On maximal inequalities for purely discontinuous martingales in infinite dimensions

Carlo Marinelli[*] Michael Röckner[†]

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Abstract

The purpose of this paper is to give a survey of a class of maximal inequalities for purely discontinuous martingales, as well as for stochastic integral and convolutions with respect to Poisson measures, in infinite dimensional spaces. Such maximal inequalities are important in the study of stochastic partial differential equations with noise of jump type.

1 Introduction

The purpose of this work is to collect several proofs, in part revisited and extended, of a class of maximal inequalities for stochastic integrals with respect to compensated random measures, including Poissonian integrals as a special case. The precise formulation of these inequalities can be found in Sections 3 to 5 below. Their main advantage over the maximal inequalities of Burkholder, Davis and Gundy is that their right-hand side is expressed in terms of predictable "ingredients", rather than in terms of the quadratic variation. Since our main motivation is the application to stochastic partial differential equations (SPDE), in particular to questions of existence, uniqueness, and regularity of solutions (cf. [21, 22, 23, 25, 26]), we focus on processes in continuous time taking values in infinite-dimensional spaces. Corresponding estimates for finite-dimensional processes have been used in many areas, for instance in connection to Malliavin calculus for processes with jumps, flow properties of solutions to SDEs, and numerical schemes for Lévy-driven SDEs (see e.g. [2, 14, 16]). Very recent extensions to vector-valued settings have been used to develop the theory of stochastic integration with jumps in (certain) Banach spaces (see [7] and references therein).

We have tried to reconstruct the historical developments around this class of inequalities (an investigation which les us to quite a few surprises), together with relevant references, and we hope that our account could at least serve to correct some terminology that seems not appropriate. In fact, while we refer to Section 6 below for details, it seems important to remark already at this stage that the estimates which we termed "Bichteler-Jacod's inequalities" in our previous article [23] should have probably more

[*]Department of Mathematics, University College London, Gower Street, London WC1E 6BT, United Kingdom. URL: `http://goo.gl/4GKJP`

[†]Fakultät für Mathematik, Universität Bielefeld, Germany.

rightfully been baptized as "Novikov's inequalities", in recognition of the contribution [29].

Let us conclude this introductory section with a brief outline of the remaining content: after fixing some notation and collecting a few elementary (but useful) results in Section 2, we state and prove several upper and lower bounds for purely discontinuous Hilbert-space-valued continuous-time martingales in Section 3. We actually present several proofs, adapting, simplifying, and extending arguments of the existing literature. The proofs in Subsections 3.2 and 3.3 might be, at least in part, new. On the issue of who proved what and when, however, we refer to the (hopefully) comprehensive discussion in Section 6. Section 4 deals with L_q-valued processes that can be written as stochastic integrals with respect to compensated Poisson random measures. Unfortunately, to keep this survey within a reasonable length, it has not been possible to reproduce the proof, for which we refer to the original contribution [7]. The (partial) extension to the case of stochastic convolutions is discussed in Section 5.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the "usual" conditions, on which all random elements will be defined, and H a real (separable) Hilbert space with norm $\|\cdot\|$. If ξ is an E-valued random variable, with E a normed space, and $p > 0$, we shall use the notation

$$\big\|\xi\big\|_{\mathbb{L}_p(E)} := \big(\mathbb{E}\|\xi\|_E^p\big)^{1/p}.$$

Let μ be a random measure on a measurable space $(Z, \mathcal{Z})$, with dual predictable projection (compensator) ν. We shall use throughout the paper the symbol M to denote a martingale of the type $M = g\star\bar{\mu}$, where $\bar{\mu} := \mu - \nu$ and g is a vector-valued (predictable) integrand such that the stochastic integral

$$(g \star \bar{\mu})_t := \int_{(0,t]}\int_Z g(s,z)\,\bar{\mu}(ds,dz)$$

is well defined. We shall deal only with the case that g (hence M) takes values in H or in an L_q space. Integrals with respect to μ, ν and $\bar{\mu}$ will often be written in abbreviated form, e.g. $\int_0^t g\,d\bar{\mu} := (g\star\bar{\mu})_t$ and $\int g\,d\bar{\mu} := (g\star\bar{\mu})_\infty$. If M is H-valued, the following well-known identities hold for the quadratic variation $[M, M]$ and the Meyer process $\langle M, M\rangle$:

$$[M,M]_T = \sum_{s\leq T}\|\Delta M_s\|^2 = \int_0^T \|g\|^2\,d\mu, \qquad \langle M,M\rangle_T = \int_0^T \|g\|^2\,d\nu$$

for any stopping time T. Moreover, we shall need the fundamental Burkholder-Davis-Gundy's (BDG) inequality:

$$\big\|M_\infty^*\big\|_{\mathbb{L}_p} \eqsim \big\|[M,M]_\infty^{1/2}\big\|_{\mathbb{L}_p} \qquad \forall p \in [1,\infty[,$$

where $M_\infty^* := \sup_{t\geq 0}\|M_t\|$. An expression of the type $a \lesssim b$ means that there exists a (positive) constant N such that $a \leq Nb$. If N depends on the parameters $p_1, \dots, p_n$, we shall write $a \lesssim_{p_1,\dots,p_n} b$. Moreover, if $a \lesssim b$ and $b \lesssim a$, we shall write $a \eqsim b$.

The following lemma about (Fréchet) differentiability of powers of the norm of a Hilbert space is elementary and its proof is omitted.

Lemma 2.1. *Let $\phi : H \to \mathbb{R}$ be defined as $\phi : x \mapsto \|x\|^p$, with $p > 0$. Then $\phi \in C^\infty(H \setminus \{0\})$, with first and second Fréchet derivatives*

$$\phi'(x) : \eta \mapsto p\|x\|^{p-2}\langle x, \eta\rangle, \tag{2.1}$$

$$\phi''(x) : (\eta, \zeta) \mapsto p(p-2)\|x\|^{p-4}\langle x, \eta\rangle\langle x, \zeta\rangle + p\|x\|^{p-2}\langle \eta, \zeta\rangle. \tag{2.2}$$

In particular, $\phi \in C^1(H)$ if $p > 1$, and $\phi \in C^2(H)$ if $p > 2$.

It should be noted that, here and in the following, for $p \in [1, 2[$ and $p \in [2, 4[$, the linear form $\|x\|^{p-2}\langle x, \cdot\rangle$ and the bilinear form $\|x\|^{p-4}\langle x, \cdot\rangle\langle x, \cdot\rangle$, respectively, have to be interpreted as the zero form if $x = 0$.

The estimate contained in the following lemma is simple but perhaps not entirely trivial.

Lemma 2.2. *Let $1 \leq p \leq 2$. One has, for any $x, y \in H$,*

$$0 \leq \|x + y\|^p - \|x\|^p - p\|x\|^{p-2}\langle x, y\rangle \lesssim_p \|y\|^p. \tag{2.3}$$

Proof. Let x, $y \in H$. We can clearly assume x, $y \neq 0$, otherwise (2.3) trivially holds. Since the function $\phi : x \mapsto \|x\|^p$ is convex and Fréchet differentiable on $H \setminus \{0\}$ for all $p \geq 1$, one has

$$\phi(x + y) - \phi(x) \geq \langle \nabla\phi(x), y\rangle,$$

hence, by (2.1),

$$\|x + y\|^p - \|x\|^p - p\|x\|^{p-2}\langle x, y\rangle \geq 0.$$

To prove the upper bound we distinguish two cases: if $\|x\| \leq 2\|y\|$, it is immediately seen that (2.3) is true; if $\|x\| > 2\|y\|$, Taylor's formula applied to the function $[0, 1] \ni t \mapsto \|x + ty\|^p$ implies

$$\|x + y\|^p - \|x\|^p - p\|x\|^{p-2}\langle x, y\rangle \lesssim_p \|x + \theta y\|^{p-2}\|y\|^2$$

for some $\theta \in]0, 1[$ (in particular $x + \theta y \neq 0$). Moreover, we have

$$\|x + \theta y\| \geq \|x\| - \|y\| > 2\|y\| - \|y\| = \|y\|,$$

hence, since $p - 2 \leq 0$, $\|x + \theta y\|^{p-2} \leq \|y\|^{p-2}$. □

For the purposes of the following lemma only, let $(X, \mathcal{A}, m)$ be a measure space, and denote $L_p(X, \mathcal{A}, m)$ simply by L_p.

Lemma 2.3. *Let $1 < q < p$. For any $\alpha \geq 0$, one has*

$$\|f\|_{L_q}^\alpha \leq \|f\|_{L_2}^\alpha + \|f\|_{L_p}^\alpha$$

Proof. By a well-known consequence of Hölder's inequality one has

$$\|f\|_{L_q} \leq \|f\|_{L_2}^r \|f\|_{L_p}^{1-r},$$

for some $0 < r < 1$. Raising this to the power α and applying Young's inequality with conjugate exponents $s := 1/r$ and $s' := 1/(1 - r)$ yields

$$\|f\|_{L_q}^\alpha \leq \|f\|_{L_2}^{r\alpha} \|f\|_{L_p}^{(1-r)\alpha} \leq r\|f\|_{L_2}^\alpha + (1 - r)\|f\|_{L_p}^\alpha \leq \|f\|_{L_2}^\alpha + \|f\|_{L_p}^\alpha. \quad □$$

3 Inequalities for martingales with values in Hilbert spaces

The following domination inequality, due to Lenglart [17], will be used several times.

Lemma 3.1. *Let X and A be a positive adapted right-continuous process and an increasing predictable process, respectively, such that $\mathbb{E}[X_T|\mathcal{F}_0] \leq \mathbb{E}[A_T|\mathcal{F}_0]$ for any bounded stopping time. Then one has*

$$\mathbb{E}(X^*_\infty)^p \lesssim_p \mathbb{E}A^p_\infty \qquad \forall p \in]0,1[.$$

Theorem 3.2. *Let $\alpha \in [1,2]$. One has*

$$\mathbb{E}(M^*_\infty)^p \lesssim_{\alpha,p} \begin{cases} \mathbb{E}\Big(\int \|g\|^\alpha \, d\nu\Big)^{p/\alpha} & \forall p \in \,]0,\alpha]\,, \\ \mathbb{E}\Big(\int \|g\|^\alpha \, d\nu\Big)^{p/\alpha} + \mathbb{E}\int \|g\|^p \, d\nu & \forall p \in [\alpha,\infty[\,, \end{cases} \tag{BJ}$$

and

$$\mathbb{E}(M^*_\infty)^p \gtrsim_{\alpha,p} \mathbb{E}\Big(\int \|g\|^2 \, d\nu\Big)^{p/2} + \mathbb{E}\int \|g\|^p \, d\nu \qquad \forall p \in [2,\infty[\,. \tag{3.1}$$

Sometimes we shall use the notation $\mathsf{BJ}_{\alpha,p}$ to denote the inequality BJ with parameters α and p.

Several proofs of BJ will be given below. Before doing that, a few remarks are in order. Choosing $\alpha = 2$ and $\alpha = p$, respectively, one obtains the probably more familiar expressions

$$\mathbb{E}(M^*_\infty)^p \lesssim_p \begin{cases} \mathbb{E}\Big(\int \|g\|^2 \, d\nu\Big)^{p/2} & \forall p \in \,]0,2]\,, \\ \mathbb{E}\int \|g\|^p \, d\nu & \forall p \in [1,2], \\ \mathbb{E}\Big(\int \|g\|^2 \, d\nu\Big)^{p/2} + \mathbb{E}\int \|g\|^p \, d\nu & \forall p \in [2,\infty[\,. \end{cases}$$

In more compact notation, BJ may equivalent be written as

$$\big\|M^*_\infty\big\|_{\mathbb{L}_p} \lesssim_{\alpha,p} \begin{cases} \big\|g\big\|_{\mathbb{L}_p(L_\alpha(\nu))} & \forall p \in \,]0,\alpha]\,, \\ \big\|g\big\|_{\mathbb{L}_p(L_\alpha(\nu))} + \big\|g\big\|_{\mathbb{L}_p(L_p(\nu))} & \forall p \in [\alpha,\infty[\,, \end{cases}$$

where

$$\big\|g\big\|_{\mathbb{L}_p(L_\alpha(\nu))} := \big\|\|g\|_{L_\alpha(\nu)}\big\|_{\mathbb{L}_p}, \qquad \|g\|_{L_\alpha(\nu)} := \Big(\int \|g\|^\alpha \, d\nu\Big)^{1/\alpha}.$$

This notation is convenient but slightly abusive, as it is not standard (nor clear how) to define L_p spaces with respect to a random measure. However, if μ is a Poisson measure, then ν is "deterministic" (i.e. it does not depend on $\omega \in \Omega$), and the above notation is thus perfectly lawful. In particular, if ν is deterministic, it is rather straightforward to see that the above estimates imply

$$\big\|M^*_\infty\big\|_{\mathbb{L}_p} \lesssim_p \inf_{g_1+g_2=g} \big\|g_1\big\|_{\mathbb{L}_p(L_2(\nu))} + \big\|g_2\big\|_{\mathbb{L}_p(L_p(\nu))} =: \big\|g\big\|_{\mathbb{L}_p(L_2(\nu))+\mathbb{L}_p(L_p(\nu))}, \qquad 1 \leq p \leq 2,$$

as well as

$$\big\|M^*_\infty\big\|_{\mathbb{L}_p} \lesssim_p \max\Big(\big\|g\big\|_{\mathbb{L}_p(L_2(\nu))}, \big\|g\big\|_{\mathbb{L}_p(L_p(\nu))}\Big) =: \big\|g\big\|_{\mathbb{L}_p(L_2(\nu))\cap\mathbb{L}_p(L_p(\nu))}, \qquad p \geq 2$$

(for the notions of sum and intersection of Banach spaces see e.g. [15]). Moreover, since the dual space of $\mathbb{L}_p(L_2(\nu)) \cap \mathbb{L}_p(L_p(\nu))$ is $\mathbb{L}_{p'}(L_2(\nu)) + \mathbb{L}_{p'}(L_{p'}(\nu))$ for any $p \in [1,\infty[$, where $1/p + 1/p' = 1$, by a duality argument one can obtain the lower bound

$$\big\|M^*_\infty\big\|_{\mathbb{L}_p} \gtrsim \big\|g\big\|_{\mathbb{L}_p(L_2(\nu))+\mathbb{L}_p(L_p(\nu))} \qquad \forall p \in]1,2].$$

One thus has

$$\big\|M^*_\infty\big\|_{\mathbb{L}_p} \eqsim_p \begin{cases} \big\|g\big\|_{\mathbb{L}_p(L_2(\nu))+\mathbb{L}_p(L_p(\nu))} & \forall p \in]1,2], \\ \big\|g\big\|_{\mathbb{L}_p(L_2(\nu))\cap\mathbb{L}_p(L_p(\nu))} & \forall p \in [2,\infty[. \end{cases}$$

By virtue of the Lévy-Itô decomposition and of the BDG inequality for stochastic integrals with respect to Wiener processes, the above maximal inequalities admit corresponding versions for stochastic integrals with respect to Lévy processes (cf. [13, 21]). We do not dwell on details here.

3.1 Proofs

We first prove the lower bound (3.1). The proof is taken from [20] (we recently learned, however, cf. Section 6 below, that the same argument already appeared in [9]).

Proof of (3.1). Since $p/2 > 1$, one has

$$\mathbb{E}[M,M]^{p/2}_\infty = \mathbb{E}\Big(\sum\|\Delta M\|^2\Big)^{p/2} \geq \mathbb{E}\sum\|\Delta M\|^p = \mathbb{E}\int\|g\|^p\,d\mu = \mathbb{E}\int\|g\|^p\,d\nu,$$

as well as, since $x \mapsto x^{p/2}$ is convex,

$$\mathbb{E}[M,M]^{p/2}_\infty \geq \mathbb{E}\langle M,M\rangle^{p/2}_\infty = \mathbb{E}\Big(\int\|g\|^2\,d\nu\Big)^{p/2},$$

see e.g. [18]. Therefore, recalling the BDG inequality,

$$\mathbb{E}(M^*_\infty)^p \gtrsim \mathbb{E}[M,M]^{p/2}_\infty \gtrsim \mathbb{E}\Big(\int\|g\|^2\,d\nu\Big)^{p/2} + \mathbb{E}\int\|g\|^p\,d\nu. \qquad \square$$

We now give several alternative arguments for the upper bounds.

The first proof we present is based on Itô's formula and Lenglart's domination inequality. It does not rely, in particular, on the BDG inequality, and it is probably, in this sense, the most elementary.

First proof of BJ. Let $\alpha \in]1,2]$, and $\phi : H \ni x \mapsto \|x\|^\alpha = h(\|x\|^2)$, with $h : y \mapsto y^{\alpha/2}$. Furthermore, let $(h_n)_{n\in\mathbb{N}}$ be a sequence of functions of class $C^\infty_c(\mathbb{R})$ such that $h_n \to h$

pointwise, and define $\phi_n : x \mapsto h_n(\|x\|^2)$, so that $\phi_n \in C^2_b(H)$ [1]. Itô's formula (see e.g. [27]) then yields

$$\phi_n(M_\infty) = \int_0^\infty \phi_n'(M_-)\,dM + \sum \bigl(\phi_n(M_- + \Delta M) - \phi_n(M_-) - \phi_n'(M_-)\Delta M\bigr).$$

Taking expectation and passing to the limit as $n \to \infty$, one has, by estimate (2.3) and the dominated convergence theorem,

$$\begin{aligned}\mathbb{E}\|M_\infty\|^\alpha &\leq \mathbb{E}\sum\bigl(\|M_- + \Delta M\|^\alpha - \|M_-\|^\alpha - \alpha\|M_-\|^{\alpha-2}\langle M_-, \Delta M\rangle\bigr)\\ &\lesssim_\alpha \mathbb{E}\sum\|\Delta M\|^\alpha = \mathbb{E}\int\|g\|^\alpha\,d\mu = \mathbb{E}\int\|g\|^\alpha\,d\nu,\end{aligned}$$

which implies, by Doob's inequality,

$$\mathbb{E}(M^*_\infty)^\alpha \lesssim_\alpha \mathbb{E}\int\|g\|^\alpha\,d\nu.$$

If $\alpha = 1$ we cannot use Doob's inequality, but we can argue by a direct calculation:

$$\begin{aligned}\mathbb{E}M^*_\infty = \mathbb{E}\sup_{t\geq 0}\left\|\int_0^t g\,d\bar{\mu}\right\| &\leq \mathbb{E}\sup_{t\geq 0}\left\|\int_0^t g\,d\mu\right\| + \mathbb{E}\sup_{t\geq 0}\left\|\int_0^t g\,d\nu\right\|\\ &\leq \mathbb{E}\sup_{t\geq 0}\int_0^t\|g\|\,d\mu + \mathbb{E}\sup_{t\geq 0}\int_0^t\|g\|\,d\nu\\ &\leq 2\mathbb{E}\int\|g\|\,d\nu.\end{aligned}$$

An application of Lenglart's domination inequality finishes the proof of the case $\alpha \in [1,2]$, $p \in]0,\alpha]$.

Let us now consider the case $\alpha = 2$, $p > 2$. We apply Itô's formula to a C^2_b approximation of $x \mapsto \|x\|^p$, as in the first part of the proof, then take expectation and pass to the limit, obtaining

$$\mathbb{E}\|M_\infty\|^p \leq \mathbb{E}\sum\bigl(\|M_- + \Delta M\|^p - \|M_-\|^p - p\|M_-\|^{p-2}\langle M_-, \Delta M_s\rangle\bigr).$$

Applying Taylor's formula to the function $t \mapsto \|x + ty\|$ we obtain, in view of (2.2),

$$\begin{aligned}&\|M_- + \Delta M\|^p - \|M_-\|^p - p\|M_-\|^{p-2}\langle M_-, \Delta M\rangle\\ &\quad= \frac{1}{2}p(p-2)\bigl\|M_- + \theta\Delta M\bigr\|^{p-4}\langle M_- + \theta\Delta M, \Delta M\rangle^2\\ &\qquad+ \frac{1}{2}p\bigl\|M_- + \theta\Delta M\bigr\|^{p-2}\|\Delta M\|^2\\ &\quad\leq \frac{1}{2}p(p-1)\bigl\|M_- + \theta\Delta M\bigr\|^{p-2}\|\Delta M\|^2,\end{aligned}$$

where $\theta \equiv \theta_s \in]0,1[$. Since $\|M_- + \theta\Delta M\| \leq \|M_-\| + \|\Delta M\|$, we also have

$$\bigl\|M_- + \theta\Delta M\bigr\|^{p-2} \lesssim_p \|M_-\|^{p-2} + \|\Delta M\|^{p-2} \leq (M^*_-)^{p-2} + \|\Delta M\|^{p-2}.$$

[1] The subscript $\cdot_c$ means "with compact support", and $C^2_b(H)$ denotes the set of twice continuously differentiable functions $\varphi : H \to \mathbb{R}$ such that φ, φ' and φ'' are bounded.

Appealing to Doob's inequality, one thus obtains

$$\begin{aligned}\mathbb{E}\big(M^*_\infty\big)^p \lesssim_p \mathbb{E}\|M_\infty\|^p &\lesssim_p \mathbb{E}\sum\big((M^*_-)^{p-2}\|\Delta M\|^2 + \|\Delta M\|^p\big)\\ &= \mathbb{E}\int\big((M^*_-)^{p-2}\|g\|^2 + \|g\|^p\big)\,d\mu\\ &= \mathbb{E}\int\big((M^*_-)^{p-2}\|g\|^2 + \|g\|^p\big)\,d\nu\\ &\leq \mathbb{E}(M^*_\infty)^{p-2}\int\|g\|^2\,d\nu + \mathbb{E}\int\|g\|^p\,d\nu.\end{aligned}$$

By Young's inequality in the form

$$ab \leq \varepsilon a^{\frac{p}{p-2}} + N(\varepsilon)b^{p/2},$$

we are left with

$$\mathbb{E}(M^*_\infty)^p \leq \varepsilon N(p)\mathbb{E}(M^*_\infty)^p + N(\varepsilon,p)\mathbb{E}\Big(\int\|g\|^2\,d\nu\Big)^{p/2} + \mathbb{E}\int\|g\|^p\,d\nu.$$

The proof of the case $p > \alpha = 2$ is completed choosing ε small enough.

We are thus left with the case $\alpha \in [1,2[$, $p > \alpha$. Note that, by Lemma 2.3,

$$\|\cdot\|_{L_2(\nu)} \leq \|\cdot\|_{L_2(\nu)} + \|\cdot\|_{L_p(\nu)} \lesssim \|\cdot\|_{L_\alpha(\nu)} + \|\cdot\|_{L_p(\nu)},$$

hence the desired result follows immediately by the cases with $\alpha = 2$ proved above. □

Remark 3.3. The proof of $\mathsf{BJ}_{2,p}$, $p \geq 2$, just given is a (minor) adaptation of the proof in [23], while the other cases are taken from [20]. However (cf. Section 6 below), essentially the same result with a very similar proof was already given by Novikov [29]. In the latter paper the author treats the finite-dimensional case, but the constants are explicitly dimension-free. Moreover, he deduces the case $p < \alpha$ from the case $p = \alpha$ using the extrapolation principle of Burkholder and Gundy [6], where we used instead Lenglart's domination inequality. However, the proof of the latter is based on the former.

Second proof of $\mathsf{BJ}_{\alpha,p}$ *($p \leq \alpha$).* An application of the BDG inequality to M, taking into account that $\alpha/2 \leq 1$, yields

$$\mathbb{E}(M^*_T)^\alpha \lesssim_\alpha \mathbb{E}\Big(\sum\nolimits_{\leq T}\|\Delta M\|^2\Big)^{\alpha/2} \leq \mathbb{E}\sum\nolimits_{\leq T}\|\Delta M\|^\alpha = \mathbb{E}\big(\|g\|^\alpha \star \mu\big)_T = \mathbb{E}\big(\|g\|^\alpha \star \nu\big)_T$$

for any stopping time T. The result then follows by Lenglart's domination inequality. □

We are now going to present several proofs for the case $p > \alpha$. As seen at the end of the first BJ, it suffices to consider the case $p > \alpha = 2$.

Second proof of $\mathsf{BJ}_{2,p}$ *($p > 2$).* Let us show that $\mathsf{BJ}_{2,2p}$ holds if $\mathsf{BJ}_{2,p}$ does: the identity

$$[M,M] = \|g\|^2 \star \mu = \|g\|^2 \star \bar{\mu} + \|g\|^2 \star \nu,$$

the BDG inequality, and $\mathsf{BJ}_{2,p}$ imply

$$\begin{aligned}\mathbb{E}(M^*_\infty)^{2p} &\lesssim_p \mathbb{E}[M,M]^p_\infty \lesssim \mathbb{E}\big|\big(\|g\|^2 \star \bar\mu\big)_\infty\big|^p + \mathbb{E}\big(\|g\|^2 \star \nu\big)^p_\infty \\ &\lesssim_p \mathbb{E}\int \|g\|^{2p}\,d\nu + \mathbb{E}\Big(\int \|g\|^4\,d\nu\Big)^{p/2} + \mathbb{E}\Big(\int \|g\|^2\,d\nu\Big)^{\frac{1}{2}2p} \\ &= \|g\|^{2p}_{L_{2p}(\nu)} + \|g\|^{2p}_{L_4(\nu)} + \|g\|^{2p}_{L_2(\nu)}\end{aligned} \tag{3.2}$$

Since $2 < 4 < 2p$, one has, by Lemma 2.3,

$$\|g\|^{2p}_{L_4(\nu)} \leq \|g\|^{2p}_{L_{2p}(\nu)} + \|g\|^{2p}_{L_2(\nu)},$$

which immediately implies that $\mathsf{BJ}_{2,2p}$ holds true. Let us now show that $\mathsf{BJ}_{2,p}$ implies $\mathsf{BJ}_{2,2p}$ also for any $p \in [1,2]$. Recalling that $\mathsf{BJ}_{2,p}$ does indeed hold for $p \in [1,2]$, this proves that $\mathsf{BJ}_{2,p}$ holds for all $p \in [2,4]$, hence for all $p \geq 2$, thus completing the proof. In fact, completely similarly as above, one has, for any $p \in [1,2]$,

$$\begin{aligned}\mathbb{E}(M^*_\infty)^{2p} &\lesssim_p \mathbb{E}\big|\big(\|g\|^2 \star \bar\mu\big)_\infty\big|^p + \mathbb{E}\big(\|g\|^2 \star \nu\big)^p_\infty \\ &\lesssim_p \mathbb{E}\int \|g\|^{2p}\,d\nu + \mathbb{E}\Big(\int \|g\|^2\,d\nu\Big)^{\frac{1}{2}2p}.\end{aligned} \qquad \square$$

Remark 3.4. The above proof, with $p > 2$, is adapted from [3], where the authors assume $H = \mathbb{R}$ and $p = 2^n$, $n \in \mathbb{N}$, mentioning that the extension to any $p \geq 2$ can be obtained by an interpolation argument.

Third proof of $\mathsf{BJ}_{2,p}$ *($p > 2$).* Let $k \in \mathbb{N}$ be such that $2^k \leq p < 2^{k+1}$. Applying the BDG inequality twice, one has

$$\mathbb{E}\big\|(g \star \bar\mu)_\infty\big\|^p \lesssim_p \mathbb{E}\big(\|g\|^2 \star \mu\big)^{p/2}_\infty \lesssim_p \mathbb{E}\big|\big(\|g\|^2 \star \bar\mu\big)_\infty\big|^{p/2} + \mathbb{E}\big(\|g\|^2 \star \nu\big)^{p/2}_\infty,$$

where

$$\mathbb{E}\big|\big(\|g\|^2 \star \bar\mu\big)_\infty\big|^{p/2} \lesssim_p \mathbb{E}\big(\|g\|^2 \star \mu\big)^{p/4}_\infty \lesssim_p \mathbb{E}\big|\big(\|g\|^4 \star \bar\mu\big)_\infty\big|^{p/4} + \mathbb{E}\big(\|g\|^4 \star \nu\big)^{p/4}_\infty.$$

Iterating we are left with

$$\mathbb{E}\big\|(g \star \bar\mu)_\infty\big\|^p \lesssim_p \mathbb{E}\big(\|g\|^{2^{k+1}} \star \mu\big)^{p/2^{k+1}}_\infty + \sum_{i=1}^k \mathbb{E}\Big(\int \|g\|^{2^i}\,d\nu\Big)^{p/2^i},$$

where, recalling that $p/2^{k+1} < 1$,

$$\begin{aligned}\mathbb{E}\big(\|g\|^{2^{k+1}} \star \mu\big)^{p/2^{k+1}}_\infty &= \mathbb{E}\Big(\sum \|\Delta M\|^{2^{k+1}}\Big)^{p/2^{k+1}} \\ &\leq \mathbb{E}\sum \|\Delta M\|^p = \mathbb{E}\int \|g\|^p\,d\mu = \mathbb{E}\int \|g\|^p\,d\nu.\end{aligned}$$

The proof is completed observing that, since $2 \leq 2^i \leq p$ for all $1 \leq i \leq k$, one has, by Lemma 2.3,

$$\begin{aligned}\mathbb{E}\Big(\int \|g\|^{2^i}\,d\nu\Big)^{p/2^i} &= \mathbb{E}\|g\|^p_{L_{2^i}(\nu)} \leq \mathbb{E}\|g\|^p_{L_2(\nu)} + \mathbb{E}\|g\|^p_{L_p(\nu)} \\ &= \mathbb{E}\Big(\int \|g\|^2\,d\nu\Big)^{p/2} + \mathbb{E}\int \|g\|^p\,d\nu.\end{aligned} \qquad \square$$

Remark 3.5. The above proof, which can be seen as a variation of the previous one, is adapted from [33, Lemma 4.1] (which was translated to the H-valued case in [21]). In [33] the interpolation step at the end of the proof is obtained in a rather tortuous (but interesting way), which is not reproduced here.

The next proof is adapted from [14].

Fourth proof of $\mathsf{BJ}_{2,p}$ *($p > 2$).* Let us start again from the BDG inequality:

$$\mathbb{E}(M^*_\infty)^p \lesssim_p \mathbb{E}[M,M]^{p/2}_\infty.$$

Since $[M,M]$ is a real, positive, increasing, purely discontinuous process with $\Delta[M,M] = \|\Delta M\|^2$, one has

$$\begin{aligned}[M,M]^{p/2}_\infty &= \sum \bigl([M,M]^{p/2} - [M,M]^{p/2}_-\bigr)\\ &= \sum \Bigl(\bigl([M,M]_- + \|\Delta M\|^2\bigr)^{p/2} - [M,M]^{p/2}_-\Bigr).\end{aligned}$$

For any a, $b \geq 0$, the mean value theorem applied to the function $x \mapsto x^{p/2}$ yields the inequality

$$(a+b)^{p/2} - a^{p/2} = (p/2)\xi^{p/2-1}b \leq (p/2)(a+b)^{p/2-1}b \leq (p/2)2^{p/2-1}(a^{p/2-1}b + b^{p/2}),$$

where $\xi \in \,]a,b[$, hence also

$$\bigl([M,M]_- + \|\Delta M\|^2\bigr)^{p/2} - [M,M]^{p/2}_- \lesssim_p [M,M]^{p/2-1}_- \|\Delta M\|^2 + \|\Delta M\|^p.$$

This in turn implies

$$\begin{aligned}\mathbb{E}[M,M]^{p/2}_\infty &\lesssim_p \sum \Bigl([M,M]^{p/2-1}_- \|\Delta M\|^2 + \|\Delta M\|^p\Bigr)\\ &= \mathbb{E}\int \Bigl([M,M]^{p/2-1}_- \|g\|^2 + \|g\|^p\Bigr)\,d\mu\\ &= \mathbb{E}\int \Bigl([M,M]^{p/2-1}_- \|g\|^2 + \|g\|^p\Bigr)\,d\nu\\ &\leq \mathbb{E}[M,M]^{p/2-1}_\infty \int \|g\|^2\,d\nu + \mathbb{E}\int \|g\|^p\,d\nu.\end{aligned}$$

By Young's inequality in the form

$$a^{p/2-1}b \leq \varepsilon a^{p/2} + N(\varepsilon)b^{p/2}, \qquad a,\, b \geq 0,$$

one easily infers

$$\mathbb{E}[M,M]^{p/2}_\infty \lesssim_p \mathbb{E}\Bigl(\int \|g\|^2\,d\nu\Bigr)^{p/2} + \mathbb{E}\int \|g\|^p\,d\nu,$$

thus concluding the proof. □

3.2 A (too?) sophisticated proof

In this subsection we prove a maximal inequality valid for any H-valued local martingale M (that is, we do not assume that M is purely discontinuous), from which $\mathsf{BJ}_{2,p}$, $p > 2$, follows immediately.

Theorem 3.6. *Let M be* any *local martingale with values in H. One has, for any $p \geq 2$,*

$$\mathbb{E}(M_\infty^*)^p \lesssim_p \mathbb{E}\langle M, M\rangle_\infty^{p/2} + \mathbb{E}\big((\Delta M)_\infty^*\big)^p.$$

Proof. We are going to use Davis' decomposition (see [28] for a very concise proof in the case of real martingales, a detailed "transliteration" of which to the case of Hilbert-space-valued martingales can be found in [24]): setting $S := (\Delta M)^*$, one has $M = L+K$, where L and K are martingales satisfying the following properties:

(i) $\|\Delta L\| \lesssim S_-$;

(ii) K has integrable variation and $K = K^1 + \widetilde{K^1}$, where $\widetilde{K^1}$ is the predictable compensator of K^1 and $\int |dK^1| \lesssim S_\infty$.

Since $M^* \leq L^* + K^*$, we have

$$\|M_\infty^*\|_{\mathbb{L}_p} \leq \|L_\infty^*\|_{\mathbb{L}_p} + \|K_\infty^*\|_{\mathbb{L}_p},$$

where, by the BDG inequality, $\|K_\infty^*\|_{\mathbb{L}_p} \lesssim_p \|[K, K]^{1/2}\|_{\mathbb{L}_p}$. Moreover, by the maximal inequality for martingales with predictably bounded jumps in [18, p. 37][2] and the elementary estimate $\langle L, L\rangle^{1/2} \leq \langle M, M\rangle^{1/2} + \langle K, K\rangle^{1/2}$, one has

$$\begin{aligned}\|L_\infty^*\|_{\mathbb{L}_p} &\lesssim_p \|\langle L, L\rangle_\infty^{1/2}\|_{\mathbb{L}_p} + \|S_\infty\|_{\mathbb{L}_p}\\ &\leq \|\langle M, M\rangle_\infty^{1/2}\|_{\mathbb{L}_p} + \|\langle K, K\rangle_\infty^{1/2}\|_{\mathbb{L}_p} + \|(\Delta M)_\infty^*\|_{\mathbb{L}_p}.\end{aligned}$$

Since $p \geq 2$, the inequality between moments of a process and of its dual predictable projection in [18, Theoreme 4.1] yields $\|\langle K, K\rangle^{1/2}\|_{\mathbb{L}_p} \lesssim_p \|[K, K]^{1/2}\|_{\mathbb{L}_p}$. In particular, we are left with

$$\|M_\infty^*\|_{\mathbb{L}_p} \lesssim_p \|\langle M, M\rangle_\infty^{1/2}\|_{\mathbb{L}_p} + \|(\Delta M)_\infty^*\|_{\mathbb{L}_p} + \|[K, K]_\infty^{1/2}\|_{\mathbb{L}_p}.$$

Furthermore, applying a version of Stein's inequality between moments of a process and of its predictable projection (see e.g. [24], and [35, p. 103] for the original formulation), one has, for $p \geq 2$,

$$\|[\widetilde{K^1}, \widetilde{K^1}]^{1/2}\|_{\mathbb{L}_p} \lesssim_p \|[K^1, K^1]^{1/2}\|_{\mathbb{L}_p},$$

hence, recalling property (ii) above and that the quadratic variation of a process is bounded by its first variation, we are left with

$$\begin{aligned}\|[K, K]^{1/2}\|_{\mathbb{L}_p} &\leq \|[K^1, K^1]^{1/2}\|_{\mathbb{L}_p} + \|[\widetilde{K^1}, \widetilde{K^1}]^{1/2}\|_{\mathbb{L}_p}\\ &\lesssim_p \|[K^1, K^1]^{1/2}\|_{\mathbb{L}_p} \leq \Big\|\int |dK^1|\Big\|_{\mathbb{L}_p} \lesssim \|(\Delta M)_\infty^*\|_{\mathbb{L}_p}.\end{aligned}$$

□

[2] One can verify that the proof in [18] goes through without any change also for Hilbert-space-valued martingales.

It is easily seen that Theorem 3.6 implies $\mathsf{BJ}_{2,p}$ (for $p \geq 2$): in fact, one has

$$\mathbb{E}\big((\Delta M)^*\big)^p \leq \mathbb{E}\sum \|\Delta M\|^p = \mathbb{E}\int \|g\|^p \, d\nu.$$

Remark 3.7. The above proof is a simplified version of an argument from [20]. As we recently learned, however, a similar argument was given in [9]. As a matter of fact, their proofs is somewhat shorter than ours, as they claim that $[K, L] = 0$. Unfortunately, we have not been able to prove this claim.

3.3 A conditional proof

The purpose of this subsection is to show that if $\mathsf{BJ}_{2,p}$, $p \geq 2$, holds for real (local) martingales, then it also holds for (local) martingales with values in H. For this we are going to use Khinchine's inequality: let $x \in H$, and $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of H. Setting $x_k := \langle x, e_k \rangle$. Then one has

$$\|x\| = \Big(\sum_k x_k^2\Big)^{1/2} = \Big\|\sum_k x_k \varepsilon_k\Big\|_{L_2(\bar\Omega)} \eqsim \Big\|\sum_k x_k \varepsilon_k\Big\|_{L_p(\bar\Omega)},$$

where $(\bar\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ is an auxiliary probability space, on which a sequence (ε_k) of i.i.d Rademacher random variables are defined.

Writing

$$M_k := \langle M, e_k \rangle = g_k \star \bar\mu, \qquad g_k := \langle g, e_k \rangle,$$

one has $\sum_k M_k \varepsilon_k = \Big(\sum_k g_k \varepsilon_k\Big) \star \bar\mu$, hence Khinchine's inequality, Tonelli's theorem, and Theorem 3.2 for real martingales yield

$$\begin{aligned}
\mathbb{E}\|M\|^p &\eqsim \mathbb{E}\Big\|\Big(\sum g_k\varepsilon_k\Big) \star \bar\mu\Big\|^p_{L_p(\bar\Omega)} = \bar{\mathbb{E}}\,\mathbb{E}\Big|\Big(\sum g_k\varepsilon_k\Big) \star \bar\mu\Big|^p \\
&\lesssim_p \bar{\mathbb{E}}\,\mathbb{E}\Big(\int \Big|\sum g_k\varepsilon_k\Big|^2 d\nu\Big)^{p/2} + \bar{\mathbb{E}}\,\mathbb{E}\int \Big|\sum g_k\varepsilon_k\Big|^p d\nu \\
&=: I_1 + I_2.
\end{aligned}$$

Tonelli's theorem, together with Minkowski's and Khinchine's inequalities, yield

$$\begin{aligned}
I_1 = \mathbb{E}\,\bar{\mathbb{E}}\Big(\int \Big|\sum g_k\varepsilon_k\Big|^2 d\nu\Big)^{p/2} &= \mathbb{E}\Big\|\int \Big|\sum g_k\varepsilon_k\Big|^2 d\nu\Big\|^{p/2}_{L_{p/2}(\bar\Omega)} \\
&\leq \mathbb{E}\Big(\int \Big\|\Big|\sum g_k\varepsilon_k\Big|^2\Big\|_{L_{p/2}(\bar\Omega)} d\nu\Big)^{p/2} \\
&= \mathbb{E}\Big(\int \Big\|\sum g_k\varepsilon_k\Big\|^2_{L_p(\bar\Omega)} d\nu\Big)^{p/2} \eqsim \Big(\int \|g\|^2 \, d\nu\Big)^{p/2}.
\end{aligned}$$

Similarly, one has

$$I_2 = \mathbb{E}\int \Big\|\sum g_k\varepsilon_k\Big\|^p_{L_p(\bar\Omega)} d\nu \eqsim \mathbb{E}\int \|g\|^p \, d\nu.$$

The proof is completed appealing to Doob's inequality. □

Remark 3.8. This conditional proof has probably not appeared in published form, although the idea is contained in [32].

4 Inequalities for Poisson stochastic integrals with values in L_q spaces

Even though there exist in the literature some maximal inequalities for stochastic integrals with respect to compensated Poisson random measures and Banach-space-valued integrands, here we limit ourselves to reporting about (very recent) two-sided estimates in the case of L_q-valued integrands. Throughout this section we assume that μ is a Poisson random measure, so that its compensator ν is of the form $\mathrm{Leb}\otimes\nu_0$, where Leb stands for the one-dimensional Lebesgue measure and ν_0 is a (non-random) σ-finite measure on Z. Let $(X, \mathcal{A}, n)$ be a measure space, and denote L_q spaces on X simply by L_q, for any $q \geq 1$. Moreover, let us introduce the following spaces, where $p_1, p_2, p_3 \in [1, \infty[$:

$$L_{p_1,p_2,p_3} := \mathbb{L}_{p_1}(L_{p_2}(\mathbb{R}_+ \times Z \to L_{p_3}(X))), \qquad \tilde{L}_{p_1,p_2} := \mathbb{L}_{p_1}(L_{p_2}(X \to L_2(\mathbb{R}_+ \times Z))).$$

Then one has the following result, due to Dirksen [7]:

$$\Big\| \sup_{t\geq 0} \|(g \star \bar{\mu})_t\|_{L_q} \Big\|_{\mathbb{L}_p} \eqsim_{p,q} \|g\|_{\mathcal{I}_{p,q}},$$

where

$$\mathcal{I}_{p,q} := \begin{cases} L_{p,p,q} + L_{p,q,q} + \tilde{L}_{p,q}, & 1 < p \leq q \leq 2, \\ (L_{p,p,q} \cap L_{p,q,q}) + \tilde{L}_{p,q}, & 1 < q \leq p \leq 2, \\ L_{p,p,q} \cap (L_{p,q,q} + \tilde{L}_{p,q}), & 1 < q < 2 \leq p, \\ L_{p,p,q} + (L_{p,q,q} \cap \tilde{L}_{p,q}), & 1 < p < 2 \leq q, \\ (L_{p,p,q} + L_{p,q,q}) \cap \tilde{L}_{p,q}, & 2 \leq p \leq q, \\ L_{p,p,q} \cap L_{p,q,q} \cap \tilde{L}_{p,q}, & 2 \leq q \leq p. \end{cases} \tag{4.1}$$

The proof of this result is too long to be included here. We limit instead ourselves to briefly recalling what the main "ingredients" are: one first establishes extensions of the classical Rosenthal inequality

$$\mathbb{E}\Big| \sum \xi_k \Big|^p \lesssim_p \max\Big(\mathbb{E} \sum |\xi_k|^p, \Big(\mathbb{E} \sum |\xi_k|^2 \Big)^{p/2} \Big),$$

where $p \geq 2$ and $\xi = (\xi_k)_k$ is any (finite) sequence of independent real random variables. In particular, several extensions are obtained in cases where the independent random variables ξ_k take values in Banach spaces satisfying certain geometric properties; further extensions are proved, by duality arguments, to the case where $p \in]1, 2[$. Particularly "nice" versions are then derived assuming that the random variables ξ_k take values in L_q spaces, thanks to their rich geometric structure. Finally, it is shown that, using decoupling techniques, such inequalities can be extended from sequences of independent random variables to stochastic integrals of step processes with respect to compensated Poisson random measures.

5 Inequalities for stochastic convolutions

In this section we show how one can extend, under certain assumptions, maximal inequalities from stochastic integrals to stochastic convolutions using dilations of semigroups.

As is well known, stochastic convolutions are in general not semimartingales, hence establishing maximal inequalities for them is, in general, not an easy task. Usually one tries to approximate stochastic convolutions by processes which can be written as solutions to stochastic differential equations in either a Hilbert or a Banach space, for which one can (try to) obtain estimates using tools of stochastic calculus. As a final step, one tries to show that such estimates can be transfered to stochastic convolutions as well, based on establishing suitable convergence properties. At present it does not seem possible to claim that any of the two methods is superior to the other (cf., e.g., the discussion in [37]). We choose to concentrate on the dilation technique for its simplicity and elegance.

We shall say that a linear operator A on a Banach space E, such that $-A$ is the infinitesimal generator of a strongly continuous semigroup S, is of class D if there exist a Banach space $\bar{E}$, an isomorphic embedding $\iota : E \to \bar{E}$, a projection $\pi : \bar{E} \to \iota(E)$, and a strongly continuous bounded group $(U(t))_{t\in\mathbb{R}}$ on $\bar{E}$ such that the following diagram commutes for all $t > 0$:

$$\begin{array}{ccc} E & \xrightarrow{S(t)} & E \\ {\scriptstyle \iota}\big\downarrow & & \big\uparrow{\scriptstyle \iota^{-1}\circ\pi} \\ \bar{E} & \xrightarrow[U(t)]{} & \bar{E} \end{array}$$

As far as we know there is no general characterization of operators of class D.[3] Several sufficient conditions, however, are known.

We begin with the classical dilation theorem by Sz.-Nagy (see e.g. [36]).

Proposition 5.1. *Let A be a linear operator m-accretive operator on a Hilbert space H. Then A is of class D.*

The next result, due to Fendler [10], is analogous to Sz.-Nagy's dilation theorem in the context of L_q spaces, although it requires an extra positivity assumption.

Proposition 5.2. *Let $E = L_q(X)$, where X is any measure space and $q \in \,]1,\infty[$. Assume that A is a linear m-accretive operator on E such that $S(t) := e^{-tA}$ is positivity preserving for all $t > 0$. Then A is of class D, with $\bar{E} = L_q(Y)$, where Y is another measure space.*

The following very recent result, due to Fröhlich and Weis [11], allows one to consider classes of operators that are not necessarily accretive (for many interesting examples, see e.g. [37]). For all unexplained notions of functional calculus for operators we refer to, e.g., [38].

Proposition 5.3. *Let $E = L_q(X,m)$, with $q \in \,]1,\infty[$, and assume that A is sectorial and admits a bounded H^∞-calculus with $\omega_{H^\infty}(A) < \pi/2$. Then A is of class D, and one can choose $\bar{E} = L_q([0,1] \times X, \mathrm{Leb} \otimes m)$.*

Let us now show how certain maximal estimates for stochastic integrals yield maximal estimates for convolutions involving the semigroup generated by an operator of class D.

[3] The definition of class D is not standard and it is introduced just for the sake of concision.

Note that, since the operator norms of π and $U(t)$ are less than or equal to one, one has

$$\begin{aligned}
&\mathbb{E}\sup_{t\geq 0}\Big\|\int_0^t\!\!\int_Z S(t-s)g(s,z)\,\bar{\mu}(ds,dz)\Big\|_E^p\\
&\quad= \mathbb{E}\sup_{t\geq 0}\Big\|\pi U(t)\int_0^t\!\!\int_Z U(-s)\iota(g(s,z))\,\bar{\mu}(ds,dz)\Big\|_{\bar{E}}^p\\
&\quad\leq \|\pi\|_\infty^p\,\sup_{t\geq 0}\|U(t)\|_\infty^p\,\mathbb{E}\sup_{t\geq 0}\Big\|\int_0^t\!\!\int_Z U(-s)\iota(g(s,z))\,\bar{\mu}(ds,dz)\Big\|_{\bar{E}}^p\\
&\quad\leq \mathbb{E}\sup_{t\geq 0}\Big\|\int_0^t\!\!\int_Z U(-s)\iota(g(s,z))\,\bar{\mu}(ds,dz)\Big\|_{\bar{E}}^p, \qquad (5.1)
\end{aligned}$$

where $\|\cdot\|_\infty$ denotes the operator norm. We have thus reduced the problem to finding a maximal estimate for a stochastic integral, although involving a different integrand and on a larger space.

If E is a Hilbert space we can proceed rather easily.

Proposition 5.4. *Let A be of class D on a Hilbert space E. Then one has, for any $\alpha \in [1,2]$,*

$$\mathbb{E}\sup_{t\geq 0}\Big\|\int_0^t S(t-\cdot)g\,d\bar{\mu}\Big\|_E^p \lesssim_{\alpha,p} \begin{cases} \mathbb{E}\Big(\int \|g\|^\alpha\,d\nu\Big)^{p/\alpha} & \forall p\in\,]0,\alpha]\,,\\ \mathbb{E}\Big(\int \|g\|^\alpha\,d\nu\Big)^{p/\alpha} + \mathbb{E}\int\|g\|^p\,d\nu & \forall p\in[\alpha,\infty[\,.\end{cases}$$

Proof. We consider only the case $p>\alpha$, as the other one is actually simpler. The estimate $\mathsf{BJ}_{\alpha,p}$ and (5.1) yield

$$\begin{aligned}
&\mathbb{E}\sup_{t\geq 0}\Big\|\int_0^t S(t-\cdot)g\,d\bar{\mu}\Big\|_E^p\\
&\quad\lesssim_{\alpha,p} \mathbb{E}\int\big\|U(-\cdot)\iota\circ g\big\|_{\bar{E}}^p\,d\nu + \mathbb{E}\Big(\int\big\|U(-\cdot)\iota\circ g\big\|_{\bar{E}}^\alpha\,d\nu\Big)^{p/\alpha}\\
&\quad\leq \mathbb{E}\int\|g\|_E^p\,d\nu + \mathbb{E}\Big(\int\|g\|_E^\alpha\,d\nu\Big)^{p/\alpha},
\end{aligned}$$

because U is a unitary group and the embedding ι is isometric. □

If $E = L_q(X)$, the transposition of maximal inequalities from stochastic integrals to stochastic convolution is not so straightforward. In particular, (5.1) implies that the corresponding upper bounds will be functions of the norms of $U(-\cdot)\,\iota\circ g$ in three spaces of the type $L_{p,p,q}$, $L_{p,q,q}$ and $\tilde{L}_{p,q}$ (with X replaced by a different measure space Y, so that $\bar{E} = L_q(Y)$). In analogy to the previous proposition, it is not difficult to see that

$$\big\|U(-\cdot)\,\iota\circ g\big\|_{\mathbb{L}_{p_1}L_{p_2}(\mathbb{R}_+\times Z\to L_{p_3}(Y))} \leq \big\|g\big\|_{\mathbb{L}_{p_1}L_{p_2}(\mathbb{R}_+\times Z\to L_{p_3}(X))}. \qquad (5.2)$$

However, estimating the norm of $U(-\cdot)\,\iota\circ g$ in $\tilde{L}_{p,q}(Y)$ in terms of the norm of g in $\tilde{L}_{p,q}$ does not seem to be possible without further assumptions. Nonetheless, the following sub-optimal estimates can be obtained.

Proposition 5.5. *Let A be of class D on $E = L_q := L_q(X)$ and μ a Poisson random measure. Then one has*

$$\mathbb{E}\sup_{t\geq 0}\biggl\|\int_0^t S(t-\cdot)g\,d\bar{\mu}\biggr\|_{L_q}^p \lesssim_{p,q} \|g\|_{\mathcal{J}_{p,q}},$$

where

$$\mathcal{J}_{p,q} := \begin{cases} L_{p,p,q} + L_{p,q,q}, & 1 < p \leq q \leq 2,\\ L_{p,p,q} \cap L_{p,q,q}, & 1 < q \leq p \leq 2,\\ L_{p,p,q} \cap L_{p,q,q}, & 1 < q < 2 \leq p,\\ L_{p,p,q} + (L_{p,q,q} \cap L_{p,2,q}), & 1 < p < 2 \leq q,\\ (L_{p,p,q} + L_{p,q,q}) \cap L_{p,2,q}, & 2 \leq p \leq q,\\ L_{p,p,q} \cap L_{p,q,q} \cap L_{p,2,q}, & 2 \leq q \leq p. \end{cases}$$

Proof. Note that, if $q < 2$, one has, by definition, $\|\cdot\|_{\mathcal{I}_{p,q}} \leq \|\cdot\|_{\mathcal{J}p,q}$ (where the spaces $\mathcal{I}_{p,q}$ have been defined in (4.1)); if $q \geq 2$, by Minkowski's inequality,

$$\biggl\|\Bigl(\int |g|^2\,d\nu\Bigr)^{1/2}\biggr\|_{L_q} = \biggl\|\int |g|^2\,d\nu\biggr\|_{L_{q/2}}^{1/2} \leq \Bigl(\int \|g\|_{L_q}^2\,d\nu\Bigr)^{1/2},$$

that is, $\|\cdot\|_{\tilde{L}_{p,q}} \leq \|\cdot\|_{L_{p,2,q}}$. This implies $\|\cdot\|_{\mathcal{I}_{p,q}} \leq \|\cdot\|_{\mathcal{J}p,q}$ for all $q \geq 2$, hence for all (admissible) values of p and q. Therefore (5.1) and the maximal estimate (4.1) yield the desired result. □

Remark 5.6. The above maximal inequalities for stochastic convolutions continue to hold if A is only quasi-m-accretive and g has compact support in time. In this case the inequality sign $\lesssim_{p,q}$ has to be replaced by $\lesssim_{p,q,\eta,T}$, where T is a finite time horizon. One simply has to repeat the same arguments using the m-accretive operator $A + \eta I$, for some $\eta > 0$.

6 Historical and bibliographical remarks

In this section we try to reconstruct, at least in part, the historical developments around the maximal inequalities presentend above. Before doing that, however, let us briefly explain how we became interested in the class of maximal inequalities: the first-named author used in [21] a Hilbert-space version of a maximal inequality in [33] to prove well-posedness for a Lévy-driven SPDE arising in the modeling of the term structure of interest rates. The second-named author pointed out that such an inequality, possibly adapted to the more general case of integrals with respect to compensated Poisson random measures (rather than with respect to Lévy processes), was needed to solve a problem he was interested in, namely to establish regularity of solutions to SPDEs with jumps with respect to initial conditions: our joint efforts led to the results in [23], where we proved a slightly less general version of the inequality $\mathsf{BJ}_{2,p}$, $p \geq 2$, using an argument involving only Itô's formula. At the time of writing [23] we did not realize that, as demonstrated in the present paper, it would have been possible to obtain the same result adapting one of the two arguments (for Lévy-driven integrals) we were aware of, i.e. those in [3] and [33].

The version in [23] of the inequality $\mathsf{BJ}_{2,p}$, $p \geq 2$, was called in that paper "Bichteler-Jacod inequality", as we believed it appeared (in dimension one) for the first time in [3]. This is actually what we believed until a few days ago (this explains the label BJ), when, after this paper as well as the first drafts of [19] and [20] were completed, we found a reference to [29] in [40]. This is one of the surprises we alluded to in the introduction. Namely, Novikov proved (in 1975, hence well before Bichteler and Jacod, not to mention how long before ourselves) the upper bound $\mathsf{BJ}_{\alpha,p}$ for all values of α and p, assuming $H = \mathbb{R}^n$, but with constants that are independent of the dimension. For this reason it seems that, if one wants to give a name (as we do) to the inequality BJ and its extensions, they should be called Novikov's inequalities.[4] Unfortunately Novikov's paper [29] was probably not known also to Kunita, who proved in [16] (in 2004) a slightly weaker version of $\mathsf{BJ}_{2,p}$, $p \geq 2$, in $H = \mathbb{R}^n$, also using Itô's formula. Moreover, Applebaum [1] calls these inequalities "Kunita's estimates", but, again, they are just a version of what we called (and are going to call) Novikov's inequality.

Even though the proofs in [2, 3] are only concerned with the real-valued case, the authors explicitly say that they knew how to get the constant independent of the dimension (see, in particular, [2, Lemma 5.1 and remark 5.2]). The proofs in [14, 33] are actually concerned with integrals with respect to Lévy processes, but the adaptation to the more general case presented here is not difficult. Moreover, the inequalities in [2, 3, 14, 33] are of the type

$$\mathbb{E}\sup_{t\leq T} \|(g\star\bar{\mu})_t\|^p \lesssim_{p,d,T} \mathbb{E}\int_0^T \biggl(\int_Z \|g((s,\cdot)\|^2\,dm\biggr)^{p/2} ds + \mathbb{E}\int_0^T\!\!\int_Z \|g((s,\cdot)\|^p\,d\nu_0\,ds,$$

where μ is a Poisson random measure with compensator $\nu = \mathrm{Leb}\otimes\nu_0$. Our proofs show that all their arguments can be improved to yield a constant depending only on p and that the first term on the right-hand side can be replaced by $\mathbb{E}\bigl(\|g\|^2\star\nu\bigr)_T^{p/2}$.

Again through [40] we also became aware of the Novikov-like inequality by Dzhaparidze and Valkeila [9], where Theorem 3.6 in proved with $H = \mathbb{R}$. It should be observed that the inequality in the latter theorem is apparently more general than, but actually equivalent to BJ (cf. [20]).

Another method to obtain Novikov-type inequalities, also in vector-valued settings, goes through their analogs in discrete time, i.e. the Burkholder-Rosenthal inequality. We have not touched upon this method, as we are rather interested in "direct" methods in continuous time. We refer the interested reader to the very recent preprints [7, 8], as well as to [31, 39] and references therein.

The idea of using dilation theorems to extend results from stochastic integrals to stochastic convolutions has been introduced, to the best of our knowledge, in [12]. This method has then been generalized in various directions, see e.g. [13, 21, 37]. In this respect, it should be mentioned that the "classical" direct approach, which goes through approximations by regular processes and avoid dilations (here "classical" stands for

[4] It should be mentioned that there are discrete-time real-valued analogs of $\mathsf{BJ}_{2,p}$, $p \geq 2$, that go under the name of Burkholder-Rosenthal (in alphabetical but reverse chronological order: Rosenthal [34] proved it for sequences of independent random variables in 1970, then Burkholder [5] extended it to discrete-time (real) martingales in 1973), and some authors speak of continuous-time Burkholder-Rosenthal inequalities. One may then also propose to use the expression Burkholder-Rosenthal-Novikov inequality, that, however, seems too long.

equations on Hilbert spaces driven by Wiener process), has been (partially) extended to Banach-space valued stochastic convolutions with jumps in [4]. The former and the latter methods are complementary, in the sense that none is more general than the other. Furthermore, it is well known (see e.g. [30]) that the factorization method breaks down when applied to stochastic convolutions with respect to jump processes.

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