On the existence of the dual right Markov process and applications

Lucian Beznea[1] and Michael Röckner[2]

Abstract. We show that given a Borel right process there exists a dual process which is also a right Markov process. However, it is necessary to enlarge the initial space to a new Lusin topological space and the dual process is a right process with respect to a second Lusin topology. As a result both processes can be identified as solutions to martingale problems. Another application is the proof that the Riesz decomposition holds (in potential and harmonic components) for the excessive functions and the set of all potentials becomes solid in order.

Keywords. Dual process, weak duality hypothesis, moderate dual, excessive function, potential, martingale problem, Riesz decomposition.

1 Introduction

The duality theory for a Markov process is usually developed under the so called "weak duality hypothesis". The main additional assumption is the existence on the given topological state space of a second right process whose transition function is in duality with the transition function of the initial process, with respect to a fixed duality measure m; see e.g., the articles [GeSh 84], [BeBo 03], and the monographs [ChuWa 05], [DeMe 87], [BeBo 04b], and [CheFu 11]. Recall that in particular, the pair of standard processes in duality associated to a quasi-regular (non-symmetric) Dirichlet form fits in the weak duality frame; cf. [MaRö 92], see also [BeBo 06] for the semi-Dirichlet forms context.

In a recent work [FiGe 08] P.J. Fitzsimmons and R.K. Getoor developed the potential theory of the moderate dual of a Borel right (Markov) process. They obtain results for a right process using as main tool its moderate dual process. The basic remark is the identification of the excessive functions and the excessive measures of the moderate dual process. In particular, it is proved a Riesz decomposition of an excessive function in potential and harmonic components. The moderate dual always exists, however it is a left continuous process but

[1]Simion Stoilow Institute of Mathematics of the Romanian Academy, Research unit No. 2, P.O. Box 1-764, RO-014700 Bucharest, Romania, and University of Bucharest, Faculty of Mathematics and Computer Science (e-mail: lucian.beznea@imar.ro)

[2]Fakultät für Mathematik, Universität Bielefeld, Postfach 100 131, D-33501 Bielefeld, Germany (e-mail: roeckner@math.uni-bielefeld.de)

not necessary a strong Markov process. Note that the set of all potential excessive functions is not solid in order (i.e., it is not true that an excessive function dominated by a potential is again a potential).

In this paper we show that given a Borel right process with state space E, a Lusin topological space, there exists a dual process which is a right process (hence in particular, it is a strong Markov process). However, it is necessary to enlarge the initial space E to a new Lusin topological space $\overline{E}$ and the dual process is a right one with respect to a second Lusin topology on $\overline{E}$. As a consequence, we obtain the Riesz decomposition for the excessive functions in potential and harmonic components on $\overline{E}$. It turns out that in our frame the set of all potentials on $\overline{E}$ becomes solid in order.

Our approach uses a refinement of a technique from [BeBoRö 06], developed there in order to associate a right (Markov) process to every strongly continuous resolvent of sub-Markovian contractions on $L^p(E, m)$. Actually, we prove here that with such an L^p-resolvent we can associate a pair of two processes which are both right continuous in two (maybe different) Lusin topologies on a bigger common state space $\overline{E}$, and the resolvents of the processes are in duality with respect to the extension of the measure m to $\overline{E}$ with zero on $\overline{E} \setminus E$.

Another notable consequence is that we obtain dual processes of reasonable regularity to solutions of stochastic differential equations on infinite dimensional state spaces, in particular stochastic partial differential equations (see the final remark of this paper).

2 Dual processes associated to an L^p-resolvent

Let $(E, \mathcal{B})$ be a Lusin measurable space. We consider a sub-Markovian resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$. Recall that a positive, numerical, $\mathcal{B}$-measurable function v on E (resp. a positive σ-finite measure ξ on $(E, \mathcal{B})$) is called *$\mathcal{U}$-excessive* provided that $\alpha U_\alpha v \leq v$ and $\sup_{\alpha>0} \alpha U_\alpha v = v$ pointwise (resp. $\xi \circ \alpha U_\alpha \leq \xi$ for all $\alpha > 0$). We denote by $\mathcal{E}(\mathcal{U})$ (resp. $Exc(\mathcal{U})$) the set of all $\mathcal{U}$-excessive functions (resp. $\mathcal{U}$-excessive measures). If we have only $\alpha U_\alpha v \leq v$ for all $\alpha > 0$ then the function v is called *$\mathcal{U}$-supermedian.* If v is $\mathcal{U}$-supermedian, then its *$\mathcal{U}$-excessive regularization* $\widehat{v}$ is defined by

$$\widehat{v}(x) := \sup_{\alpha>0} \alpha U_\alpha v(x), \quad x \in E.$$

We denote by $\mathcal{D}_{\mathcal{U}}$ the set of all non-branch points with respect to $\mathcal{U}$,

$$\mathcal{D}_{\mathcal{U}} := \{x \in E : \ \inf(u, v)(x) = \widehat{\inf(u, v)}(x) \text{ for all } u, v \in \mathcal{E}(\mathcal{U})\}.$$

A $\mathcal{U}$-excessive measure ξ is called *potential* if there exists a σ-finite measure μ on $(\mathcal{E}, \mathcal{B})$ such that $\xi = \mu \circ U$; we denote by $Pot(\mathcal{U})$ the set of all potentials from $Exc(\mathcal{U})$.

If $\beta > 0$ then the family $\mathcal{U}_\beta = (U_{\beta+\alpha})_{\alpha>0}$ is also a sub-Markovian resolvent of kernels on $(E, \mathcal{B})$.

The *fine topology* is the topology on E generated by all $\mathcal{U}_\beta$-excessive functions for one $\beta > 0$. A Lusin topology on E which is smaller than the fine topology and $\mathcal{B}$ is its Borel σ-algebra is called *natural*.

(2.1) The following assertions are equivalent for a resolvent $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$ (cf. [BeBo 04b] and [BeBoRö 06]):

(a) For one (and therefore for all $\beta > 0$) we have

$(a.1)$ $\mathcal{D}_{\mathcal{U}_\beta} = E$, $\sigma(\mathcal{E}(\mathcal{U}_\beta)) = \mathcal{B}$ and $1 \in \mathcal{E}(\mathcal{U}_\beta)$.

$(a.2)$ An $\mathcal{U}_\beta$-excessive measure dominated by a potential $\mathcal{U}_\beta$-excessive measure is also a potential.

(b) There exist a Lusin topology $\mathcal{T}$ on E such that its Borel σ-algebra is $\mathcal{B}$ and a Borel right process with state space $(E, \mathcal{T})$, having $\mathcal{U}$ as associated resolvent.

(c) For every natural topology $\mathcal{T}$ on E there exists a Borel right process with state space $(E, \mathcal{T})$, having $\mathcal{U}$ as associated resolvent.

Remark 2.1. *If (a) holds then every $\mathcal{U}$-excessive measure dominated by a potential is also a potential (i.e., assertion $(a.2)$ also holds for $\beta = 0$, with the convention $\mathcal{U}_0 := \mathcal{U}$).*

Proof. Let $\xi, \mu \circ U \in Exc(\mathcal{U})$ with $\xi \leq \mu \circ U$. Then for all $\beta > 0$ we have $Exc(\mathcal{U}_\beta) \ni \xi \leq \mu_\beta \circ U_\beta$, where $\mu_\beta := \mu + \beta\xi$. Hence by $(a.2)$ there exists a measure ν_β on $(E, \mathcal{B})$ such that $\xi = \nu_\beta \circ U_\beta$. The uniqueness of mass implies that if $\alpha, \beta > 0$, $\alpha < \beta$ then $\nu_\beta = \nu_\alpha + (\beta - \alpha)\xi$. We define the measure ν by $\nu(f) = \lim_{\beta\to 0} \nu_\beta(f) = \inf_{\beta>0} \nu_\beta(f)$, $f \in p\mathcal{B}$. If f is such that $\mu(Uf) < \infty$ then clearly $\lim_{\beta\to 0} \xi(\beta U_\beta f) \leq \lim_{\beta\to 0} \mu(\beta U_\beta U f) = 0$. From $\xi = \nu_\alpha \circ U_\beta + (\beta - \alpha)\xi \circ U_\beta$, letting $\alpha \to 0$, we get $\xi = \nu \circ U_\beta + \xi \circ \beta U_\beta$. Consequently, letting now $\beta \to 0$, we conclude that $\xi = \nu \circ U$. □

Let m be a fixed σ-finite measure on $(E, \mathcal{B})$ and $\mathcal{U}^* = (U^*_\alpha)_{\alpha>0}$ a second sub-Markovian resolvent of kernels on $(E, \mathcal{B})$. We say that $\mathcal{U}$ and $\mathcal{U}^*$ are in *duality with respect to m* if

$$\int_E f U_\alpha g dm = \int_E g U^*_\alpha f dm \text{ for all } \alpha > 0 \text{ and } f, g \in p\mathcal{B}.$$

Remark 2.2. *If $\mathcal{U}$ and $\mathcal{U}^*$ are in duality with respect to m, then the following assertions hold:*

(i) *The measure m is simultaneously $\mathcal{U}$-excessive and $\mathcal{U}^*$-excessive.*

(ii) Let $\beta > 0$ and ξ be a measure on $(E, \mathcal{B})$ such that $\xi \ll m$. Then $\xi \in Exc(\mathcal{U}_\beta)$ if and only if there exists $v^* \in \mathcal{E}(\mathcal{U}^*_\beta)$ such that $\xi = v^* \cdot m$.

(2.2) **Trivial modification**

(2.2.1) Let $(F, \mathcal{B}_1)$ be a second Lusin measurable space such that $E \in \mathcal{B}_1$ and $\mathcal{B} = \mathcal{B}_1|_E$. For every $\alpha > 0$ we consider the kernel U^1_α on $(F, \mathcal{B}_1)$ defined by

$$U^1_\alpha g := 1_E U_\alpha(g|_E) + \frac{1}{\alpha} g 1_{F\setminus E}, g \in p\mathcal{B}_1.$$

Then the family $\mathcal{U}^1 = (U^1_\alpha)$ is a sub-Markovian resolvent of kernels on $(F, \mathcal{B}_1)$, called *trivial extension of* $\mathcal{U}$ *from* E *to* F.

(2.2.2) A function $u \in p\mathcal{B}_1$ belongs to $\mathcal{E}(\mathcal{U}^1_\beta)$ if and only if $u|_E$ belongs to $\mathcal{E}(\mathcal{U}_\beta)$.

(2.2.3) Let ξ be a measure on $(F, \mathcal{B}_1)$. Then $\xi \in Exc(\mathcal{U}^1)$ (resp. $\xi \in Pot(\mathcal{U}^1)$) if and only if $\xi|_E \in Exc(\mathcal{U})$ (resp. $\xi|_E \in Pot(\mathcal{U})$).

(2.2.4) The resolvents $\mathcal{U}$ and $\mathcal{U}^1$ satisfy simultaneously conditions $(a.1)$ and $(a.2)$ respectively.

(2.2.5) Let $M \in \mathcal{B}$ be such that $U_\alpha(1_M) = 0$ on $E \setminus M$. Then for every $\alpha > 0$ we may consider the kernel U'_α on $E \setminus M$ defined by

$$U'_\alpha f := U_\alpha \overline{f}|_{E\setminus M}, f \in p\mathcal{B}|_{E\setminus M}$$

where $\overline{f} \in p\mathcal{B}$, $f = \overline{f}|_{E\setminus M}$. Then the family of $\mathcal{U}' = (U'_\alpha)_{\alpha>0}$ is a sub-Markovian resolvent of kernels on $(E \setminus M, \mathcal{B}|_{E\setminus M})$, called the *restriction of* $\mathcal{U}$ *to* $E \setminus M$. The trivial extension of $\mathcal{U}'$ to E is called the *trivial modification of* $\mathcal{U}$ *on* M.

(2.2.6) Recall (see e.g., Subsection 3.2 in [BeRö 11]) that a set $M \in \mathcal{B}$ is called m-*inessential* (with respect to $\mathcal{U}$) provided that it is m-negligible and $R^M_\beta 1 = 0$ on $E \setminus M$ for some $\beta > 0$ (and therefore for all $\beta > 0$); if $u \in \mathcal{E}(\mathcal{U}_\beta)$, then $R^M_\beta u$ denotes the *reduced function* (with respect to $\mathcal{U}_\beta$) *of* u *on* M, defined as

$$R^M_\beta u := \inf\{v \in \mathcal{E}(\mathcal{U}_\beta) : v \geq u \text{ on } M\} .$$

One can see that if M is m-inessential then it is finely closed, m-polar (i.e., $R^M 1 = 0$ m-a.e.), and $U_\alpha(1_M) = 0$ on $E \setminus M$ for all $\alpha > 0$. Therefore, by (2.2.5) we may consider the restriction of $\mathcal{U}$ to $E \setminus M$. If in addition $\mathcal{U}$ is the resolvent of a right process with state space E, then it is possible to consider also the restriction of the process to $E \setminus M$ and its resolvent is precisely the restriction of $\mathcal{U}$ to $E \setminus M$ (see [BeRö 11] (3.3)). Consequently, trivial modification of $\mathcal{U}$ on the m-inessential set M is also the resovent of a right process with state space E.

(2.3) **Fine densities**

Recall (see e.g. [BeBo 04a]) that a positive (nearly) Borel measurable function f on E is called *m-fine* provided there exists an m-inessential set M such that f is finely continuous on the (finely open) set $E \setminus M$. An *m-fine version* of a positive function f on E is a function that is m-fine and equals f m-a.e.

The following result on the existence of the m-fine versions for densities of excessive measures holds (cf. Theorem 4.1 in [BeBo 04a]):

Assume that $\mathcal{U}$ is the resolvent of a right process with state space E. Let η be a $\mathcal{U}_\beta$-excessive measure, $\eta \ll m$, then there exists an m-fine version of the Radon-Nikodym derivative $dm/d\eta$. In particular, for each bounded $\mathcal{U}^*_\beta$-excessive function v^* the function $v^{*\prime}$ defined as

$$v^{*\prime} := \liminf_n nU_{\beta+n} v^*$$

is a $\mathcal{B}$-measurable m-fine version of v^*.

The last assertion follows since the measure $v^* \cdot m$ is $\mathcal{U}_\beta$-excessive and using Proposition 4.4.3 from [BeBo 04b]. Note that due to a result of G. Mokobodzki the set $[v^* \neq v^{*\prime}]$ is $\mathcal{U}$-negligible (see Ch. XII, Section 3 in [DeMe 87]).

(2.4) Assume that the resolvent $\mathcal{U} = (\mathcal{U}_\alpha)_{\alpha>0}$ satisfies condition $(a.1)$. Then there exist a Lusin topological space $(E_1, \mathcal{T}_1)$ with $E \subset E_1$, $E \in \mathcal{B}_1$ (:= the Borel σ-algebra on E_1), $\mathcal{B} = \mathcal{B}_1|_E$, and a Borel right process with state space E_1 such that if $\mathcal{U}^1 = (U^1_\alpha)_{\alpha>0}$ denotes the resolvent of the process, then $U^1_\alpha(1_{E_1\setminus E}) = 0$ and the restriction of $\mathcal{U}^1$ to E is $\mathcal{U}$. In particular, every $\mathcal{U}_\beta$-excessive function has a unique extension to E_1 by continuity in the fine topology generated by $\mathcal{U}^1$ (cf. [BeBoRö 06]).

Theorem 2.3. *Let $p \in (1, \infty)$ and $(V_\alpha)_{\alpha>0}$ a strongly continuous resolvent of contractions on $L^p(E, m)$, where m is a σ-finite measure on $(E, \mathcal{B})$. Assume that $(V_\alpha)_{\alpha>0}$ is sub-Markovian (i.e., $0 \leq \alpha V_\alpha f \leq 1$ for all $\alpha > 0$, provided that $f \in L^p(E, m)$, $0 \leq f \leq 1$) and that m is sub-invariant (that is, if $f \in L^1_+(E, m) \cap L^p(E, m)$ then $m(\alpha V_\alpha f) \leq m(f)$). Then the following assertions hold.*

(i) (cf. [BeBoRö 06]) There exist two sub-Markovian resolvents of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$ and $\mathcal{U}^* = (U^*_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$ such that:

- both resolvents $\mathcal{U}$ and $\mathcal{U}^*$ satisfy condition $(a.1)$;

- $\mathcal{U}$ and $\mathcal{U}^*$ are in duality with respect to m;

- $U_\alpha = V_\alpha$ as operators on $L^p(E, m)$ for all $\alpha > 0$.

(ii) There exists a larger Lusin measurable space $(\overline{E}, \overline{\mathcal{B}})$, with $E \subset \overline{E}$, $E \in \overline{\mathcal{B}}$, $\mathcal{B} = \overline{\mathcal{B}}|_E$, such that if $\overline{m}$ is the extension of m from E to $\overline{E}$ with zero on $\overline{E} \setminus E$ (i.e., $\overline{m}$ is the measure on $(\overline{E}, \overline{\mathcal{B}})$ such that $\overline{m}|_E = m$ and $\overline{m}(\overline{E} \setminus E) = 0$) then:

- There exist two Lusin topologies $\mathcal{T}$ and $\mathcal{T}^*$ on $\overline{E}$ with $\sigma(\mathcal{T}) = \sigma(\mathcal{T}^\star) = \overline{\mathcal{B}}$ and two processes $\overline{X}$ and $\overline{X}^*$ with common state space $\overline{E}$, such that $\overline{X}$ (resp. $\overline{X}^*$) is a Borel right process when $\overline{E}$ is endowed with the topology $\mathcal{T}$ (resp. $\mathcal{T}^*$);

- There exists an m-negligible set $N \in \overline{\mathcal{B}}$ such that the trace of $\mathcal{T}^*$ on $\overline{E} \setminus N$ is smaller than the trace on $\overline{E} \setminus N$ of the fine topology induced by $\overline{X}$.

- The resolvents of $\overline{X}$ and $\overline{X}^*$ are in duality with respect to $\overline{m}$;

Proof. Assertion (i) follows from Theorem 2.2 and Corollary 2.4 in [BeBoRö 06], since $(V^*_\alpha)_{\alpha>0}$ is also a resolvent of contractions on $L^q(E,m)$ which is strongly continuous and sub-Markovian, where V^*_α is the adjoint operator of V_α.

(ii) Step I. We consider the resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$ given in assertion (i). By (2.4) there exists a Lusin topological space $(E_1, \mathcal{T}_1)$ with $E \subset E_1$, $E \in \mathcal{B}_1$ (:= the Borel σ-algebra on E_1), $\mathcal{B} = \mathcal{B}_1|_E$, and a Borel right process X_1 with state space E_1, such that if $\mathcal{U}^1 = (U^1_\alpha)_{\alpha>0}$ denotes the resolvent family of X_1, then $U^1_\alpha(1_{E_1\setminus E}) = 0$ and the restriction of $\mathcal{U}^1$ to E is precisely $\mathcal{U}$. Clearly we have in particular $U^1_\alpha = V_\alpha$, regarded as operators on $L^p(E_1, m_1)$ for all $\alpha > 0$, where m_1 is the extension of m from E to E_1, with zero on $E_1 \setminus E$.

Step II. Let $\mathcal{U}^* = (U^*_\alpha)_{\alpha>0}$ be the dual resolvent family of kernels on $(E, \mathcal{B})$, given by assertion (i). We consider its trivial extension from E to E_1 and we shall denote it also by $\mathcal{U}^*$. From (2.2.4) we get that $\mathcal{U}^*$ satisfies condition $(a.1)$ on E_1.

Let now $\mathcal{R}^*$ be a Ray cone associated with $\mathcal{U}^*_\beta$ and $\mathcal{R}^*_o$ a countable dense (in the supremum norm) subset of $\mathcal{R}^*$. By (2.3) there exist an m-inessential set $M \in \mathcal{B}_1$ and an m-negligible set $N \in \mathcal{B}_1$, such that for each $v^* \in \mathcal{R}^*_o$ there exists a function $v^{*\prime} \in bp\mathcal{B}$ which is finely continuous on $E_1 \setminus M$ and equals v^* on $E_1 \setminus N$. Modifying trivially the process X_1 and its resolvent $\mathcal{U}^1$ on the m-inessential set M (cf. (2.2.6)), we may assume that each $v^{*\prime}$ is finely continuous on E.

By Step I there exists a Lusin topological space $(\overline{E}, \mathcal{T}^*)$ such that $E_1 \in \overline{\mathcal{B}}$ (= the Borel σ-algebra on $\overline{E}$), $\mathcal{B}_1 = \overline{\mathcal{B}}|_{E_1}$ and a Borel right process $\overline{X}^*$ with state space $(\overline{E}, \mathcal{T}^*)$, such that $\mathcal{U}^*$ is the restriction to E_1 of its resolvent. Each $\mathcal{U}^*_\beta$-excessive function v^* has a unique extension $\widetilde{v^*}$ from E_1 to $\overline{E}$. Actually, one can take $(\overline{E}, \mathcal{T}^*)$ as being the topology on $\overline{E}$ generated by the set $\{\widetilde{v^*} : \ v^* \in \mathcal{R}^*_o\}$.

We consider now the trivial extension of $\mathcal{U}^1$ from E_1 to $\overline{E}$, denoted also by $\mathcal{U}^1$. Because $\mathcal{U}^1$ is the resolvent of a right process with state space E_1, it follows by (2.1) (the implication $(b) \Longrightarrow (a)$) that $\mathcal{U}^1$ satisfies conditions $(a.1)$ and $(a.2)$ on E_1. By (4.4) we deduce now that $(a.1)$ and $(a.2)$ are satisfied by $\mathcal{U}^1$ on $\overline{E}$. Again by (2.1) (the implication $(a) \Longrightarrow (b)$) we obtain the existence of the claimed Lusin topology on $\overline{E}$ and of the right process $\overline{X}$ with state space $(\overline{E}, \mathcal{T})$, having the resolvent $\mathcal{U}^1$. Since for every $v^* \in \mathcal{R}^*_o$ one has $v^* = v^{*\prime}$ on $E_1 \setminus N$, it follows that $\widetilde{v^*}|_{\overline{E}\setminus N}$ is finely continuous and thus the trace of $\mathcal{T}^*$ on $\overline{E} \setminus N$ is smaller than the trace on $\overline{E} \setminus N$ of the fine toplogy induced by $\overline{X}$. Therefore assertion (ii) holds. □

Corollary 2.4. *Assume that $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is the resolvent of a right process X with state space E, endowed with a Lusin topology $\mathcal{T}_0$, having $\mathcal{B}$ as Borel σ-algebra, and let m be a fixed $\mathcal{U}$-excessive measure. Then the following assertions hold:*
(i) (cf. [BeBoRö 06]) There exists a second sub-Markovian resolvent of kernels $\mathcal{U}^* = (U^*_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$ satisfying condition $(a.1)$ and such that $\mathcal{U}$ and $\mathcal{U}^*$ are in duality with respect to m.
(ii) There exist a larger Lusin measurable space $(\overline{E}, \overline{\mathcal{B}})$, with $E \subset \overline{E}$, $E \in \mathcal{B}$, $\mathcal{B} = \overline{\mathcal{B}}|_E$, two topologies $\mathcal{T}$ and $\mathcal{T}^*$ on $\overline{E}$ with $\sigma(\mathcal{T}) = \sigma(\mathcal{T}^*) = \overline{\mathcal{B}}$, $\mathcal{T}_0 \subset \mathcal{T}|_E$, and two processes $\overline{X}$ and $\overline{X}^*$ with common state space $\overline{E}$, such that $\overline{X}$ (resp. $\overline{X}^*$) is a Borel right process when $\overline{E}$ is endowed with the topology $\mathcal{T}$ (resp. $\mathcal{T}^*$), the restriction of $\overline{X}$ to E is precisely X, and the resolvents of $\overline{X}$ and $\overline{X}^*$ are in duality with respect to $\overline{m}$, where $\overline{m}$ is the extension of m from E to $\overline{E}$ with zero on $\overline{E} \setminus E$.

In addition, the set $\overline{E} \setminus E$ is $\overline{m}$-polar with respect to $\overline{X}$ and the β-level excessive function with respect to $\overline{X}^*$ are precisely the unique extensions by continuity in the fine topology generated by $\overline{X}^*$ of the $\mathcal{U}^*_\beta$-excessive functions (in particular, the set E is dense in $\overline{E}$ in the fine topology of $\overline{X}^*$).

Proof. Note that if $p > 1$ then $(\mathcal{U}_\alpha)_{\alpha>0}$ induces a strongly continuous resolvent of sub-Markovian contractions on $L^p(E, m)$. Assertion (i) follows by Theorem 2.3 $i)$ (see also [BeBoRö 06]).
(ii) The space $\overline{E}$ and the process $\overline{X}^*$ will be obtained applying (2.4) for the resolvent of kernels $\mathcal{U}^* = (U^*_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$. We consider the trivial extension of $\mathcal{U}$ to $\overline{E}$, which will be denoted by $\overline{\mathcal{U}} = (\bar{U}_\alpha)_{\alpha>0}$. By (2.2.4) it follows that $\mathcal{U}$ satisfies conditions $(a.1)$ and $(a.2)$ on $\overline{E}$. Let $\beta > 0$ and $\mathcal{R}$ be a Ray cone of $\mathcal{U}_\beta$-excessive functions such that the Ray topology $\mathcal{T}_\mathcal{R}$ generated by $\mathcal{R}$ is finer than $\mathcal{T}_0$. Let $\mathcal{R}_0$ be a countable subset of $\mathcal{R}$ which is dense in $\mathcal{R}$ in the sup norm and $\overline{\mathcal{R}_0}$ be a countable family of Borel extensions to $\overline{E}$ of the functions of $\mathcal{R}_0$. Let $\overline{\mathcal{R}}$ be a Ray cone with respect to $\mathcal{E}(\overline{\mathcal{U}}_\beta)$ such that $\overline{\mathcal{R}_0} \subset \overline{\mathcal{R}}$ and $\mathcal{T}$ be the topology on $\overline{E}$ generated by $\overline{\mathcal{R}}$. Then clearly $\mathcal{T}$ is a natural topology on $\overline{E}$ with respect to $\overline{\mathcal{U}}$ and $\mathcal{T}_0 \subset \mathcal{T}_\mathcal{R} \subset \mathcal{T}|_E$. The process $\overline{X}$ is given by the implication $(a) \Rightarrow (c)$ of (2.1). □

3 Applications

1. The martingale problem. Under the assumptions of Theorem 2.3 let L be the infinitesimal generator of $(V_\alpha)_{\alpha>0}$, having the domain $D(\mathsf{L}) = V_\beta\big(L^p(E, m)\big)$, $\mathsf{L}(V_\alpha f) = \alpha V_\alpha f - f$ for all $f \in L^p(E, m)$. Let further $(\mathsf{L}^*, D(\mathsf{L}^*))$ be the infinitesimal generator of the strongly continuous resolvent of contractions on $L^p(E,m)$ induced by the dual resolvent $\mathcal{U}^*$.

For a probability measure η on $(E,\mathcal{B})$ having density with respect to m, $\frac{d\eta}{dm} \in L^{p'}(E,m)$, where $\frac{1}{p}+\frac{1}{p'}=1$, consider the probability P^η of the process $\overline{X}$ (resp. $P^{*\eta}$ of the process $\overline{X}^*$) with the initial distribution η.

The next result shows that the processes X and $\overline{X}$ given by Theorem 2.3 solve the martingale problems of $(\mathsf{L}, D(\mathsf{L}))$ and respectively $(\mathsf{L}^*, D(\mathsf{L}^*))$, on a common larger state space $\overline{E}$

Corollary 3.1. *The following assertions hold.*

(i) The processes

$$u(X_t)-\int_0^t \mathsf{L}u(X_s)\mathrm{d}s, t\geq 0, u\in D(\mathsf{L}), \text{ and } u(\overline{X}^*_t)-\int_0^t \mathsf{L}^*u(\overline{X}^*_s)\mathrm{d}s, t\geq 0, u\in D(\mathsf{L}^*),$$

are martingales under P^η with respect to the filtration of $\overline{X}$ and respectively under $P^{*\eta}$ with respect to the filtration of $\overline{X}^*$.

(ii) If u is L-harmonic, i.e., $u \in D(\mathsf{L})$ and $\mathsf{L}u = 0$, then $u(X_t)$ and $u(\overline{X}^*_t)$ are martingales with respect to the corresponding probabilities and filtrations mentioned above.

Proof. Assertion (i) follows applying the general results on existence of solutions to martingale problems, obtained in [BeBoRö 06a] (see Theorem 2.3 and Proposition 1.4)

(ii) is a consequence (i) and of Proposition 3.2 from [BeCîRö 13], which states that in this situation L-harmonic is equivalent with L^*-harmonic. □

2. Riesz decomposition for excessive functions. Suppose further that $\mathcal{U}=(U_\alpha)_{\alpha>0}$ is the resolvent of a right (Markov) process with state space E. Recall that every $\xi \in Exc(\mathcal{U})$ has a unique decomposition, $\xi = \xi_1+\xi_2$, where $\xi_1 \in Pot(\mathcal{U})$ and $\xi_2 \in Har(\mathcal{U}) := \{\eta \in Exc(\mathcal{U}) : Pot(\mathcal{U}) \ni \nu \leq \eta \Rightarrow \nu = 0\}$.

Let m be a fixed $\mathcal{U}$-excessive measure and $\mathcal{U}^*=(U^*_\alpha)_{\alpha>0}$ be a second resolvent of kernels on $(E,\mathcal{B})$ satisfying conditions $(a.1)$ and such that $\mathcal{U}$ and $\mathcal{U}^*$ are in duality with respect to m; cf. assertion (i) of Corollary 2.4.

We denote by $\mathcal{E}_m(\mathcal{U})$ (resp. $Exc_m(\mathcal{U})$) the set of all $\mathcal{U}$-excessive functions which are finite m-a.e. (resp. the set of all $\mathcal{U}$-excessive measures which are absolutely continuous with respect to m). Notice that if $u,v \in \mathcal{E}_m(\mathcal{U})$ and $u=v$ m-a.e., then $u=v$ m-quasi everywhere (i.e., the set $[u \neq v]$ is m-polar). We shall identify the elements of $\mathcal{E}_m(\mathcal{U})$ which are equal m-a.e.

Recall that the map $\mathcal{E}_m(\mathcal{U}) \ni u \longmapsto u\cdot m \in Exc_m(\mathcal{U}^*)$ is a bijection between $\mathcal{E}_m(\mathcal{U})$ and $Exc_m(\mathcal{U}^*)$; see assertion (ii) of Remark 2.2. Taking into account this remark, a function $u \in \mathcal{E}_m(\mathcal{U})$ is called *potential* (resp. *harmonic*) provided that the $\mathcal{U}^*$-excessive measure $u\cdot m$ belongs to $\mathrm{Pot}(\mathcal{U}^*)$ (resp. $\mathrm{Har}(\mathcal{U}^*)$).

We consider the larger space $\overline{E}$ given by assertion (ii) of Corollary 2.4 and let $\overline{\mathcal{U}} = (\overline{\mathcal{U}}_\alpha)_{\alpha>0}$ (resp. $\overline{\mathcal{U}}^* = (\overline{\mathcal{U}}^*_\alpha)_{\alpha>0}$) be the resolvent of $\overline{X}$ (resp. of $\overline{X}^*$). We clearly have $\mathcal{E}_m(\mathcal{U}) \equiv \mathcal{E}_{\overline{m}}(\overline{\mathcal{U}})$ and $Exc_m(\mathcal{U}^*) = Exc_{\overline{m}}(\overline{\mathcal{U}}^*)$.

Corollary 3.2. *(Riesz decomposition). Every function $u \in \mathcal{E}_m(\mathcal{U})$ has a unique decomposition, $u = u_1 + u_2$, where u_1 is a potential and u_2 is harmonic with respect to $\overline{\mathcal{U}}$. The set of all potentials with respect to $\overline{\mathcal{U}}$ is a solid subset of $\mathcal{E}_m(\mathcal{U})$, i.e., if $u, v \in \mathcal{E}_m(\mathcal{U})$, $v \leq u$, and u is a potential then v is also a potential with respect to $\overline{\mathcal{U}}$.*

Proof. The assertions follow by the above considerations and the identification between $\mathcal{E}_m(\mathcal{U})$ and $Exc_{\overline{m}}(\mathcal{U}^*)$. The solidity of potentials is a consequence of Remark 2.1. □

3. Final remark. Since unique solutions to stochastic differential equations (SDE) give rise to Markov processes, Corollary 3.2 implies that we always have existence of a respective dual process enlarging the state space properly, which is still regular enough (i.e. right continuous and strong Markov). Such SDE include also those on infinite dimensional state space, so stochastic partial differential equations (SPDE) of evolutionary type are included. Prominent examples as the stochastic Navier-Stokes, the stochastic Burgers or the stochastic porous media equations are covered (see e.g. [DaZa 92], [PrRö 07] or [LiuRö 14] and the references therein). By Corollary 3.1 these dual processes solve a martingale problem in the usual way (w.r.t. the dual operator L^*), but it is not clear whether they also satisfy an S(P)DE, because for this one has to gain more knowledge about the domain $D(\mathsf{L}^*)$ of L^*.

References

[BeBo 03] Beznea, L. and Boboc, N., Sub-Markovian resolvents under weak duality hypothesis, *Probab. Theory Related Fields* **126** (2003), 339–363.

[BeBo 04a] Beznea, L. and Boboc, N., Fine densities for excessive measures and the Revuz correspondence, *Potential Anal.* **20** (2004), 61–83.

[BeBo 04b] Beznea, L. and Boboc, N., *Potential Theory and Right Processes*, Springer Series, Mathematics and Its Applications (572), Kluwer, Dordrecht, 2004.

[BeBo 06] Beznea, L. and Boboc, N., Weak Duality and the Dual Process for a Semi-Dirichlet form, *Infin. Dimen. Anal. Quantum Probab. Relat. Top.* **9** (2006), no.1, 27–46.

[BeBoRö 06] Beznea, L., Boboc, N., and Röckner, M., Quasi-regular Dirichlet forms and L^p-resolvents on measurable spaces, *Potential Anal.* **25** (2006), 269–282.

[BeBoRö 06a] Beznea, L., Boboc, N., and Röckner, M.: Markov processes associated with L^p-resolvents and applications to stochastic differential equations on Hilbert space. *J. Evol. Eq.*, **6** (2006), 745–772.

[BeCîRö 13] Beznea, L., Cîmpean, I., and Röckner, M.: *Transience, recurrence and irreducibility of a sub-Markovian resolvent of kernels*, 2013 (preprint).

[BeRö 11] Beznea, L. and Röckner, M., From resolvents to càdlàg processes through compact excessive functions and applications to singular SDE on Hilbert spaces, *Bull. Sci. Math.* **135** (2011), 844-870.

[CheFu 11] Chen, Z.-Q. and Fukushima, M., *Symmetric Markov Processes, Time Change, and Boundary Theory* (LMS-35), Princeton Univ. Press, 2011.

[ChuWa 05] Chung, K. L. and Walsh, J.B., *Markov Processes, Brownian Motion, and Time Symmetry* (Second edition), Springer, 2005.

[DaZa 92] Da Prato, G. and Zabczyk, J., *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, 1992.

[DeMe 87] Dellacherie, C., Meyer, P.A.: *Probabilités et potentiel. ch. XII-XVI.* Hermann, Paris, 1987

[FiGe 08] Fitzsimmons, P. J. and Getoor, R.K., Potential Theory of moderate Markov dual process, *Potential Anal.* **31** (2009), no.3, 275–310.

[Ge 90] Getoor, R.K., *Excessive measures*, Birkhäuser, 1990.

[GeSh 84] Getoor, R.K. and Sharpe, M.J., Naturality, standardness and weak duality for Markov processes, *Z. Wahrsch. Th. verw. Geb.* **67** (1984), 1–62.

[LiuRö 14] Liu, W. and Röckner, M., *Extension of [PrRö 07]* (in preparation).

[MaRö 92] Ma, Z.-M. and Röckner, M., *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms.* Universitext. Springer-Verlag, 1992.

[PrRö 07] Prévôt, C. and Röckner, M., *A Concise Course on Stochastic Partial Differential Equations*, Springer LNM **1905**, Springer, Berlin, 2007.