = 1 + \frac{4}{d-2},\ d \geq 3.$$

Inspired by the deterministic case, the local well-posedness of (2.5) depends crucially on the dispersive properties of the linear part in (2.5). Hence, in order to prove Proposition 2.5, let us first introduce the evolution operators and Strichartz estimates in Sobolev spaces.

Lemma 2.6 *For* $\mathbb{P}-a.e.\omega$*, the operator* $A(t)$ *defined in* (2.6) *generates evolution operators* $U(t,s) = U(t,s,\omega)$ *in the space* $H^1(\mathbb{R}^d)$*,* $0 \leq s \leq t \leq T$*. Moreover, for each* $x \in H^1(\mathbb{R}^d)$ *and* $s \in [0,T]$*, the process* $[s,T] \ni t \to U(t,s)x$ *is continuous and* $(\mathscr{F}_t)$*-adapted, hence progressively measurable with respect to the filtration* $(\mathscr{F}_t)_{t\geq s}$*.*

Proof. This lemma is based on [8] and can be proved analogously as Lemma 3.3 in [1] (see also [14]). □

Lemma 2.7 *Assume (H1). Then for any* $T > 0$*,* $u_0 \in H^1$ *and* $f \in L^{q_2'}(0,T;W^{1,p_2'})$*, the solution of*

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)ds, 0 \leq t \leq T, \tag{2.9}$$

satisfies the estimates

$$\|u\|_{L^{q_1}(0,T;L^{p_1})} \leq C_T(|u_0|_2 + \|f\|_{L^{q_2'}(0,T;L^{p_2'})}), \tag{2.10}$$

and

$$\|u\|_{L^{q_1}(0,T;W^{1,p_1})} \leq C_T(|u_0|_{H^1} + \|f\|_{L^{q_2'}(0,T;W^{1,p_2'})}), \tag{2.11}$$

where (p_1, q_1) *and* (p_2, q_2) *are Strichartz pairs, namely*

$$(p_i, q_i) \in [2,\infty] \times [2,\infty] : \frac{2}{q_i} = \frac{d}{2} - \frac{d}{p_i},\ if\ d \neq 2,$$

or

$$(p_i, q_i) \in [2,\infty) \times (2,\infty] : \frac{2}{q_i} = \frac{d}{2} - \frac{d}{p_i},\ if\ d = 2,$$

Furthermore, the process C_t, $t \geq 0$, can be taken to be $(\mathscr{F}_t)$-progressively measurable, increasing and continuous.

(See the Appendix for the proof.)

Proof of Proposition 2.5. It is equivalent to solve the weak equation (2.7) in the mild sense, namely

$$y = U(t,0)x - \lambda i \int_0^t U(t,s) e^{(\alpha-1)ReW(s)} g(y(s))ds, \tag{2.12}$$

where $g(y) = |y|^{\alpha-1}y$. The following fixed point arguments are standard in the deterministic case (see e.g. [9] and [11]). However, we emphasize that we have to secure the $(\mathscr{F}_t)$-adaptedness of the solutions, which allows us later to apply Itô's formula to obtain a priori estimates (see also [1]).

Let us first consider the case $d \geq 3$. Choose the Strichartz pair $(p,q) = (\frac{d(\alpha+1)}{d+\alpha-1}, \frac{4(\alpha+1)}{(d-2)(\alpha-1)})$, set $\mathcal{X} = C([0,T];L^2) \cap L^q(0,T;L^p)$, $\mathcal{Y} = C([0,T];H^1) \cap L^q(0,T;W^{1,p})$, and consider the integral operator

$$F(y)(t) = U(t,0)x - \lambda i \int_0^t U(t,s)(e^{(\alpha-1)ReW(s)} g(y(s)))ds,\ t \in [0,T], \tag{2.13}$$

defined for $y \in \mathcal{Y}$.

We claim that

$$F(\mathcal{Y}) \subseteq \mathcal{Y}. \tag{2.14}$$

In fact, by the Strichartz estimates in Lemma 2.7

$$\|F(y)\|_{L^q(0,T;W^{1,p})} \leq C_T \left[|x|_{H^1} + \|e^{(\alpha-1)ReW} g(y)\|_{L^{q'}(0,T;W^{1,p'})} \right]. \tag{2.15}$$

To estimate the right-hand side, we have that

$$\begin{aligned} &\|e^{(\alpha-1)ReW} g(y)\|_{L^{q'}(0,T;W^{1,p'})} \\ \leq &D_1(T) \left(\||y|^{\alpha-1}y\|_{L^{q'}(0,T;L^{p'})} + \||y|^{\alpha-1}|\nabla y|\|_{L^{q'}(0,T;L^{p'})} \right), \end{aligned} \tag{2.16}$$

where in the last inequality we have used $|\nabla g(y)| \leq \alpha |y|^{\alpha-1}|\nabla y|$, $|\nabla(e^{(\alpha-1)ReW}g(y))| \leq |e^{(\alpha-1)W}|\,[(\alpha-1)|\nabla W||g(y)| + |\nabla g(y)|]$ and $D_1(T) := \alpha(|\nabla W|_{L^\infty(0,T;L^\infty)} + 2)e^{(\alpha-1)|W|_{L^\infty(0,T;L^\infty)}}$.

With our choice of (p,q), it is easy to verify that $(\frac{1}{p'}, \frac{\alpha}{q}) = (\alpha-1)(\frac{1}{(\alpha-1)l}, \frac{1}{q}) + (\frac{1}{p}, \frac{1}{q})$, where $\frac{1}{l} = \frac{1}{p'} - \frac{1}{p}$, satisfying $\frac{1}{(\alpha-1)l} = \frac{1}{p} - \frac{1}{d}$. Hence, from Hölder's inequality and the Sobolev imbedding $|y|_{L^{(\alpha-1)l}} \leq D|y|_{W^{1,p}}$ it follows that

$$\begin{aligned} \||y|^{\alpha-1}y\|_{L^{q'}(0,T;L^{p'})} \leq & T^\theta \||y|^{\alpha-1}y\|_{L^{\frac{q}{\alpha}}(0,T;L^{p'})} \\ \leq & T^\theta \|y\|^{\alpha-1}_{L^q(0,T;L^{(\alpha-1)l})}\|y\|_{L^q(0,T;L^p)} \\ \leq & D^{\alpha-1}T^\theta \|y\|^{\alpha-1}_{L^q(0,T;W^{1,p})}\|y\|_{L^q(0,T;L^p)}, \end{aligned} \tag{2.17}$$

with $\theta = \frac{1}{q'} - \frac{\alpha}{q} \geq 0$, and also

$$\||y|^{\alpha-1}|\nabla y|\|_{L^{q'}(0,T;L^{p'})} \leq D^{\alpha-1}T^\theta \|y\|^{\alpha-1}_{L^q(0,T;W^{1,p})}\|\nabla y\|_{L^q(0,T;L^p)}. \tag{2.18}$$

Thus, inserting (2.17), (2.18) into (2.16) and (2.15) yields that for $y \in \mathcal{Y}$

$$\|F(y)\|_{L^q(0,T;W^{1,p})} \leq C_T \left[|x|_{H_1} + D_2(T)T^\theta \|y\|^\alpha_{L^q(0,T;W^{1,p})}\right], \tag{2.19}$$

with $D_2(T) = D_1(T)D^{\alpha-1}$. Similarly,

$$\|F(y)\|_{L^\infty(0,T;H^1)} \leq C_T \left[|x|_{H_1} + D_2(T)T^\theta \|y\|^\alpha_{L^q(0,T;W^{1,p})}\right]. \tag{2.20}$$

Hence (2.19) and (2.20) yield (2.14), as claimed.

We now start to construct the strong solutions of (2.5) by similar arguments as in [1].

Step 1. Fix $\omega \in \Omega$ and consider F on the set

$$\begin{aligned} \mathcal{Y}^{\tau_1}_{M_1} = & \{y \in C([0,\tau_1]; H^1) \cap L^q(0,\tau_1; W^{1,p}); \\ & \sup_{0\leq t\leq \tau_1} |y(t) - U(t,0)x|_{H^1} + \|y\|_{L^q(0,\tau_1;W^{1,p})} \leq M_1\}, \end{aligned}$$

where $\tau_1 = \tau_1(\omega) \in (0,T]$ and $M_1 = M_1(\omega) > 0$ are random variables.

For $y \in \mathcal{Y}^{\tau_1}_{M_1}$ by estimates (2.19) and (2.20)

$$\|F(y) - U(\cdot,0)x\|_{L^\infty(0,\tau_1;H^1)} + \|F(y)\|_{L^q(0,\tau_1;W^{1,p})} \leq \varepsilon_1(\tau_1) + 2C_{\tau_1}D_2(\tau_1)\tau_1^\theta M_1^\alpha,$$

where $\varepsilon_1(t) := \|U(\cdot,0)x\|_{L^q(0,t;W^{1,p})}$ is $(\mathscr{F}_t)$-adapted. By Lemma 2.7, $\varepsilon_1(t) = \|\mathbb{1}_{(0,t)}(\cdot)U(\cdot,0)x\|_{L^q(0,T;W^{1,p})} \leq C_T|x|_{H^1} < \infty$, and $\mathbb{1}_{(0,t)}(\cdot)U(\cdot,0)x \to 0$, as $t \to 0^+$. This implies

$$\varepsilon_1(t) \to 0, \quad as\ t \to 0^+$$

In order to obtain $F(\mathcal{Y}^{\tau_1}_{M_1}) \subset \mathcal{Y}^{\tau_1}_{M_1}$, we shall choose M_1 and τ_1 in such a way that

$$\varepsilon_1(\tau_1) + 2C_{\tau_1}D_2(\tau_1)\tau_1^\theta M_1^\alpha \leq M_1.$$

To this end, we define the real-valued continuous, $(\mathscr{F}_t)$-adapted process

$$Z^{(1)}_t = 2^\alpha C_t D_2(t)\varepsilon_1^{\alpha-1}(t)t^\theta, \quad t \in [0,T],$$

choose the $(\mathscr{F}_t)$-stopping time

$$\tau_1 = \inf\left\{t \in [0,T], Z^{(1)}_t > \frac{1}{2}\right\} \wedge T$$

and set $M_1 = 2\varepsilon_1(\tau_1)$. Then it follows that $Z^{(1)}_{\tau_1} \leq \frac{1}{2}$ and $F(\mathcal{Y}^{\tau_1}_{M_1}) \subset \mathcal{Y}^{\tau_1}_{M_1}$.

Moreover, the estimates as in the proof of (2.19) show that for $y_1, y_2 \in \mathcal{Y}^{\tau_1}_{M_1}$

$$\begin{aligned}
&\|F(y_1) - F(y_2)\|_{L^\infty(0,\tau_1;L^2)} + \|F(y_1) - F(y_2)\|_{L^q(0,\tau_1;L^p)} \\
\leq& 2C_{\tau_1}\|\lambda e^{(\alpha-1)ReW}(g(y_1) - g(y_2))\|_{L^{q'}(0,\tau_1;L^{p'})} \\
\leq& C_{\tau_1}D_1(\tau_1)\|(|y_1|^{\alpha-1} + |y_2|^{\alpha-1})|y_1 - y_2|\|_{L^{q'}(0,\tau_1;L^{p'})} \\
\leq& C_{\tau_1}D_1(\tau_1)D^{\alpha-1}\tau_1^\theta\left(\|y_1\|^{\alpha-1}_{L^q(0,\tau_1;W^{1,p})} + \|y_2\|^{\alpha-1}_{L^q(0,\tau_1;W^{1,p})}\right)\|y_1 - y_2\|_{L^q(0,\tau_1;L^p)} \\
\leq& 2C_{\tau_1}D_2(\tau_1)M_1^{\alpha-1}\tau_1^\theta\|y_1 - y_2\|_{L^q(0,\tau_1;L^p)} \\
\leq& \frac{1}{2}\|y_1 - y_2\|_{L^q(0,\tau_1;L^p)},
\end{aligned} \tag{2.21}$$

which implies that F is a contraction in $C([0,\tau_1];L^2) \cap L^q(0,\tau_1;L^p)$.

Since $\mathcal{Y}^{\tau_1}_{M_1}$ is a complete metric subspace in $C([0,\tau_1];L^2) \cap L^q(0,\tau_1;L^p)$, Banach's fixed point theorem yields a unique $y \in \mathcal{Y}^{\tau_1}_{M_1}$ with $y = F(y)$ on $[0,\tau_1]$.

Consequently, setting $y_1(t) := y(t \wedge \tau_1)$, $t \in [0,T]$, and using similar arguments as in the proof of Step 1 in Lemma 4.2 in [1], we deduce that (y_1, τ_1) is a strong solution of (2.5), such that $y_1(t) = y_1(t \wedge \tau_1)$, $t \in [0,T]$, and $y_1|_{[0,\tau_1]} \in C([0,\tau_1];H^1) \cap L^q(0,\tau_1;W^{1,p})$.

Step 2. Suppose that at the n^{th} step we have a strong solution (y_n, τ_n) of (2.5), such that $\tau_n \geq \tau_{n-1}$, $y_n(t) = y_n(t \wedge \tau_n)$, $t \in [0,T]$, and $y_n|_{[0,\tau_n]} \in C([0,\tau_n]; H^1) \cap L^q(0,\tau_n; W^{1,p})$.

Set

$$\begin{aligned}\mathcal{Y}^{\sigma_n}_{M_{n+1}} =&\{z \in C([0,\sigma_n]; H^1) \cap L^q(0,\sigma_n; W^{1,p});\\ &\sup_{0\leq t\leq \sigma_n} |z(t) - U(t+\tau_n, \tau_n)y_n(\tau_n)|_{H^1} + \|z\|_{L^q(0,\sigma_n;W^{1,p})} \leq M_{n+1}\},\end{aligned}$$

and define the integral operator F_n on $\mathcal{Y}$ by

$$F_n(z)(t) = U(\tau_n + t, \tau_n)y_n(\tau_n) - \lambda i \int_0^t U(\tau_n + t, \tau_n + s)e^{(\alpha-1)ReW(\tau_n+s)}g(z(s))ds,$$
$$t \in [0,T], \quad z \in \mathcal{Y}. \tag{2.22}$$

Analogous calculations as in Step 1 show that for $z \in \mathcal{Y}^{\sigma_n}_{M_{n+1}}$

$$\begin{aligned}&\|F_n(z) - U(\cdot + \tau_n, \tau_n)y_n(\tau_n)\|_{L^\infty(0,\sigma_n;H^1)} + \|F_n(z)\|_{L^q(0,\sigma_n;W^{1,p})}\\ \leq&\varepsilon_{n+1}(\sigma_n) + 2C_{\tau_n+\sigma_n}D_2(\tau_n+\sigma_n)\sigma_n^\theta M^\alpha_{n+1},\end{aligned}$$

and for $z_1, z_2 \in \mathcal{Y}^{\sigma_n}_{M_{n+1}}$

$$\begin{aligned}&\|F(z_1) - F(z_2)\|_{L^\infty(0,\sigma_n;L^2)} + \|F(z_1) - F(z_2)\|_{L^q(0,\sigma_n;L^p)}\\ \leq&2C_{\tau_n+\sigma_n}D_2(\tau_n+\sigma_n)M^{\alpha-1}_{n+1}\sigma_n^\theta\|z_1 - z_2\|_{L^q(0,\sigma_n;L^p)}.\end{aligned}$$

where $\varepsilon_{n+1}(t) := \|U(\tau_n + \cdot, \tau_n)y_n(\tau_n)\|_{L^q(0,t;W^{1,p})}$ is $(\mathscr{F}_{\tau_n+t})$-adapted and

$$\varepsilon_{n+1}(t) \to 0, \quad as\ t \to 0.$$

Similarly, we define the continuous $(\mathscr{F}_{\tau_n+t})$-adapted process

$$Z^{(n)}_t := 2^\alpha C_{\tau_n+t}D_2(\tau_n+t)\varepsilon^{\alpha-1}_{n+1}(t)t^\theta,\ t \in [0,T],$$

set

$$\sigma_n = \inf\left\{t \in [0, T-\tau_n] : Z^{(n)}_t > \frac{1}{2}\right\} \wedge (T - \tau_n)$$

and choose $M_{n+1} = 2\varepsilon_{n+1}(\sigma_n)$. It follows that $F_n(\mathcal{Y}^{\sigma_n}_{M_{n+1}}) \subset \mathcal{Y}^{\sigma_n}_{M_{n+1}}$ and F_n is a contraction in $C([0,\sigma_n]; L^2) \cap L^q(0,\sigma_n; L^p)$. Hence, because $\mathcal{Y}^{\sigma_n}_{M_{n+1}}$ is a complete metric subspace in $C([0,\sigma_n]; L^2) \cap L^q(0,\sigma_n; L^p)$, Banach's fixed point theorem implies that there is a unique $z_{n+1} \in \mathcal{Y}^{\sigma_n}_{M_{n+1}}$ such that $z_{n+1} = F_n(z_{n+1})$ on $[0, \sigma_n]$.

Then, set $\tau_{n+1} = \tau_n + \sigma_n$ and define

$$y_{n+1}(t) = \begin{cases} y_n(t), & t \in [0, \tau_n]; \\ z_{n+1}((t - \tau_n) \wedge \sigma_n), & t \in (\tau_n, T]. \end{cases}$$

It follows from the definitions of F and F_n that $y_{n+1} = F(y_{n+1})$ on $[0, \tau_{n+1}]$, implying y_{n+1} is a solution to (2.5) on $[0, \tau_{n+1}]$. Moreover, using similar arguments as in the proof of Step 2 in Lemma 4.2 and of Lemma 6.2 in [1], we deduce that τ_{n+1} is an $(\mathscr{F}_t)$-stopping time and y_{n+1} is adapted to $(\mathscr{F}_t)$ in H^1. Hence, (y_{n+1}, τ_{n+1}) is a strong solution of (2.5), such that $y_{n+1}(t) = y_{n+1}(t \wedge \tau_{n+1})$, $t \in [0, T]$, and $y_{n+1}|_{[0,\tau_{n+1}]} \in C([0, \tau_{n+1}]; H^1) \cap L^q(0, \tau_{n+1}; W^{1,p})$.

Step 3. Starting from Step 1 and repeating the procedure in Step 2, we finally construct a sequence of strong solutions (y_n, τ_n), $n \in \mathbb{N}$, where τ_n are increasing stopping times and $y_{n+1} = y_n$ on $[0, \tau_n]$.

The integrability property $y \in L^\gamma(0, \tau_n; W^{1,\rho})$ for any Strichartz pair (ρ, γ) follows easily from Lemma 2.7 and the estimate (2.19).

To prove the uniqueness, for any two strong solutions $(\widetilde{y}_i, \sigma_i)$, $i = 1, 2$, define $\varsigma = \sup\{t \in [0, \sigma_1 \wedge \sigma_2] : \widetilde{y}_1 = \widetilde{y}_2 \ on\ [0, t]\}$. Suppose that $\mathbb{P}(\varsigma < \sigma_1 \wedge \sigma_2) > 0$. For $\omega \in \{\varsigma < \sigma_1 \wedge \sigma_2\}$, we have $\widetilde{y}_1(\omega) = \widetilde{y}_2(\omega)$ on $[0, \varsigma(\omega)]$ by the continuity in H^1, and for $t \in [0, \sigma_1 \wedge \sigma_2(\omega) - \varsigma(\omega))$

$$\begin{aligned} &\|\widetilde{y}_1(\omega) - \widetilde{y}_2(\omega)\|_{L^q(\varsigma(\omega),\varsigma(\omega)+t;L^p)} \\ \leq &2C_{\varsigma(\omega)+t} D_2(\varsigma(\omega) + t) \widetilde{M}(t) t^\theta \|\widetilde{y}_1(\omega) - \widetilde{y}_2(\omega)\|_{L^q(\varsigma(\omega),\varsigma(\omega)+t;L^p)}, \end{aligned}$$

where $\widetilde{M}(t) := \|\widetilde{y}_1(\omega)\|^{\alpha-1}_{L^q(\varsigma(\omega),\varsigma(\omega)+t;W^{1,p})} + \|\widetilde{y}_2(\omega)\|^{\alpha-1}_{L^q(\varsigma(\omega),\varsigma(\omega)+t;W^{1,p})} \to 0$ as $t \to 0$. Therefore, with t small enough we deduce that $\widetilde{y}_1(\omega) = \widetilde{y}_2(\omega)$ on $[\varsigma(\omega), \varsigma(\omega) + t]$, hence $\widetilde{y}_1(\omega) = \widetilde{y}_2(\omega)$ on $[0, \varsigma(\omega) + t]$, which contradicts the definition of ς.

Now, we are left with proving the blowup alternative. Let us consider the subcritical and critical cases respectively.

(i). The subcritical case $1 < \alpha < 1 + \frac{4}{d-2}$, $\theta > 0$: Suppose that $\mathbb{P}(M^* < \infty; \tau_n < \tau^*(x), \forall n \in \mathbb{N}) > 0$, where $M^* := \sup_{t \in [0, \tau^*(x))} |y(t)|_{H^1}$. Define

$$Z_t := 2^\alpha (M^*)^{\alpha-1} C_{T+t}^\alpha D_2(T+t) t^\theta, \ t \in [0, T],$$

and

$$\sigma := \inf \left\{ t \in [0, T] : Z_t > \frac{1}{4} \right\} \wedge T.$$

For $\omega \in \{M^* < \infty; \tau_n < \tau^*(x), \forall n \in \mathbb{N}\}$, since $\tau_n(\omega) < T$, $\forall n \in \mathbb{N}$, by the definition of σ_n in Step 2, we have

$$\sigma_n(\omega) = \inf \left\{ t \in [0, T - \tau_n(\omega)] : Z_t^{(n)}(\omega) > \frac{1}{2} \right\}.$$

Notice that, for every $n \geq 1$, $\varepsilon_{n+1}(t) \leq C_{\tau_n + t} M^*$ due to the Strichartz estimate (2.11). Moreover, $|y(\tau_n(\omega))|_{H^1} \leq M^*$, $C_{\tau_n(\omega)+t} \leq C_{T+t}$ and $D_2(\tau_n(\omega) + t) \leq D_2(T+t)$. It follows that $Z_t(\omega) \geq Z_t^{(n)}(\omega)$, therefore $\sigma_n(\omega) > \sigma(\omega) > 0$. Hence $\tau_{n+1}(\omega) = \tau_n(\omega) + \sigma_n(\omega) > \tau_n(\omega) + \sigma(\omega)$, which implies $\tau_{n+1}(\omega) > \tau_1(\omega) + n\sigma(\omega)$ for every $n \geq 1$, contradicting the fact that $\tau_n(\omega) \leq T$. Therefore, we have shown the blow-up alternative in the subcritical case.

(ii). The critical case $\alpha = 1 + \frac{4}{d-2}$ with $d \geq 3$, $\theta = 0$: We will adapt the arguments from [7] and [6]. Set $q_1 = \frac{2(d+2)}{d-2}$. Besides the Strichartz pair $(p, q) = (\frac{2d^2}{d^2-2d+4}, \frac{2d}{d-2})$, let us choose another Strichartz pair $(p_2, p_2) = (2 + \frac{4}{d}, 2 + \frac{4}{d})$. Then $\frac{1}{p_2'} = \frac{\alpha-1}{q_1} + \frac{1}{p_2}$.

Suppose that $\mathbb{P}(\|y\|_{L^{q_1}(0,\tau^*(x);L^{q_1})} < \infty;\ \tau_n < \tau^*(x), \forall n \in \mathbb{N}) > 0$. For $\omega \in \{\|y\|_{L^{q_1}(0,\tau^*(x);L^{q_1})} < \infty;\ \tau_n < \tau^*(x), \forall n \in \mathbb{N}\}$, we have $\sigma_n(\omega) = \inf\{t \in [0, T - \tau_n(\omega)]; Z_t^{(n)}(\omega) > \frac{1}{2}\}$ and $Z_{\sigma_n(\omega)}^{(n)}(\omega) = \frac{1}{2}$. For convenience, we omit the dependence on ω below.

From the definition of F_n and the construction of y, one can check that for every $n \geq 1$ and $t \in [0, \tau^*(x) - \tau_n)$

$$y(\tau_n + t) = U(\tau_n + t, \tau_n) y(\tau_n) - \lambda i \int_{\tau_n}^{\tau_n + t} U(\tau_n + t, s) e^{(\alpha-1)ReW(s)} g(y(s)) ds.$$

Then by Lemma 2.7 and Höder's inequality, for every $n \geq 1$, $t \in [\tau^*(x) - \tau_n)$

$$\begin{aligned}\|y\|_{L^{p_2}(\tau_n,\tau_n+t;W^{1,p_2})} \leq & C_T |y(\tau_n)|_{H^1} + C_T \|e^{(\alpha-1)ReW(s)} g(y(s))\|_{L^{p_2'}(\tau_n,\tau_n+t;W^{1,p_2'})} \\ \leq & C_T |y(\tau_n)|_{H^1} + C_T D_1(T) \|y\|_{L^{q_1}(\tau_n,\tau^*(x);L^{q_1})}^{\alpha-1} \|y\|_{L^{p_2}(\tau_n,\tau_n+t;W^{1,p_2})}.\end{aligned}$$

Since $\|y\|_{L^{q_1}(0,\tau^*(x);L^{q_1})} < \infty$, we have $\|y\|_{L^{q_1}(\tau_n,\tau^*(x);L^{q_1})} \to 0$ as $n \to \infty$. Hence, choosing n large enough, such that $C_T D_1(T)\|y\|^{\alpha-1}_{L^{q_1}(\tau_n,\tau^*(x);L^{q_1})} < \frac{1}{2}$, we have for $t \in [0, \tau^*(x) - \tau_n)$, $\|y\|_{L^{p_2}(\tau_n,\tau_n+t;W^{1,p_2})} \leq 2C_T|y(\tau_n)|_{H^1}$, yielding

$$\|y\|_{L^{p_2}(0,\tau^*(x);W^{1,p_2})} < \infty.$$

Therefore,

$$\begin{aligned}\|y\|_{L^q(0,\tau^*(x);W^{1,p})} \leq & C_T|x|_{H^1} + C_T\|e^{(\alpha-1)ReW(s)}g(y(s))\|_{L^{p_2'}(0,\tau^*(x);W^{1,p_2'})} \\ \leq & C_T|x|_{H^1} + C_T D_1(T)\|y\|^{\alpha-1}_{L^{q_1}(0,\tau^*(x);L^{q_1})}\|y\|_{L^{p_2}(0,\tau^*(x);W^{1,p_2})} < \infty.\end{aligned}$$

Now, we note that for every $n \geq 1$ and $t \in [0, \sigma_n]$

$$\epsilon_{n+1}(t) = \|U(\tau_n + \cdot, \tau_n)y(\tau_n)\|_{L^q(0,t;W^{1,p})} \leq \widetilde{M}_n^* + C_T D_2(T)(\widetilde{M}_n^*)^\alpha,$$

where

$$\widetilde{M}_n^*(\omega) := \|y(\omega)\|_{L^q(\tau_n(\omega),\tau^*(x)(\omega);W^{1,p})} \to 0,\ as\ n \to \infty.$$

Then we choose n large enough such that

$$\widetilde{Z}^{(n)}(\omega) := 2^\alpha C_T(\omega)D_2(T)(\omega)[\widetilde{M}_n^*(\omega) + C_T(\omega)D_2(T)(\omega)(\widetilde{M}_n^*)^\alpha(\omega)]^{\alpha-1} < \frac{1}{6}.$$

But this implies $\frac{1}{6} > \widetilde{Z}^{(n)}(\omega) > Z_t^{(n)}(\omega)$ for any $t \in [0, \sigma_n(\omega)]$, in particular, $\frac{1}{6} > \widetilde{Z}^{(n)}(\omega) > Z^{(n)}_{\sigma_n(\omega)}(\omega) = \frac{1}{2}$, yielding a contradiction. Therefore, we have proved the blowup alternative in the critical case and completed the proof of Proposition 2.5 for the case $d \geq 3$.

For the case $d = 1, 2$, we modify the Strichartz pair (p, q) by $p = \alpha + 1$ and $q = \frac{4(\alpha+1)}{d(\alpha-1)}$. Note that $(\frac{1}{p'}, \frac{1}{q}) = (\alpha - 1)(\frac{1}{p}, 0) + (\frac{1}{p}, \frac{1}{q})$ and $2 < p < \infty$. Hölder's inequality and Sobolev's imbedding $|y|_{L^p} \leq D|y|_{H^1}$ give

$$\||y|^{\alpha-1}y\|_{L^{q'}(0,T;L^{p'})} \leq D^{\alpha-1}T^\theta\|y\|^{\alpha-1}_{L^\infty(0,T;H^1)}\|y\|_{L^q(0,T;L^p)}, \tag{2.23}$$

where $\theta = 1 - \frac{2}{q} > 0$, and

$$\||y|^{\alpha-1}\nabla y\|_{L^{q'}(0,T;L^{p'})} \leq D^{\alpha-1}T^\theta\|y\|^{\alpha-1}_{L^\infty(0,T;H^1)}\|\nabla y\|_{L^q(0,T;L^p)}. \tag{2.24}$$

Hence the estimates (2.19) and (2.20) are accordingly modified by

$$\|F(y)\|_{L^q(0,T;W^{1,p})} \le C_T \left[|x|_{H_1} + D_2(T)T^\theta \|y\|^{\alpha-1}_{L^\infty(0,T;H^1)} \|y\|_{L^q(0,T;W^{1,p})} \right], \tag{2.25}$$

and

$$\|F(y)\|_{L^\infty(0,T;H^1)} \le C_T \left[|x|_{H_1} + D_2(T)T^\theta \|y\|^{\alpha-1}_{L^\infty(0,T;H^1)} \|y\|_{L^q(0,T;W^{1,p})} \right]. \tag{2.26}$$

Similarly to (2.21), we get

$$\begin{aligned} &\|F(y_1) - F(y_2)\|_{L^\infty(0,T;L^2)} + \|F(y_1) - F(y_2)\|_{L^q(0,T;L^p)} \\ \le& C_T D_2(T) T^\theta \left(\|y_1\|^{\alpha-1}_{L^\infty(0,T;H^1)} + \|y_2\|^{\alpha-1}_{L^\infty(0,T;H^1)} \right) \|y_1 - y_2\|_{L^q(0,T;L^p)}. \end{aligned} \tag{2.27}$$

Therefore, similar arguments as those after (2.19) and (2.20) yield the asserted results in the case $d = 1, 2$. This completes the proof of Proposition 2.5. □

From the blowup alternative in Theorem 2.1 and Remark 2.2, we see that global existence follows from an a priori estimate for the energy, which will be derived from the Hamiltonian in the next section.

3 A priori estimate of the energy

Define the Hamiltonian H

$$H(u) = \frac{1}{2} \int |\nabla u|^2 d\xi - \frac{\lambda}{\alpha+1} \int |u|^{\alpha+1} d\xi, \quad u \in H^1, \tag{3.1}$$

for $1 < \alpha < 1 + \frac{4}{(d-2)_+}$, $d \ge 1$. Note that H is well defined by the Sobolev imbedding theorem.

Let X, $\tau^*(x)$ be as in Theorem 2.1. The evolution formula for $H(X)$ is given in Theorem 3.1 below.

Theorem 3.1 *Let α satisfy* (1.3)*. Set $\phi_j = \mu_j e_j$, $j = 1, ..., N$. Then $\mathbb{P}$-a.s*

$$\begin{aligned}
&H(X(t)) \\
=&H(x) + \int_0^t Re\,\langle -\nabla(\mu X(s)), \nabla X(s)\rangle_2\, ds + \frac{1}{2}\sum_{j=1}^N \int_0^t |\nabla(X(s)\phi_j)|_2^2 ds \\
&- \frac{1}{2}\lambda(\alpha-1)\sum_{j=1}^N \int_0^t \int (Re\phi_j)^2 |X(s)|^{\alpha+1} d\xi ds \\
&+ \sum_{j=1}^N \int_0^t Re\,\langle \nabla(\phi_j X(s)), \nabla X(s)\rangle_2\, d\beta_j(s) \\
&- \lambda \sum_{j=1}^N \int_0^t \int Re\phi_j |X(s)|^{\alpha+1} d\xi d\beta_j(s), \quad 0 \le t < \tau^*(x).
\end{aligned}$$

Remark 3.2 *In the deterministic case $\mu_j = 0$, $1 \le j \le N$, the Hamiltonian is conserved, i.e. $H(X(t)) = H(x)$. In the stochastic conservative case $\mu_j = -i\widetilde{\mu}_j$, $\widetilde{\mu}_j \in \mathbb{R}$, $1 \le j \le N$, the above evolution formula for $H(X(t))$ coincides with* (4.26) *in [4].*

Proof. This formula follows heuristically by applying Itô's formula to the integrands in $H(X(t))$ with the variable ξ fixed and then integrating over $\mathbb{R}^d$. But the spaces L^2, L^p consist of equivalent classes of functions, the delicate problem here is to find a suitable version such that for every ξ fixed, $(X(t,\xi))_{t\in[0,T]}$ is a continuous semimartingale, which may not exist. Therefore, we proceed by approximation to give a rigorous proof.

We introduce the operators Θ_m, $m \in \mathbb{N}$, used in [4] and defined for any $f \in \mathcal{S}$ by

$$\Theta_m f := \left(\theta(\frac{|\cdot|}{m})\right)^\vee * f \quad (= m^d \theta^\vee(m\cdot) * f),$$

where $\theta \in C_c^\infty$ is real-valued, nonnegative and $\theta(x) = 1$ for $|x| \le 1$, $\theta(x) = 0$ for $|x| > 2$.

By Hausdorf-Young's inequality, since $\int \theta^\vee d\xi = 1$, we have for any $p \in [1, \infty)$

$$\|\Theta_m\|_{L^p \to L^p} \le C, \tag{3.2}$$

where $C=C(p)$ is independent of m and

$$\Theta_m f \to f \quad in\ L^p,\ as\ m\to\infty. \tag{3.3}$$

Moreover, for any $f\in L^{\frac{\alpha+1}{\alpha}}$ we have

$$\Theta_m f \in L^{\alpha+1}, \tag{3.4}$$

$$Re\int if(\xi)\overline{\Theta_m f(\xi)}d\xi = 0. \tag{3.5}$$

(See the Appendix for the proof.)

Consider the approximating equation

$$\begin{aligned} &idX_m = \Delta X_m dt - i\mu X_m dt + \lambda\Theta_m(g(X_m))dt + iX_m dW, t\in(0,T),\\ &X_m(0)=x, \end{aligned} \tag{3.6}$$

where $g(X_m)=|X_m|^{\alpha-1}X_m$. Since the bound in (3.2) is independent of m, the arguments in the proof of Proposition 2.5 show that there exist unique strong solutions $(X_{m,n},\tau_n)$ of (3.6), $n\in\mathbb{N}$, where τ_n are increasing stopping times, independent of m. Define

$$X_m := \lim_{n\to\infty} X_{m,n}\mathbb{1}_{[0,\tau^*(x))} \tag{3.7}$$

with $\tau^*(x)=\lim_{n\to\infty}\tau_n$. Set $q=\frac{4(\alpha+1)}{d(\alpha-1)}$. We have $\mathbb{P}-a.s.$

$$R(t) := \sup_{m\geq 1}(\|X_m\|_{C([0,t];H^1)} + \|X_m\|_{L^q(0,t;W^{1,\alpha+1})}) < \infty, \quad t<\tau^*(x). \tag{3.8}$$

Moreover, it follows from Lemma 5.1 and Lemma 5.2 in Section 5 that

$$
\begin{aligned}
&H(X_m(t))\\
=&H(x)+\int_0^t Re\left\langle -\nabla(\mu X_m),\nabla(X_m)\right\rangle_2 dt+\frac{1}{2}\sum_{j=1}^N\int_0^t|\nabla(X_m(s)\phi_j)|_2^2ds\\
&-\frac{1}{2}\lambda(\alpha-1)\sum_{j=1}^N\int_0^t\int(Re\phi_j)^2|X_m(s)|^{\alpha+1}d\xi ds\\
&-\lambda\int_0^t Re\int i\nabla[(\Theta_m-1)g(X_m)]\nabla\overline{X_m}d\xi ds \qquad (3.9)\\
&+\sum_{j=1}^N\int_0^t Re\left\langle\nabla(\phi_j X_m(s)),\nabla(X_m(s))\right\rangle_2 d\beta_j(s)\\
&-\lambda\sum_{j=1}^N\int_0^t\int Re\phi_j|X_m(s)|^{\alpha+1}d\xi d\beta_j(s).
\end{aligned}
$$

In order to pass to the limit in (3.9), we note that $\mathbb{P}$-a.s. for $t<\tau^*(x)$

$$X_m\to X,\ in\ L^\infty(0,t;H^1)\cap L^q(0,t;W^{1,\alpha+1}) \qquad (3.10)$$

(see Section 5 for the proof).

Let us consider the fifth term in the right hand side of (3.9) for example. We will show that $\mathbb{P}$-a.s. for $t<\tau^*(x)$

$$\lambda\int_0^t Re\int i\nabla[(\Theta_m-1)g(X_m)]\nabla\overline{X_m}d\xi ds\to 0,\ as\ m\to\infty. \qquad (3.11)$$

Indeed, because of (3.10) it suffices to show that $\mathbb{P}$-a.s.

$$\nabla[(\Theta_m-1)g(X_m)]\to 0,\ in\ L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}}) \qquad (3.12)$$

for $t<\tau^*(x)$. We note that by (3.2)

$$
\begin{aligned}
&\|\nabla[(\Theta_m-1)g(X_m)]\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})}\\
\le&\|(\Theta_m-1)(\nabla g(X_m)-\nabla g(X))\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})}+\|(\Theta_m-1)\nabla g(X)\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})}\\
\le&C\|\nabla g(X_m)-\nabla g(X)\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})}+\|(\Theta_m-1)\nabla g(X)\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})},
\end{aligned}
$$

where C is independent of m. Using the arguments after (4.8) below we deduce that the first term tends to 0. Moreover, the second term also converges to 0, due to (3.3) and (3.2). Therefore, we obtain (3.11), as claimed.

One easily verifies that we can also take the limit for the remaining terms in (3.9) using (3.10). Consequently, we complete the proof. □

We next prove the a priori estimate of the energy in Theorem 3.6 below. Before that, let us first state and prove some technical lemmas.

Lemma 3.3 *Let $Y \geq 0$ be a real-valued progressively measurable process. We have*

$$\mathbb{E}\left(\int_0^t Y^2(s)ds\right)^{\frac{1}{2}} \leq \epsilon \mathbb{E}\sup_{s\leq t} Y(s) + C_\epsilon \int_0^t \mathbb{E}\sup_{r\leq s} Y(r)ds.$$

Proof. This lemma follows easily from the fact that $\int_0^t Y(s)^2 ds \leq \sup_{s\leq t} Y(s) \int_0^t Y(s)ds$ and Cauchy's inequality. □

Lemma 3.4 *For $1 < \alpha < 1 + \frac{4}{d}$, $d \geq 1$, we have*

$$|X|^{\alpha+1}_{L^{\alpha+1}} \leq C_\epsilon |X|_2^p + \epsilon |\nabla X|_2^2, \tag{3.13}$$

where $p > 2$.

Proof. From the Gagliardo-Nirenberg inequality it follows that $|X|^{\alpha+1}_{L^{\alpha+1}} \leq C|X|_2^\beta |\nabla X|_2^\gamma$, where $\beta = (1-\theta)(\alpha+1)$ and $\gamma = \theta(\alpha+1) \in (0,2)$ with $\theta = \frac{d(\alpha-1)}{2(\alpha+1)} \in (0,1)$. Then, (3.13) follows immediately from Young's inequality $ab \leq C_\epsilon a^\rho + \epsilon b^\delta$, $\frac{1}{\rho} + \frac{1}{\delta} = 1$, by choosing $\gamma\delta = 2$. □

Unlike in the conservative case, $|X(t)|_2^2$ is no longer independent of t, but a general martingale (see Lemma 4.3 in [1]). After applying Lemma 3.4 to control $|X(t)|^{\alpha+1}_{L^{\alpha+1}}$, we also need Lemma 3.5 below to bound the p-power of $|X(t)|_2$. Its proof is postponed to the Appendix.

Lemma 3.5 *Let $p \geq 2$. Then there exists $\widetilde{C}(T) < \infty$ such that*

$$\mathbb{E} \sup_{t\in[0,\tau^*(x))} |X(t)|_2^p \leq \widetilde{C}(T) < \infty.$$

With the above preliminaries, we are now ready to prove the main a priori estimate for the solution X given by Theorem 2.1.

Theorem 3.6 *Under condition* (1.3) *or* (1.4)*, there exists* $\widetilde{C}(T) < \infty$*, such that*

$$\mathbb{E}\left[\sup_{t\in[0,\tau^*(x))}\left(|\nabla X(t)|_2^2 + |X(t)|_{L^{\alpha+1}}^{\alpha+1}\right)\right] \le \widetilde{C}(T) < \infty. \tag{3.14}$$

Proof. (i) First assume that $\lambda = 1$. From the definition of H in (3.1) and Theorem 3.1, it follows that $\mathbb{P}$-a.s. for every $n \ge 1$ and $t \in [0,T]$

$$\begin{aligned}
&\frac{1}{2}|\nabla X(t\wedge\tau_n)|_2^2 \\
=&H(x) + \frac{1}{\alpha+1}|X(t\wedge\tau_n)|_{L^{\alpha+1}}^{\alpha+1} \\
&+ \int_0^{t\wedge\tau_n}\left[Re\left\langle -\nabla(\mu X(s)), \nabla X(s)\right\rangle_2 + \frac{1}{2}\sum_{j=1}^N |\nabla(X(s)\phi_j)|_2^2\right] ds \\
&- \frac{1}{2}(\alpha-1)\sum_{j=1}^N \int_0^{t\wedge\tau_n}\int (Re\phi_j)^2|X(s)|^{\alpha+1} d\xi ds \\
&+ \sum_{j=1}^N \int_0^{t\wedge\tau_n} Re\left\langle \nabla(\phi_j X(s)), \nabla X(s)\right\rangle_2 d\beta_j(s) \\
&- \sum_{j=1}^N \int_0^{t\wedge\tau_n}\int Re\phi_j |X(s)|^{\alpha+1} d\xi d\beta_j(s) \\
=&H(x) + \frac{1}{\alpha+1}|X(t\wedge\tau_n)|_{L^{\alpha+1}}^{\alpha+1} + J_1(t\wedge\tau_n) + J_2(t\wedge\tau_n) + J_3(t\wedge\tau_n) + J_4(t\wedge\tau_n),
\end{aligned} \tag{3.15}$$

where τ_n is as in Theorem 2.1 and $\phi_j = \mu_j e_j$, $1 \le j \le N$.

To estimate the second term in the right hand side of (3.15), we note that, from (3.13) and Lemma 3.5 it follows that

$$\begin{aligned}
\frac{1}{\alpha+1}\mathbb{E}\sup_{s\le t\wedge\tau_n}|X(s)|_{L^{\alpha+1}}^{\alpha+1} \le& \frac{1}{\alpha+1}C_\epsilon \mathbb{E}\sup_{s\le t\wedge\tau_n}|X(s)|_2^p + \epsilon\frac{1}{\alpha+1}\mathbb{E}\sup_{s\le t\wedge\tau_n}|\nabla X(s)|_2^2 \\
\le& \frac{1}{\alpha+1}C_\epsilon\widetilde{C}_T + \epsilon\frac{1}{\alpha+1}\mathbb{E}\sup_{s\le t\wedge\tau_n}|\nabla X(s)|_2^2.
\end{aligned} \tag{3.16}$$

Concerning $J_1(t\wedge\tau_n)$, we note that

$$J_1(t) \le C\int_0^t |\nabla X(s)|_2^2 + |X(s)|_2^2 ds,$$

where C depends on $|\phi_j|_\infty$ and $|\nabla\phi_j|_\infty$, $1\le j\le N$. Hence by Lemma 3.5

$$\mathbb{E}\sup_{s\le t\wedge\tau_n} |J_1(s)| \le C\widetilde{C}(T)t + C\int_0^t \mathbb{E}\sup_{r\le s\wedge\tau_n} |\nabla X(r)|_2^2 ds. \tag{3.17}$$

Moreover, since

$$\mathbb{E}\sup_{s\le t\wedge\tau_n} |J_2(s)| \le (\alpha-1)|\mu|_{L^\infty}\int_0^t \mathbb{E}\sup_{r\le s\wedge\tau_n} |X(r)|_{L^{\alpha+1}}^{\alpha+1} ds, \tag{3.18}$$

using the estimate (3.16) we have that

$$\begin{aligned}\mathbb{E}\sup_{s\le t\wedge\tau_n} |J_2(s)| \le &(\alpha-1)|\mu|_{L^\infty} C_\epsilon\widetilde{C}(T)t \\ &+ \epsilon(\alpha-1)|\mu|_{L^\infty}\int_0^t \mathbb{E}\sup_{r\le s\wedge\tau_n} |\nabla X(r)|_2^2 ds. \end{aligned}\tag{3.19}$$

For J_3, the Burkholder-Davis-Gundy inequality yields that

$$\begin{aligned}\mathbb{E}\sup_{s\le t\wedge\tau_n} |J_3(s)| &\le C\mathbb{E}\left[\int_0^{t\wedge\tau_n}\sum_{j=1}^N \left(Re\left\langle\nabla(\phi_j X(s)),\nabla X(s)\right\rangle_2\right)^2 ds\right]^{\frac{1}{2}} \\ &\le C\mathbb{E}\left(\int_0^{t\wedge\tau_n} |X(s)|_2^4 ds\right)^{\frac{1}{2}} + C\mathbb{E}\left(\int_0^{t\wedge\tau_n} |\nabla X(s)|_2^4 ds\right)^{\frac{1}{2}},\end{aligned}$$

where C depends on $|\phi_j|_\infty$, $|\nabla\phi_j|_\infty$, $1\le j\le N$. It follows from Lemma 3.3 with Y replaced by $|X(s)|_2^2$ and $|\nabla X(s)|_2^2$ respectively and Lemma 3.5 that

$$\begin{aligned}\mathbb{E}\sup_{s\le t\wedge\tau_n} |J_3(s)| \le &\epsilon C\widetilde{C}(T) + CC_\epsilon\widetilde{C}(T)t + \epsilon C\mathbb{E}\sup_{s\le t\wedge\tau_n} |\nabla X(s)|_2^2 \\ &+ CC_\epsilon\int_0^t \mathbb{E}\sup_{r\le s\wedge\tau_n} |\nabla X(r)|_2^2 ds. \end{aligned}\tag{3.20}$$

For the remaining term J_4, it follows similarly from the Burkholder-Davis-Gundy inequality and Lemma 3.3 with Y replaced by $|X|_{L^{\alpha+1}}^{\alpha+1}$ that

$$\begin{aligned}\mathbb{E}\sup_{s\leq t\wedge\tau_n}|J_4(s)| \leq & C\mathbb{E}\left[\int_0^{t\wedge\tau_n}\sum_{j=1}^{N}\left(\int Re\phi_j|X(s)|^{\alpha+1}d\xi\right)^2 ds\right]^{\frac{1}{2}} \\ \leq & C\mathbb{E}\left(\int_0^{t\wedge\tau_n}|X(s)|_{L^{\alpha+1}}^{2(\alpha+1)}ds\right)^{\frac{1}{2}} \\ \leq & \epsilon C\mathbb{E}\sup_{s\leq t\wedge\tau_n}|X(s)|_{L^{\alpha+1}}^{\alpha+1} + CC_\epsilon\int_0^t\mathbb{E}\sup_{r\leq s\wedge\tau_n}|X(r)|_{L^{\alpha+1}}^{\alpha+1}ds.\end{aligned} \tag{3.21}$$

Then (3.16) implies that

$$\begin{aligned}\mathbb{E}\sup_{s\leq t\wedge\tau_n}|J_4(s)| \leq & CC_\epsilon(\epsilon\widetilde{C}(T)+C_\epsilon\widetilde{C}(T)t)+\epsilon^2 C\mathbb{E}\sup_{s\leq t\wedge\tau_n}|\nabla X(s)|_2^2 \\ & +\epsilon CC_\epsilon\int_0^t\mathbb{E}\sup_{r\leq s\wedge\tau_n}|\nabla X(r)|_2^2 ds.\end{aligned} \tag{3.22}$$

Now, taking (3.16)-(3.22) into (3.15) and summing up the respective terms, we conclude that

$$\begin{aligned}\frac{1}{2}\mathbb{E}\sup_{s\leq t\wedge\tau_n}|\nabla X(s)|_2^2 \leq & C_1(T)+\epsilon C_2(T)\mathbb{E}\sup_{s\leq t\wedge\tau_n}|\nabla X(s)|_2^2 \\ & + C_3(T)\int_0^t\mathbb{E}\sup_{r\leq s\wedge\tau_n}|\nabla X(r)|_2^2 ds,\end{aligned}$$

where the constants $C_k(T)$, $1\leq k\leq 3$, depend on T, $H(x)$, α, $|\phi_j|_\infty$, $|\nabla\phi_j|_\infty$, $1\leq j\leq N$, and $\mathbb{E}\sup_{t\in[0,\tau^*(x))}|X(t)|_2^p$ with $p\geq 2$. Then, choosing a sufficiently small ϵ and using Gronwall's lemma, we obtain

$$\mathbb{E}\sup_{t\in[0,\tau_n]}|\nabla X(t)|_2^2\leq\widetilde{C}(T)<\infty.$$

Finally, taking $n\to\infty$ and appylying Fatou's lemma, we obtain

$$\mathbb{E}\sup_{t\in[0,\tau^*(x))}|\nabla X(t)|_2^2\leq\widetilde{C}(T)<\infty,$$

which implies (3.14) by (3.13) and Lemma 3.5.

(*ii*) In the defocusing case $\lambda = -1$, the positivity of the Hamiltonian simplifies many estimates in the previous case (*i*), without using Lemma 3.4 and Lemma 3.5, and the condition on α is less restrictive.

More precisely, taking (3.17), (3.18), (3.20) and (3.21) into Theorem 3.1 and summing up the respective terms, we derive that

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\sup_{s\leq t\wedge\tau_n}|\nabla X(s)|_2^2+\frac{1}{\alpha+1}\mathbb{E}\sup_{s\leq t\wedge\tau_n}|X(s)|_{L^{\alpha+1}}^{\alpha+1}\\
\leq &C_1(T)+\epsilon C_2(T)\ \mathbb{E}\sup_{s\leq t\wedge\tau_n}(|\nabla X(s)|_2^2+|X(s)|_{L^{\alpha+1}}^{\alpha+1})\\
&+C_3(T)\int_0^t\mathbb{E}\sup_{r\leq s\wedge\tau_n}(|\nabla X(r)|_2^2+|X(r)|_{L^{\alpha+1}}^{\alpha+1})ds,
\end{aligned}$$

where the constants $C_k(T)$, $1 \leq k \leq 3$, depend on T, $H(x)$, α, $|\phi_j|_\infty$, $|\nabla\phi_j|_\infty$, $1 \leq j \leq N$, and $\mathbb{E}\sup_{t\in[0,\tau^*(x))}|X(t)|_2^2$.

Therefore, similar arguments as at the end of the previous case yield (3.14). This completes the proof of Theorem 3.6. $\square$

4 Proof of Theorem 1.2.

By Lemma 3.5, Theorem 3.6 and the fact that $\|e^{-W}\|_{L^\infty(0,T;W^{1,\infty})} < \infty$, $\mathbb{P}$-a.s, it follows that

$$\sup_{0\leq t<\tau^*(x)}|y(t)|_{H^1}^2<\infty, a.s. \tag{4.1}$$

Therefore, $\tau^*(x) = T$, $\mathbb{P}$-a.s, due to the blowup alternative in Proposition 2.5 (see also Remark 2.2). Modifying the definition of y by $y := \lim_{n\to\infty} y_n$, we deduce that (y, T) is the unique strong solution of (2.5). Therefore, letting $X = e^W y$, we conclude that (X, T) is the desired unique strong solution of (1.1).

The integrability (1.6) follows from Lemma 3.5 and Theorem 3.6, and (1.7) follows from (2.8).

It remains to prove the continuous dependence on initial data. Again it is equivalent to prove this for the random equation (2.5), and by Lemma 2.7 we only need to show it for the Strichartz pair $(p, q) = (\alpha + 1, \frac{4(\alpha+1)}{d(\alpha-1)})$.

Suppose that $x_m \to x$ in H^1. Let (y_m, T) be the unique strong solutions of (2.5) corresponding to the initial data x_m, $m \geq 1$. Since $|x_m|_{H^1} \leq |x|_{H^1}+1$ for $m \geq m_1$ with m_1 large enough, we modify $\tau_1(\leq T)$ in the proof of Proposition 2.5 by

$$\tau_1 = \inf\{t \in [0,T] : 2^\alpha(|x|_{H^1}+1)^{\alpha-1}C_t^\alpha D_2(t)t^\theta > \frac{1}{2}\} \wedge T,$$

such that τ_1 is independent for $m \geq m_1$. Hence, the contraction arguments there and the uniqueness yield that

$$\widetilde{R} := \sup_{m \geq m_1} (\|y_m\|_{L^\infty(0,\tau_1;H^1)} + \|y_m\|_{L^q(0,\tau_1;W^{1,p})}) < \infty, \quad \mathbb{P}-a.s.$$

Let us first prove the continuous dependence on initial data on the interval $[0,\tau_1]$. Analogous calculations as in (2.21) show that

$$\begin{aligned} &\|y_m - y\|_{L^\infty(0,t;L^2)} + \|y_m - y\|_{L^q(0,t;L^p)} \\ &\leq 2C_T|x_m - x|_2 + 2C_T D_2(T)\widetilde{R}^{\alpha-1}t^\theta \|y_m - y\|_{L^q(0,t;L^p)}, \end{aligned} \tag{4.2}$$

where $\theta = 1 - \frac{2}{q} > 0$. Then taking t small and independent of $m(\geq m_1)$, we have

$$\|y_m - y\|_{L^\infty(0,t;L^2)} + \|y_m - y\|_{L^q(0,t;L^p)} \to 0, \ as \ m \to \infty. \tag{4.3}$$

In particular, it follows that

$$y_m \to y, \quad in \ measure \ dt \times d\xi, \quad as \ m \to \infty. \tag{4.4}$$

Next, to obtain that

$$\|y_m - y\|_{L^\infty(0,t;H^1)} + \|y_m - y\|_{L^q(0,t;W^{1,p})} \to 0, \tag{4.5}$$

we use equation (6.3) in the Appendix to derive that for $m \geq m_1$

$$\begin{aligned} \nabla(y_m - y) =& U(t,0)\nabla(x_m - x) + \int_0^t U(t,s)\Big\{ i(D_j\nabla\widetilde{b}^j + \nabla\widetilde{b}^j D_j + \nabla\widetilde{c})(y_m - y) \\ &- \lambda i \nabla\left[e^{(\alpha-1)ReW(s)}\left(g(y_m(s)) - g(y(s))\right)\right]\Big\} ds, \end{aligned} \tag{4.6}$$

where $g(y) = |y|^{\alpha-1}y$.

We note that, by Proposition 2.3(a) in [12] and (6.1) in the Appendix, using a similar estimate as in (2.23), we obtain

$$
\begin{aligned}
&\|i(D_j\nabla\widetilde{b}^j+\nabla\widetilde{b}^jD_j+\nabla\widetilde{c})(y_m-y)\|_{\widetilde{X}'_{[0,t]}}\\
\le&\kappa_T\|y_m-y\|_{\widetilde{X}_{[0,t]}}\\
\le&\kappa_T C_T|x_m-x|_2+\kappa_T C_T\|e^{(\alpha-1)ReW}(g(y_m)-g(y))\|_{L^{q'}(0,t;L^{p'})}\\
\le&C(T)|x_m-x|_2+C(T)t^\theta\|y_m-y\|_{L^q(0,t;L^p)},
\end{aligned}
\tag{4.7}
$$

where $\theta=1-\frac{2}{q}>0$, $\widetilde{X}_{[0,t]}$ is the local smoothing space defined in [12] and $C(T)$ depends on κ_T, C_T, $\|e^{(\alpha-1)W}\|_{L^\infty(0,T;L^\infty)}$ and $\widetilde{R}$.

Then, applying (6.1) to (4.6), we derive by (4.7) and a similar estimate as in (2.23) that

$$
\begin{aligned}
&\|\nabla y_m-\nabla y\|_{L^\infty(0,t;L^2)}+\|\nabla y_m-\nabla y\|_{L^q(0,t;L^p)}\\
\le&2C_T|\nabla x_m-\nabla x|_2+2C_T\|i(D_j\nabla\widetilde{b}^j+\nabla\widetilde{b}^jD_j+\nabla\widetilde{c})(y_m-y)\|_{\widetilde{X}'_{[0,t]}}\\
&+2C_T\|\lambda i\nabla[e^{(\alpha-1)ReW}(g(y_m)-g(y))]\|_{L^{q'}(0,t;L^{p'})}\\
\le&C(T)|x_m-x|_{H^1}+C(T)t^\theta\|y_m-y\|_{L^q(0,t;L^p)}\\
&+C(T)\|\nabla g(y_m)-\nabla g(y)\|_{L^{q'}(0,t;L^{p'})},
\end{aligned}
\tag{4.8}
$$

where C_T depends on κ_T, C_T, $\|e^{(\alpha-1)W}\|_{L^\infty(0,T;W^{1,\infty})}$ and $\widetilde{R}$.

As regards the last term in the right hand side of (4.8), we note that $\nabla g(y)=F_1(y)\nabla y+F_2(y)\nabla\overline{y}$, where $F_1(y)=\frac{\alpha+1}{2}|y|^{\alpha-1}$ and $F_2(y)=\frac{\alpha-1}{2}|y|^{\alpha-3}y^2$. Then

$$
\begin{aligned}
\nabla g(y_m)-\nabla g(y)=&F_1(y_m)[\nabla y_m-\nabla y]+[F_1(y_m)-F_1(y)]\nabla y\\
&+F_2(y_m)[\nabla\overline{y_m}-\nabla\overline{y}]+[F_2(y_m)-F_2(y)]\nabla\overline{y}\\
=&I_1+I_2+I_3+I_4.
\end{aligned}
\tag{4.9}
$$

Since $|I_1|+|I_3|\le\alpha|y_m|^{\alpha-1}|\nabla y_m-\nabla y|$, (2.24) yields

$$
\|I_1+I_3\|_{L^{q'}(0,t;L^{p'})}\le\alpha D^{\alpha-1}\widetilde{R}^{\alpha-1}t^\theta\|y_m-y\|_{L^q(0,t;W^{1,p})}. \tag{4.10}
$$

Thus plugging (4.9) and (4.10) into (4.8), together with (4.2), we derive that

$$
\begin{aligned}
&\|y_m-y\|_{L^\infty(0,t;H^1)}+\|y_m-y\|_{L^q(0,t;W^{1,p})}\\
\le&C(T)|x_m-x|_{H^1}+C(T)t^\theta\|y_m-y\|_{L^q(0,t;W^{1,p})}\\
&+C(T)\|I_2+I_4\|_{L^{q'}(0,t;L^{p'})}.
\end{aligned}
\tag{4.11}
$$

Therefore, choosing t small and independent of $m(\geq m_1)$, we deduce that (4.5) holds once we prove that

$$\|I_2 + I_4\|_{L^{q'}(0,t;L^{p'})} \to 0, \ \ as\ m \to \infty. \tag{4.12}$$

In order to prove (4.12), by (4.3) we have for dt-a.e. $s \in [0,t]$, as $m \to \infty$

$$|F_1(y_m(s))|_{L^{\frac{p}{p-2}}} \to |F_1(y(s))|_{L^{\frac{p}{p-2}}},$$

which, by (4.4), implies that for dt-a.e. $s \in [0,t]$

$$F_1(y_m(s)) \to F_1(y(s)), \ \ in\ L^{\frac{p}{p-2}},$$

then

$$[F_1(y_m(s)) - F_1(y(s))]\nabla y(s) \to 0, \ \ in\ L^{p'}.$$

Moreover, for dt-a.e. $s \in [0,t]$,

$$\begin{aligned} &|[F_1(y_m(s)) - F_1(y(s))]\nabla y(s)|_{L^{p'}} \\ \leq &\frac{\alpha+1}{2} D^{\alpha-1}(\|y_m\|^{\alpha-1}_{L^\infty(0,t;H^1)} + \|y\|^{\alpha-1}_{L^\infty(0,t;H^1)})\|y(s)\|_{W^{1,p}} \\ \leq &\frac{\alpha+1}{2} D^{\alpha-1}\widetilde{R}\ \|y(s)\|_{W^{1,p}} \quad \in L^{q'}(0,t), \end{aligned}$$

Thus, by Lebesgue's dominated convergence theorem, we obtain

$$\|I_2\|_{L^{q'}(0,t;L^{p'})} \to 0, \ \ as\ m \to \infty.$$

The proof for I_4 is similar. Therefore, we have proved (4.12) hence also (4.5) for t small enough and independent of $m(\geq m_1)$. Reiterating this procedure in finite steps we obtain (4.5) on $[0,\tau_1]$.

Now, since $y_m(\tau_1) \to y(\tau_1)$ in H^1, similarly we can extend the above results to $[0,\tau_2]$ with τ_2 depending on $|y(\tau_1)|_{H^1}$ and $\tau_1 \leq \tau_2 \leq T$. Reiterating this procedure, we then obtain increasing stopping times τ_n, $n \in \mathbb{N}$, depending on $|y(\tau_{n-1})|_{H^1}$, such that (4.5) holds on every $[0,\tau_n]$. Finally, as $\sup\limits_{t\in[0,T]} |y(t)|_2 < \infty$, $\mathbb{P}$-a.s, using the proof of the blowup alternative in Proposition 2.5, we deduce that for $\mathbb{P}$-a.e. ω there exists $n(\omega) < \infty$ such that $\tau_{n(\omega)}(\omega) = T$. This implies the continuous dependence on initial data on $[0,T]$ and consequently completes the proof of Theorem 1.2. □

5 Itô-formulae for L^p- and H^1- norms

This section contains the Itô-formulae for $|X_m(t)|^{\alpha+1}_{L^{\alpha+1}}$ and $|\nabla X_m(t)|_2^2$, as well as the asymptotic formula (3.10), which are used in the proof of Theorem 3.1 in Section 3.

Let us start with Itô's formula for $|X_m(t)|^{\alpha+1}_{L^{\alpha+1}}$. First, we note that Theorem 2.1 in [10] is not applicable here, as we do not have $X \in L^{\alpha+1}(0,t;W^{1,\alpha+1})$ and $|X|^{\alpha-1}X \in L^{\alpha+1}(0,t;L^{\alpha+1})$ from Theorem 2.1. However, for the nonlinearity in the approximating equation (3.6), by (3.4) and (3.5) we have $\Theta_m(g(X_m)) \in L^{\alpha+1}$ and $Re\int ig(X_m)\overline{\Theta_m(g(X_m))}d\xi = 0$, which allow to use the technique from [10] to obtain the Itô formula.

Let us adapt the same notation from [10]. Set $h^\epsilon = h * \psi_\epsilon$ for any locally integrable function h mollified by ψ_ϵ, where $\psi_\epsilon = \epsilon^{-d}\psi(\frac{x}{\epsilon})$ and $\psi \in C_c^\infty(\mathbb{R}^d)$ is a real-valued nonnegative function with unit integral. Recall that $|h^\epsilon|_{L^p} \leq |h|_{L^p}$ and if $h \in L^p$, then $h^\epsilon \to h$ in L^p as $\epsilon \to 0$, $p > 1$, which will be used in the later estimates.

Lemma 5.1 *Let X_m be as in* (3.7). *Set $p = \alpha + 1$ with $1 < \alpha < 1 + \frac{4}{(d-2)_+}$, $d \geq 1$. We have $\mathbb{P}$-a.s.*

$$\begin{aligned}
|X_m(t)|^p_{L^p} =&|x|^p_{L^p} - p\int_0^t Re\int i\nabla g(X_m)(s)\nabla\overline{X_m}(s)d\xi ds \\
&+ \frac{1}{2}p(p-2)\sum_{j=1}^N\int_0^t\int (Re\phi_j)^2|X_m(s)|^p d\xi ds \\
&+ p\sum_{j=1}^N\int_0^t\int Re\phi_j|X_m(s)|^p d\xi d\beta_j(s), \quad 0 \leq t < \tau^*(x).
\end{aligned} \tag{5.1}$$

Here $g(X_m) = |X_m|^{p-2}X_m$ and $\phi_j = \mu_j e_j$, $1 \leq j \leq N$.

Proof. By (3.6) we have $\mathbb{P}$-a.s. that

$$\begin{aligned}
X_m(t) =&x(t) + \int_0^t \left[-i\Delta X_m(s) - \mu X_m(s) - \lambda i g_m(s)\right] ds \\
&+ \int_0^t X_m(s)\phi_j d\beta_j(s), \qquad t < \tau^*(x),
\end{aligned} \tag{5.2}$$

where $g_m(s) = \Theta_m(g(X_m(s)))$, (5.2) is considered as an Itô equation in H^{-1} and we used the summation convention over repeated indices for simplicity.

Taking convolution of both sides of (5.2) with the mollifiers ψ_ϵ, we have for every $\xi \in \mathbb{R}^d$ that

$$(X_m(t))^\epsilon(\xi) = x^\epsilon(\xi) + \int_0^t [-i\Delta(X_m(s))^\epsilon(\xi) - (\mu X_m(s))^\epsilon(\xi) - \lambda i(g_m(s))^\epsilon(\xi)]\,ds \\ + \int_0^t (X_m(s)\phi_j)^\epsilon(\xi) d\beta_j(s), \quad t < \tau^*(x), \tag{5.3}$$

which holds on a set $\Omega_\xi \in \mathscr{F}$ with $\mathbb{P}(\Omega_\xi) = 1$.

In order to find $\widetilde{\Omega} \in \mathcal{F}$ with $\mathbb{P}(\widetilde{\Omega}) = 1$ such that (5.3) holds on $\widetilde{\Omega}$ for all $\xi \in \mathbb{R}^d$, we need the continuity in ξ of all terms in (5.3). Let us check this for the stochastic integral term in (5.3). Set $\sigma_{n,l} = \inf\{s \in [0, \tau_n] : |X_m(s)|_{H^1} > l\} \wedge \tau_n$. Since the function $\xi \to (X_m(s)\phi_j)^\epsilon(\xi)$ is continuous and

$$\mathbb{E}\Big|\sum_{j=1}^N \int_0^{t\wedge\sigma_{n,l}} (X_m(s)\phi_j)^\epsilon(\xi) d\beta_j(s)\Big|^2 \le (\sum_{j=1}^N |\phi_j|^2_{L^\infty})|\psi_\epsilon|_2^2 \; l^2 t < \infty,$$

it follows that $\xi \to \int_0^t (X_m(s)\phi_j)^\epsilon(\xi) d\beta_j(s)$ is continuous on $\{t \le \sigma_{n,l}\}$. But since $\sup_{t\in[0,\tau_n]} |X_m(t)|_{H^1} < \infty$, $\mathbb{P}$-a.s, for $\mathbb{P}$-a.e $\omega \in \Omega$ there exists $l(\omega) \in \mathbb{N}$ such that $\sigma_{n,l}(\omega) = \tau_n(\omega)$ for all $l \ge l(\omega)$. Therefore,

$$\bigcup_{l\in\mathbb{N}} \{t \le \sigma_{n,l}\} = \{t \le \tau_n\}, \tag{5.4}$$

implying that $\xi \to \int_0^t (X_m(s)\phi_j)^\epsilon(\xi) d\beta_j(s)$ is continuous on $\{t \le \tau_n\}$ hence on $\{t < \tau^*(x)\}$. One can also check the continuity in ξ for the drift terms in (5.3).

Therefore, we conclude that (5.3) holds on a full probability set $\widetilde{\Omega} \in \mathscr{F}$ and $\widetilde{\Omega}$ is independent of $\xi \in \mathbb{R}^d$.

Now, we set for simplicity $X_m^\epsilon(t) = (X_m(t))^\epsilon(\xi)$ and correspondingly for the respective other terms. Then by Itô's formula we have $\mathbb{P}$-a.s.

$$
\begin{aligned}
|X_m^\epsilon(t)|^p =&|x^\epsilon|^p - p\int_0^t Re(ig(\overline{X_m^\epsilon})(s)\Delta X_m^\epsilon(s))ds - p\int_0^t Re(g(\overline{X_m^\epsilon})(s)(\mu X_m)^\epsilon(s))ds \\
&- \lambda p\int_0^t Re(ig(\overline{X_m^\epsilon})(s)g_m^\epsilon(s))ds + \frac{p}{2}\int_0^t |X_m^\epsilon(s)|^{p-2}|(X_m\phi_j)^\epsilon(s)|^2 ds \\
&+ \frac{1}{2}p(p-2)\int_0^t |X_m^\epsilon(s)|^{p-4}[Re(\overline{X_m^\epsilon}(s)(X_m\phi_j)^\epsilon(s))]^2 ds \\
&+ p\int_0^t Re(g(\overline{X_m^\epsilon})(s)(X_m\phi_j)^\epsilon(s))d\beta_j(s), \quad t<\tau^*(x). \qquad (5.5)
\end{aligned}
$$

We next integrate (5.5) over $\mathbb{R}^d$, and it is not difficult to justify the interchange of integrals by the deterministic and stochastic Fubini theorem. We refer to [14] for more details. Therefore, we obtain that

$$
\begin{aligned}
|X_m^\epsilon(t)|_{L^p}^p =&|x^\epsilon|_{L^p}^p - p\int_0^t Re\int i\nabla g(X_m^\epsilon)(s)\nabla\overline{X_m^\epsilon}(s)d\xi ds \\
&- p\int_0^t Re\int (\mu X_m)^\epsilon(s)g(\overline{X_m^\epsilon})(s)d\xi ds \\
&- \lambda p\int_0^t Re\int ig(\overline{X_m^\epsilon})(s)g_m^\epsilon(s)d\xi ds \\
&+ \frac{p}{2}\int_0^t\int |X_m^\epsilon(s)|^{p-2}|(X_m\phi_j)^\epsilon(s)|^2 d\xi ds \\
&+ \frac{1}{2}p(p-2)\int_0^t\int |X_m^\epsilon(s)|^{p-4}[Re(\overline{X_m^\epsilon}(s)(X_m\phi_j)^\epsilon(s))]^2 d\xi ds \\
&+ p\int_0^t Re\int g(\overline{X_m^\epsilon})(s)(X_m\phi_j)^\epsilon(s)d\xi d\beta_j(s) \\
=&|x^\epsilon|_{L^p}^p + K_1 + K_2 + K_3 + K_4 + K_5 + K_6. \qquad (5.6)
\end{aligned}
$$

Now, we can take the limit $\epsilon \to 0$ in (5.6). Below we only do that for K_1, K_3 and K_6. The other terms can be treated similarly.

First, note that as $\epsilon \to 0^+$

$$X_m^\epsilon \to X_m, \quad in\ L^q(0,t;W^{1,p}), \qquad (5.7)$$

in particularly,

$$X_m^\epsilon \to X_m,\ \nabla X_m^\epsilon \to \nabla X_m \quad in\ measure\ dt\times d\xi. \qquad (5.8)$$

In order to take the limit for K_1, it suffices to show that

$$\|\nabla g(X_m^\epsilon) - \nabla g(X_m)\|_{L^{q'}(0,t;L^{p'})} \to 0. \tag{5.9}$$

To this end, direct calculations show that

$$\nabla g(X_m^\epsilon) = \frac{p-2}{2}|X_m^\epsilon|^{p-4}(X_m^\epsilon)^2\nabla\overline{X_m^\epsilon} + \frac{p}{2}|X_m^\epsilon|^{p-2}\nabla X_m^\epsilon. \tag{5.10}$$

To treat the first term in the right hand side above, observe that for $dt-a.e$ $s \in [0,t]$ as $\epsilon \to 0$

$$\begin{aligned}&||X_m^\epsilon|^{p-4}(s)(X_m^\epsilon)^2(s)|_{L^{\frac{p}{p-2}}} = ||X_m^\epsilon|^{p-2}(s)|_{L^{\frac{p}{p-2}}} = |X_m^\epsilon(s)|_{L^p}^{p-2}\\ \to&|X_m(s)|_{L^p}^{p-2} = ||X_m|^{p-4}(s)(X_m)^2(s)|_{L^{\frac{p}{p-2}}},\end{aligned}$$

and $|\nabla\overline{X_m^\epsilon}(s)|_{L^p} \to |\nabla\overline{X_m}(s)|_{L^p}$, which yields by (5.8) that, as $\epsilon \to 0$

$$\frac{p-2}{2}|X_m^\epsilon|^{p-4}(s)(X_m^\epsilon)^2(s)\nabla\overline{X_m^\epsilon}(s) \to \frac{p-2}{2}|X_m|^{p-4}(s)(X_m)^2(s)\nabla\overline{X_m}(s),\ in\ L^{p'}.$$

Similar results hold also for the second term in the right hand side of (5.10). Thus for $dt-a.e$ $s \in [0,t]$ as $\epsilon \to 0$

$$\nabla g(X_m^\epsilon)(s) \to \nabla g(X_m)(s),\ in\ L^{p'}. \tag{5.11}$$

Moreover,

$$\begin{aligned}&|\nabla g(X_m^\epsilon)(s) - \nabla g(X_m)(s)|_{L^{p'}}\\ \le&2(p-1)|X_m(s)|_{L^p}^{p-2}|\nabla X_m(s)|_{L^p} \in L^{q'}(0,t),\end{aligned} \tag{5.12}$$

which implies (5.9) by Lebesgue's dominated convergence theorem. Hence

$$\lim_{\epsilon\to 0} K_1 = -p\int_0^t Re\int i\nabla g(X_m)(s)\nabla\overline{X_m}(s)d\xi ds.$$

Concerning the term K_3 with g_m^ϵ in (5.6), first observe that

$$|g(\overline{X_m^\epsilon})(s) - g(\overline{X_m})(s)|_{L^{p'}} \to 0,\quad |g_m^\epsilon(s) - g_m(s)|_{L^p} \to 0,\quad s \in [0,t],$$

thus as $\epsilon \to 0$

$$Re\int ig(\overline{X_m^\epsilon})(s)g_m^\epsilon(s)d\xi \to Re\int ig(\overline{X_m})(s)g_m(s)d\xi.$$

Moreover, by Hölder's inequality, (3.4) and Sobolev's imbedding theorem we have

$$\begin{aligned}\big|Re\int ig(\overline{X_m^\epsilon})(s)g_m^\epsilon(s)d\xi\big| \leq & |g(\overline{X_m^\epsilon})(s)|_{L^{p'}}|g_m^\epsilon(s)|_{L^p} \\ \leq & C|X_m(s)|_{L^p}^{2(p-1)} \\ \leq & C\sup_{s\in[0,t]}|X_m(s)|_{H^1}^{2(p-1)} < \infty,\end{aligned} \tag{5.13}$$

which, by Lebesgue's dominated convergence theorem and (3.5), implies that

$$\lim_{\epsilon\to 0} K_3 = -\lambda p\int_0^t Re\int ig(\overline{X_m})(s)g_m(s)d\xi ds = 0.$$

Finally, as regards the last stochastic term K_6 in (5.6), we first prove that for $\sigma_{n,l}$ defined above, as $\epsilon \to 0$

$$\mathbb{E}\int_0^{t\wedge\sigma_{n,l}} Re\left[\int g(\overline{X_m^\epsilon})(s)(X_m\phi_j)^\epsilon(s)d\xi - \int g(\overline{X_m})(s)(X_m\phi_j)(s)d\xi\right]^2 ds \to 0. \tag{5.14}$$

In fact, using similar arguments as above, we have for $s \in [0, t\wedge\sigma_{n,l}]$

$$Re\int g(\overline{X_m^\epsilon})(s)(X_m\phi_j)^\epsilon(s)d\xi - Re\int g(\overline{X_m})(s)(X_m\phi_j)(s)d\xi \to 0. \tag{5.15}$$

Furthermore, as in estimate (5.13), for $s \in [0, t\wedge\sigma_{n,l}]$

$$\big|\int g(\overline{X_m^\epsilon})(s)(X_m\phi_j)^\epsilon(s)d\xi\big|^2 \leq C\sup_{s\in[0,t\wedge\sigma_{n,l}]}|X_m(s)|_{H^1}^{2p} < Cl^{2p}, \tag{5.16}$$

which yields (5.14) by Lebesgue's dominated convergence theorem. Hence

$$K_6 \to p\int_0^t\int Re\phi_j|X_m(s)|^p d\xi d\beta_j(s) \tag{5.17}$$

in $\mathbb{P}$-measure on $\{t \leq \sigma_{n,l}\}$ as $\epsilon \to 0$, which implies by (5.4) that (5.17) holds on $\{t \leq \tau_n\}$. Therefore, as $\tau_n \to \tau^*(x)$ $\mathbb{P}$-a.s, we conclude that (5.17) holds $\mathbb{P}$-a.s. for $t < \tau^*(x)$.

Therefore, we can pass to the limit $\epsilon \to 0$ in (5.6). As K_2 and K_4 are canceled after taking the limit, we finally obtain the desired formula (5.1) . □

Next, we prove the Itô formula for $|\nabla X_m|_2^2$.

Lemma 5.2 *Assume the conditions in Lemma 5.1 to hold. We have $\mathbb{P}$-a.s. for $t < \tau^*(x)$*

$$\begin{aligned}|\nabla X_m(t)|_2^2 =& |\nabla x|_2^2 + 2\int_0^t Re\,\langle -\nabla(\mu X_m)(s), \nabla X_m(s)\rangle_2\, ds \\ &+ \sum_{j=1}^N \int_0^t |\nabla(X_m(s)\phi_j)|_2^2 ds - 2\lambda \int_0^t Re \int i\nabla g_m(s)\nabla\overline{X_m}(s)d\xi ds \\ &+ 2\sum_{j=1}^N \int_0^t Re\,\langle \nabla(\phi_j X_m(s)), \nabla X_m(s)\rangle_2\, d\beta_j(s). \end{aligned} \tag{5.18}$$

Proof. We follow the ideas from the proof of (4.14) in [1] to derive (5.18). Let $\{f_k | k \in \mathbb{N}\} \subset H^2$ be an orthonormal basis of L^2, set $J_\epsilon = (I - \epsilon\Delta)^{-1}$ and $h_\epsilon := J_\epsilon(h) \in H^1$ for any $h \in H^{-1}$. Then we have from equation (3.6) that $\mathbb{P}$-a.s. for $t \in (0, \tau^*(x))$

$$\begin{aligned} &idX_{m,\epsilon} = \Delta X_{m,\epsilon}dt - i(\mu X_m)_\epsilon dt + \lambda g_{m,\epsilon}dt + i(X_m\phi_j)_\epsilon d\beta_j, \\ &X_{m,\epsilon}(0) = x_\epsilon, \end{aligned} \tag{5.19}$$

where $g_{m,\epsilon} = [\Theta_m(g(X_m))]_\epsilon$ and we used the summation convention.

Since $\partial_l f_k \in H^1$ for each f_k, $1 \le l \le d$, $k \in \mathbb{N}$, it follows from (5.19) and Fubini's theorem that $\mathbb{P}$-a.s. for $t \in (0, \tau^*(x))$

$$\begin{aligned} &\langle X_{m,\epsilon}(t), \partial_l f_k\rangle_2 \\ =& \langle x_\epsilon, \partial_l f_k\rangle_2 + \int_0^t \langle -i\Delta X_{m,\epsilon}(s), \partial_l f_k\rangle_2\, ds + \int_0^t \langle -(\mu X_m)_\epsilon(s), \partial_l f_k\rangle_2\, ds \\ &+ \int_0^t \langle -\lambda i g_{m,\epsilon}(s), \partial_l f_k\rangle_2\, ds + \int_0^t \langle (X_m(s)\phi_j)_\epsilon, \partial_l f_k\rangle_2\, d\beta_j. \end{aligned}$$

Applying Itô's product rule and integrating by parts, we deduce that

$$\begin{aligned}
&|\langle X_{m,\epsilon}(t), \partial_l f_k\rangle_2|^2 \\
=&|\langle \partial_l x_\epsilon, f_k\rangle_2|^2 + 2Re\int_0^t \overline{\langle \partial_l X_{m,\epsilon}(s), f_k\rangle_2}\langle -i\partial_l \Delta X_{m,\epsilon}(s), f_k\rangle_2 ds \\
&+ 2Re\int_0^t \overline{\langle \partial_l X_{m,\epsilon}(s), f_k\rangle_2}\langle -\partial_l(\mu X_m)_\epsilon(s), f_k\rangle_2 ds \\
&+ 2Re\int_0^t \overline{\langle \partial_l X_{m,\epsilon}(s), f_k\rangle_2}\langle -\lambda i\partial_l g_{m,\epsilon}(s), f_k\rangle_2 ds \\
&+ 2Re\int_0^t \overline{\langle \partial_l X_{m,\epsilon}(s), f_k\rangle_2}\langle \partial_l(X_m(s)\phi_j)_\epsilon, f_k\rangle_2 d\beta_j(s) \\
&+ \int_0^t |\langle \partial_l(X_m(s)\phi_j)_\epsilon, f_k\rangle_2|^2 ds, \quad t < \tau^*(x), \quad \mathbb{P}-a.s.
\end{aligned}$$

We note that $\Delta X_{m,\epsilon}$ and $g_{m,\epsilon}$ are in H^1, thus the above integrals make sense. This is the reason why we have introduced the operator J_ϵ.

Now summing over $k \in \mathbb{N}$ and interchanging infinite sum and integrals (which can be justified easily), we obtain $\mathbb{P}$-a.s. for all $t \in (0, \tau^*(x))$

$$\begin{aligned}
&|\partial_l X_{m,\epsilon}(t)|_2^2 \\
=&\sum_{k=1}^{\infty} |\langle X_{m,\epsilon}(t), \partial_l f_k\rangle_2|^2 \\
=&|\partial_l x_\epsilon|_2^2 + 2\int_0^t Re\left\langle i\Delta X_{m,\epsilon}(s), \partial_l^2 X_{m,\epsilon}(s)\right\rangle_2 ds \\
&+ 2\int_0^t Re\langle -\partial_l(\mu X_m)_\epsilon(s), \partial_l X_{m,\epsilon}(s)\rangle_2 ds + \int_0^t |\partial_l(X_m(s)\phi_j)_\epsilon|_2^2 ds \\
&- 2\lambda\int_0^t Re\langle i\partial_l g_{m,\epsilon}(s), \partial_l X_{m,\epsilon}(s)\rangle_2 ds + 2\int_0^t Re\langle \partial_l(X_m(s)\phi_j)_\epsilon, \partial_l X_{m,\epsilon}(s)\rangle_2 d\beta_j(s).
\end{aligned}$$

Finally, summing over $l : 1 \le l \le d$ and using the fact that $f_\epsilon \to f$ in H^k and $|f_\epsilon|_{H^k} \le |f|_{H^k}$ for $k = -1, 0, 1$, we can pass to the limit $\epsilon \to 0$ in the above equality and then obtain the evolution formula (5.18). □

We conclude this section with the proof of the asymptotic formula (3.10).

Proof of (3.10)**.** This proof is analogous to that of continuous dependence on initial data in Theorem 1.2, hence we only give a sketch of it. Set

$q = \frac{4(\alpha+1)}{d(\alpha-1)}$. By the rescaling transformation $X_m = e^W y_m$, it suffices to prove that $\mathbb{P}$-a.s.

$$y_m \to y, \quad in\ L^\infty(0,t;H^1) \cap L^q(0,t;W^{1,\alpha+1}), \quad t < \tau^*(x).$$

Notice that, (3.6) implies that

$$y_m = U(t,0)x - \lambda i \int_0^t U(t,s)e^{-W(s)}\Theta_m(g(e^{W(s)}y_m(s)))ds. \tag{5.20}$$

By (3.8), (2.8) and since $\|W\|_{L^\infty(0,T;W^{1,\infty})} < \infty$, we have $\mathbb{P}$-a.s. for $t < \tau^*(x)$

$$\begin{aligned} \widetilde{R}(t) := &\sup_{m\geq 1}(\|y_m\|_{C([0,t];H^1)} + \|y_m\|_{L^q(0,t;W^{1,\alpha+1})}) \\ &+ (\|y\|_{C([0,t];H^1)} + \|y\|_{L^q(0,t;W^{1,\alpha+1})}) < \infty. \end{aligned} \tag{5.21}$$

Moreover, combining (2.12) and (5.20), we have

$$y_m - y = -\lambda i \int_0^t U(t,s)e^{-W(s)}\left[\Theta_m(g(e^{W(s)}y_m(s))) - g(e^{W(s)}y(s))\right] ds. \tag{5.22}$$

Now, we first claim that there exists t small enough and independent of m, such that

$$\|y_m - y\|_{L^\infty(0,t;L^2)} + \|y_m - y\|_{L^q(0,t;L^{\alpha+1})} \to 0, \ as\ m \to \infty, \tag{5.23}$$

in particularly,

$$y_m \to y \ \ in\ measure\ dt \times d\xi. \tag{5.24}$$

Indeed, applying Strichartz estimate (2.10) to (5.22) we have

$$\begin{aligned} &\|y_m - y\|_{L^\infty(0,t;L^2)} + \|y_m - y\|_{L^q(0,t;L^{\alpha+1})} \\ \leq &2C_T\|e^{-W}\|_{L^\infty(0,T;L^\infty)}\|\Theta_m(g(e^W y_m)) - g(e^W y)\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})} \\ \leq &C(T)\|\Theta_m[g(e^W y_m) - g(e^W y)]\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})} + C(T)\|(\Theta_m - 1)g(e^W y)\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})} \\ \leq &C(T)t^\theta\|y_m - y\|_{L^q(0,t;L^{\alpha+1})} + C(T)\|(\Theta_m - 1)g(e^W y)\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})}, \end{aligned} \tag{5.25}$$

where we used (3.2) and estimates as in (2.23) in the last inequality. Here $\theta = 1 - \frac{2}{q} > 0$, $C(T)$ depends on C_T, $\|W\|_{L^\infty(0,T;L^\infty)}$ and $\widetilde{R}(t^*)$ with any fixed $t^* \in (t, \tau^*(x))$. Choosing t small enough and then using (3.3), we consequently obtain (5.23), as claimed.

Next, we prove that for t sufficiently small and independent of m

$$\|y_m - y\|_{L^\infty(0,t;H^1)} + \|y_m - y\|_{L^q(0,t;W^{1,\alpha+1})} \to 0, \ \textit{as } m \to \infty. \tag{5.26}$$

Indeed, from (6.3) in the Appendix it follows that

$$\begin{aligned}\nabla(y_m - y) = \int_0^t U(t,s)\Big\{ & i(D_j \nabla \widetilde{b}^j + \nabla \widetilde{b}^j D_j + \nabla \widetilde{c})(y_m - y) \\ & - \lambda i \nabla \left[e^{-W} \left(\Theta_m(g(e^W y_m)) - g(e^W y) \right) \right] \Big\} ds.\end{aligned} \tag{5.27}$$

Using estimate as in (4.7), together with (5.25), we have that

$$\begin{aligned}&\|i(D_j \nabla \widetilde{b}^j + \nabla \widetilde{b}^j D_j + \nabla \widetilde{c})(y_m - y)\|_{\widetilde{X}'_{[0,t]}} \\ \leq & C(T) t^\theta \|y_m - y\|_{L^q(0,t;L^{\alpha+1})} + C(T) \|(\Theta_m - 1) g(e^W y)\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})},\end{aligned} \tag{5.28}$$

where $\theta = 1 - \frac{2}{q} > 0$, and $C(T)$ depends on κ_T, C_T, $\|W\|_{L^\infty(0,T;L^\infty)}$ and $\widetilde{R}(t^*) < \infty$, $\mathbb{P}$-a.s.

Then, similarly to (4.8), we have for $m \geq 1$

$$\begin{aligned}&\|\nabla y_m - \nabla y\|_{L^\infty(0,t;L^2)} + \|\nabla y_m - \nabla y\|_{L^q(0,t;L^{\alpha+1})} \\ \leq & C(T) t^\theta \|y_m - y\|_{L^q(0,t;L^{\alpha+1})} + C(T) \|\nabla g(e^W y_m) - \nabla g(e^W y)\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})} \\ & + C(T) \|(\Theta_m - 1) g(e^W y)\|_{L^{q'}(0,t;W^{1,\frac{\alpha+1}{\alpha}})},\end{aligned} \tag{5.29}$$

where $C(T)$ is independent of t and m.

Therefore, applying analogous arguments as those after (4.8) to control the second term, and then using (3.3) to take the limit in the last term, we deduce that (5.26) holds for t small enough and independent of m. Reiterating this procedure with estimates as above we conclude (3.10) for any $t < \tau^*(x)$. □

6 Appendix

Proof of Lemma 2.7. Estimate (2.10) is already proved in Lemma 4.1 in [1]. We can use the same arguments there to derive that

$$\|u\|_{L^{q_1}(0,T;L^{p_1})\cap \widetilde{X}_{[0,T]}} \leq C_T(|u_0|_2 + \|f\|_{L^{q_2'}(0,T;L^{p_2'})+\widetilde{X}'_{[0,T]}}), \tag{6.1}$$

where $\widetilde{X}_{[0,T]}$ is the local smoothing space introduced in [12] up to time T and (q_i, p_i), $i = 1, 2$, are Strichartz pairs.

Next, we prove the estimate (2.11). Since the proof relies on Theorem 1.13 and Proposition 2.3 (a) in [12], we adapt the notations there $D_t := -i\partial_t$, $D_j := -i\partial_j$, $1 \leq j \leq d$, to rewrite (2.9) in the form

$$D_t u = (D_j a^{jk} D_k + D_j \widetilde{b}^j + \widetilde{b}^j D_j + \widetilde{c})u - if$$

with $a^{jk} = \delta_{jk}$, $\widetilde{b}^j = -i\partial_j W_t$ and $\widetilde{c} = -\sum_{j=1}^{d}(\partial_j W)^2 + (\mu + \widetilde{\mu})i$, $1 \leq j, k \leq d$.

Direct computations show

$$\begin{aligned} D_t \nabla u =& (-\Delta + D_j \widetilde{b}^j + \widetilde{b}^j D_j + \widetilde{c})\nabla u \\ &+ (D_j \nabla \widetilde{b}^j + \nabla \widetilde{b}^j D_j + \nabla \widetilde{c})u - i\nabla f. \end{aligned} \tag{6.2}$$

We regard (6.3) as the equation for the unknown ∇u and treat the lower order term $(D_j \nabla \widetilde{b}^j + \nabla \widetilde{b}^j D_j + \nabla \widetilde{c})u$ as equal terms with ∇f. This leads to

$$\nabla u(t) = U(t,0)\nabla u_0 + \int_0^t U(t,s)\Big[i(D_j \nabla \widetilde{b}^j(s) + \nabla \widetilde{b}^j(s) D_j + \nabla \widetilde{c}(s))u(s) + \nabla f(s)\Big] ds. \tag{6.3}$$

Hence applying (6.1) to (6.3) and then using Proposition 2.3 (a) in [12] to control the lower order term, we derive that

$$\begin{aligned} &\|\nabla u\|_{L^{q_1}(0,T;L^{p_1})\cap \widetilde{X}_{[0,T]}} \\ \leq& C_T\Big[|\nabla u_0|_2 + \|i(D_j \nabla \widetilde{b}^j + \nabla \widetilde{b}^j D_j + \nabla \widetilde{c})u\|_{\widetilde{X}'_{[0,T]}} + \|\nabla f\|_{L^{q_2'}(0,T;L^{p_2'})}\Big] \\ \leq& C_T\Big[|\nabla u_0|_2 + \kappa_T \|u\|_{\widetilde{X}_{[0,T]}} + \|\nabla f\|_{L^{q_2'}(0,T;L^{p_2'})}\Big] \\ \leq& C_T\Big[|\nabla u_0|_2 + C_T\kappa_T(|u_0|_2 + \|f\|_{L^{q_2'}(0,T;L^{p_2'})}) + \|\nabla f\|_{L^{q_2'}(0,T;L^{p_2'})}\Big] \\ =& C_T(C_T\kappa_T + 1)\Big[|u_0|_{H^1} + \|f\|_{L^{q_2'}(0,T;W^{1,p_2'})}\Big], \end{aligned} \tag{6.4}$$

where we also used (2.10) to estimate $\|u\|_{\widetilde{X}_{[0,T]}}$ in the last two inequalities. This together with (2.10) yields the estimate (2.11).

Now, set

$$\begin{aligned}C_t =& \sup\{\|U(\cdot,0)u_0\|_{L^{q_1}(0,t;W^{1,p_1})}; |u_0|_{H^1} \le 1\} \\ &+ \sup\left\{\left\|\int_0^{\cdot} U(\cdot,s)f(s)ds\right\|_{L^{q_1}(0,t;W^{1,p_1})}; \|f\|_{L^{q_2'}(0,t;W^{1,p_2'})} = 1\right\}. \quad (6.5)\end{aligned}$$

Then the asserted properties of C_t, $t \ge 0$, follow analogously as in the proof of Lemma 4.1 in [1] (see also [14]). This completes the proof of Lemma 2.7. □

Proof of (3.4)**.** Hausdorf-Young's inequality shows that

$$|\Theta_m f|_{L^{\alpha+1}} = |(\theta(\frac{|\cdot|}{m}))^\vee * f|_{L^{\alpha+1}} \le |(\theta(\frac{|\cdot|}{m}))^\vee|_{L^{\frac{\alpha+1}{2}}} |f|_{L^{\frac{\alpha+1}{\alpha}}}.$$

As $\theta(\frac{|\cdot|}{m}) \in C_c^\infty \subset \mathcal{S}$, $(\theta(\frac{|\cdot|}{m}))^\vee \in \mathcal{S} \subset L^{\frac{\alpha+1}{2}}$, which implies $|\Theta_m f|_{L^{\alpha+1}} < \infty$. □

Proof of (3.5)**.** For $f \in L^{\frac{\alpha+1}{\alpha}} \cap L^1$, (3.5) follows from Fourier's inversion formula and Fubini's theorem. The general case $f \in L^{\frac{\alpha+1}{\alpha}}$ follows from a standard approximating procedure. □

Proof of Lemma 3.5. As in the proof of Lemma 4.3 in [1], we have

$$|X(t)|_2^2 = |x|_2^2 + 2\sum_{j=1}^N \int_0^t Re\mu_j < X(s), X(s)e_j >_2 d\beta_j(s), \quad t < \tau^*(x), \ \mathbb{P}-a.s., \quad (6.6)$$

where $\tau^*(x)$ is as in Theorem 2.1. Then, Itô's formula implies

$$\begin{aligned}|X(t)|_2^p =& |x|_2^p + p\int_0^t |X(s)|_2^{p-2} \sum_{j=1}^N Re\mu_j \langle X(s), X(s)e_j\rangle_2 \, d\beta_j(s) \\ &+ \frac{1}{2}p(p-2)\int_0^t |X(s)|_2^{p-4} \sum_{j=1}^N (Re\mu_j)^2 \langle X(s), X(s)e_j\rangle_2^2 \, ds, \quad t < \tau^*(x).\end{aligned}$$

Hence, by the Burkholder-Davis-Gundy inequality and Lemma 3.3 with Y replaced by $|X|_2^{2p}$, we derive that for every $n \in \mathbb{N}$

$$
\begin{aligned}
&\mathbb{E} \sup_{s\in[0,t\wedge\tau_n]} |X(s)|_2^p \\
\leq & |x|_2^p + \sqrt{2|\mu|_\infty} pC\mathbb{E}\left[\int_0^{t\wedge\tau_n} |X(s)|_2^{2p} ds\right]^{\frac{1}{2}} + 2p(p-2)|\mu|_\infty \mathbb{E}\int_0^{t\wedge\tau_n} |X(s)|_2^p ds \\
\leq & |x|_2^p + \epsilon\sqrt{2|\mu|_\infty} pC\mathbb{E} \sup_{s\in[0,t\wedge\tau_n]} |X(s)|_2^p + C_\epsilon \sqrt{2|\mu|_\infty} pC \int_0^t \mathbb{E} \sup_{r\in[0,s\wedge\tau_n]} |X(r)|_2^p ds \\
& + 2p(p-2)|\mu|_\infty \int_0^t \mathbb{E} \sup_{r\in[0,s\wedge\tau_n]} |X(r)|_2^p ds.
\end{aligned}
$$

Therefore, similar arguments as at the end of the proof of Theorem 3.6 yield Lemma 3.5. □

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