

# On the maximal inequalities of Burkholder, Davis and Gundy

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12 August 2013

## Abstract

We give a proof of the maximal inequalities of Burkholder, Davis and Gundy for real as well as Hilbert-space-valued local martingales using almost only stochastic calculus. Some parts of the exposition, especially in the infinite dimensional case, appear to be original.

*Keywords and phrases:* maximal inequalities, infinite dimensional stochastic analysis, semimartingales.

*2010 Mathematics Subject Classification:* 60G44

## 1 Introduction

The aim of this work is to provide a self-contained proof of the Burkholder-Davis-Gundy (BDG) inequality for local càdlàg martingales, both in finite and infinite dimension, using only stochastic calculus and functional-analytic arguments. In particular, in the case of real local martingales we provide a proof entirely based on stochastic calculus for semimartingales, and, in the case of Hilbert-space-valued local martingales, the proof uses some (relatively elementary) techniques from duality of Banach spaces, interpolation of operators, and vector measures. We also include a (known) proof for continuous local martingales, entirely based on stochastic calculus, which serves as motivation for the general case. Even though we do not claim to have any original result, some of the proofs appear to be new.

Let  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  be a filtered probability space and  $\mathbf{H}$  a real separable Hilbert space with norm  $\|\cdot\|$  and inner product  $(\cdot, \cdot)$ . The goal is to prove the following theorem.

**Theorem 1.1.** *Let  $M$  be an  $\mathbf{H}$ -valued  $\mathbb{F}$ -local càdlàg martingale with  $M_0 = 0$ . Then one has, for any  $p \in [1, \infty[$  and for any  $\mathbb{F}$ -stopping time  $\tau$ ,*

$$\mathbb{E}[M, M]_{\tau}^{p/2} \lesssim_p \mathbb{E} \sup_{t \leq \tau} \|M_t\|^p \lesssim_p \mathbb{E}[M, M]_{\tau}^{p/2}. \quad (1.1)$$

Moreover, if  $M$  is continuous, then (1.1) also holds for  $p \in ]0, 2[$ , i.e.

$$\mathbb{E} \sup_{t \leq \tau} \|M_t\|^p \approx_p \mathbb{E}[M, M]_{\tau}^{p/2} = \mathbb{E}\langle M, M \rangle_{\tau}^{p/2} \quad (1.2)$$

for all  $p \in ]0, \infty[$ .

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Here  $[M, M]$  and  $\langle M, M \rangle$  denote the (scalar) quadratic variation and Meyer process of  $M$ , respectively. Throughout the paper we use standard notation and terminology from the general theory of processes, our main reference being [11]. Setting  $\mathbb{L}_p := L_p(\Omega, \mathbb{P})$  for convenience of notation, one can equivalently write (1.1) as

$$\|M_\tau^*\|_{\mathbb{L}_p} \approx_p \|[M, M]_\tau^{1/2}\|_{\mathbb{L}_p},$$

where  $M_t^* := \sup_{s \leq t} \|M_s\|$ .

A proof of the Davis inequality (i.e. of (1.1) with  $p = 1$ ) for real martingales that uses only stochastic calculus was given by Meyer [12], adapting the corresponding proof for real continuous martingales by Gettoor and Sharpe [5], the key tool being a continuous-time version of Davis' decomposition, also proved in [12]. The contribution of this paper is to extend the stochastic calculus method of Meyer to the full range  $p \in [1, \infty[$  and to the Hilbert-space-valued case. As we are going to see, a proof based only on stochastic calculus is available for real martingales, but (unfortunately, perhaps) we have not been able to provide such a proof for Hilbert-valued martingales. Additional functional-analytic tools seem necessary. In particular, we could not extend an  $\mathbb{L}_p$  ( $p > 1$ ) inequality for compensators of processes with integrable variation to the Hilbert-valued case, although this problem was circumvented using a different estimate (which involves the quadratic rather than the first variation of compensators) and a simple duality argument.

While we are not aware of any standard reference where the BDG inequalities for general local martingales (i.e. not necessarily pathwise continuous) are proved by means of stochastic calculus only, a proof based on Garsia-Neveu-type lemmata can be found, for instance, in [3, 8, 10]. The latter proof covers also the more general case (not considered here) where the  $\mathbb{L}_p$  norm is replaced by an Orlicz norm.

We conclude this section with a few words about notation: we shall write  $a \lesssim b$  to mean that there exists a positive constant  $N$  such that  $a \leq Nb$ . If the constant  $N$  depends on the parameters  $p_1, \dots, p_n$ , we shall also write  $N = N(p_1, \dots, p_n)$  and  $\lesssim_{p_1, \dots, p_n}$ . The expression  $a \approx b$  is equivalent to  $a \lesssim b \lesssim a$ . The operator norm of an operator  $T : E \rightarrow F$ , with  $E$  and  $F$  two Banach spaces, will be denoted by  $\|T\|_{E \rightarrow F}$ . The spaces of bounded linear and bilinear maps from  $E$  to  $F$  will be denoted by  $\mathcal{L}(E, F)$  and  $\mathcal{L}_2(E, F)$ , respectively. Furthermore, every (local) martingale appearing below is assumed to be càdlàg.

## 2 First steps

Let us start with a simple but helpful remark: given a local  $\mathbf{H}$ -valued martingale with respect to the filtration  $\mathbb{F}$ , one has, for any  $\mathbb{F}$ -stopping time  $\tau$ ,  $[M^\tau, M^\tau]_\infty = [M, M]_\tau$  and  $M_\tau^* = (M^\tau)_\infty^*$ , where  $M^\tau$  denotes the stopped martingale  $t \mapsto M_{t \wedge \tau}$ . Therefore, by the monotone convergence theorem, proving (1.1) is equivalent to proving the inequalities

$$\|M_\infty^*\|_{\mathbb{L}_p} \lesssim_p \|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p} \lesssim_p \|M_\infty^*\|_{\mathbb{L}_p}. \quad (2.1)$$

for any  $\mathbf{H}$ -valued martingale  $M$  such that  $M_0 = 0$ . (We shall assume throughout, without explicit mention, that all (local) martingale start at zero). For convenience, we

shall call “upper bound” and “lower bound” the left-hand and the right-hand inequality in (2.1), respectively.

In this section we are going to show, using only stochastic calculus, that the upper bound holds for any  $p \geq 2$ , and that the lower bound holds in  $\mathbb{L}_{2p}$  provided the upper bound holds in  $\mathbb{L}_p$ .

**Proposition 2.1** (Upper bound,  $p > 2$ ). *Let  $M$  be an  $\mathbf{H}$ -valued martingale. One has, for any  $p \in ]2, \infty[$ ,*

$$\|M_\infty^*\|_{\mathbb{L}_p} \lesssim_p \|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p}. \quad (2.2)$$

*Proof.* Let us introduce a sequence of stopping times  $(T_n)_{n \in \mathbb{N}}$  defined by

$$T_n := \inf \{t \geq 0 : \|M_t\| > n\}.$$

Let  $n \in \mathbb{N}$  be fixed for the time being, and consider the stopped martingale  $M^{T_n}$ .

The first and second Fréchet derivatives of the function  $\phi : \mathbf{H} \ni x \mapsto \|x\|^p \in \mathbb{R}$  at a generic point  $x \in \mathbf{H}$  are given, respectively, by

$$\mathbf{H} \simeq \mathcal{L}(\mathbf{H}, \mathbb{R}) \ni D\phi(x) : u \mapsto p\|x\|^{p-2}(x, u)$$

and

$$\mathbf{H} \otimes \mathbf{H} \simeq \mathcal{L}_2(\mathbf{H}, \mathbb{R}) \ni D^2\phi(x) : (u, v) \mapsto p(p-1)\|x\|^{p-4}(x, u)(x, v).$$

Itô’s formula (in the form stated in [11, Thm. 27.2, p. 190]) yields, denoting the bilinear form  $(x, \cdot)(x, \cdot)$  by  $x \otimes x$ , and writing  $T$  instead of  $T_n$  for simplicity,

$$\begin{aligned} \|M_T\|^p &= p \int_0^T \|M_{s-}\|^{p-2} M_{s-} dM_s \\ &\quad + \frac{1}{2}p(p-1) \int_0^T \|M_s\|^{p-4} (M_s \otimes M_s) d[[M, M]_s^c] \\ &\quad + \sum_{s \leq T} \left( \|M_s\|^p - \|M_{s-}\|^p - p\|M_{s-}\|^{p-2} (M_{s-}, \Delta M_s) \right). \end{aligned} \quad (2.3)$$

Let  $(e_i)_{i \in \mathbb{N}}$  be an orthonormal basis of  $\mathbf{H}$ , and set  $M^i = \langle M, e_i \rangle$  for all  $i \in \mathbb{N}$ . Then we have, thanks to Kunita-Watanabe’s inequality,

$$\begin{aligned} &\int_0^T \|M_s\|^{p-4} (M_s \otimes M_s) d[[M, M]_s^c] \\ &= \sum_{i,j} \int_0^T \|M_s\|^{p-4} M_s^i M_s^j d[M^i, M^j]_s^c \\ &\leq \sum_{i,j} \left( \int_0^T \|M_s\|^{p-4} (M_s^i)^2 d[M^j, M^j]_s^c \right)^{1/2} \left( \int_0^T \|M_s\|^{p-4} (M_s^j)^2 d[M^i, M^i]_s^c \right)^{1/2} \\ &=: \sum_{i,j} a_{ij}^{1/2} b_{ij}^{1/2} = \|a^{1/2} b^{1/2}\|_{\ell_1} \leq \|a^{1/2}\|_{\ell_2} \|b^{1/2}\|_{\ell_2} \\ &= \left( \sum_{i,j} \int_0^T \|M_s\|^{p-4} (M_s^i)^2 d[M^j, M^j]_s^c \right)^{1/2} \left( \sum_{i,j} \int_0^T \|M_s\|^{p-4} (M_s^j)^2 d[M^i, M^i]_s^c \right)^{1/2} \\ &= \int_0^T \|M_s\|^{p-2} d[M, M]_s^c \leq (M_T^*)^{p-2} [M, M]_T^c. \end{aligned}$$

Using Taylor's formula with remainder in Lagrange form, one obtains

$$\begin{aligned} & \sum_{s \leq T} (\|M_s\|^p - \|M_{s-}\|^p - p\|M_{s-}\|^{p-2} \langle M_{s-}, \Delta M_s \rangle) \\ & \leq \frac{p(p-1)}{2} \sup_{s \leq T} \|M_{s-}\|^{p-2} \sum_{s \leq T} \|\Delta M_s\|^2 \leq \frac{p(p-1)}{2} (M_T^*)^{p-2} [M, M]_T^d, \end{aligned}$$

hence also

$$\begin{aligned} & \frac{1}{2} p(p-1) \int_0^T \|M_s\|^{p-4} (M_s \otimes M_s) d[M, M]_s^c \\ & \quad + \sum_{s \leq T} (\|M_s\|^p - \|M_{s-}\|^p - p\|M_{s-}\|^{p-2} \langle M_{s-}, \Delta M_s \rangle) \\ & \leq \frac{p(p-1)}{2} (M_T^*)^{p-2} ([M, M]_T^c + [M, M]_T^d) = \frac{p(p-1)}{2} (M_T^*)^{p-2} [M, M]_T. \end{aligned}$$

Since  $M_T^* \leq n + [M, M]_T^{1/2}$  and  $[M, M]_T^{1/2} \in \mathbb{L}_p$  imply  $M_T^* \in \mathbb{L}_p$ , the first term on the right-hand side of (2.3) is a martingale, hence its expectation is zero. Therefore, taking expectation in (2.3), Doob's and Hölder's inequalities yield

$$\mathbb{E}(M_T^*)^p \lesssim_p \mathbb{E}\|M_T\|^p \lesssim_p (\mathbb{E}(M_T^*)^p)^{\frac{p-2}{p}} (\mathbb{E}[M, M]_T^{p/2})^{2/p},$$

which is equivalent to

$$\|M_{T_n}^*\|_{\mathbb{L}_p} \lesssim_p \|[M, M]_{T_n}^{1/2}\|_{\mathbb{L}_p}.$$

Passing to the limit as  $n \rightarrow \infty$ , the proof is completed thanks to the monotone convergence theorem.  $\square$

**Corollary 2.2** (Lower bounds by upper bounds). *Let  $M$  be an  $\mathbf{H}$ -valued martingale, and assume that the upper bound (2.2) holds for some  $p \geq 1$ . Then one has*

$$\|[M, M]_\infty^{1/2}\|_{\mathbb{L}_{2p}} \lesssim_p \|M_\infty^*\|_{\mathbb{L}_{2p}}.$$

*Proof.* The integration by parts formula  $[M, M] = \|M\|^2 - 2M_- \cdot M$  yields

$$[M, M]_\infty^{1/2} \leq M_\infty^* + \sqrt{2} \sqrt{(M_- \cdot M)_\infty^*},$$

which implies, by the previous proposition,

$$\begin{aligned} \|[M, M]_\infty^{1/2}\|_{\mathbb{L}_{2p}} & \leq \|M_\infty^*\|_{\mathbb{L}_{2p}} + \sqrt{2} \|(M_- \cdot M)_\infty^*\|_{\mathbb{L}_p}^{1/2} \\ & \lesssim_p \|M_\infty^*\|_{\mathbb{L}_{2p}} + \|[M_- \cdot M, M_- \cdot M]_\infty^{1/2}\|_{\mathbb{L}_p}^{1/2}. \end{aligned}$$

One easily sees that, for any predictable process  $\Phi$  with values in  $\mathcal{L}(\mathbf{H}, \mathbb{R}) \simeq \mathbf{H}$ , one has  $[\Phi \cdot M, \Phi \cdot M] \leq \|\Phi\| \cdot [M, M]$ . Therefore, by Young's inequality, for any  $\varepsilon > 0$  there exists  $N = N(\varepsilon)$  such that

$$\begin{aligned} [M_- \cdot M, M_- \cdot M]_\infty^{1/2} & \leq (\|M_-\|^2 \cdot [M, M])_\infty^{1/2} \\ & \leq M_\infty^* [M, M]_\infty^{1/2} \leq N(M_\infty^*)^2 + \varepsilon [M, M]_\infty, \end{aligned}$$

whence

$$\begin{aligned} \|[M_- \cdot M, M_- \cdot M]_\infty^{1/2}\|_{\mathbb{L}_p} &\leq N \|(M_\infty^*)^2\|_{\mathbb{L}_p} + \varepsilon \|[M, M]_\infty\|_{\mathbb{L}_p} \\ &= N \|M_\infty^*\|_{\mathbb{L}_{2p}}^2 + \varepsilon \|[M, M]_\infty^{1/2}\|_{\mathbb{L}_{2p}}^2, \end{aligned}$$

which in turn implies

$$\|[M, M]_\infty^{1/2}\|_{\mathbb{L}_{2p}} \lesssim_p (1 + \sqrt{N}) \|M_\infty^*\|_{\mathbb{L}_{2p}} + \sqrt{\varepsilon} \|[M, M]_\infty^{1/2}\|_{\mathbb{L}_{2p}}.$$

The proof is completed by choosing  $\varepsilon$  sufficiently small.  $\square$

*Remark 2.3.* We learned about the simple arguments of the above proofs in [11], where the upper bound is stated as exercise 6.E.3. Later we found that the argument used in the proof of the lower bound is the same used by Gettoor and Sharpe [5] (in the simpler case of real continuous martingale), who in turn attribute it to Garsia (probably in the form of a personal communication or an unpublished manuscript). Metivier actually writes that the lower bound is “easy” for any  $p \geq 2$ . Unfortunately we have not been able to find an easy proof for the case  $2 \leq p < 4$ . Let us also mention that the proof of the above corollary appears also in, e.g., [13] (for continuous real martingales, but the argument above is almost literally the same).

### 3 Continuous local martingales

We provide a proof of (1.2) based only on stochastic calculus, following [5] (this proof has been reproduced *verbatim* in some textbooks, see e.g. [6, 14]). This way one can clearly see the arguments that will be used in Section 5 to treat discontinuous martingales, without the many complications that appear in the general case.

The main idea is that, since we do not have “tools” to estimate the  $\mathbb{L}_p$ -norms of  $M_\infty^*$  and of  $[M, M]_\infty^{1/2}$ , we try to reduce to a situation where estimates of  $\mathbb{L}_2$ -norms of the maximum and of the quadratic variation of an auxiliary local martingale  $N$  (that are already known to hold) would suffice. Reading the proofs in [5], it might not be clear, at least at a first sight, how the auxiliary martingales  $N$  are chosen. Our (very minor) contribution is to show why it is natural to choose precisely those  $N$ .

The proof of (1.2) is split in several propositions.

**Proposition 3.1** (Upper bound,  $p < 2$ ). *Let  $M$  be an  $\mathbf{H}$ -valued continuous martingale. One has, for any  $p \in ]0, 2]$ ,*

$$\|M_\infty^*\|_{\mathbb{L}_p} \lesssim_p \|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p} = \|\langle M, M \rangle_\infty^{1/2}\|_{\mathbb{L}_p}.$$

*Proof.* We apply the “principle” outlined above, that is, we look for an auxiliary (local) martingale  $N$  which is more manageable than  $M$ , and we exploit the inequality  $\mathbb{E}(N_\infty^*)^2 \lesssim \mathbb{E}[N, N]_\infty$ . Let us set  $N = H \cdot M$ , where the integrand  $H$  is a predictable real process to be determined. Note that the identity  $[N, N] = H^2 \cdot [M, M]$  holds and, by the fundamental theorem of calculus,

$$[M, M]_\infty^{p/2} = \frac{p}{2} \int_0^\infty [M, M]_s^{p/2-1} d[M, M]_s.$$

It is thus natural to choose  $H = \sqrt{p/2}(\varepsilon + [M, M])^{p/4-1/2}$ , so that  $[N, N] = (\varepsilon + [M, M])^{p/2}$ , where  $\varepsilon > 0$  is introduced to avoid singularities. Let us now try to obtain a relation between  $N_\infty^*$  and  $M_\infty^*$ . Observe that, by the associativity property of the stochastic integral, we have  $M = H^{-1} \cdot N$ , as well as, by the integration by parts formula,

$$M_\infty = (H^{-1} \cdot N)_\infty = H_\infty^{-1} N_\infty - \int_0^\infty N_s dH_s^{-1},$$

which implies (taking into account that  $s \mapsto H_s$  is decreasing, hence  $s \mapsto H_s^{-1}$  is increasing)

$$\|M_\infty\| \leq H_\infty^{-1} N_\infty^* + N_\infty^* \int_0^\infty dH_s^{-1} = 2H_\infty^{-1} N_\infty^*,$$

thus also  $M_\infty^* \leq 2H_\infty^{-1} N_\infty^*$ , as well as  $\|M_\infty^*\|_{\mathbb{L}_p} \leq 2\|H_\infty^{-1} N_\infty^*\|_{\mathbb{L}_p}$ . In order to obtain an expression involving the  $\mathbb{L}_2$  norm of  $N_\infty^*$ , we apply Hölder's inequality in the form

$$\|XY\|_{\mathbb{L}_p} \leq \|X\|_{\mathbb{L}_2} \|Y\|_{\mathbb{L}_q}, \quad \frac{1}{p} = \frac{1}{2} + \frac{1}{q},$$

which yields

$$\|M_\infty^*\|_{\mathbb{L}_p} \leq 2\|H_\infty^{-1} N_\infty^*\|_{\mathbb{L}_p} \leq 2\|H_\infty^{-1}\|_{\mathbb{L}_q} \|N_\infty^*\|_{\mathbb{L}_2}.$$

By the definition of  $H$  and the identity  $q = 2p/(2-p)$ , one has

$$\|H_\infty^{-1}\|_{\mathbb{L}_q} = \sqrt{2/p} \|(\varepsilon + [M, M])_\infty^{1/2}\|_{\mathbb{L}_p}^{1-p/2},$$

as well as

$$\|N_\infty^*\|_{\mathbb{L}_2} \leq 2\|[N, N]_\infty^{1/2}\|_{\mathbb{L}_2} = 2\|(\varepsilon + [M, M])_\infty^{1/2}\|_{\mathbb{L}_p}^{p/2},$$

which allows us to conclude that

$$\|M_\infty^*\|_{\mathbb{L}_p} \leq 4\sqrt{2/p} \|(\varepsilon + [M, M])_\infty^{1/2}\|_{\mathbb{L}_p}.$$

The proof is finished by observing that  $\varepsilon > 0$  is arbitrary, hence the previous inequality also holds with  $\varepsilon = 0$ .  $\square$

**Proposition 3.2** (Lower bound,  $p > 2$ ). *Let  $M$  be an  $\mathbf{H}$ -valued continuous martingale. One has, for any  $p \in ]2, \infty]$ ,*

$$\|\langle M, M \rangle_\infty^{1/2}\|_{\mathbb{L}_p} = \|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p} \lesssim_p \|M_\infty^*\|_{\mathbb{L}_p}.$$

*Proof.* Let us set, in analogy to the proof of the previous proposition,

$$N := H \cdot M, \quad H := \sqrt{p/2}[M, M]^{p/4-1/2},$$

so that  $[N, N] = [M, M]^{p/2}$ , which implies

$$\|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p}^{p/2} = \|[N, N]_\infty^{1/2}\|_{\mathbb{L}_2} = \|N_\infty\|_{\mathbb{L}_2(H)} \leq \|N_\infty^*\|_{\mathbb{L}_2}.$$

The integration-by-parts formula yields

$$N_\infty = (H \cdot M)_\infty = H_\infty M_\infty - \int_0^\infty M_s dH_s,$$

from which one infers, since  $s \mapsto H_s$  is increasing (because  $p/2 - 1 > 0$ ), that  $\|N_\infty\| \leq N_\infty^* \leq 2H_\infty M_\infty^*$ . This in turn implies

$$\|N_\infty^*\|_{\mathbb{L}_2} \leq 2\|H_\infty M_\infty^*\|_{\mathbb{L}_2} \leq 2\|H_\infty\|_{\mathbb{L}_q} \|M_\infty^*\|_{\mathbb{L}_p},$$

where  $1/2 = p^{-1} + q^{-1}$ , i.e.  $q = 2p/(p-2)$ . Using the definition of  $H$ , one has

$$\|H_\infty\|_{\mathbb{L}_q} = \sqrt{p/2} \|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p}^{p/2-1},$$

hence

$$\|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p}^{p/2} \leq \|N_\infty^*\|_{\mathbb{L}_2} \leq \sqrt{p/2} \|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p}^{p/2-1} \|M_\infty^*\|_{\mathbb{L}_p},$$

which implies the results by simplifying and rearranging terms.  $\square$

Note that both inequalities proved in the last two propositions relied on (essentially) the same auxiliary local martingale. However, unfortunately it seems difficult to use once again the same construction to prove the lower bound in the case  $p \in ]0, 2[$ . One can immediately convince himself about this by inspection of the proof of Proposition 3.1. On the other hand, a similar proof will still do, provided a different auxiliary martingale is used.

**Proposition 3.3** (Lower bound,  $p < 2$ ). *Let  $M$  be an  $\mathbf{H}$ -valued continuous martingale. One has, for any  $p \in ]0, 2[$ ,*

$$\|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p} \lesssim_p \|M_\infty^*\|_{\mathbb{L}_p}.$$

*Proof.* We introduce once again an auxiliary local martingale  $N := H \cdot M$ , and then we compare the  $\mathbb{L}_2$ -norms of  $[N, N]_\infty^{1/2}$  and  $N_\infty^*$ . It is (intuitively) clear that, in order to exploit the inequality  $\|[N, N]_\infty^{1/2}\|_{\mathbb{L}_2} \lesssim \|N_\infty^*\|_{\mathbb{L}_2}$ , one would need to establish an upper bound for  $N_\infty^*$  in terms of  $M_\infty^*$ . For this purpose, let us use once again the integration-by-parts formula, assuming that  $H$  is a real predictable process with finite variation which will be defined later. Then one has

$$N_t = (H \cdot M)_t = H_t M_t + \int_0^t M_s d(-H_s) \quad \forall t > 0.$$

This “starting point” already suggests how to choose  $H$ : in fact, neglecting the integral on the right hand side, we see that  $\mathbb{E}\|N_t\|^2$  should be of the order of  $\mathbb{E}H_t^2(M_t^*)^2$ , and we would like this term to be of the order of  $\mathbb{E}(M_t^*)^p$ , which suggests that we may try taking  $H$  of the order of  $(M^*)^{p/2-1}$ . Let us then set

$$H := (\varepsilon + M^*)^{p/2-1},$$

where  $\varepsilon > 0$  is arbitrary and is introduced to avoid singularities. The identity  $[N, N] = H^2 \cdot [M, M]$  implies  $[N, N]_\infty \geq H_\infty^2 [M, M]_\infty$ , because  $s \mapsto H_s^2$  is decreasing. Similarly, the integration-by-parts formula, the definition of  $H$ , and elementary calculus imply the

estimate

$$\begin{aligned}
N_\infty^* &\leq H_\infty M_\infty^* + \int_0^t M_s^* d(-H_s) \\
&\leq (\varepsilon + M_\infty^*)^{p/2} + \int_0^\infty (\varepsilon + M_s^*) d(-(\varepsilon + M_s^*)^{p/2-1}) \\
&\leq (\varepsilon + M_\infty^*)^{p/2} + (1 - p/2) \int_0^\infty (\varepsilon + M_s^*)^{p/2-1} d(\varepsilon + M_s^*) \\
&\leq \frac{2}{p} (\varepsilon + M_\infty^*)^{p/2}.
\end{aligned}$$

Collecting estimates and taking  $\mathbb{L}_2$ -norms, we have

$$\begin{aligned}
\|H_\infty[M, M]_\infty^{1/2}\|_{\mathbb{L}_2} &\leq \|[N, N]_\infty^{1/2}\|_{\mathbb{L}_2} \\
&\leq \|N_\infty^*\|_{\mathbb{L}_2} \leq \frac{2}{p} \|(\varepsilon + M_\infty^*)^{p/2}\|_{\mathbb{L}_2} = \frac{2}{p} \|\varepsilon + M_\infty^*\|_{\mathbb{L}_p}^{p/2}.
\end{aligned}$$

In order to obtain a term depending on the  $\mathbb{L}_p$  norm of  $[M, M]_\infty^{1/2}$  on the left-hand side, we proceed as follows: let  $q > 0$  be defined by the relation  $p^{-1} = 1/2 + q^{-1}$ , i.e.  $q = 2p/(2 - p)$ , and write, using Hölder's inequality,

$$\begin{aligned}
\|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p} &= \|H_\infty^{-1} H_\infty[M, M]_\infty^{1/2}\|_{\mathbb{L}_p} \leq \|H_\infty^{-1}\|_{\mathbb{L}_q} \|H_\infty[M, M]_\infty^{1/2}\|_{\mathbb{L}_2} \\
&\leq \frac{2}{p} \|H_\infty^{-1}\|_{\mathbb{L}_q} \|\varepsilon + M_\infty^*\|_{\mathbb{L}_p}^{p/2},
\end{aligned}$$

where, by the definition of  $H$  and elementary computations,  $\|H_\infty^{-1}\|_{\mathbb{L}_q} = \|\varepsilon + M_\infty^*\|_{\mathbb{L}_p}^{1-p/2}$ .

We have thus proved the inequality  $\|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p} \leq (2/p) \|\varepsilon + M_\infty^*\|_{\mathbb{L}_p}^{p/2}$ , which is valid also for  $\varepsilon = 0$ , since  $\varepsilon$  is arbitrary. The proof is thus finished.  $\square$

*Remark 3.4.* It is possible to give an alternative very short proof of (1.2) that involves little more than just Itô's formula. In fact, appealing to Lenglart's domination inequality (see [9]), one can show that once either the (lower or upper) bound holds in  $\mathbb{L}_p$ , then it holds in  $\mathbb{L}_q$  for all  $q \in ]0, p[$ . In particular, the bounds of Section 2 are enough to prove Theorem 1.1 for continuous local martingales (cf. [13] for more detail). This method, however, does not work for general local martingales.

## 4 Auxiliary results

### 4.1 Calculus for functions of finite variation

We shall denote the variation of a function  $f : \mathbb{R}_+ \rightarrow H$  by  $\int_0^\infty |df|$ . Recall that, if  $f$  has finite variation and  $f(0) = 0$ , then  $f^*$  is bounded by the variation of  $f$  (in fact,  $f^*$  is bounded by the oscillation of  $f$ , which is in turn bounded by the variation of  $f$ ).

Let  $U$  and  $V$  be two  $\mathbf{H}$ -valued functions with finite variation. Then the following integration-by-parts formula holds

$$\langle U, V \rangle = U_- \cdot V + V_- \cdot U + \sum (\Delta U, \Delta V).$$



Since the series in the previous expression can be written as  $(V - V_-) \cdot U$ , one also has

$$(U, V) = U_- \cdot V + V \cdot U \equiv \int U_- dV + \int V dU. \quad (4.1)$$

Using the integration-by-parts formula for semimartingales, it is immediately seen that (4.1) still holds if only one of  $U$  and  $V$  is a process with finite variation and the other one is a semimartingale (and appropriate measurability conditions are satisfied).

Calculus rules for functions (and processes) of finite variation may differ substantially from the “classical” calculus rules for continuous functions. In this section we collect some elementary identities that will be needed in the sequel. In particular, if  $U \equiv V$ , (4.1) yields

$$dU^2 = (U_- + U) dU, \quad (4.2)$$

which also implies, if  $U > 0$ ,

$$d(U^{1/2}) = \frac{1}{U_-^{1/2} + U^{1/2}} dU. \quad (4.3)$$

Assume now  $U \geq \delta$  for some  $\delta > 0$ , and set  $V = 1/U$ . Then (4.1) yields

$$1 = U/U = - \int U_- d(-1/U) + \int (1/U) dU,$$

or equivalently

$$d(-1/U) = \frac{1}{UU_-} dU. \quad (4.4)$$

**Lemma 4.1.** *Let  $V$  be an increasing function with  $V_0 = 0$ . Then one has*

$$\int_0^t V_{s-} d(V_s^{q-1}) \leq \frac{q-1}{q} V_t^q \quad \forall q \in ]1, \infty[$$

and

$$\int_0^t V_{s-} d(-V_s^{q-1}) \leq \frac{1-q}{q} V_t^q \quad \forall q \in ]0, 1[.$$

*Proof.* Let  $0 = t_0 < t_1 < \dots < t_n = t$  be a finite partition of  $[0, t]$ . For  $q \geq 1$ , one has

$$\int_0^t V_{s-} d(V_s^{q-1}) = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} V_{t_i} (V_{t_{i+1}}^{q-1} - V_{t_i}^{q-1}),$$

hence the first inequality is proved if we can show that

$$x_1 (x_2^{q-1} - x_1^{q-1}) \leq \frac{q-1}{q} (x_2^q - x_1^q) \quad \forall 0 \leq x_1 \leq x_2.$$

In fact, by the fundamental theorem of calculus, one has  $x_2^q - x_1^q \leq q(x_2 - x_1)x_2^{q-1}$ , which implies, after a few elementary computations,

$$x_2^q \geq -\frac{1}{q-1} x_1^q + \frac{q}{q-1} x_1 x_2^{q-1},$$

hence also

$$\frac{q-1}{q}(x_2^q - x_1^q) \geq x_1(x_2^{q-1} - x_1^{q-1}).$$

Let us now turn to the case  $0 < q < 1$ . Note that, in principle, we cannot write

$$\int_0^t V_{s-} d(-V_s^{q-1}) = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} V_{t_i} (V_{t_i}^{q-1} - V_{t_{i+1}}^{q-1})$$

because  $V_0 = 0$ . However, a simple regularization of the type  $V_0 = \varepsilon > 0$  and then passing to the limit as  $\varepsilon \rightarrow 0$  at the end of the computations would suffice. Hence we can proceed in a slightly formal (but harmless) way accepting the previous identity as true, and, in analogy to the case  $q > 1$ , it is enough to show that

$$V_{t_i} (V_{t_i}^{q-1} - V_{t_{i+1}}^{q-1}) \leq \frac{1-q}{q} (V_{t_{i+1}}^q - V_{t_i}^q) \quad \forall i \in \{0, 1, \dots, n-1\}.$$

The latter inequality certainly holds true if one has

$$x_2^q - x_1^q \geq \frac{q}{1-q} x_1 (x_1^{q-1} - x_2^{q-1}) \quad \forall 0 \leq x_1 \leq x_2.$$

Let  $0 \leq x_1 \leq x_2$ . Since  $x \mapsto x^{q-1}$  is decreasing, the fundamental theorem of calculus yields

$$x_2^q - x_1^q = q \int_{x_1}^{x_2} y^{q-1} dy \geq qx_2^q - qx_1x_2^{q-1}.$$

Rearranging terms, this implies

$$x_2^q \geq \frac{1}{1-q} x_1^q - \frac{q}{1-q} x_1 x_2^{q-1},$$

hence also

$$x_2^q - x_1^q \geq \frac{q}{1-q} (x_1^q - x_1 x_2^{q-1}),$$

which is the desired inequality.  $\square$

We shall also need some  $\mathbb{L}_p$  estimates for compensators of processes with integrable variation.

**Proposition 4.2.** *Let  $V$  be a real increasing process with compensator  $\tilde{V}$ . Then*

$$\left\| \int_0^\infty |d\tilde{V}| \right\|_{\mathbb{L}_p} \leq p \left\| \int_0^\infty |dV| \right\|_{\mathbb{L}_p} \quad \forall 1 \leq p < \infty.$$

*Proof.* Since  $V$  is increasing, then  $\tilde{V}$  is also increasing<sup>1</sup>, hence we only have to prove  $\|\tilde{V}_\infty\|_{\mathbb{L}_p} \leq p \|V_\infty\|_{\mathbb{L}_p}$ . We have

$$\|\tilde{V}_\infty\|_{\mathbb{L}_p} = \sup_{\xi \in B_1(\mathbb{L}_q)} \mathbb{E} \xi \tilde{V}_\infty,$$

<sup>1</sup>An overkill proof of this fact is that  $V$  is a submartingale, and  $V = (V - \tilde{V}) + \tilde{V}$  is its Doob-Meyer decomposition.

where  $q$  is the conjugate exponent of  $p$  and  $B_1(\mathbb{L}_q)$  stands for the unit ball of  $\mathbb{L}_q$ . Let  $\xi \in B_1(\mathbb{L}_q)$  be arbitrary but fixed, and introduce the martingale  $N$  defined by  $N_t := \mathbb{E}[\xi | \mathcal{F}_t]$ ,  $t \geq 0$ ,  $N_\infty := \xi$ . The integration-by-parts formula yields

$$\xi \tilde{V}_\infty = N_\infty \tilde{V}_\infty = (\tilde{V} \cdot N)_\infty + (N_- \cdot \tilde{V})_\infty,$$

hence, using the definition of compensator, the fact that  $V$  is increasing, and Hölder's inequality, one obtains

$$\mathbb{E} \xi \tilde{V}_\infty = \mathbb{E}(N_- \cdot V)_\infty \leq \mathbb{E} N_\infty^* V_\infty \leq \|N_\infty^*\|_{\mathbb{L}_q} \|V_\infty\|_{\mathbb{L}_p}.$$

Since, by Doob's inequality, one has

$$\|N_\infty^*\|_{\mathbb{L}_q} \leq p \|N_\infty\|_{\mathbb{L}_q} = p \|\xi\|_{\mathbb{L}_q} \leq p,$$

the conclusion follows because  $\xi$  is arbitrary.  $\square$

The following proposition extends, in the case  $p = 1$ , the previous inequality to Hilbert-space-valued processes.

**Proposition 4.3.** *Let  $X : \mathbb{R}_+ \rightarrow H$  be a right-continuous measurable process such that  $\mathbb{E} \int |dX| < \infty$ . Then  $X$  admits a dual predictable projection (compensator)  $\tilde{X}$ , which satisfies*

$$\mathbb{E} \int_0^\infty |d\tilde{X}| \leq \mathbb{E} \int_0^\infty |dX|. \quad (4.5)$$

*Proof.* The existence of  $\tilde{X}$ , as a process with integrable variation, follows by a result due to Dinculeanu (see [4, Thm. 22.8, p. 278]). Let  $\mathcal{M} := \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ . By [4, Thm. 19.8, p. 220], there exists a  $\sigma$ -additive measure  $\mu_X : \mathcal{M} \rightarrow H$  with finite variation  $|\mu_X|$  such that

$$\mu_X(A) = \mathbb{E} \int_0^\infty 1_A dX \quad \forall A \in \mathcal{M}$$

and

$$|\mu_X|(A) = \mathbb{E} \int_0^\infty 1_A |dX| \quad \forall A \in \mathcal{M}.$$

The dual predictable projection  $\mu_X^p$  of  $\mu_X$  is defined by

$$\mu_X^p : \mathcal{M} \ni A \mapsto \int_{\mathbb{R}_+ \times \Omega} {}^p 1_A d\mu_X,$$

where  ${}^p Y$  denotes the predictable projection of a measurable process  $Y$ . The dual predictable projection (compensator)  $\tilde{X}$  is constructed as the unique process associated to the measure  $\mu_{\tilde{X}}^p =: \mu_{\tilde{X}}$ . We can thus write

$$\mathbb{E} \int_0^\infty |d\tilde{X}| = \int_{\mathbb{R}_+ \times \Omega} 1 d|\mu_{\tilde{X}}| = \int_{\mathbb{R}_+ \times \Omega} 1 d|\mu_X^p|,$$

where, by [4, Thm. 22.1, p. 272],  $|\mu_X^p| \leq |\mu_X|^p$ . Taking into account that, if there exists a constant  $N$  such that  $|Y| \leq N$  then  ${}^p Y \leq N$  outside an evanescent set (see e.g. [3, §VI.43]), one has, by the definition of dual predictable projection of a measure,

$$\mathbb{E} \int_0^\infty |d\tilde{X}| \leq \int_{\mathbb{R}_+ \times \Omega} 1 d|\mu_X|^p \leq \int_{\mathbb{R}_+ \times \Omega} 1 d|\mu_X| = \mathbb{E} \int_0^\infty |dX|. \quad \square$$

## 4.2 An extension of the Riesz-Thorin interpolation theorem

We quote, omitting the proof, a generalization of the Riesz-Thorin interpolation theorem, dealing with  $L_p$  spaces with mixed norm.

Let  $(X_1, \mu_1), (X_2, \mu_2), \dots, (X_n, \mu_n)$  be measure spaces, and  $p_1, p_2, \dots, p_n \in [1, \infty]$ . Setting  $\mathbf{p} = (p_1, p_2, \dots, p_n)$  and  $1/\mathbf{p} := (1/p_1, 1/p_2, \dots, 1/p_n)$  for convenience of notation, let us define the following spaces of integrable functions with mixed norm:

$$L_{\mathbf{p}} := L_{p_1, \dots, p_n} := L_{p_1}(X_1 \rightarrow L_{p_2, \dots, p_n}, \mu_1), \quad \dots, \quad L_{p_n} := L_{p_n}(X_n, \mu_n).$$

The following result is due to Benedek and Panzone [1, p. 316].

**Theorem 4.4.** *Let  $T$  be a linear operator such that*

$$\|T\|_{L_{\mathbf{p}_0} \rightarrow L_{\mathbf{q}_0}} \leq M_0, \quad \|T\|_{L_{\mathbf{p}_1} \rightarrow L_{\mathbf{q}_1}} \leq M_1,$$

with  $\mathbf{p}_0, \mathbf{p}_1, \mathbf{q}_0, \mathbf{q}_1 \in [1, \infty]^n$ . Let  $\theta \in ]0, 1[$  and define  $\mathbf{p}, \mathbf{q}$  through

$$\frac{\theta}{\mathbf{p}_1} + \frac{1-\theta}{\mathbf{p}_0} = \frac{1}{\mathbf{p}}, \quad \frac{\theta}{\mathbf{q}_1} + \frac{1-\theta}{\mathbf{q}_0} = \frac{1}{\mathbf{q}}.$$

Then one has

$$\|T\|_{L_{\mathbf{p}} \rightarrow L_{\mathbf{q}}} \leq M_0^{1-\theta} M_1^\theta.$$

Let  $(X_1, \mu_1), (X_2, \mu_2)$  be two measure spaces, and  $H$  a real separable Hilbert space. We shall say that a map  $T : L_0(X_1 \rightarrow H, \mu_1) \rightarrow L_0(X_2, \mu_2)$  is *sublinear* if

- (a)  $|T(f_1 + f_2)| \leq |Tf_1| + |Tf_2|$  for all  $f_1, f_2 \in L_0(X_1 \rightarrow H, \mu_1)$ ;
- (b)  $|T(\alpha f)| = |\alpha| |Tf|$  for all  $\alpha \in \mathbb{R}$  and  $f \in L_0(X_1 \rightarrow H, \mu_1)$ .

The following result can be deduced by the previous theorem identifying  $H$  with  $\ell_2$ , and by using the linearization method of [7] (cf. also [2]) to cover the case of sublinear operators.

**Theorem 4.5.** *Let  $T$  be a sublinear operator such that*

$$\|T\|_{L_{p_0}(X_1 \rightarrow H) \rightarrow L_{q_0}(X_2)} \leq M_0, \quad \|T\|_{L_{p_1}(X_1 \rightarrow H) \rightarrow L_{q_1}(X_2)} \leq M_1,$$

with  $p_0, p_1, q_0, q_1 \in [1, \infty]$ . Let  $\theta \in ]0, 1[$  and define  $p, q$  through

$$\frac{\theta}{p_1} + \frac{1-\theta}{p_0} = \frac{1}{p}, \quad \frac{\theta}{q_1} + \frac{1-\theta}{q_0} = \frac{1}{q}.$$

Then one has

$$\|T\|_{L_p(X_1 \rightarrow H) \rightarrow L_q(X_2)} \leq M_0^{1-\theta} M_1^\theta.$$

### 4.3 Stein's estimate for predictable projections of discrete-time processes

In the proof of the BDG inequality for Hilbert-space-valued martingales we shall use a slightly extended version of an  $\mathbb{L}_p$  estimate for the quadratic variation of the predictable projection of an arbitrary discrete-time process, due to Stein (cf. [15, Thm. 8, p. 103]). For the sake of completeness, we include its simple and elegant proof, which relies on Theorem 4.4 above.

**Theorem 4.6** (Stein). *Let  $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \in \mathbb{N}}, \mathbb{P})$  be a discrete-time stochastic basis and  $(f_n)_{n \in \mathbb{N}}$  an  $\mathbf{H}$ -valued process. For any  $p \in ]1, \infty[$  and any sequence  $(n_k)_{k \in \mathbb{N}}$  of positive integers, denoting conditional expectation with respect to  $\mathcal{F}_n$  by  $\mathbb{E}_n$ , one has*

$$\left\| \left( \sum_k \|\mathbb{E}_{n_k} f_k\|^2 \right)^{1/2} \right\|_{\mathbb{L}_p} \lesssim_p \left\| \left( \sum_k \|f_k\|^2 \right)^{1/2} \right\|_{\mathbb{L}_p},$$

$$i. e. \quad \left\| (\mathbb{E}_{n_k} f_k)_k \right\|_{\mathbb{L}_p(\ell_2(\mathbf{H}))} \lesssim_p \|f\|_{\mathbb{L}_p(\ell_2(\mathbf{H}))}.$$

*Proof.* Define the linear operator

$$T : (f_k) \mapsto (\mathbb{E}_{n_k} f_k).$$

Let us show that  $T$  is bounded on  $\mathbb{L}_p(\ell_p(\mathbf{H}))$ : one has, by the conditional Jensen inequality and Tonelli's theorem,

$$\begin{aligned} \|Tf\|_{\mathbb{L}_p(\ell_p(\mathbf{H}))}^p &\equiv \|(\mathbb{E}_{n_k} f_k)\|_{\mathbb{L}_p(\ell_p(\mathbf{H}))}^p = \mathbb{E} \sum_k \|\mathbb{E}_{n_k} f_k\|^p \\ &\leq \mathbb{E} \sum_k \mathbb{E}_{n_k} \|f_k\|^p = \sum_k \mathbb{E} \mathbb{E}_{n_k} \|f_k\|^p \leq \sum_k \mathbb{E} \|f_k\|^p \\ &= \mathbb{E} \sum_k \|f_k\|^p = \|f\|_{\mathbb{L}_p(\ell_p(\mathbf{H}))}^p. \end{aligned}$$

Let us also show that  $T$  is bounded on  $\mathbb{L}_p(\ell_\infty(\mathbf{H}))$ : one has

$$\begin{aligned} \|Tf\|_{\mathbb{L}_p(\ell_\infty(\mathbf{H}))}^p &\equiv \mathbb{E} \sup_k \|\mathbb{E}_{n_k} f_k\|^p \leq \mathbb{E} \sup_k (\mathbb{E}_{n_k} \|f_k\|)^p \\ &\leq \mathbb{E} \sup_n (\mathbb{E}_n \sup_k \|f_k\|)^p = \mathbb{E} \sup_n |\xi_n^k|^p, \end{aligned}$$

where  $n \mapsto \xi_n := \mathbb{E}_n \xi_\infty$ , with  $\xi_\infty := \sup_k \|f_k\|$ , is a real-valued martingale. Doob's maximal inequality then yields

$$\mathbb{E} \sup_n |\xi_n|^p \lesssim_p \mathbb{E} |\xi_\infty|^p = \mathbb{E} (\sup_k \|f_k\|)^p = \|f\|_{\mathbb{L}_p(\ell_\infty(\mathbf{H}))}^p.$$

Identifying  $\mathbf{H}$  with  $\ell_2$ , we have shown that

$$\|T\|_{L_{p,p,2} \rightarrow L_{p,p,2}} < \infty, \quad \|T\|_{L_{p,\infty,2} \rightarrow L_{p,\infty,2}} < \infty,$$

hence Theorem 4.4 implies that

$$\|T\|_{\mathbb{L}_p(\ell_q(\mathbf{H})) \rightarrow \mathbb{L}_p(\ell_q(\mathbf{H}))} = \|T\|_{L_{p,q,2} \rightarrow L_{p,q,2}} < \infty \quad \forall 1 < p \leq q < \infty.$$

In particular, this proves the theorem in the case  $1 < p \leq 2$ . Let us show that  $T$  is a bounded endomorphism of  $\mathbb{L}_p(\ell_2(\mathbf{H}))$  also if  $p > 2$ . Let  $p > 2$  and  $f \in \mathbb{L}_p(\ell_2(\mathbf{H}))$ . Then one has, denoting the duality form between  $\mathbb{L}_p(\ell_2(\mathbf{H}))$  and  $\mathbb{L}_{p'}(\ell_2(\mathbf{H}))$  by  $\langle \cdot, \cdot \rangle$  and the unit ball of  $\mathbb{L}_{p'}(\ell_2(\mathbf{H}))$  by  $B_1$ , taking into account that  $T$  is self-adjoint on  $\mathbb{L}_2(\ell_2(\mathbf{H}))$ ,

$$\begin{aligned} \|Tf\|_{\mathbb{L}_p(\ell_2(\mathbf{H}))} &= \sup_{g \in B_1} \langle Tf, g \rangle = \sup_{g \in B_1} \langle f, Tg \rangle \\ &\leq \sup_{g \in B_1} \|f\|_{\mathbb{L}_p(\ell_2(\mathbf{H}))} \|Tg\|_{\mathbb{L}_{p'}(\ell_2(\mathbf{H}))} \\ &\leq \|f\|_{\mathbb{L}_p(\ell_2(\mathbf{H}))} \|T\|_{\mathbb{L}_{p'}(\ell_2(\mathbf{H})) \rightarrow \mathbb{L}_{p'}(\ell_2(\mathbf{H}))}, \end{aligned}$$

where the operator norm of  $T$  in the last term is finite because, as already proved,  $T$  is a bounded endomorphism of  $\mathbb{L}_{p'}(\ell_2(\mathbf{H}))$ .  $\square$

## 5 General case

### 5.1 Predictably bounded jumps

**Proposition 5.1** (Conditional lower bound,  $p = 1$ ). *Let  $M$  be an  $\mathbf{H}$ -valued martingale and  $D$  an increasing adapted process such that  $\|\Delta M_-\| \leq D_-$ . Then one has*

$$\|[M, M]_\infty^{1/2}\|_{\mathbb{L}_1} \lesssim \|M_\infty^* + D_\infty\|_{\mathbb{L}_1}.$$

*Proof.* For a fixed  $\varepsilon > 0$ , let us introduce the local martingale  $N := H_- \cdot M$ , where

$$H := (\varepsilon + D + M^*)^{-1/2}.$$

We are going to compare the  $\mathbb{L}_2$  norms of  $[N, N]_\infty$  and of  $\|N_\infty\|^2$ . To this purpose, note that, since  $s \mapsto H_s$  is decreasing, one has

$$[N, N]_\infty = H_-^2 \cdot [M, M] \geq H_\infty^2 [M, M]_\infty,$$

hence also  $\|H_\infty [M, M]_\infty^{1/2}\|_{\mathbb{L}_2} \leq \|[N, N]_\infty^{1/2}\|_{\mathbb{L}_2}$ . Moreover, the Cauchy-Schwarz inequality yields

$$\begin{aligned} \|[M, M]_\infty^{1/2}\|_{\mathbb{L}_1} &= \|[M, M]_\infty^{1/2} H_\infty H_\infty^{-1}\|_{\mathbb{L}_1} \leq \|[M, M]_\infty^{1/2} H_\infty\|_{\mathbb{L}_2} \|H_\infty^{-1}\|_{\mathbb{L}_2} \\ &\leq \|[N, N]_\infty^{1/2}\|_{\mathbb{L}_2} \|H_\infty^{-1}\|_{\mathbb{L}_2} = \|N_\infty\|_{\mathbb{L}_2(H)} \|H_\infty^{-1}\|_{\mathbb{L}_2}, \end{aligned} \tag{5.1}$$

where

$$\|H_\infty^{-1}\|_{\mathbb{L}_2} = \|(\varepsilon + D_\infty + M_\infty^*)^{1/2}\|_{\mathbb{L}_2} = \|\varepsilon + D_\infty + M_\infty^*\|_{\mathbb{L}_1}^{1/2}$$

Thanks to the integration-by-parts formula  $H_- \cdot M = HM - M \cdot H$ , one has

$$\|N_\infty\| \leq H_\infty M_\infty^* + \int_0^\infty \|M_s\| dH_s,$$

which implies, by the inequality  $\|M_s\| \leq \|M_{s-}\| + \|\Delta M_s\| \leq \varepsilon + M_{s-}^* + D_{s-}$  and the definition of  $H$ ,

$$\|N_\infty\| \leq (\varepsilon + D_\infty + M_\infty^*)^{1/2} + \int_0^\infty (\varepsilon + M_{s-}^* + D_{s-}) d(-(\varepsilon + D_s + M_s^*)^{-1/2}).$$

Setting  $U := H_-^{-1} \equiv (\varepsilon + D + M^*)^{1/2}$ , the integral on the right hand side can be written as

$$\int_0^\infty U_-^2 d(-1/U) = \int_0^\infty \frac{U_-^2}{\overline{U}U_-} dU \leq \int_0^\infty dU = (\varepsilon + D_\infty + M_\infty^*)^{1/2},$$

hence

$$\|N_\infty\|_{\mathbb{L}_2(H)} \leq \|(\varepsilon + D_\infty + M_\infty^*)^{1/2}\|_{\mathbb{L}_2} = \|\varepsilon + D_\infty + M_\infty^*\|_{\mathbb{L}_1}^{1/2}.$$

Taking (5.1) into account and recalling that  $\varepsilon > 0$  is arbitrary, the proof is completed.  $\square$

**Proposition 5.2** (Conditional upper bound,  $p = 1$ ). *Let  $M$  be an  $\mathbf{H}$ -valued martingale and  $D$  an increasing adapted process such that  $\|\Delta M_-\| \leq D_-$ . Then one has*

$$\|M_\infty^*\|_{\mathbb{L}_1} \lesssim \|[M, M]_\infty^{1/2} + D_\infty\|_{\mathbb{L}_1}.$$

*Proof.* We adapt the proof of Proposition 3.1. Let

$$H := \sqrt{1/2}([M, M] + D^2)^{-1/4}, \quad N := H_- \cdot M,$$

so that  $M = H_-^{-1} \cdot M$  and, by the integration-by-parts formula,

$$M_\infty = (H_-^{-1} \cdot N)_\infty = H_\infty^{-1} N_\infty - (N \cdot H^{-1})_\infty.$$

Taking norm on both sides and recalling that  $s \mapsto H_s^{-1}$  is increasing, one has

$$M_\infty^* \leq H_\infty^{-1} N_\infty^* + N_\infty^* \int_0^\infty d(H_s^{-1}) = 2H_\infty^{-1} N_\infty^*,$$

hence also, using the Cauchy-Schwarz and Doob inequalities,

$$\begin{aligned} \|M_\infty^*\|_{\mathbb{L}_1} &\leq 2\|H_\infty^{-1} N_\infty^*\|_{\mathbb{L}_1} \leq 2\|H_\infty^{-1}\|_{\mathbb{L}_2} \|N_\infty^*\|_{\mathbb{L}_2} \\ &\leq 4\|H_\infty^{-1}\|_{\mathbb{L}_2} \|N_\infty\|_{\mathbb{L}_2(H)} = 4\|H_\infty^{-1}\|_{\mathbb{L}_2} \|[N, N]_\infty^{1/2}\|_{\mathbb{L}_2}, \end{aligned}$$

where

$$\|H_\infty^{-1}\|_{\mathbb{L}_2} = \sqrt{2} \|(\varepsilon + [M, M]_\infty + D_\infty^2)^{1/4}\|_{\mathbb{L}_2} = \sqrt{2} \|(\varepsilon + [M, M]_\infty + D_\infty^2)^{1/2}\|_{\mathbb{L}_1}.$$

Moreover, by definition of  $N$ , one has

$$[N, N]_\infty = \frac{1}{2} \int_0^\infty ([M, M]_{s-} + D_{s-}^2)^{-1/2} d[M, M]_s,$$

hence, noting that  $[M, M]_{s-} + D_{s-}^2 \geq [M, M]_{s-} + \|\Delta M_s\|^2 = [M, M]_s$  for all  $s \geq 0$ ,

$$\begin{aligned} [N, N]_\infty &\leq \int_0^\infty \frac{1}{2[M, M]_s^{1/2}} d[M, M]_s \\ &\leq \int_0^\infty \frac{1}{[M, M]_{s-}^{1/2} + [M, M]_s^{1/2}} d[M, M]_s = [M, M]_\infty^{1/2}, \end{aligned}$$

where we have used (4.3), as well as the fact that  $[M, M]$  is an increasing process and  $x \mapsto x^{-1/2}$  is decreasing. This implies

$$\|[N, N]_\infty^{1/2}\|_{\mathbb{L}_2} = \|[M, M]_\infty^{1/4}\|_{\mathbb{L}_2} = \|[M, M]_\infty^{1/2}\|_{\mathbb{L}_1}^{1/2} \leq \|(\varepsilon + [M, M]_\infty + D_\infty^2)^{1/2}\|_{\mathbb{L}_1},$$

and the proof is finished by combining the estimates.  $\square$

We now consider the case  $p \in ]1, 2[$ , whose proof proceeds along the lines of the previous one, but is technically more complicated.

**Proposition 5.3** (Conditional lower bound,  $1 < p < 2$ ). *Let  $M$  be an  $\mathbf{H}$ -valued martingale and  $D$  an increasing adapted process such that  $\|\Delta M_-\| \leq D_-$ . Then one has, for any  $p \in ]1, 2[$ ,*

$$\|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p} \lesssim_p \|M_\infty^* + D_\infty\|_{\mathbb{L}_p}.$$

*Proof.* Let us introduce the local martingale  $N = H_- \cdot M$ , with

$$H := (\varepsilon + M^* + D)^{p/2-1}.$$

Defining  $q > 0$  by  $p^{-1} = 1/2 + q^{-1}$ , i.e.  $q = 2p/(2-p)$ , Hölder's inequality yields

$$\|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p} = \|[M, M]_\infty^{1/2} H_\infty H_\infty^{-1}\|_{\mathbb{L}_p} \leq \|[M, M]_\infty^{1/2} H_\infty\|_{\mathbb{L}_2} \|H_\infty^{-1}\|_{\mathbb{L}_q},$$

where, by definition of  $H$  and elementary computations,

$$\|H_\infty^{-1}\|_{\mathbb{L}_q} = \|\varepsilon + M_\infty^* + D_\infty\|_{\mathbb{L}_p}^{1-p/2}.$$

Moreover, since  $s \mapsto H_s$  is decreasing, one has  $[N, N]_\infty = (H_-^2 \cdot [M, M])_\infty \geq H_\infty^2 [M, M]_\infty$ , which implies  $\|[M, M]_\infty^{1/2} H_\infty\|_{\mathbb{L}_2} \leq \|[N, N]_\infty^{1/2}\|_{\mathbb{L}_2}$ , hence also

$$\|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p} \leq \|\varepsilon + M_\infty^* + D_\infty\|_{\mathbb{L}_p}^{1-p/2} \|[N, N]_\infty^{1/2}\|_{\mathbb{L}_2} = \|\varepsilon + M_\infty^* + D_\infty\|_{\mathbb{L}_p}^{1-p/2} \|N_\infty\|_{\mathbb{L}_2(H)}.$$

The integration-by-parts formula

$$N_\infty = (H_- \cdot M)_\infty = H_\infty M_\infty + \int_0^\infty M_s d(-H_s)$$

and the inequality  $\|M_s\| \leq \|M_{s-}\| + \|\Delta M_s\| \leq \varepsilon + M_{s-}^* + D_{s-}$  (valid for all  $s \geq 0$ ) yield

$$\begin{aligned} N_\infty^* &= (\varepsilon + D_\infty + M_\infty^*)^{p/2-1} M_\infty + \int_0^\infty M_s d(-(\varepsilon + D_s + M_s^*)^{p/2-1}) \\ &\leq (\varepsilon + D_\infty + M_\infty^*)^{p/2} + \int_0^\infty (\varepsilon + M_{s-}^* + D_{s-}) d(-(\varepsilon + D_s + M_s^*)^{p/2-1}). \end{aligned}$$

Setting  $V := \varepsilon + M^* + D$  and appealing to Lemma 4.1, one obtains the estimate

$$\begin{aligned} &\int_0^\infty (\varepsilon + M_{s-}^* + D_{s-}) d(-(\varepsilon + D_s + M_s^*)^{p/2-1}) \\ &= \int_0^\infty V_- d(-V^{p/2-1}) \leq \frac{2-p}{p} V_\infty^{p/2} = \frac{2-p}{p} (\varepsilon + M_\infty^* + D_\infty)^{p/2}, \end{aligned}$$

which yields

$$\|N_\infty\|_{\mathbb{L}_2(H)} \leq \|N_\infty^*\|_{\mathbb{L}_2} \leq \frac{2}{p} \|(\varepsilon + M_\infty^* + D_\infty)^{p/2}\|_{\mathbb{L}_2} = \frac{2}{p} \|\varepsilon + M_\infty^* + D_\infty\|_{\mathbb{L}_p}^{p/2}.$$

Collecting estimates, one concludes that

$$\|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p} \leq \frac{2}{p} \|\varepsilon + M_\infty^* + D_\infty\|_{\mathbb{L}_p}.$$

The proof is completed by recalling that  $\varepsilon > 0$  is arbitrary, hence the last inequality holds also with  $\varepsilon = 0$ .  $\square$



**Proposition 5.4** (Conditional upper bound,  $1 < p < 2$ ). *Let  $M$  be an  $\mathbf{H}$ -valued martingale and  $D$  an increasing adapted process such that  $\|\Delta M_-\| \leq D_-$ . Then one has, for any  $p \in ]1, 2[$ ,*

$$\|M_\infty^*\|_{\mathbb{L}_p} \lesssim_p \|[M, M]_\infty^{1/2} + D_\infty\|_{\mathbb{L}_p}.$$

*Proof.* Let  $\varepsilon > 0$  be arbitrary, and define the processes

$$H := \sqrt{p/2} (\varepsilon + [M, M] + D^2)^{p/4-1/2}, \quad N := H_- \cdot M,$$

so that  $M = H_-^{-1} \cdot N$ , hence also, exactly as in the proof of Proposition 5.2,  $M_\infty^* \leq 2H_\infty^{-1} N_\infty^*$ , which in turn implies, by Hölder's inequality,

$$\|M_\infty^*\|_{\mathbb{L}_p} \leq 2\|H_\infty^{-1} N_\infty^*\|_{\mathbb{L}_p} \leq 2\|H_\infty^{-1}\|_{\mathbb{L}_q} \|N_\infty^*\|_{\mathbb{L}_2},$$

where  $q > 0$  is defined by  $p^{-1} = 1/2 + q^{-1}$ , i.e.  $q = 2p/(2-p)$ . By definition of  $H$ , one has

$$\begin{aligned} \|H_\infty^{-1}\|_{\mathbb{L}_q} &= \sqrt{2/p} \|(\varepsilon + [M, M]_\infty + D_\infty^2)^{1/2(1-p/2)}\|_{\mathbb{L}_q} \\ &\leq \sqrt{2/p} \|(\varepsilon + [M, M]^{1/2} + D)\|_{\mathbb{L}_q}^{1-p/2} = \sqrt{2/p} \|\varepsilon + [M, M]^{1/2} + D\|_{\mathbb{L}_p}^{1-p/2}. \end{aligned}$$

Moreover, since

$$[N, N]_\infty = (H_-^2 \cdot [M, M])_\infty = \frac{p}{2} \int_0^\infty (\varepsilon + [M, M]_{s-} + D_{s-}^2)^{p/2-1} d[M, M]_s$$

and  $[M, M]_{s-} + D_{s-}^2 \geq [M, M]_{s-} + \|\Delta M_s\|^2 = [M, M]_s$ , one has

$$[N, N]_\infty \leq \frac{p}{2} \int_0^\infty (\varepsilon + [M, M]_s)^{p/2-1} d(\varepsilon + [M, M]_s),$$

because  $x \mapsto x^{p/2-1}$  is decreasing (recall that  $p/2 - 1 < 0$ ). Setting  $V = \varepsilon + [M, M]$  and recalling the integration-by-parts formula (4.1) as well as Lemma 4.1, one has

$$\begin{aligned} \int_0^\infty (\varepsilon + [M, M]_s)^{p/2-1} d(\varepsilon + [M, M]_s) \\ = V_\infty^{p/2} + \int_0^\infty V_{s-} d(-V_s^{p/2-1}) \leq \frac{2}{p} V^{p/2} = \frac{2}{p} (\varepsilon + [M, M]_\infty)^{p/2}, \end{aligned}$$

in particular  $[N, N]_\infty^{1/2} \leq (\varepsilon + [M, M]_\infty^{1/2})^{p/2}$ . Doob's inequality then yields

$$\begin{aligned} \|N_\infty^*\|_{\mathbb{L}_2} &\leq 2\|N_\infty\|_{\mathbb{L}_2(H)} = 2\|[N, N]_\infty^{1/2}\|_{\mathbb{L}_2} \\ &\leq 2\|(\varepsilon + [M, M]_\infty^{1/2})^{p/2}\|_{\mathbb{L}_2} \leq 2\|\varepsilon + [M, M]_\infty^{1/2} + D_\infty\|_{\mathbb{L}_p}^{p/2}. \end{aligned}$$

Collecting estimates, one ends up with

$$\|M_\infty^*\|_{\mathbb{L}_p} \leq 4\sqrt{2/p} \|\varepsilon + [M, M]_\infty^{1/2} + D_\infty\|_{\mathbb{L}_p}.$$

The proof is completed, once again, simply by recalling that  $\varepsilon > 0$  is arbitrary.  $\square$

**Proposition 5.5** (Conditional lower bound,  $p > 2$ ). *Let  $M$  be an  $\mathbf{H}$ -valued martingale and  $D$  an increasing adapted process such that  $\|\Delta M_-\| \leq D_-$ . Then one has, for any  $p \in ]2, \infty[$ ,*

$$\|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p} \lesssim_p \|M_\infty^*\|_{\mathbb{L}_p} + \|D_\infty\|_{\mathbb{L}_p}$$

*Proof.* Let us set

$$H = \sqrt{p/2} ([M, M] + D^2)^{p/4-1/2}, \quad N = H_- \cdot M,$$

so that

$$[N, N]_\infty = \frac{p}{2} \int_0^\infty ([M, M]_{s-} + D_{s-}^2)^{p/2-1} d[M, M]_s \geq \frac{p}{2} \int_0^\infty [M, M]_s^{p/2-1} d[M, M]_s,$$

where we have used the inequality  $[M, M]_- + D_-^2 \geq [M, M]_- + \|\Delta M\|^2 = [M, M]$ . Let us show that  $[N, N]_\infty \geq [M, M]_\infty^{p/2}$ : setting  $V := [M, M]$  and  $q := p/2$  for notational convenience, one has, using the integration-by-parts formula (4.1),

$$[N, N]_\infty \geq q \int_0^\infty V^{q-1} dV = q \left( V_\infty^q - \int_0^\infty V_- d(V^{q-1}) \right).$$

Lemma 4.1 then yields

$$[N, N]_\infty \geq q \left( V_\infty^q - \int_0^\infty V_- d(V^{q-1}) \right) \geq q(1 - (q-1)/q)V_\infty^q = [M, M]_\infty^{p/2},$$

hence, immediately,

$$\|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p} \leq \|[N, N]_\infty^{1/2}\|_{\mathbb{L}_2}^{2/p} = \|N_\infty\|_{\mathbb{L}_2(H)}^{2/p} \leq \|N_\infty^*\|_{\mathbb{L}_2}^{2/p}.$$

Moreover, integrating by parts, one has

$$N_\infty = (H_- \cdot M)_\infty = H_\infty M_\infty - \int_0^\infty M_t dH_t,$$

which implies  $N_\infty^* \leq 2H_\infty M_\infty^*$ , thus also, setting  $q := 2p/(p-2)$ ,

$$\|N_\infty^*\|_{\mathbb{L}_2} \leq 2\|H_\infty M_\infty^*\|_{\mathbb{L}_2} \leq 2\|H_\infty\|_{\mathbb{L}_q} \|M_\infty^*\|_{\mathbb{L}_p} \leq \sqrt{2p} \left\| ([M, M]_\infty + D_\infty^2)^{1/2} \right\|_{\mathbb{L}_p}^{p/2-1} \|M_\infty^*\|_{\mathbb{L}_p}.$$

The last two estimates and the elementary inequality  $(a+b)^{1/2} \leq a^{1/2} + b^{1/2}$  yield

$$\|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p} \leq (2p)^{1/p} \left\| [M, M]_\infty^{1/2} + D_\infty \right\|_{\mathbb{L}_p}^{1-2/p} \|M_\infty^*\|_{\mathbb{L}_p}^{2/p}.$$

Let us apply Young's inequality in the form

$$ab \leq \varepsilon \frac{a^q}{q} + N(\varepsilon) \frac{b^{q'}}{q'}, \quad \frac{1}{q} + \frac{1}{q'} = 1, \quad N(\varepsilon) = \varepsilon^{-1/(q-1)},$$

choosing  $q = p/(p-2)$ , hence  $q' = p/2$ , and  $\varepsilon = \frac{q}{2}(2p)^{-1/p}$ : we obtain

$$\begin{aligned} \|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p} &\leq \frac{1}{2} \left\| [M, M]_\infty^{1/2} + D_\infty \right\|_{\mathbb{L}_p} + N(p) \|M_\infty^*\|_{\mathbb{L}_p} \\ &\leq \frac{1}{2} \left\| [M, M]_\infty^{1/2} \right\|_{\mathbb{L}_p} + \frac{1}{2} \|D_\infty\|_{\mathbb{L}_p} + N(p) \|M_\infty^*\|_{\mathbb{L}_p}, \end{aligned}$$

with  $N(p) = 2^{p/2+1}$  (which is not the optimal constant), hence, collecting terms,

$$\|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p} \leq 2^{p/2+2} \|M_\infty^*\|_{\mathbb{L}_p} + \|D_\infty\|_{\mathbb{L}_p}. \quad \square$$

## 5.2 Davis' decomposition

The following result, known as Davis' decomposition (whose continuous-time adaptation is due to Meyer [12]), is the key to extend the maximal estimates of the previous subsection to general martingales without any assumption on their jumps. The proof we give here follows closely [12].

**Lemma 5.6** (Davis' decomposition). *Let  $M$  be an  $\mathbf{H}$ -valued martingale, and define  $S := (\Delta M)^*$ , i.e.  $S_t := \sup_{s \leq t} \|\Delta M_s\|$  for all  $t \geq 0$ . Then there exist martingales  $L$  and  $K$  such that*

- (i)  $M = L + K$ ;
- (ii)  $\|\Delta L\| \leq 4S_-$ ;
- (iii)  $K = K^1 - K^2$ , where  $\int_0^\infty |dK^1| \leq 2S_\infty$  and  $K^2$  is the compensator of  $K^1$ .

*Proof.* Let us define the process  $K^1$  by

$$K_t^1 := \sum_{s \leq t} \Delta M_s 1_{\{\|\Delta M_s\| \geq 2S_{t-}\}} \quad \forall t \geq 0. \quad (5.2)$$

Note that, if  $t$  is such that  $\|\Delta M_t\| \geq 2S_{t-}$ , one has

$$\|\Delta M_t\| + 2S_{t-} \leq 2\|\Delta M_t\| = 2S_t,$$

hence  $\|\Delta M_t\| \leq 2(S_t - S_{t-})$ , which implies that the process  $K^1$  has variation bounded by  $2S_\infty$ . Let  $K = K^1 - K^2$ , where  $K^2$  is the (predictable) compensator of  $K^1$ , so that, in particular,  $K$  is a martingale. Furthermore, let us set  $L^1 := M - K^1$ ,  $L^2 := -K^2$ , and  $L = L^1 - L^2$ . In particular,  $L = M - K$  is a martingale. It remains only to prove (ii): since  $L = L^1 + K^2$  and  $K^2$  is predictable, it follows that, for any totally inaccessible jump time  $T$ , one has  $\Delta L_T = \Delta L_T^1$  (e.g. by [11, Prop. 7.7]), hence also, by (5.2) and by definition of  $L^1$ ,  $\|\Delta L_T\| \leq 2S_{T-}$ . Moreover, for any predictable jump time  $T$ , one has  $\mathbb{E}[\Delta L_T | \mathcal{F}_{T-}] = 0$  because  $L$  is a martingale, and  $\mathbb{E}[\Delta L_T^2 | \mathcal{F}_{T-}] = \Delta L_T^2$  because  $L^2$  is predictable, thus also  $\Delta L_T^2 = -\mathbb{E}[\Delta L_T^1 | \mathcal{F}_{T-}]$ . This implies, again by (5.2) and by definition of  $L^1$ , that  $\|\Delta L_T^2\| \leq 2\mathbb{E}[S_{T-} | \mathcal{F}_{T-}] = 2S_{T-}$ , hence also that  $\|\Delta L_T\| \leq \|\Delta L_T^1\| + \|\Delta L_T^2\| \leq 4S_{T-}$ . By general decomposition results for stopping times (see e.g. [11, §7]), it follows that  $\|\Delta L\| \leq 4S_-$ .  $\square$

## 5.3 Real martingales

Throughout this section we shall use the symbols  $L$ ,  $K$  and  $S$  as they have been defined in Lemma 5.6 above.

We start with a simple corollary of Davis' decomposition and Proposition 4.2.

**Lemma 5.7.** *Let  $M$  be a real martingale with Davis' decomposition  $M = K + L$ . Then one has  $\|\int_0^\infty |dK|\|_{\mathbb{L}_p} \lesssim_p \|S_\infty\|_{\mathbb{L}_p}$ .*

*Proof.* Note that we can write  $K^1 = K^{1+} + K^{1-}$ , where  $K^{1+}$  has only positive jumps and  $K^{1-}$  has only negative jumps. Then it is immediate to see that  $K^2 = \widetilde{K}^{1+} + \widetilde{K}^{1-}$ , which, taking Proposition 4.2 into account, implies that

$$\begin{aligned} \left\| \int_0^\infty |dK| \right\|_{\mathbb{L}_p} &\leq \left\| \int_0^\infty |dK^{1+}| \right\|_{\mathbb{L}_p} + \left\| \int_0^\infty |d\widetilde{K}^{1+}| \right\|_{\mathbb{L}_p} + \left\| \int_0^\infty |dK^{1-}| \right\|_{\mathbb{L}_p} + \left\| \int_0^\infty |d\widetilde{K}^{1-}| \right\|_{\mathbb{L}_p} \\ &\leq (p+1) \left\| \int_0^\infty |dK^{1+}| \right\|_{\mathbb{L}_p} + (p+1) \left\| \int_0^\infty |dK^{1-}| \right\|_{\mathbb{L}_p}. \end{aligned}$$

Since the total variation of both  $K^{1+}$  and  $K^{1-}$  is bounded by  $2S_\infty$ , we conclude that

$$\left\| \int_0^\infty |dK| \right\|_{\mathbb{L}_p} \leq 4(p+1) \|S_\infty\|_{\mathbb{L}_p}. \quad \square$$

*Proof of Theorem 1.1 ( $\mathbf{H} = \mathbb{R}$ ).* Note that the upper bound for  $p \geq 2$  has already been addressed. We treat the remaining three cases separately.

LOWER BOUND,  $1 \leq p < 2$ . Let  $M = L + K$  be the Davis' decomposition of the martingale  $M$ . Then one has  $[M, M]_\infty^{1/2} \leq [L, L]_\infty^{1/2} + [K, K]_\infty^{1/2}$ , hence also

$$\|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p} \leq \|[L, L]_\infty^{1/2}\|_{\mathbb{L}_p} + \|[K, K]_\infty^{1/2}\|_{\mathbb{L}_p},$$

where, by Propositions 5.1 and 5.3,

$$\|[L, L]_\infty^{1/2}\|_{\mathbb{L}_p} \lesssim_p \|L_\infty^*\|_{\mathbb{L}_p} + \|S_\infty\|_{\mathbb{L}_p}.$$

Note that  $M = L + K$  also implies  $L^* \leq M^* + K^*$ , and, by definition of  $S$ , it is immediate that  $S_\infty \leq 2M_\infty^*$ . Therefore, taking Lemma 5.7 into account, one has

$$\begin{aligned} \|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p} &\lesssim_p \|M_\infty^*\|_{\mathbb{L}_p} + \|K_\infty^*\|_{\mathbb{L}_p} + \|[K, K]_\infty^{1/2}\|_{\mathbb{L}_p} \leq \|M_\infty^*\|_{\mathbb{L}_p} + 2 \left\| \int_0^\infty |dK| \right\|_{\mathbb{L}_p} \\ &\lesssim_p \|S_\infty\|_{\mathbb{L}_p} \lesssim \|M_\infty^*\|_{\mathbb{L}_p}. \quad \square \end{aligned}$$

UPPER BOUND,  $1 \leq p < 2$ . Taking into account Propositions 5.2 and 5.4, the estimates  $\|M_\infty^*\|_{\mathbb{L}_p} \leq \|L_\infty^*\|_{\mathbb{L}_p} + \|K_\infty^*\|_{\mathbb{L}_p}$  and  $[L, L]_\infty^{1/2} \leq [M, M]_\infty^{1/2} + [K, K]_\infty^{1/2}$  yield

$$\begin{aligned} \|M_\infty^*\|_{\mathbb{L}_p} &\lesssim_p \|[L, L]_\infty^{1/2}\|_{\mathbb{L}_p} + \|S_\infty\|_{\mathbb{L}_p} + \|K_\infty^*\|_{\mathbb{L}_p} \\ &\leq \|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p} + \|S_\infty\|_{\mathbb{L}_p} + \|[K, K]_\infty^{1/2}\|_{\mathbb{L}_p} + \|K_\infty^*\|_{\mathbb{L}_p}. \end{aligned}$$

Noting that

$$S_\infty = (\Delta M)^* = ((|\Delta M|^2)^*)^{1/2} \leq \left( \sum |\Delta M|^2 \right)^{1/2} \leq [M, M]_\infty^{1/2},$$

hence  $\|S_\infty\|_{\mathbb{L}_p} \leq \|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p}$ , we are left with

$$\|M_\infty^*\|_{\mathbb{L}_p} \lesssim_p \|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p} + \|[K, K]_\infty^{1/2}\|_{\mathbb{L}_p} + \|K_\infty^*\|_{\mathbb{L}_p},$$

where, repeating an argument used in the previous part of the proof,

$$\|[K, K]_\infty^{1/2}\|_{\mathbb{L}_p} + \|K_\infty^*\|_{\mathbb{L}_p} \lesssim_p \left\| \int_0^\infty |dK| \right\|_{\mathbb{L}_p} \lesssim \|S_\infty\|_{\mathbb{L}_p} \leq \|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p}.$$

Collecting estimates, the proof is completed.

LOWER BOUND,  $p > 2$ . Davis' decomposition  $M = L + K$  implies  $[M, M]_\infty^{1/2} \leq [L, L]_\infty^{1/2} + [K, K]_\infty^{1/2}$ , hence also, taking Proposition 5.5 into account,

$$\|[M, M]_\infty^{1/2}\|_{\mathbb{L}_p} \leq \|[L, L]_\infty^{1/2}\|_{\mathbb{L}_p} + \|[K, K]_\infty^{1/2}\|_{\mathbb{L}_p} \lesssim_p \|L_\infty^*\|_{\mathbb{L}_p} + \|S_\infty\|_{\mathbb{L}_p} + \|[K, K]_\infty^{1/2}\|_{\mathbb{L}_p}.$$

We can further estimate the terms on the right-hand side as follows:

$$\begin{aligned} \|L_\infty^*\|_{\mathbb{L}_p} &\leq \|M_\infty^*\|_{\mathbb{L}_p} + \|K_\infty^*\|_{\mathbb{L}_p} \leq \|M_\infty^*\|_{\mathbb{L}_p} + \left\| \int_0^\infty |dK| \right\|_{\mathbb{L}_p}, \\ \|S_\infty\|_{\mathbb{L}_p} &\leq 2\|M_\infty^*\|_{\mathbb{L}_p}, \\ \|[K, K]_\infty^{1/2}\|_{\mathbb{L}_p} &\leq \left\| \int_0^\infty |dK| \right\|_{\mathbb{L}_p}, \end{aligned}$$

hence the proof is complete, recalling that by Lemma 5.7

$$\left\| \int_0^\infty |dK| \right\|_{\mathbb{L}_p} \lesssim_p \|S_\infty\|_{\mathbb{L}_p} \lesssim \|M_\infty^*\|_{\mathbb{L}_p}.$$

## 5.4 Hilbert-space-valued martingales

**Lemma 5.8** (Upper bounds by lower bounds). *Let  $p \in ]1, \infty[$  and assume that*

$$\|[N, N]_\infty^{1/2}\|_{\mathbb{L}_p} \lesssim_p \|N_\infty^*\|_{\mathbb{L}_p}$$

*for any  $\mathbf{H}$ -valued martingale  $N$ . Then, for any  $\mathbf{H}$ -valued martingale  $M$ , one has*

$$\|M_\infty^*\|_{\mathbb{L}_{p'}} \lesssim_{p'} \|[M, M]_\infty^{1/2}\|_{\mathbb{L}_{p'}},$$

*where  $p'$  is the conjugate exponent of  $p$ , i.e.  $1/p + 1/p' = 1$ .*

*Proof.* Note that the (topological and algebraic) dual of  $\mathbb{L}_p(\mathbf{H})$  is  $\mathbb{L}_{p'}(\mathbf{H})$ , with duality form  $\mathbb{E}(\cdot, \cdot)$ . Therefore, denoting the unit ball of  $\mathbb{L}_p(\mathbf{H})$  by  $B_1$ , one has, recalling Doob's inequality,

$$\|M_\infty^*\|_{\mathbb{L}_{p'}} \lesssim_{p'} \|M_\infty\|_{\mathbb{L}_{p'}(\mathbf{H})} = \sup_{\xi \in B_1} \mathbb{E}(M_\infty, \xi).$$

Let  $\xi \in \mathbb{L}_p(\mathbf{H})$ ,  $\mathbb{E}\|\xi\|^p \leq 1$ , be arbitrary, and consider the martingale  $t \mapsto N_t := \mathbb{E}[\xi | \mathcal{F}_t]$ ,  $N_\infty := \xi$ . Kunita-Watanabe's and Hölder's inequalities yield

$$\begin{aligned} \mathbb{E}(M_\infty, \xi) &\equiv \mathbb{E}(M_\infty, N_\infty) = \mathbb{E}[M, N]_\infty \\ &\leq \mathbb{E}[M, M]_\infty^{1/2} [N, N]_\infty^{1/2} \leq \|[M, M]_\infty^{1/2}\|_{\mathbb{L}_{p'}} \|[N, N]_\infty^{1/2}\|_{\mathbb{L}_p}. \end{aligned}$$

By the hypothesis and Doob's inequality, one also has

$$\|[N, N]_\infty^{1/2}\|_{\mathbb{L}_p} \lesssim_p \|N_\infty\|_{\mathbb{L}_p(\mathbf{H})} \equiv \|\xi\|_{\mathbb{L}_p(\mathbf{H})} \leq 1,$$

therefore we are left with  $\mathbb{E}(M_\infty, \xi) \lesssim_p \|[M, M]_\infty^{1/2}\|_{\mathbb{L}_{p'}}$ . Since  $\xi \in B_1$  is arbitrary, the claim is proved.  $\square$

**Lemma 5.9** (Interpolation of lower bounds). *Let  $1 < p_1 < p_2 < \infty$ , and assume that, for any  $\mathbf{H}$ -valued martingale  $M$ , one has*

$$\| [M, M]_\infty^{1/2} \|_{\mathbb{L}_{p_i}} \lesssim_{p_i} \| M_\infty^* \|_{\mathbb{L}_{p_i}} \quad \forall i \in \{1, 2\}.$$

*Then one has  $\| [M, M]_\infty^{1/2} \|_{\mathbb{L}_p} \lesssim_p \| M_\infty^* \|_{\mathbb{L}_p}$  for all  $p \in [p_1, p_2]$ .*

*Proof.* Thanks to the martingale convergence theorem and Doob's inequality, we can identify the space of  $\mathbf{H}$ -valued martingales such that  $\| M_\infty^* \|_{\mathbb{L}_p} < \infty$ ,  $p > 1$ , with  $L_p(\Omega \rightarrow \mathbf{H}, \mathcal{F}_\infty, \mathbb{P})$ . For  $i \in \{1, 2\}$ , let us define, with a slight abuse of notation, the operator

$$\begin{aligned} T : \mathbb{L}_{p_i}(\mathbf{H}) &\rightarrow \mathbb{L}_{p_i} \\ M_\infty &\mapsto [M, M]_\infty^{1/2}, \end{aligned}$$

which is immediately seen to be sublinear in the sense of §4.2. Taking Doob's inequality into account, one has

$$\| TM \|_{\mathbb{L}_{p_i}} \equiv \| [M, M]_\infty^{1/2} \|_{\mathbb{L}_{p_i}} \lesssim_{p_i} \| M_\infty^* \|_{\mathbb{L}_{p_i}} \lesssim_{p_i} \| M_\infty \|_{\mathbb{L}_{p_i}(\mathbf{H})} \quad \forall i \in \{1, 2\},$$

i.e.  $T$  is bounded from  $\mathbb{L}_{p_1}(\mathbf{H})$  to  $\mathbb{L}_{p_1}$ , and from  $\mathbb{L}_{p_2}(\mathbf{H})$  to  $\mathbb{L}_{p_2}$ . Therefore, by Theorem 4.5 we infer that  $T$  is bounded from  $\mathbb{L}_p(\mathbf{H})$  to  $\mathbb{L}_p$  for all  $p \in ]p_1, p_2[$ .  $\square$

*Proof of Theorem 1.1.* If  $p = 1$ , the proof of the previous section still works, if one appeals to Proposition 4.3, which implies that

$$\left\| \int_0^\infty |dK| \right\|_{\mathbb{L}_1} \lesssim \left\| \int_0^\infty |dK^1| \right\|_{\mathbb{L}_1} \lesssim \| S_\infty \|_{\mathbb{L}_1} \lesssim \| M_\infty^* \|_{\mathbb{L}_1}.$$

By the results in Section 2 we know that the upper bound holds for all  $p \geq 2$ , and that the lower bound holds for  $p = 2$  and for all  $p \geq 4$ . Therefore, by Lemma 5.9, the lower bound in fact holds for all  $p \geq 2$ . In turn, Lemma 5.8 yields the validity of the upper bound for all  $p \in ]1, 2[$  (hence for all  $p \geq 1$ ), provided the lower bound holds for all  $p \in ]1, 2[$ , which we shall prove now: letting  $M = L + K$  be the Davis decomposition of  $M$  and proceeding as in the proof for real-valued martingales of the previous section, one obtains

$$\| [M, M]_\infty^{1/2} \|_{\mathbb{L}_p} \lesssim_p \| M_\infty^* \|_{\mathbb{L}_p} + \| K_\infty^* \|_{\mathbb{L}_p} + \| [K, K]_\infty^{1/2} \|_{\mathbb{L}_p}.$$

Applying the upper bound to the martingale  $K$ , one has  $\| K_\infty^* \|_{\mathbb{L}_p} \lesssim_p \| [K, K]_\infty^{1/2} \|_{\mathbb{L}_p}$ , thus also

$$\| [M, M]_\infty^{1/2} \|_{\mathbb{L}_p} \lesssim_p \| M_\infty^* \|_{\mathbb{L}_p} + \| [K, K]_\infty^{1/2} \|_{\mathbb{L}_p}.$$

To complete the proof, we only have to show that the second term on the right-hand side is controlled by the  $\mathbb{L}_p$ -norm of  $M_\infty^*$ . To this purpose, recall that  $K = K^1 - K^2$ , where  $K^1, K^2$  have integrable variation and  $K^2$  is the (predictable) compensator of  $K^1$ . Therefore one has

$$[K, K]_\infty^{1/2} \leq [K^1, K^1]_\infty^{1/2} + [K^2, K^2]_\infty^{1/2}$$

and, by a reasoning already used in the proof of Davis' decomposition,

$$[K^2, K^2]_\infty = \sum_{n \in \mathbb{N}} \|\Delta K_{T_n}^2\|^2,$$

where  $\Delta K_{T_n}^2 = \mathbb{E}[\Delta K_{T_n}^1 | \mathcal{F}_{T_n-}]$  and  $(T_n)_{n \in \mathbb{N}}$  is a sequence of predictable stopping times. Applying Theorem 4.6 to the discrete-time process  $n \mapsto K_{T_n}^1$ , one obtains  $\|[K^2, K^2]_\infty^{1/2}\|_{\mathbb{L}_p} \lesssim_p \|[K^1, K^1]_\infty^{1/2}\|_{\mathbb{L}_p}$ , hence also

$$\begin{aligned} \|[K, K]_\infty^{1/2}\|_{\mathbb{L}_p} &\lesssim \|[K^1, K^1]_\infty^{1/2}\|_{\mathbb{L}_p} + \|[K^2, K^2]_\infty^{1/2}\|_{\mathbb{L}_p} \\ &\lesssim_p \|[K^1, K^1]_\infty^{1/2}\|_{\mathbb{L}_p} \leq \left\| \int_0^\infty |dK^1| \right\|_{\mathbb{L}_p} \lesssim \|M_\infty^*\|_{\mathbb{L}_p}. \quad \square \end{aligned}$$

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