An analytic approach to infinite-dimensional continuity and Fokker–Planck–Kolmogorov equations

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Abstract. We prove a new uniqueness result for solutions to Fokker–Planck–Kolmogorov (FPK) equations for probability measures on infinite-dimensional spaces. We consider infinite-dimensional drifts that admit certain finite-dimensional approximations. In contrast to most of the previous work on FPK-equations in infinite dimensions, we include cases with non-constant coefficients in the second order part and also include degenerate cases where these coefficients can even be zero. Also a new existence result is proved. Some applications to Fokker–Planck–Kolmogorov equations associated with SPDEs are presented.

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Introduction

In this paper we study the Cauchy problem for infinite-dimensional Fokker–Planck–Kolmogorov equations of the form $\partial_t\mu = L^*\mu$ for bounded Borel measures μ on the space $\mathbb{R}^\infty \times (0, T_0)$, where $\mathbb{R}^\infty$ is the countable power of $\mathbb{R}$ with the product topology, and second order operators

$$L\varphi = \sum_{i,j} a^{ij}\partial_{x_i}\partial_{x_j}\varphi + \sum_i B^i\partial_{x_i}\varphi$$

defined on smooth functions of finitely many variables. Such equations arise in many applications and have been intensively studied in the last decades. In particular, they are satisfied by transition probabilities of infinite-dimensional diffusions, which is an important motivation for this paper. The finite-dimensional case has been studied in depth by many authors (see the recent surveys [10] and [12]), in particular, there is an extensive literature on regularity and uniqueness of solutions to Fokker–Planck–Kolmogorov equations for measures on finite-dimensional spaces, see [3], [8], [9], [10], [12], [18], [27], [33], and the references there. The infinite-dimensional case is considerably less studied, although there is also a vast literature devoted to this case (see, e.g., [5], [6], [7], [15], [24], [30], and the references there).

The organization of the paper is as follows. In Section 1 we introduce a general class of Fokker–Planck–Kolmogorov equations in infinite dimensions and prove some preliminary results. In Section 2 we prove uniqueness of probability solutions for these equations under a certain approximative condition (which is a condition on all components of the drift term in a certain uniform way), which considerably generalizes our previous uniqueness results in [6] and [7]. The main difference with the finite-dimensional case is that in the latter the global integrability of the coefficients a^{ij} and b^i with respect to the solution ensures its uniqueness, but there is no infinite-dimensional analog of this simple sufficient condition. What we prove is only a partial analog (Example 2.1(ii) formally gives a full analog, but the condition on the norm of the whole drift is very restrictive in infinite dimensions). More precisely, we establish two uniqueness results: Theorem 2.3 (nondegenerate diffusion matrices) and Theorem 2.5 that applies also to degenerate equations, in particular to fully degenerate transport (or continuity) equations including the continuity equation associated to $2d$-Navier–Stokes equation.

In Section 3 we address the question of existence of solutions to our general FPK-equations and prove Theorem 3.1 which implies existence under quite broad assumptions, in particular, for stochastic Navier–Stokes equations over domains in $\mathbb{R}^d$ for all dimensions d. In Section 2 and Section 3 we also consider examples that include two other types of SPDEs, namely, stochastic reaction diffusion equations on a bounded domain in $\mathbb{R}^d$ (Example 2.9) and Burgers equation (Example 2.10) on the interval $(0, 1)$; their mixture is considered in Example 2.11.

The approach and assumptions in this work differ from those in our earlier paper [5], where probabilistic tools were employed. Here we develop a purely analytic approach without stochastic analysis and (for the first time in infinite dimension) also include the case of nonconstant diffusion matrices. The techniques are also different from the ones in [5], [6], and [16], where measures on Hilbert spaces were considered, but the essential difference is not the type of infinite-dimensional spaces, but rather the method of proof which could be called *approximative Holmgren method*, the idea of which is to multiply the original equation by a solution of a certain equation approximating the adjoint equation (but not the exact adjoint equation as in Holmgren's method) and obtain after integration certain estimates (which replace exact equalities in the classical Holmgren method).

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1. Framework and preliminaries

Let us describe our framework. Let $B = (B^i(x,t))$ be a sequence of Borel functions on $\mathbb{R}^\infty \times (0,T_0)$, where $T_0 > 0$ is fixed, and let a^{ij} be Borel functions on $\mathbb{R}^\infty \times (0,T_0)$. Let us consider the Cauchy problem

$$\begin{cases} \partial_t\mu = L^*\mu, \\ \mu|_{t=0} = \nu, \end{cases} \tag{1.1}$$

where L^* is the formal adjoint operator for a differential operator L defined by

$$L\varphi(x,t) = \sum_{i,j=1}^{\infty} a^{ij}(x,t)\partial_{e_i}\partial_{e_j}\varphi(x,t) + \sum_{i=1}^{\infty} B^i(x,t)\partial_{e_i}\varphi(x,t)$$

for every smooth function φ depending on finitely many coordinates of x, $\partial_{e_i}\varphi$ denotes the partial derivative along the vector $e_i = (0,\ldots,0,1,0,\ldots)$. Equations of this form are usually called Fokker–Planck–Kolmogorov equations.

Throughout this paper a measure means a bounded signed measure (not necessarily nonnegative, although our principal results will be concerned with probability measures). The total variation of a measure μ is denoted by $|\mu|$. Let J be an interval in $[0,+\infty)$. We use the standard notation $C(\mathbb{R}^k \times J)$ and $C^{2,1}(\mathbb{R}^k \times J)$ for the class of real continuous functions on $\mathbb{R}^k \times J$ and its subclass consisting of all functions f having continuous partial derivatives $\partial_t f$, $\partial_{x_i} f$ and $\partial^2_{x_ix_j} f$. Let $C_b(\mathbb{R}^k \times J)$ and $C_b^{2,1}(\mathbb{R}^k \times J)$ denote the subclasses in these classes consisting of bounded functions and functions f with bounded derivatives $\partial_t f$, $\partial_{x_i} f$ and $\partial^2_{x_ix_j} f$, respectively, and $C_0^{2,1}(\mathbb{R}^k \times J)$ is the subspace in $C_b^{2,1}(\mathbb{R}^k \times J)$ consisting of functions with compact support in $\mathbb{R}^k \times J$.

The inner product in $\mathbb{R}^d$ will be denoted by $\langle\,\cdot\,,\,\cdot\,\rangle$; in the case of L^2-spaces we write $\langle\,\cdot\,,\,\cdot\,\rangle_2$ for its inner product and the corresponding norm is denoted by $\|\cdot\|_2$. The L^p-norm will be denoted by $\|\cdot\|_p$. The norm $\|\cdot\|_{p,k}$ in the Sobolev space $H^{p,k}(U)$ of all functions on a domain U belonging to $L^p(U)$ along with their generalized partial derivatives up to order k is defined as the sum of the L^p-norms of all partial derivatives up to order k (including $k=0$).

Let $P_N\colon \mathbb{R}^\infty \to \mathbb{R}^N$, $P_N x = (x_1,\ldots,x_N)$. Given a function φ on $\mathbb{R}^k$ we denote by the same symbol the function on $\mathbb{R}^\infty$ defined by $\varphi(x) := \varphi(P_k x)$.

We shall say that a bounded Borel measure $\mu = \mu_t(dx)\,dt$ on $\mathbb{R}^\infty \times (0,T_0)$, where $(\mu_t)_{0<t<T_0}$ is a family of bounded Borel measures on $\mathbb{R}^\infty$, satisfies the equation

$$\partial_t\mu = L^*\mu$$

if the functions a^{ij}, B^i are integrable with respect to the variation $|\mu|$ of μ and for every $k \ge 1$ and every function $\varphi \in C_0^{2,1}(\mathbb{R}^k \times (0,T_0))$ we have

$$\int_0^{T_0}\int_{\mathbb{R}^\infty} \Big[\partial_t\varphi + \sum_{i,j=1}^{\infty} a^{ij}\partial_{x_i}\partial_{x_j}\varphi + \sum_{i=1}^{\infty} B^i\partial_{x_i}\varphi\Big]\,d\mu_t\,dt = 0.$$

It is obvious that it is enough to have this identity for all $\varphi \in C_0^\infty(\mathbb{R}^k \times (0,T_0))$.

Let ν be a bounded Borel measure on $\mathbb{R}^\infty$. We say that the measure μ satisfies the initial condition $\mu|_{t=0} = \nu$ if for every $k \ge 1$ and $\zeta \in C_0^2(\mathbb{R}^k)$ we have

$$\lim_{t\to 0}\int_{\mathbb{R}^\infty} \zeta(x)\,\mu_t(dx) = \int_{\mathbb{R}^d} \zeta(x)\,\nu(dx).$$

Clearly, if $\sup_t \|\mu_t\| < \infty$, it suffices to have this equality for all $\zeta \in C_0^\infty(\mathbb{R}^k)$.

We need the following auxiliary lemma.

Lemma 1.1. *Let $\mu = \mu_t(dx)\,dt$ be a solution to* (1.1) *such that $\sup_{t\in(0,1)}\|\mu_t\| < \infty$. Assume that $B^k \in L^1(|\mu|)$ for every $k \in \mathbb{N}$ and let $0 < T < T_0$. Then for every number $k \ge 1$ and every function $\varphi \in C_b(\mathbb{R}^k \times [0,T]) \bigcap C_b^{2,1}(\mathbb{R}^k \times (0,T))$ the equality*

$$\int_{\mathbb{R}^\infty} \varphi(x,t)\,\mu_t(dx) = \int_{\mathbb{R}^\infty} \varphi(x,0)\,\nu(dx) + \int_0^t\int_{\mathbb{R}^\infty} [\partial_s\varphi + L\varphi]\,d\mu_s\,ds \tag{1.2}$$

holds for almost every $t \in [0,T]$. Conversely, (1.2) *implies* (1.1).

Proof. It is enough to prove this equality in the case where $\varphi(z,t)=0$ if $|z|>R>0$ for almost every $t\in[0,T]$. Let $\eta\in C_0^\infty((0,T))$. According to our definition we have

$$\int_0^T\int_{\mathbb{R}^\infty}[\partial_t(\varphi\eta)+L(\varphi\eta)]\,d\mu_t\,dt=0.$$

Thus, we obtain

$$-\int_0^T\eta'(t)\int_{\mathbb{R}^\infty}\varphi(x,t)\,\mu_t(dx)\,dt=\int_0^T\eta(t)\int_{\mathbb{R}^\infty}[\partial_t\varphi+L\varphi]\,d\mu_t\,dt.$$

Hence the function

$$t\mapsto\int_{\mathbb{R}^\infty}\varphi(x,t)\,\mu_t(dx)$$

on $(0,T)$ has an absolutely continuous version for which

$$\frac{d}{dt}\int_{\mathbb{R}^\infty}\varphi(x,t)\,\mu_t(dx)=\int_{\mathbb{R}^\infty}[\partial_t\varphi+L\varphi]\,d\mu_t.$$

Therefore, for some constant $C\in\mathbb{R}$ the equality

$$\int_{\mathbb{R}^\infty}\varphi(x,t)\,\mu_t(dx)=C+\int_0^t\int_{\mathbb{R}^\infty}[\partial_s\varphi+L\varphi]\,\mu_s\,ds$$

holds for almost every $t\in[0,T]$. Note that $\varphi(x,t)$ converges uniformly to $\varphi(x,0)$ as $t\to0$. Moreover, we have

$$\lim_{t\to0}\int_{\mathbb{R}^\infty}\varphi(x,0)\,\mu_t(\,dx)=\int_{\mathbb{R}^\infty}\varphi(x,0)\,\nu(dx).$$

It follows that

$$C=\int_{\mathbb{R}^\infty}\varphi(x,0)\,\nu(dx),$$

which completes the proof of one implication. The converse is, however, obvious. $\square$

Remark 1.2. Let $k\in\mathbb{N}$. If $\varphi(\,\cdot\,,t)=\psi\in C_b^2(\mathbb{R}^k)$ for every $t\in[0,T]$, $T<T_0$, then by (1.2) we have

$$\int_{\mathbb{R}^\infty}\psi(x)\,\mu_t(dx)=\int_{\mathbb{R}^\infty}\psi(x)\,\nu(dx)+\int_0^t\int_{\mathbb{R}^\infty}L\psi(x,s)\,\mu_s(dx)\,ds \tag{1.3}$$

for almost all $t\in[0,T]$. Moreover, if J_ψ^μ denotes the set of all $t\in[0,T]$ such that equality (1.3) holds, then the closure of J_ψ^μ coincides with $[0,T]$ and the restriction of the mapping

$$t\mapsto\int_{\mathbb{R}^\infty}\psi(x)\,\mu_t(dx)$$

to J_ψ^μ is continuous, since the right-hand side of (1.3) is continuous in t.

Remark 1.3. Let φ be as in Lemma 1.1. and assume that $T\in J^\mu_{\varphi(\,\cdot\,,T)}$. Then equality (1.2) holds with $t=T$. Indeed, $\varphi(x,t)$ converges uniformly to $\varphi(x,T)$ as $t\to T$. Let I be the set of all $t\in[0,T]$ such that equality (1.2) holds. Let us take a sequence $t_n\in J^\mu_{\varphi(\,\cdot\,,T)}\bigcap I$ such that $\lim\limits_{n\to\infty}t_n=T$. Then we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^\infty}\varphi(x,t_n)\,\mu_{t_n}(dx)=\int_{\mathbb{R}^\infty}\varphi(x,T)\,\mu_T(dx)$$

and equality (1.2) holds for each t_n. Letting $n\to\infty$, we obtain equality (1.2) with $t=T$.

2. Uniqueness of probability solutions

In this section two establish two different uniqueness results: first we consider nondegenerate diffusion matrices and then turn to the general case that includes fully degenerate equations. We start with stating our assumptions about A and b.

(A) $a^{ij}=a^{ji}$, each function a^{ij} depends only on the variables $t,x_1,x_2,\dots,x_{\max\{i,j\}}$ and is continuous and for every natural number N the matrix $A_N=(a^{ij})_{1\le i,j\le N}$ satisfies the following condition:

there exist positive numbers γ_N, λ_N and $\beta_N\in(0,1]$ such that for all $x,y\in\mathbb{R}^N$ and $t\in[0,T_0]$ one has

$$\gamma_N|y|^2\le\langle A_N(x,t)y,y\rangle\le\gamma_N^{-1}|y|^2,\quad \|A_N(x,t)-A_N(y,t)\|\le\lambda_N|x-y|^{\beta_N},$$

where $\|\cdot\|$ is the operator norm and $|\cdot|$ is the standard Euclidean norm.

Let ν be a Borel probability measure on $\mathbb{R}^\infty$ and let $\mathcal{P}_\nu$ be some *convex* set of probability solutions $\mu=\mu_t(dx)\,dt$ to (1.1), i.e., $\mu_t\ge0$ and $\mu_t(\mathbb{R}^\infty)=1$ for every $t\in(0,T_0)$, such that $|B^k|\in L^2(\mu)$ for each $k\in\mathbb{N}$ and the following condition holds:

(B) for every $\varepsilon > 0$ and every natural number d there exist a natural number $N \geq d$ and a $C_b^{2,1}$-mapping $(b^k)_{k=1}^N \colon \mathbb{R}^N \times [0, T_0] \to \mathbb{R}^N$ such that

$$\int_0^{T_0} \int_{\mathbb{R}^\infty} |A_N(x,t)^{-1/2}(B_N(x,t) - b(x_1,\ldots,x_N,t))|^2 \, \mu_t(dx)\, dt < \varepsilon,$$

where $B_N = (B^1, \ldots, B^N)$.

Let us illustrate condition (B) by several examples.

Example 2.1. (i) Let B^k depend only on the variables $t, x_1, x_2, \ldots, x_k$. Then in order to ensure our condition (B) we need only the inclusion $|B^k| \in L^2(\mu)$ for all $k \geq 1$. Indeed, we set $N = d$ and approximate each function B^k separately.

(ii) Let α_k be a positive number for each $k \in \mathbb{N}$ and

$$l^2_{1/\alpha} = \Bigl\{(z_k) \colon \sum_{k=1}^\infty \alpha_k^{-1} z_k^2 < \infty\Bigr\}, \quad \|x\|_{1/\alpha} = \Bigl(\sum_{k=1}^\infty \alpha_k^{-1} z_k^2\Bigr)^{1/2}.$$

Suppose that a^{ij} satisfy condition (A) and there exists a positive number C independent of N such that

$$|A_N(x,t)^{-1/2} y| \leq C \|y\|_{l_{1/\alpha}}$$

for all x, t and $y = (y_1, y_2, \ldots, y_N, 0, 0, \ldots)$. For example, this is true if $a^{ij} = 0$ for $i \neq j$ and $a^{ii} = \alpha_i$.

Let $(B^k(x,t)) \in l^2_{1/\alpha}$ for μ-almost every (x,t) and let $\|B\|_{1/\alpha} \in L^2(\mu)$. For every $\varepsilon > 0$ and every natural number d we pick a number $M > d$ such that

$$\sum_{k=M+1}^\infty \int_0^{T_0} \int_{\mathbb{R}^\infty} \alpha_k^{-1} |B^k|^2 \, d\mu_t \, dt < \varepsilon/2.$$

Then for every B^k we find a smooth function b^k depending on the first n_k variables such that

$$\int_0^{T_0} \int_{\mathbb{R}^\infty} \alpha_k^{-1} |B^k - b^k|^2 \, d\mu_t \, dt < \varepsilon (2M)^{-1}, \quad k = 1, \ldots, M.$$

Set $N = \max\{M, n_1, n_2, \ldots, n_M\}$ and $b^k \equiv 0$ for $k > M$. Then

$$\begin{aligned}\sum_{k=1}^N \int_0^{T_0} \int_{\mathbb{R}^\infty} & \alpha_k^{-1} |B^k - b^k|^2 \, d\mu_t \, dt \\ &= \sum_{k=1}^M \int_0^{T_0} \int_{\mathbb{R}^\infty} \alpha_k^{-1} |B^k - b^k|^2 \, d\mu_t \, dt + \sum_{k=M+1}^N \int_0^{T_0} \int_{\mathbb{R}^\infty} \alpha_k^{-1} |B^k|^2 \, d\mu_t \, dt < \varepsilon.\end{aligned}$$

(iii) Finally, for a^{ij} as in (ii), we can combine both examples. Let $B = G + F$, where $G^k, F^k \in L^2(\mu)$, $G^k(x,t) = G^k(x_1, x_2, \ldots, x_k, t)$, $F(x,t) \in l^2_{1/\alpha}$ and $\|F\|_{1/\alpha} \in L^2(\mu)$. Obviously, for given B^k of this type the set of all probability solutions $\mu = \mu_t(dx)dt$ to (1.1) satisfying the previous integrability conditions is convex.

Remark 2.2. (i) Obviously, condition (B) is equivalent to the following: there exist an increasing sequence $N_l \to +\infty$ and $C_b^{2,1}$-mappings $b^l = (b^{l,k})_{k=1}^{N_l}$ on $\mathbb{R}^{N_l} \times [0, T_0]$ such that

$$\lim_{l \to \infty} \int_0^{T_0} \int_{\mathbb{R}^\infty} |A_{N_l}(x,t)^{-1/2}(B_{N_l}(x,t) - b^l(x_1,\ldots,x_{N_l},t))|^2 \, \mu_t(dx)\, dt = 0.$$

(ii) Assume that $a^{ij} = \delta^{ij}$. Let $\widetilde{P}_N(x,t) = (P_N x, t)$ and let $\mathbb{E}_\mu[\,\cdot\,|\widetilde{P}_N = (x,t)]$ be the corresponding conditional expectation. Then condition (B) is equivalent to the following: for every $\varepsilon > 0$ and every natural number d there exists a natural number $N \geq d$ such that

$$\int_0^{T_0} \int_{\mathbb{R}^\infty} \sum_{k=1}^N \Bigl| B^k(x,t) - \mathbb{E}_\mu[B^k | \widetilde{P}_N = (x,t)] \Bigr|^2 \, \mu_t(dx)\, dt < \varepsilon.$$

This condition is known in Euclidian quantum field theory as the Høegh-Krohn condition (see [1]) and has been used, e.g., to prove Markov uniqueness for semigroups (see [31]).

Theorem 2.3. *Assume that condition* (A) *holds. Then the set $\mathcal{P}_\nu$ contains at most one element.*

Proof. Assume that two measures $\sigma^1 = \sigma^1_t dt$ and $\sigma^2 = \sigma^2_t dt$ belong to $\mathcal{P}_\nu$. By our assumption about $\mathcal{P}_\nu$, $\sigma = (\sigma^1 + \sigma^2)/2 \in \mathcal{P}_\nu$. Let $d \in \mathbb{N}$, $\psi \in C_0^\infty(\mathbb{R}^d)$ and $|\psi(x)| \le 1$ for all $x \in \mathbb{R}^d$. By condition (B) for every $\varepsilon > 0$ there exist a natural number $N \ge d$ and a $C_b^{2,1}$-mapping $(b^k)_{k=1}^N$ on $\mathbb{R}^N \times [0, T_0]$ such that

$$\int_0^{T_0} \int_{\mathbb{R}^\infty} |A_N^{-1/2}(x,s)(B_N(x,s) - b(x_1,\dots,x_N,s))|^2 \, \sigma_s(dx)\, ds < \varepsilon.$$

Fix $t \in J_\psi^{\sigma^1} \bigcap J_\psi^{\sigma^2} \bigcap J_{\psi^2}^{\sigma^1} \bigcap J_{\psi^2}^{\sigma^2}$. Let f be a solution to the finite-dimensional Cauchy problem

$$\begin{cases} \partial_t f + \sum_{i,j=1}^N a^{ij}\partial_{x_i}\partial_{x_j} f + \sum_{i=1}^N b^i \partial_{x_i} f = 0 & \text{on } \mathbb{R}^N \times (0,t), \\ f(t,x) = \psi(x). \end{cases} \tag{2.1}$$

It is known (see, e.g., [29, Theorem 1.3] and also [17], [22], and [34]) that a solution exists and belongs to the class $C_b(\mathbb{R}^N \times [0,t]) \bigcap C_b^{2,1}(\mathbb{R}^N \times (0,t))$. Moreover, according to the maximum principle $|f(x,s)| \le 1$ for all $(x,s) \in \mathbb{R}^N \times [0,t]$. Set $\mu = \sigma^1 - \sigma^2$. The measure μ solves the Cauchy problem (1.1) with zero initial condition. Applying Lemma 1.1 and Remark 1.3 with $\varphi = f$, we obtain

$$\int_{\mathbb{R}^\infty} f(x,t)\, \mu_t(dx) = \int_0^t \int_{\mathbb{R}^\infty} \Big[\partial_s f + \sum_{i,j=1}^N a^{ij}\partial_{x_j}\partial_{x_i} f + \sum_{i=1}^N B^i \partial_{x_i} f\Big]\, d\mu_s\, ds.$$

Therefore,

$$\int_{\mathbb{R}^\infty} \psi \, d\mu_t = \int_0^t \int_{\mathbb{R}^\infty} \langle B - b, \nabla f \rangle \, d\mu_s\, ds. \tag{2.2}$$

Let us estimate the following expression:

$$\int_0^t \int_{\mathbb{R}^\infty} |\sqrt{A_N}\nabla f|^2 \, d\sigma_s\, ds.$$

Using (1.2) for σ and $\varphi = f^2$, taking into account that $(\partial_s + L)(f^2) = 2f(\partial_s + L)f + 2|\sqrt{A_N}\nabla f|^2$, and recalling that $t \in J_{\psi^2}^{\sigma^1} \bigcap J_{\psi^2}^{\sigma^2}$, we obtain from (2.1) (again by Remark 1.3) that

$$\int_{\mathbb{R}^\infty} \psi^2 \, d\sigma_t - \int_{\mathbb{R}^\infty} f^2(x,0)\, \nu(dx) = 2\int_0^t \int_{\mathbb{R}^\infty} \Big[|\sqrt{A_N}\nabla f|^2 + f \sum_{i=1}^N (B^i - b^i)\partial_{x_i} f\Big]\, d\sigma_s\, ds.$$

Therefore,

$$\int_0^t \int_{\mathbb{R}^\infty} |\sqrt{A_N}\nabla f|^2 \, d\sigma_s\, ds \le 2 + \int_0^{T_0} \int_{\mathbb{R}^\infty} |A_N^{-1/2}(x,s)(B_N(x,s) - b(x_1,\dots,x_N,s))|^2 \, \sigma_s(dx)\, ds.$$

Thus we obtain the estimate

$$\int_0^t \int_{\mathbb{R}^\infty} |\sqrt{A_N}\nabla f|^2 \, d\sigma_s\, ds \le 2 + \varepsilon. \tag{2.3}$$

Applying (2.2) and (2.3) and the fact that $|\mu| \le \sigma^1 + \sigma^2 = 2\sigma$ we have

$$\int_{\mathbb{R}^\infty} \psi \, d\mu_t \le 2\sqrt{\varepsilon(2+\varepsilon)}.$$

Since $\varepsilon > 0$ was arbitrary, we obtain

$$\int_{\mathbb{R}^\infty} \psi \, d\mu_t \le 0.$$

Replacing ψ with $-\psi$ we arrive at the equality

$$\int_{\mathbb{R}^\infty} \psi \, d\mu_t = 0.$$

Therefore,

$$\int_{\mathbb{R}^\infty} \psi \, d\sigma_t^1 = \int_{\mathbb{R}^\infty} \psi \, d\sigma_t^2$$

for every $t \in J_\psi^{\sigma^1} \bigcap J_\psi^{\sigma^2} \bigcap J_{\psi^2}^{\sigma^1} \bigcap J_{\psi^2}^{\sigma^2}$, hence for almost every $t \in [0, T_0]$. Thus, $\sigma^1 = \sigma^2$. □

We now consider a typical example to which the previous theorem applies, namely, the Fokker–Planck–Kolmogorov equations associated with stochastic partial differential equations of reaction diffusion type on a domain $D \subset \mathbb{R}^d$, i.e.,

$$du(t) = \sigma(u(t),t)dW(t) + B(u(t),t)dt, \ t \in [0, T_0],$$

where $\sigma\sigma^* = A$ and $u(t) \in L^2(D)$. Furthermore, $W(t)$, $t \ge 0$, is a cylindrical Wiener process in $L^2(D)$ on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathrm{P})$ and $u(0)$ has ν as given law. Below we denote by u generic elements of functional spaces such as $L^2(D)$ which we embed into $\mathbb{R}^\infty$ (e.g., by using a suitable orthonormal basis) to be able to apply our framework above.

Example 2.4. ("Reaction diffusion equations in dimension d with infinite trace") Suppose that $D \subset \mathbb{R}^d$ is an open bounded set and $\{e_k\}$ is an eigenbasis of the Laplacian on $L^2(D)$ with zero boundary condition, i.e., $\Delta e_k = -\lambda_k^2 e_k$. Let $f\colon D\times\mathbb{R}\times[0,T_0] \to \mathbb{R}$ be a Borel function. Set $B(u,t)(z) = \Delta u(z) + f(z,u(z),t)$, $z \in D$, i.e.,

$$B^i(u,t) = -\lambda_i^2 u_i + \langle f(\,\cdot\,,u(\cdot),t), e_i\rangle_2,\ u \in L^2(D),\ u_i = \langle u, e_i\rangle_2.$$

Assume that the coefficients a^{ij} satisfy (A) with $\gamma_N = \gamma > 0$ independent on N. For instance, the last assumption is true if $a^{ij} = \langle Se_i, e_j\rangle_2$ for some invertible symmetric positive operator S on $L^2(D)$.

Assume also that there exist a Borel function $C \geq 0$ and a number $m \geq 1$ such that

$$|f(z,u,t)| \leq C(t) + C(t)|u|^m.$$

Set

$$L\varphi = \sum_{i,j=1}^\infty a^{ij}\partial_{e_i}\partial_{e_j}\varphi + \sum_{i=1}^\infty B^i\partial_{e_i}\varphi.$$

Then there is at most one probability solution $\mu = \mu_t(du)\,dt$, i.e., $\mu_t \geq 0$ and $\mu_t(\mathbb{R}^\infty) = 1$ for every $t \in (0,T_0)$, to the Cauchy problem (1.1) such that

$$\int_0^{T_0} C(t)^2 \int_{L^2(D)} \|u\|_{2m}^{2m}\,\mu_t(du)\,dt < \infty.$$

Proof. The mapping $u \mapsto (u_i)$ defines an embedding $L^2(D) \to \mathbb{R}^\infty$. Extending B^i and a^{ij} to all of $\mathbb{R}^\infty \times [0,T_0]$ by zero we end up in the framework described above. Set $F^i(u,t) = \langle f(\,\cdot\,,u(\cdot),t), e_i\rangle_2$. Note that

$$\sum_{i=1}^\infty |F^i(u,t)|^2 = \|f(\,\cdot\,,u(\cdot),t)\|_{L^2}^2 \leq C(t)^2 + C(t)^2\|u\|_{2m}^{2m}.$$

Thus we have $B^i = A^i + F^i$, where $A^i(u) = -\lambda_i^2 u_i$ and $\|F\|_{l^2} \in L^2(\mu)$, and Example 2.1(iii) applies with $\alpha_k = 1$. $\square$

Let now $d = 1$, $D = (0,1)$ and $\Delta = \frac{d^2}{dz^2}$. We recall that according to [6] and [7] if $a^{ij} = \alpha\delta^{ij}$ with $\alpha > 0$ and if

$$f(z,u,t) = f_1(z,u,t) + f_2(z,u,t),$$

where $(u,t) \mapsto f_i(z,u,t)$ are continuous for each z and for some nonnegative functions $c_1, c_3 \in L^2((0,T_0))$, $c_2 \in L^1((0,T_0))$ and all t, z, u we have

(i) $|f_1(z,u,t)| \leq c_1(t)(1 + |u|^m)$,

(ii) $(f_1(z,u,t) - f_1(z,v,t))(u - v) \leq c_2(t)|u - v|^2$,

(iii) $|f_2(z,u,t)| \leq c_3(t)(1 + |u|)$,

then for every initial value ν with $\|u\|_{2m}^{2m} \in L^1(\nu)$ there exists a probability solution μ of the Cauchy problem (1.1) such that $(1 + c_1(t) + c_3(t))^2(1 + \|u\|_{2m}^{2m}) \in L^1(\mu)$. It follows from the previous example that such a solution is unique, which improves the uniqueness result from [6] and [7].

We now present another uniqueness condition that applies to degenerate (even zero) diffusion matrices. Let us list our new assumptions (A$'$) and (B$'$).

(A$'$) $A(x,t) = (a^{ij}(x,t))$, where each function a^{ij} is bounded and depends only on the variables $x_1, x_2, \ldots, x_{\max\{i,j\}}, t$ and for every natural number N the matrix A_N is symmetric nonnegative and the elements σ_N^{ij} of the matrix $\sigma_N := \sqrt{A_N}$ are in the class $C^\infty(\mathbb{R}^N \times [0,T_0])$.

Let ν be a Borel probability measure on $\mathbb{R}^\infty$ and let $\mathbb{P}_\nu$ be some *convex* set of probability solutions $\mu = \mu_t(dx)\,dt$ of (1.1), i.e., $\mu_t \geq 0$ and $\mu_t(\mathbb{R}^\infty) = 1$ for every $t \in (0,T_0)$, such that $|B^k| \in L^1(\mu)$ for each $k \in \mathbb{N}$ and the following condition holds:

(B$'$) for every $\varepsilon > 0$ and every natural number d there exist a natural number $N \geq d$, a C^∞-mapping $b = (b^k)_{k=1}^N\colon \mathbb{R}^N \times [0,T_0] \to \mathbb{R}^N$, a function θ on $\mathbb{R}^N$, a function $V \in C^2(\mathbb{R}^N)$ with $V \geq 1$, and numbers $C_0 \geq 0$ and $\delta > 0$ such that

(i) $\sqrt{V(P_N x)}$, $|B_N(x,t) - b(P_N x,t)|\sqrt{V(P_N x)} \in L^1(\mu)$ and

$$\int_0^{T_0}\int_{\mathbb{R}^\infty} |B_N(x,t) - b(P_N x,t)|\sqrt{V(P_N x)}e^{C_0(T_0-t)/2}\,\mu_t(dx)\,dt < \varepsilon,$$

where $B_N = (B^1, \ldots, B^N)$;

(ii) the matrix $\mathcal{B} = (\partial_{x_j} b^i)$ and the operator

$$L_{a,b}\varphi(x,t) = \sum_{i,j\le N} a^{ij}(x,t)\partial_{x_i}\partial_{x_j}\varphi(x,t) + \sum_{i\le N} b^i(x,t)\partial_{x_i}\varphi(x,t)$$

satisfy the estimates

$$\langle \mathcal{B}(x,t)h,h\rangle \le \theta(x)|h|^2 \quad \forall\, h\in\mathbb{R}^N, \quad L_{a,b}V(x,t) \le (C_0 - \Lambda(x,t))V(x),$$

where

$$\Lambda(x,t) := 4\sum_{i,j,k\le N} \Big|\partial_{x_k}\sigma_N^{ij}(x,t)\Big|^2 + 2\theta(x) + \delta(1+|x|^2)^{-1}|b(x,t)|^2)$$

for every $(x,t)\in\mathbb{R}^N\times[0,T_0]$.

In the notation for $N, b, \theta, V, C_0, \delta$ we omit indication of the fact that they depend on ε and d. Recall also that $|\cdot|$ is the standard Euclidean norm.

Theorem 2.5. *If* (A′) *holds, then the set* $\mathbb{P}_\nu$ *contains at most one element.*

Remark 2.6. (i) If $A = (a^{ij})$ is a constant matrix and $|b(x,t)| \le C_1(N) + C_1(N)|x|$, then condition (B′)(ii) can be replaced by

$$L_{a,b}V(x,t) \le (C_0 - 2\theta(x))V(x) \quad \forall\,(x,t)\in\mathbb{R}^N\times[0,T_0].$$

(ii) If $A = (a^{ij})$ is a constant matrix and $|b(x,t)| \le C_1(N) + C_1(N)|x|^2$, then condition (B′)(ii) can be replaced by

$$L_{a,b}V(x,t) \le (C_0 - 2\theta(x) - \delta|x|^2)V(x)$$

for every $(x,t)\in\mathbb{R}^N\times[0,T_0]$ and some $\delta>0$.

(iii) Let $a^{ij} = 0$ if $i\ne j$ and $a^{ii}(x,t) = \alpha^i(x_1,x_2,\ldots,x_i,t) \ge 0$. Suppose also that we have $|b(x,t)| \le C_1(N) + C_1(N)|x|^2$. Then condition (B′)(ii) can be replaced by

$$L_{a,b}V(x,t) \le (C_0 - \Lambda(x,t))V(x), \quad \Lambda(x,t) := 4\sum_{i=1}^N\sum_{k\le i} \frac{\big|\partial_{x_k}\alpha^i(x,t)\big|^2}{\alpha^i(x,t)} + 2\theta(x) + \delta|x|^2.$$

(iv) We note that (B′) is a substantial generalization of a corresponding condition in [30].

Let us illustrate condition (B′).

For a sequence $\{\lambda_k^2\}_k$, we write

$$\|x\|^2_{l^2_\lambda} = \sum_k \lambda_k^2 x_k^2, \quad (x,y)_{l^2_\lambda} = \sum_k \lambda_k^2 x_k y_k.$$

Example 2.7. We assume here that $A = (a^{ij})_{i,j\ge1}$ is a *constant* matrix, $A_N := (a^{ij})_{i,j\le N}$ is symmetric nonnegative.

(i) Let $b^k(x,t) = -\lambda_k^2 x_k + f^k(x,t)$, $x\in\mathbb{R}^N$. Then the estimate $\langle\mathcal{B}h,h\rangle \le \theta(x)|h|^2$, $x,h\in\mathbb{R}^N$, follows from the estimate

$$\langle \mathcal{F}(x,t)h,h\rangle \le \theta(x)|h|^2 + \|h\|^2_{l^2_\lambda}, \quad x,h\in\mathbb{R}^N,$$

where $\mathcal{F} = (\partial_{x_j} f^i)_{i,j\le N}$.

(ii) Set $V(x) = \exp\big(\kappa\sum_{k=1}^N x_k^2\big)$, where $\kappa>0$. Then the condition on θ required in (B′) is this: for some numbers C_0 and $\delta>0$ (dependent on ε and d) one has

$$\theta(x) \le C_0 - \kappa\big(\mathrm{tr}A_N + 2\kappa\langle A_N x,x\rangle + \langle b(x,t),x\rangle\big) - 2^{-1}\delta(1+|x|^2)^{-1}|b(x,t)|^2, \quad x\in\mathbb{R}^N. \tag{2.4}$$

Let us consider a more specific case: $b^k(x,t) = -\lambda_k^2 x_k + f^k(x,t)$, $f(x,t) = (f^k(x,t))_{k=1}^N$, $\langle f(x,t),x\rangle \le 0$ and $|f^k(x,t)| \le C_1 + C_2|x|^2$, where $x\in\mathbb{R}^N$. Assume that for some $\varepsilon_0>0$ and every $N\ge1$ one has

$$\varepsilon_0\big(\langle A_N x,x\rangle + |x|^2\big) \le \|x\|^2_{l^2_\lambda}, \quad x\in\mathbb{R}^N.$$

Then condition (B′)(ii) can be rewritten in the following form:

$$\langle \mathcal{F}(x,t)h,h\rangle \le \theta(x)|h|^2 + \|h\|^2_{l^2_\lambda}, \quad x,h\in\mathbb{R}^N,$$

where $\mathcal{F} = (\partial_{x_j} f^i)_{i,j\le N}$, and for every $x\in\mathbb{R}^N$

$$\Theta(x) \le C_0 - \kappa\,\mathrm{tr}A_N + 2^{-1}\kappa(\varepsilon_0-\kappa)\|x\|^2_{l^2_\lambda}.$$

Note that in this case we take $V(x)$ with $\kappa < \varepsilon_0/4$.

This assertion follows from (2.4) if we choose $\delta>0$ such that

$$\delta(1+|x|^2)^{-1}|b(x,t)|^2 \le \varepsilon_0\kappa|x|^2 + 1.$$

(iii) Let $V(x)=\exp\bigl(\kappa\|x\|^2_{l^2_\lambda}\bigr)$. Then the condition on θ required in (B′) is this: for some constants C_0 and $\delta>0$ one has

$$\theta(x)\le C_0-\kappa\Bigl(\sum_{i=1}^N a^{ii}\lambda_i^2+2\kappa\sum_{i,j\le N}a^{ij}\lambda_i^2\lambda_j^2x_ix_j+\langle b(x,t),x\rangle_{l^2_\lambda}\Bigr)-2^{-1}\delta(1+|x|^2)^{-1}|b(x,t)|^2,\quad x\in\mathbb{R}^N. \quad (2.5)$$

Let us consider a more specific case: $b^k(x,t)=-\lambda_k^2x_k+f^k(x,t)$, $f(x,t)=(f^k(x,t))_{i=1}^N$, $\langle f(x,t),x\rangle_{l^2_\lambda}\le 0$ and $|f^k(x,t)|\le C_1+C_2|x|^2$, where $x\in\mathbb{R}^N$. Assume that for some $\varepsilon_0>0$ and every $N\ge 1$ one has

$$\varepsilon_0\sum_{i,j\le N}a^{ij}\lambda_i^2\lambda_j^2x_ix_j+\varepsilon_0|x|^2\le\sum_{i\le N}\lambda_i^4x_i^2.$$

Then condition (B′)(ii) can be rewritten in the following form:

$$\langle\mathcal{F}(x,t)h,h\rangle\le\theta(x)|h|^2+\|h\|^2_{l^2_\lambda},\quad x,h\in\mathbb{R}^N,$$

where $\mathcal{F}=(\partial_{x_j}f^i)_{i,j\le N}$, and for every $x\in\mathbb{R}^N$

$$\theta(x)\le C_0-\kappa\sum_{i=1}^N a^{ii}\lambda_i^2+2^{-1}\kappa(\varepsilon_0-\kappa)\sum_{i\le N}\lambda_i^4x_i^2.$$

Note that in this case we take $V(x)$ with $\kappa<\varepsilon_0/4$.

This assertion follows (2.5) if we choose $\delta>0$ such that

$$\delta(1+|x|^2)^{-1}|b(x,t)|^2\le\varepsilon_0\kappa|x|^2+1.$$

For the proof of Theorem 2.5 we need the following lemma.

Let $\eta\in C_0^\infty(\mathbb{R}^1)$ be such that $\eta(x)=1$ if $|x|\le 1$ and $\eta(x)=0$ if $|x|>2$, $0\le\eta\le 1$ and there exists a number $C>0$ such that $|\eta'(x)|^2\eta^{-1}(x)\le C$ for every x.

Lemma 2.8. *Assume that there exist a function θ on $\mathbb{R}^N$, a function $V\in C^2(\mathbb{R}^N)$ with $V\ge 1$, and numbers $C_0\ge 0$ and $\delta>0$ such that for all $(x,t)\in\mathbb{R}^N\times[0,T_0]$, $h\in\mathbb{R}^N$ one has*

$$\langle\mathcal{B}(x,t)h,h\rangle\le\theta(x)|h|^2,\quad \mathcal{B}=(\partial_{x_j}b^i)_{i,j\le N},$$

$$L_{a,b}V(x,t)\le(C_0-\Lambda(x,t))V(x),\quad \Lambda(x,t):=4\sum_{i,j,k\le N}\bigl|\partial_{x_k}\sigma_N^{ij}(x,t)\bigr|^2+2\theta(x)+\delta(1+|x|^2)^{-1}|b(x,t)|^2).$$

Let $s\in(0,T_0)$. Then there exists a number $\kappa>0$ such that for every $M>0$ the Cauchy problem

$$\partial_tf+\zeta_ML_{a,b}f=0,\quad f|_{t=s}=\psi,$$

where $\psi\in C_b^\infty(\mathbb{R}^N)$, $\zeta_M(x)=\eta\bigl((1+|x|^2)^\kappa/M\bigr)$, has a smooth solution f such that

$$|f(x,t)|\le\max_x|\psi(x)|,\quad |\nabla f(x,t)|^2\le e^{(C_0+1)(s-t)}V(x)\max_x|\nabla\psi(x)|^2/2.$$

Proof. The existence of a smooth bounded (with bounded derivatives) solution f is well known (see [28, Theorem 2], [34, Theorem 3.2.4, Theorem 3.2.6]). The maximum principle implies that $|f(x,t)|\le\max_x|\psi(x)|$. Set $u=2^{-1}\sum_{k=1}^N|\partial_{x_k}f|^2$. Differentiating the equation $\partial_tf+\zeta_ML_{a,b}f=0$ with respect to x_k and multiplying by $\partial_{x_k}f$, we obtain

$$\partial_tu+\zeta_ML_{a,b}u+\zeta_M\langle\mathcal{B}\nabla f,\nabla f\rangle+\langle\nabla\zeta_M,\nabla f\rangle\langle b,\nabla f\rangle+\zeta_M\partial_{x_k}a^{ij}\partial^2_{x_ix_j}f\partial_{x_k}f+$$
$$+a^{ij}\partial^2_{x_ix_j}f\partial_{x_k}f\partial_{x_k}\zeta_M-\zeta_Ma^{ij}\partial^2_{x_kx_j}f\partial^2_{x_kx_j}f=0.$$

Note that $\langle\mathcal{B}\nabla f,\nabla f\rangle\le 2\theta u$ and $\langle\nabla\zeta_M,\nabla f\rangle\langle b,\nabla f\rangle\le 2|\nabla\zeta_M||b|u$. Let us consider the expression

$$\zeta_M\partial_{x_k}a^{ij}\partial^2_{x_ix_j}f\partial_{x_k}f+a^{ij}\partial^2_{x_ix_j}f\partial_{x_k}f\partial_{x_k}\zeta_M-\zeta_Ma^{ij}\partial^2_{x_kx_j}f\partial^2_{x_kx_j}f.$$

Recall that $A=\sigma_N^2$. We have

$$\sum_{i,j,k}\partial_{x_k}a^{ij}\partial^2_{x_ix_j}f\partial_{x_k}f=2\sum_{i,j,m,k}\partial_{x_k}\sigma^{im}\sigma^{mj}\partial^2_{x_ix_j}f\partial_{x_k}f\le$$
$$\le 2\sum_{i,m}\Bigl(\sum_k|\partial_{x_k}\sigma^{im}|^2\Bigr)^{1/2}\Bigl(\sum_k|\partial_{x_k}f|^2\Bigr)^{1/2}\Bigl|\sum_j\sigma^{mj}\partial^2_{x_ix_j}f\Bigr|,$$

which is estimated by

$$4u\sum_{i,m,k}|\partial_{x_k}\sigma^{im}|^2+2^{-1}\sum_{i,m}\Bigl|\sum_j\sigma^{mj}\partial^2_{x_ix_j}f\Bigr|^2.$$

Note that

$$\sum_{i,m}\Big|\sum_{j}\sigma^{mj}\partial^2_{x_ix_j}f\Big|^2=\sum_{i,j,k}a^{ij}\partial^2_{x_kx_j}f\partial^2_{x_kx_j}f.$$

Applying the inequality $xy\le(4+4\mathrm{tr}A)^{-1}x^2+(1+\mathrm{tr}A)y^2$ we obtain

$$a^{ij}\partial^2_{x_ix_j}f\partial_{x_k}f\partial_{x_k}\zeta_M\le\frac{|\nabla\zeta_M|^2}{\zeta_M}(1+\mathrm{tr}A)+(4+4\mathrm{tr}A)^{-1}\Big(a^{ij}\partial^2_{x_ix_j}f\Big)^2.$$

Note that the following inequality is true:

$$\Big(\sum_{i,j=1}^N a^{ij}\partial_{x_i}\partial_{x_j}f\Big)^2\le\Big(\sum_{i=1}^N a^{ii}\Big)\Big(\sum_{i,j,k}^N a^{ij}\partial_{x_i}\partial_{x_k}f\partial_{x_j}\partial_{x_k}f\Big).$$

This follows by the inequality

$$|\mathrm{tr}\,(AB)|^2\le\mathrm{tr}\,A\ \mathrm{tr}\,(AB^2)$$

valid for symmetric matrices A and B, where A is nonnegative. The latter is due to the Cauchy inequality applied to the inner product $\langle X,Y\rangle=\mathrm{tr}\,(XY^*)$ on the space of $N\times N$-matrices and the matrices $X=A^{1/2}$, $Y=BA^{1/2}$, for which $\mathrm{tr}\,(YY^*)=\mathrm{tr}\,(BA^{1/2}A^{1/2}B)=\mathrm{tr}\,(AB^2)$. Applying the above inequality it is easy to verify that

$$\partial_t u+\zeta_M L_{a,b}u+Qu\ge0,$$

where

$$Q=\frac{|\nabla\zeta_M|^2}{\zeta_M}(1+\mathrm{tr}A)+|\nabla\zeta_M||b|+2\zeta_M\theta+4\zeta_M\sum_{i,j,k\le N}\big|\partial_{x_k}\sigma^{ij}_N\big|^2.$$

We have

$$|\nabla\zeta_M(x)|\le4\kappa(1+|x|^2)^{-1/2}\big|\eta'\big((1+|x|^2)^\kappa/M\big)\big|.$$

Hence

$$Q\le4\kappa^2C(1+\mathrm{tr}\,A)+16\kappa C+\zeta_M\Big(4\sum_{i,j,k\le N}\big|\partial_{x_k}\sigma^{ij}_N\big|^2+2\theta+2\kappa(1+|x|^2)^{-1}|b|^2\Big).$$

Let us choose $\kappa>0$ such that

$$Q\le1+\zeta_M\Big(4\sum_{i,j,k\le N}\big|\partial_{x_k}\sigma^{ij}_N\big|^2+2\theta+\delta(1+|x|^2)^{-1}|b|^2\Big).$$

Let us set $u=wV$. Then w satisfies the inequality

$$\partial_t w+\zeta_M L_{a,\widetilde{b}}w+\widetilde{Q}w\ge0,$$

where

$$\widetilde{b}^k=b^k+2\frac{a^{kj}\partial_{x_j}V}{V},\quad\widetilde{Q}=Q+\zeta_M\frac{L_{a,b}V}{V}.$$

By our assumptions we have $\widetilde{Q}\le C_0+1$. Since $u(x,s)=|\nabla f(x,s)|^2/2=|\nabla\psi(x)|^2/2$, we have

$$w(x,s)=V(x)^{-1}|\nabla\psi(x)|^2/2\le|\nabla\psi(x)|^2/2.$$

Applying the maximum principle (see [34, Theorem 3.1.1]) we obtain

$$\max_x|w(x,t)|\le e^{(C_0+1)(s-t)}\max_x|\nabla\psi(x)|^2/2,$$

which completes the proof. □

We can now prove our theorem.

Proof. Assume that $\sigma^1=\sigma^1_t dt$ and $\sigma^2=\sigma^2_t dt$ belong to $\mathbb{P}_\nu$. By our assumption about $\mathbb{P}_\nu$ we have $\sigma=(\sigma^1+\sigma^2)/2\in\mathbb{P}_\nu$. Let $d\in\mathbb{N}$, $\psi\in C_0^\infty(\mathbb{R}^d)$ and $|\nabla\psi(x)|+|\psi(x)|\le1$ for all $x\in\mathbb{R}^d$. For every $\varepsilon>0$ and every natural number d we find a natural number $N\ge d$, a C^∞-mapping $b=(b^k)_{k=1}^N\colon\mathbb{R}^N\times[0,T_0]\to\mathbb{R}^N$, a function θ on $\mathbb{R}^N$, a function $V\in C^2(\mathbb{R}^N)$, $V\ge1$, and numbers $C_0\ge0$ and $\delta>0$ such that (i) and (ii) in condition (B$'$) are fulfilled.

Let a function $\eta\in C_0^\infty(\mathbb{R}^1)$ be such that $\eta(x)=1$ if $|x|\le1$ and $\eta(x)=0$ if $|x|>2$, $0\le\eta\le1$ and there exists a number $C>0$ such that $|\eta'(x)|^2\eta^{-1}(x)\le C$ for every x. Let $\kappa>0$ be as in Lemma 2.8. Set $\varphi_K(x)=\eta(|x|^2/K)$ and $\zeta_M(x)=\eta\big((1+|x|^2)^\kappa/M\big)$.

Let us fix a number $K>0$ and find a number M such that $\zeta_M(x)=1$ if $|x|^2<2K$.

Fix $t\in\bigcap_K(J^{\sigma^1}_{\psi\varphi_K}\bigcap J^{\sigma^2}_{\psi\varphi_K})$. Let f be a smooth bounded solution to the finite-dimensional Cauchy problem

$$\begin{cases}\partial_t f+\zeta_M(x)\sum_{i,j=1}^N a^{ij}\partial_{x_i}\partial_{x_j}f+\zeta_M(x)\sum_{i=1}^N b^i\partial_{x_i}f=0\quad\text{on }\mathbb{R}^N\times(0,t),\\ f(t,x)=\psi(x).\end{cases}$$

Set $\mu = \sigma^1 - \sigma^2$. The measure μ solves the Cauchy problem (1.1) with zero initial condition. Recall that $\zeta_M(x) = 1$ if $\varphi_K(x) \neq 0$. Therefore,

$$\int_{\mathbb{R}^\infty} \psi\varphi_K\, d\mu_t = \int_0^t \int_{\mathbb{R}^\infty} \big[\varphi_K\langle B - b, \nabla_x f\rangle + fL\varphi_K + 2\langle A\nabla_x f, \nabla_x\varphi_K\rangle\big]\, d\mu_s\, ds.$$

Applying Lemma 2.8 we have the estimate

$$|f(x,s)| \leq 1, \quad |\nabla_x f(x,s)|^2 \leq e^{(C_0+1)(T_0-s)}V(x)/2.$$

Hence

$$\int_{\mathbb{R}^\infty} \psi\, d\mu_t \leq 2\int_0^t \int_{\mathbb{R}^\infty} \Big[|B - b|V^{1/2}e^{(C_0+1)(T_0-s)/2} + |L\varphi_K| + 2|A\nabla\varphi_K|e^{(C_0+1)(T_0-s)/2}V^{1/2}\Big]\, d\sigma_s\, ds.$$

Letting $K \to +\infty$ we find that

$$\int_{\mathbb{R}^\infty} \psi\, d\mu_t \leq 2\int_0^t \int_{\mathbb{R}^\infty} |B - b|V^{1/2}e^{(C_0+1)(T_0-s)/2}\, d\sigma_s\, ds < 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we obtain

$$\int_{\mathbb{R}^\infty} \psi\, d\mu_t \leq 0.$$

Replacing ψ by $-\psi$ we arrive at the equality

$$\int_{\mathbb{R}^\infty} \psi\, d\mu_t = 0.$$

Therefore,

$$\int_{\mathbb{R}^\infty} \psi\, d\sigma_t^1 = \int_{\mathbb{R}^\infty} \psi\, d\sigma_t^2$$

for almost every t. Thus, $\sigma^1 = \sigma^2$. □

Example 2.9. ("Reaction diffusion equations") Let us return to the situation of Example 2.4, but now we assume that there exists a sequence of smooth bounded functions $f_n(z,u,t)$ such that $\lim\limits_{n\to\infty} f_n(z,u,t) = f(z,u,t)$ for every u, t, z and

$$|f_n(z,u,t)| \leq C_1 + C_1|u|^m, \quad (f_n(z,u,t) - f_n(z,v,t))(u - v) \leq C_2|u - v|^2,$$

where C_1 and C_2 do not depend on n. Assume also that $a^{ij} = (Se_i, e_j)_2$ for some symmetric nonnegative operator S on $L^2((0,1))$, which can be degenerate unlike in Example 2.4. Then there exists at most one probability solution μ of the Cauchy problem for the Fokker–Planck–Kolmogorov equation $\partial_t\mu = L^*\mu$ such that

$$\int_0^{T_0} \int_{L^2((0,1))} \|u\|_{2m}^m\, \mu_t(du)\, dt < \infty.$$

The same conclusion is true if $A = (a^{ij})$ is a nonconstant matrix satisfying condition (A$'$) and there exists a constant C_1 such that for every natural number N and every $(x,t) \in \mathbb{R}^N \times [0,T_0]$ we have

$$\sum_{i,j,k\leq N} \big|\partial_{x_k}\sigma_N^{ij}(x,t)\big|^2 \leq C_1.$$

Proof. Set $F^i(u,t) = \langle f(\,\cdot\,, u(\cdot), t), e_i\rangle_2$, $F_n^i(u,t) = \langle f_n(\,\cdot\,, u(\cdot), t), e_i\rangle_2$, $F_n(u,t) = (F_n^i(u,t))_{i=1}^\infty$, and extend all these maps to all of $\mathbb{R}^\infty \times [0,T_0]$ by zero. According to our assumptions and the dominated convergence theorem we have

$$\lim_{n\to\infty} \int_0^{T_0} \int_{L^2((0,1))} \|F(u,t) - F_n(u,t)\|_{l^2}\, \mu_t(du)\, dt = 0.$$

Let $P_N u := u_1e_1 + \ldots + u_Ne_N$. The above equality shows that for each $\varepsilon >$ and $d \geq 1$ there exist numbers n and $N > d$ such that

$$\int_0^{T_0} \int_{L^2((0,1))} \|F(u,t) - F_n(P_N u,t)\|_{l^2}\, \mu_t(du)\, dt < \varepsilon.$$

Note that the condition

$$(f_n(z,u,t) - f_n(z,v,t))(u - v) \leq C_2|u - v|^2$$

implies that

$$\sum_{i,j\leq N} \partial_{u_i}F_n^j(P_N u,t)h_ih_j \leq C_2|h|^2, \quad h = (h_i) \in \mathbb{R}^N.$$

Hence Theorem 2.5 with $V \equiv 1$ implies uniqueness. □

Below for simplicity the integral of the product of an integrable function f_1 and a bounded function f_2 is denoted by $(f_1, f_2)_2$.

Example 2.10. ("Stochastic Burgers equation") Suppose that $\{e_k\}$ is an eigenbasis of the Laplacian on $L^2[0,1]$ with zero boundary condition, i.e., $D^2 e_k = -\lambda_k^2 e_k$. Set $B(u)(z) = D^2 u(z) + D(u^2(z))$, that is,

$$B^i(u) = -\lambda_i^2 u_i - \langle u^2, De_i\rangle_2,\ u \in L^2[0,1],\ u_i = \langle u, e_i\rangle_2.$$

Assume that $a^{ij} = \langle Se_i, e_j\rangle_2$ for some symmetric nonnegative operator S on $L^2[0,1]$ with finite trace ($\mathrm{tr}S < \infty$). Set

$$L\varphi = \sum_{i,j=1}^{\infty} a^{ij}\partial_{e_i}\partial_{e_j}\varphi + \sum_{i=1}^{\infty} B^i \partial_{e_i}\varphi.$$

Let H_0^1 be the space of all absolutely continuous functions u on $[0,1]$ such that $u(0) = u(1) = 0$ and $\|u\|_{H_0^1} := \|u'\|_2 < \infty$. Then there exists at most one probability solution μ of the Cauchy problem for the Fokker–Planck–Kolmogorov equation $\partial_t\mu = L^*\mu$ such that

$$\int_0^{T_0}\int_{L^2[0,1]} \|u\|_{H_0^1}^2 e^{\delta\|u\|_2^2}\,\mu_t(du)\,dt < \infty$$

for some $\delta > 0$.

Proof. We apply Example 2.7(ii). Recall that the matrix (a^{ij}) has to satisfy the following condition for some $\varepsilon_0 > 0$:

$$\varepsilon_0\bigl(\langle A_N x, x\rangle + |x|^2\bigr) \le \|x\|_{l^2_\lambda}^2, \quad x \in \mathbb{R}^N.$$

This is equivalent to

$$\varepsilon_0\bigl(\langle Su, u\rangle_2 + \|u\|_2^2\bigr) \le \|u\|_{H_0^1}^2,$$

which is true for sufficiently small ε_0. We fix $\varepsilon_0 \in (0,\delta)$. Set $F^i(u) := \langle u^2, De_i\rangle_2$ for $u \in L^2$ and extend F^i by zero to all other $u = (u_k)$ in $\mathbb{R}^\infty$. Let $F(u) = (F^i(u))_{i=1}^\infty$, $P_N u := u_1 e_1 + \ldots + u_N e_N$,

$$b^k(u_1,\ldots,u_N) := -\lambda_k^2 u_k + F^k(P_N u), \quad k \le N.$$

Note that

$$\|F(u)\|_{l^2} = \|(u^2)'\|_2 = 2\|uu'\|_2 \le 2\|u\|_{H_0^1}^2.$$

Hence

$$\lim_{N\to\infty}\int_0^{T_0}\int_{L^2[0,1]} \|F(u) - F(P_N u)\|_{l^2} e^{\delta\|u\|_2^2}\,\mu_t(du)\,dt = 0$$

It is easy to see that $|b^k(u)| \le C_1(N) + C_2(N)\|P_N u\|_2^2$ and $\langle F(P_N u), P_N u\rangle_2 \le 0$. Moreover, for every $\gamma \in (0,1)$ we have the inequalities

$$\sum_{i,k\le N} \partial_{u_i} F^k(P_N u) h_i h_k \le \|h\|_{l^2_\lambda} + (\gamma\|P_N u\|_{H_0^1}^2 + C_\gamma)|h|^2, \quad h = (h_i) \in \mathbb{R}^N.$$

Set $\theta(P_N u) = \gamma\|P_N u\|_{H_0^1}^2 + C_\gamma$ and $C_0 = C_\gamma + \mathrm{tr}S$ (we recall that $\mathrm{tr}S < \infty$). In order to apply Example 2.7(ii) we choose $\gamma < 2^{-1}\delta(\varepsilon_0 - \delta)$. □

Example 2.11. ("Mixed Burgers/reaction diffusion type equations") (i) In the situation of the previous example we consider the operator L with the drift coefficient of the form

$$B(u)(z) = D^2 u(z) + D(u^2(z)) - u^{2m+1}(z), \quad m \in \mathbb{N},$$

that is,

$$B^i(u) = -\lambda_i^2 u_i - \langle u^2, De_i\rangle_2 - \langle u^{2m+1}, e_i\rangle_2.$$

Assume that a^{ij} satisfies the assumptions in the previous example. Then there exists at most one probability solution μ of the Cauchy problem for the Fokker–Planck–Kolmogorov equation $\partial_t\mu = L^*\mu$ such that

$$\int_0^{T_0}\int_{L^2[0,1]} \bigl[\|u\|_{4m+2}^{2m+1} + \|u\|_{H_0^1}^2\bigr] e^{\delta\|u\|_2^2}\,\mu_t(du)\,dt < \infty$$

for some $\delta > 0$.

(ii) In the situation of Example 2.10 we consider the operator L with the drift coefficient of the form

$$B(u)(z) = D^2 u(z) + D(u^m(z)) - u^{2l+1}(z), \quad 2 \le m \le l+2, \quad m, l \in \mathbb{N}$$

that is,

$$B^i(u) = -\lambda_i^2 u_i - \langle u^m, De_i\rangle_2 - \langle u^{2l+1}, e_i\rangle_2.$$

Assume also that $a^{ij}=0$ if $i\neq j$ and that $\sum_{i=1}^{\infty}a^{ii}<\infty$. Then there exists at most one probability solution μ of the Cauchy problem for the Fokker–Planck–Kolmogorov equation $\partial_t\mu=L^*\mu$ such that

$$\int_0^{T_0}\int_{L^2((0,1))}\big[\|u\|_{4l+2}^{2l+1}+\|u^m\|_{H_0^1}\big]\exp\Big(\kappa'\|u\|_{2m-2}^{2m-2}\Big)\,\mu_t(du)\,dt<\infty$$

for some $\kappa'>0$. This partially improves the results in [25].

Proof. (i) We apply Example 2.7(ii). Note that as in the above example the matrix (a^{ij}) satisfies all conditions in Example 2.7(ii). Let $\psi_M\in C^\infty(\mathbb{R}^1)$, $\psi(s)=-\psi(-s)$, $0\leq\psi'\leq 1$, $\psi_M(s)=s$ if $|s|\leq M-1$ and $\psi_M(s)=M$ if $s>M+1$. Set

$$F^i(u):=-\langle u^2,De_i\rangle_2-\langle u^{2m+1},e_i\rangle_2,\quad F^i_M(u):=-\langle u^2,De_i\rangle_2-\langle\psi_M(u)^{2m+1},e_i\rangle_2,$$

$$P_Nu:=u_1e_1+\ldots+u_Ne_N,\quad b^k(u_1,\ldots,u_N):=-\lambda_k^2u_k+F^k_M(P_Nu).$$

As above, we define all these functions by zero if u is not in $L^2[0,1]$. Note that $\|F(u)\|_{l^2}\leq 2\|u\|^2_{H_0^1}+\|u\|^{2m+1}_{4m+2}$. and the same is true for $F_M(u)$ in place of $F(u)$. Hence

$$\lim_{N\to\infty}\Big(\lim_{M\to\infty}\int_0^{T_0}\int_{L^2[0,1]}\|F(u)-F_M(P_Nu)\|_{l^2}e^{\delta\|u\|_2^2}\,\mu_t(du)\,dt\Big)=0.$$

It is easy to see that $|b^k(u)|\leq C_1(N)+C_2(N)\|P_Nu\|_2^2$. Recall that $\psi_M'\geq 0$ and $\psi_M(s)=-\psi_M(-s)$. Hence $\langle F_M(P_Nu),P_Nu\rangle_2\leq 0$. For every $\gamma\in(0,1)$ we have

$$\sum_{i,k\leq N}\partial_{u_i}F^k_M(P_Nu)h_ih_k\leq\|h\|_{l^2_\lambda}+(\gamma\|P_Nu\|^2_{H_0^1}+C_\gamma)|h|^2,\quad h=(h_i)\in\mathbb{R}^N.$$

Set $\theta(P_Nu)=\gamma\|P_Nu\|^2_{H_0^1}+C_\gamma$ and $C_0=C_\gamma+\mathrm{tr}S$ (we recall that $\mathrm{tr}S<\infty$). In order to apply Example 2.7(ii) we choose $\gamma<2^{-1}\delta(\varepsilon_0-\delta)$.

(ii) We check the condition (B$'$). Set $F^k(u)=-\langle u^m,De_k\rangle_2-\langle u^{2l+1},e_k\rangle_2$, $P_Nu=u_1e_1+\ldots+u_Ne_N$ and

$$b^k(u_1,u_2,\ldots,u_N)=-\lambda_k^2u_k+F^k(P_Nu).$$

For each $N\geq 1$ and $p\geq 1$ there exist positive numbers $C_1(N,p)$ and $C_2(N,p)$ such that

$$C_1\|P_Nu\|_2^{2p}\leq\|P_Nu\|^p_{L^p}\leq C_2\|P_Nu\|_2^{2p}.$$

Hence there exists a number $C_3(N)$ such that

$$|b(P_Nu)|(1+\|P_Nu\|_2^2)^{-1}\leq C_3(N)+C_3(N)\Big(\int_0^1|P_Nu(z)|^{m-2}\,dz+\int_0^1|P_Nu(z)|^{2l-1}\,dz\Big).$$

It is easy to see that for every $\gamma\in(0,1)$ there exists a number $C_\gamma>0$ (independent of N) such that

$$\sum_{i,k\leq N}\partial_{u_i}F^k(u^N)h_ih_k\leq\gamma\|h\|_{l^2_\lambda}+(\gamma\|(P_Nu)^{m-1}\|^2_{H_0^1}+C_\gamma)|h|^2,\quad h=(h_i)\in\mathbb{R}^N.$$

Set $V(u)=\exp(\kappa\|u\|^{2m-2}_{2m-2})$, where $0<\kappa<\kappa'$ (the number κ' comes from our assumptions). Using the inequalities $\sum_{i=1}^\infty a^{ii}u_i^2\leq c_0\sum_{i=1}^\infty$ for some $c_0>0$ and all $u\in L^2([0,1])$, $\sum_{i=1}^\infty a^{ii}<\infty$, $m\leq l+2$ and choosing a sufficiently small number κ, we obtain

$$L_{a,b}V(P_Nu)=\Big(C(m)-2^{-1}\kappa\|(P_Nu)^{m-1}\|^2_{H_0^1}-2^{-1}\kappa\|P_Nu\|^{2m+2l-2}_{2m+2l-2}\Big)V(P_Nu).$$

for some number $C(m)>0$ which does not depend on N. Note that $m-2\leq 2m+2l-2$ and $2l-1\leq 2m+2l-2$. Choosing $\gamma<\kappa/2$ we have

$$L_{a,b}V(P_Nu)\leq(C_0-\gamma\|(P_Nu)^{m-1}\|^2_{H_0^1}-\delta|b(P_Nu)|(1+\|P_Nu\|_2^2)^{-1})V(P_Nu)$$

for some $C_0>0$ and $\delta>0$. Note that C_0 does not depend on N and we can omit the term $e^{C_0(T-t)/2}$ in the condition (B$'$)(i). Finally we note that $\|F(u)\|_{l^2}\leq\|u^m\|_{H_0^1}+\|u\|^{2l+1}_{4l+2}$ and

$$\lim_{N\to\infty}\int_0^{T_0}\int_{L^2((0,1))}\|F(u)-F(P_Nu)\|_{l^2}\exp\Big(\kappa\|u\|^{2m-2}_{2m-2}\Big)\,\mu_t(du)\,dt=0.$$

Hence Theorem 2.5 implies uniqueness. $\square$

Example 2.12. ("Stochastic $2d$-Navier–Stokes equation") Let us consider the space V_2 of $\mathbb{R}^2$-valued mappings $u=(u^1,u^2)$ such that $u^j\in H_0^{2,1}(D)$ and $\operatorname{div} u=0$, where $D\subset\mathbb{R}^2$ is a bounded domain with smooth boundary. The space V_2 is equipped with its natural Hilbert norm $\|u\|_{V_2}$ defined by

$$\|u\|_{V_2}^2:=\sum_{j=1}^2\|\nabla_z u^j\|_2^2.$$

Let H be the closure of V_2 in $L^2(D,\mathbb{R}^2)$ and let P_H denote the orthogonal projector on H in $L^2(D,\mathbb{R}^2)$. It is known (see [26]) that there exists an orthonormal basis $\{\eta_n\}$ in H formed by eigenfunctions of Δ with eigenvalues $-\lambda_n^2$ such that $\eta_n\in V_2$. Recall that $\langle P_H w,\eta_n\rangle_2=\langle w,\eta_n\rangle_2$ for any $w\in L^2(D,\mathbb{R}^d)$. Set

$$B^n(u,t)=\langle u,\Delta\eta_n\rangle_2-\sum_{j=1}^2\langle P_H u^j\partial_{z_j}u,\eta_n\rangle_2=\langle u,\Delta\eta_n\rangle_2-\sum_{j=1}^2\langle\partial_{z_j}u,u^j\eta_n\rangle_2$$

whenever $u\in V_2$ and $B^n(u,t)=0$ otherwise. These functions are continuous on balls in V_2 with respect to the topology of $L^2(D,\mathbb{R}^2)$, which easily follows from the compactness of the Sobolev embedding $H^{2,1}(D)\to L^2(D)$. Consider the operator

$$L\varphi(u,t)=\sum_{i,j}^{\infty}a^{ij}\partial_{\eta_i}\partial_{\eta_j}\varphi(u,t)+\sum_{n=1}^{\infty}B^n(u,t)\partial_{\eta_n}\varphi(u,t).$$

Assume that $a^{ij}=\langle S\eta_i,\eta_j\rangle_2$ for some symmetric nonnegative bounded operator S on H. Suppose also that $\sum_i a^{ii}\lambda_i^2<\infty$. Then there exists at most one probability solution μ of the Cauchy problem for the Fokker–Planck–Kolmogorov equation $\partial_t\mu=L^*\mu$ such that for some $\delta>0$

$$\int_0^{T_0}\int_H\bigl(1+\|\Delta u\|_2^2\bigr)e^{\delta\|u\|_{V_2}^2}\,\mu_t(du)\,dt<\infty,$$

where we set $\|\Delta u\|_2=\infty$ if $u\notin H^{2,2}(D)$.

Proof. We apply Example 2.7(iii). Recall that the matrix (a^{ij}) has to satisfy the following condition for some $\varepsilon_0>0$

$$\varepsilon_0\sum_{i,j\le N}a^{ij}\lambda_i^2\lambda_j^2x_ix_j+\varepsilon_0|x|^2\le\sum_{i\le N}\lambda_i^4x_i^2$$

that is equvivalent (if we take $u=\sum_{i=1}^N\lambda_ix_ie_i$) to the estimate

$$\varepsilon_0\bigl(\langle Su,u\rangle_2+\|u\|_2^2\bigr)\le\|u\|_2^2,$$

which is true for sufficiently small ε_0. Set

$$F^n(u)=-\sum_{j=1}^2\langle\partial_{z_j}u,u^j\eta_n\rangle_2,\quad u\in V_2.$$

Note that $|F^n(u)|\le C_1(n)+C_2(n)\|u\|_2^2$, since $F^n(u)=\sum_{j=1,2}\langle u,u_j\partial_{z_j}\eta_n\rangle_2$ due to the condinition that $\operatorname{div} u=0$. It is well-known that there exists a constant $C_1>0$ such that for every function $g\in H_0^{2,1}(D)\cap H^{2,2}(D)$ we have

$$\|g\|_{2,2}\le C_1\bigl(\|\Delta g\|_2+\|g\|_2\bigr).$$

Moreover, for every $g\in H^{2,2}(D)$, every $r\ge1$ and some constant $C_2>0$ we have

$$\|g\|_r\le C_2\|g\|_{2,1}.$$

Hence

$$\begin{aligned}\|F(u)\|_{l^2}^2&\le\int_D|\nabla_z u(z)|^2|u(z)|^2\,dz\le\Bigl(\int_D|\nabla_z u(z)|^4\,dz\Bigr)^{1/2}\Bigl(\int_D|u(z)|^4\,dz\Bigr)^{1/2}\le\\&\le C_1^2C_2^4(1+\|\Delta u\|_2^2)\|u\|_{V_2}^2.\end{aligned}$$

Let $P_Nu=u_1\eta_1+\ldots+u_N\eta_N$. We have

$$\lim_{N\to\infty}\int_0^{T_0}\int_{L^2((0,1))}\|F(u)-F(P_Nu)\|_{l^2}e^{\delta\|u\|_{V_2}^2/2}\,\mu_t(du)\,dt=0.$$

It is known (see, e.g., [14, Proposition 6.3]) that in the considered case $d=2$ we have the inequality

$$\langle F(P_Nu),\Delta P_Nu\rangle_2=0$$

which gives the condition $\langle f(x,t),x\rangle_{l^2_\lambda}\le 0$ in Example 2.7(iii). In addition, for every $\gamma\in(0,1)$

$$\sum_{i,j\le N}\partial_{u_i}F^j(P_Nu)h_ih_j\le\big(C_\gamma+\gamma\|\Delta P_Nu\|^2_{l^2}\big)|h|^2+\|h\|^2_{l^2_\lambda},\quad h=(h_i).$$

Set $\theta(P_Nu)=C_\gamma+\gamma\|\Delta P_Nu\|^2_{l^2}$ and $C_0=C_\gamma+\sum_{i=1}^\infty a^{ii}\lambda_i^2$ (we recall that $\sum_{i=1}^\infty a^{ii}\lambda_i^2<\infty$). In order to apply Example 2.7(iii) we choose $\gamma<2^{-1}\delta(\varepsilon_0-\delta)$. In Example 3.5 we consider a more general equation. $\square$

It is worth noting that the last example applies to degenerate coefficients A, in particular, to A identically zero, which gives uniqueness for the so-called continuity equation corresponding to $2d$-Navier–Stokes equation.

In the next section we show that the considered classes of uniqueness are not empty.

3. Existence of solutions

First we would like to mention that if the stochastic equation associated to our Fokker–Planck–Kolmogorov equation has a solution in the sense of Stroock–Varadhan's martingale problem, then one immediately gets a solution to the FPK-equation. But uniqueness of solutions for a martingale problem does not imply uniqueness for the corresponding FPK-equation.

In this section we purely analytically prove the following existence result generalizing a result from [4] (where only a sketch of the proof of a weaker result was given).

Let $\{e_n\}$ be an orthonormal basis in l^2. The linear span of $e_1,\dots,e_n$ is denoted by H_n.

Let $T_0>0$ and let $a^{ij}\colon\mathbb{R}^\infty\times[0,T_0]\to\mathbb{R}^1$ and $B^i\colon\mathbb{R}^\infty\times[0,T_0]\to\mathbb{R}^1$ be Borel functions. Suppose that the matrices $(a^{ij})_{i,j\le n}$ are symmetric nonnegative for all n. Set

$$L\varphi(x,t):=\sum_{i,j=1}^n a^{ij}(x,t)\partial_{e_i}\partial_{e_j}\varphi(x,t)+\sum_{i=1}^n B^i(x,t)\partial_{e_i}\varphi(x,t),\ (x,t)\in\mathbb{R}^\infty\times[0,T_0]$$

for functions φ that are smooth functions of the variables $x_1,\dots,x_n,t$.

Let $B_n:=(B^1,\dots,B^n)$ and $P_nx=(x_1,\dots,x_n)$.

A Borel function $\Theta\colon\mathbb{R}^\infty\to[0,+\infty]$ such that the sublevel sets $\{\Theta\le R\}$ are compact is called a compact function. For example, one can take any numbers $\alpha_i>0$ and set $\Theta(x)=\sum_{i=1}^\infty\alpha_i^2x_i^2$.

Theorem 3.1. *Suppose that there exists a compact function $\Theta\colon\mathbb{R}^\infty\to[0,+\infty]$, finite on each H_n and such that the functions a^{ij} and B^i are continuous in x on all the sets $\{\Theta\le R\}$, and there exist numbers $M_0,C_0\ge0$ and a Borel function $V\colon\mathbb{R}^\infty\to[1,+\infty]$ whose sublevel sets $\{V\le R\}$ are compact and whose restrictions to H_n are of class C^2 and such that for all $x\in H_n$, $n\ge1$, one has*

$$\sum_{i,j=1}^n a^{ij}(x,t)\partial_{e_i}V(x)\partial_{e_j}V(x)\le M_0V(x)^2,\quad LV(x,t)\le C_0V(x)-\Theta(x).\tag{3.1}$$

Assume also that there exist constants $C_i\ge0$ and $k_i\ge0$ such that for all i and $j\le i$ one has

$$|a^{ij}(x,t)|+|B^i(x,t)|\le C_iV(x)^{k_i}(1+\delta(\Theta(x))\Theta(x)),\ (x,t)\in\mathbb{R}^\infty\times[0,T_0],\tag{3.2}$$

where δ is a bounded nonnegative Borel function on $[0,+\infty)$ with $\lim\limits_{s\to\infty}\delta(s)=0$. Then, for every Borel probability measure ν on $\mathbb{R}^\infty$ such that $W_k:=\sup_n\|V^k\circ P_n\|_{L^1(\nu)}<\infty$ for all $k\in\mathbb{N}$, the Cauchy problem (1.1) with initial distribution ν has a solution of the form $\mu=\mu_t\,dt$ with Borel probability measures μ_t on $\mathbb{R}^\infty$ such that for all $t\in[0,T_0]$

$$\int_{\mathbb{R}^\infty}V^k\,d\mu_t+k\int_0^t\int_{\mathbb{R}^\infty}V^{k-1}\Theta\,d\mu_s\,ds\le N_kW_k\quad\forall\,k\in\mathbb{N},\tag{3.3}$$

where $N_k:=M_ke^{M_k}+1$, $M_k=k(C_0+(k-1)M_0)$. In particular, $\mu_t(V<\infty)=1$ for all t and $\mu_t(\Theta<\infty)=1$ for almost all t.

Proof. For every fixed n let a_n^{ij} denote the restriction of a^{ij} to $H_n\times(0,T_0)$ and set $A_n:=(a_n^{ij})_{i,j\le n}$. Denote by ν_n the projection of ν on H_n. We show that there exist Borel probability measures $\mu_{t,n}$ on H_n such that the measure $\mu_n:=\mu_{t,n}\,dt$ solves the Cauchy problem with coefficients A_n and B_n on $H_n\times(0,T_0)$ and initial distribution ν_n. To this end we consider the Lyapunov function $V_m(x)=V(x)^m$ on H_n, where $m\ge1$. Letting $M_m:=m(C_0+(m-1)M_0)$, we obtain

$$\begin{aligned}LV_m&=mV^{m-1}\Big(LV+(m-1)V^{-1}\sum_{i,j=1}^n a^{ii}\partial_{e_i}V\partial_{e_j}V\Big)\le mV^{m-1}(C_0V-\Theta+(m-1)M_0V)\\&\le M_mV^m-mV^{m-1}\Theta.\end{aligned}$$

Since the function V_m is ν_n-integrable, we can apply the existence result from [3] and obtain the desired probability measures $\mu_{t,n}$ on H_n such that the function

$$t \mapsto \int_{H_n} \zeta(x)\, \mu_{t,n}(dx)$$

is continuous on $[0, T_0)$ for every $\zeta \in C_0^\infty(H_n)$. Moreover, by [4, Lemma 1] (see also [3, Lemma 2.2]), for each $m \geq 1$ and

$$N_m := M_m e^{M_m} + 1, \quad M_m = m(C_0 + (m-1)M_0)$$

the following estimate holds for almost all $t \in (0, T_0)$:

$$\int_{H_n} V_m(x)\, \mu_{t,n}(dx) + m \int_0^t \int_{H_n} V_{m-1}(x)\Theta(x)\, \mu_{s,n}(dx)\, ds$$
$$\leq N_m \int_{H_n} V_m(x)\, \nu_n(dx) \leq N_m + N_m W_m. \quad (3.4)$$

Therefore, by Fatou's theorem and the above stated continuity of $t \mapsto \mu_{t,n}$ it follows that (3.4) holds for all $t \in [0, T_0)$. Indeed, we replace V_m and ΘV_{m-1} in the left-hand side by $\min(k, V_m)$ and $\min(k, \Theta V_{m-1})$, obtain the desired estimate for all $t \in [0, T_0)$ keeping k fixed and then let $k \to \infty$.

Suppose now that $\zeta \in C_0^\infty(\mathbb{R}^d)$. Let us identify H_n with $\mathbb{R}^n$. If $n \geq d$, then ζ regarded as a function on $\mathbb{R}^n$ belongs to the class $C_b^\infty(\mathbb{R}^n)$. Let $m = \max(k_1, \ldots, k_d)$. Then we have the estimate

$$|L\zeta(x,t)| \leq K + KV_m(x) + KV_m(x)\delta(\Theta(x))\Theta(x), \ \ (x,t) \in \mathbb{R}^n \times [0, T_0], \quad (3.5)$$

where K is some number which depends on ζ (but is independent of n since ζ is a function of $x_1, \ldots, x_d$). Therefore, by approximation, inequality (3.4) and Lebesgue's dominated convergence theorem we have

$$\int_{H_n} \zeta(x)\, \mu_{t,n}(dx) = \int_0^t \int_{H_n} L\zeta(x,s)\, \mu_{s,n}(dx)\, ds + \int_{H_n} \zeta(x)\, \nu_n(dx), \quad (3.6)$$

because according to [3], this identity holds for all $\zeta \in C_0^\infty(\mathbb{R}^n)$, hence in our situation it remains valid also for all $\zeta \in C_b^\infty(\mathbb{R}^n)$. Letting

$$\varphi_n(t) := \int_{H_n} \zeta(x)\, \mu_{t,n}(dx), \ \ t \in [0, T_0],$$

we see from (3.4), (3.6) that the function φ_n is Lipschitzian (one can show that it is everywhere differentiable in $(0, T_0)$) and (3.5) yields that

$$|\varphi_n'(t)| \leq \int_{H_n} |L\zeta(x,t)|\, \mu_{t,n}(dx) \leq K_\zeta \int_{H_n} [1 + V_{m-1}(x)\Theta(x)]\, \mu_{t,n}(dx)$$

with some number K_ζ that does not depend on n (but only on ζ). Therefore, by (3.4) the functions φ_n possess uniformly bounded variations, hence there is a subsequence in $\{\varphi_n\}$ convergent pointwise on $[0, T_0]$. We may assume that this is true for the whole sequence. Moreover, we can do this in a such a way that this pointwise convergence holds for every function ζ from a fixed countable family $\mathcal{F}$ with the following property: the weak convergence of a uniformly tight sequence of probability measures on $\mathbb{R}^\infty$ follows from convergence of their integrals of every function in $\mathcal{F}$.

It follows from (3.4) and the compactness of the sets $\{V_m \leq R\}$ and $\{\Theta \leq R\}$ that, for every fixed $t \in (0, T_0)$, the sequence of measures $\mu_{t,n}$ is uniformly tight on $\mathbb{R}^\infty$ (see [2, Example 8.6.5]). Hence we can find a subsequence, denoted for simplicity by the same indices n, such that $\{\mu_{t,n}\}$ converges weakly on $\mathbb{R}^\infty$ for every rational $t \in (0, T_0)$. However, since we have ensured convergence of $\varphi_n(t)$ at every $t \in [0, T_0]$ for every $\zeta \in \mathcal{F}$, we see that $\{\mu_{t,n}\}$ converges weakly for every $t \in [0, T_0]$.

Estimate (3.3) follows from (3.4) taking into account that $V \geq 1$ and $\Theta \geq 0$ are lower semicontinuous, hence V^k and $V^{k-1}\Theta$ are lower continuous.

The family of measures μ_t obtained in this way is the desired solution. Indeed, let us fix $\zeta \in C_0^\infty(\mathbb{R}^d)$. We have to show that the integrals of $L\zeta(x,t)$ over $\mathbb{R}^\infty \times [0,T]$, $T < T_0$, with respect to μ_n converge to the integral with respect to $\mu = \mu_t\, dt$. This amounts to establishing such convergence for all functions $f = \partial_{x_i}\zeta B^i$ and $f = a^{ij}\partial_{x_j}\partial_{x_i}\zeta$. Suppose we are able to show this for the functions $f_N = \max(\min(f, N), -N)$. Then (3.2) and (3.4) enable us to extend the same to the original function f, because for every $\varepsilon > 0$ these estimates give a number N such that the integral of $|f|I_{|f|>N}$ with respect to $\mu_{t,n}\, dt$ is less than ε. Indeed, it suffices to show that the integral of $G := V^k(1 + \delta(\Theta)\Theta)$ over the set $\{G \geq N\}$ with respect to $\mu_{t,n}\,dt$ does not exceed ε for N sufficiently large. Take n_1 such that $1/n_1+\delta(s)<c\varepsilon$ for all $s\geq n_1$, where $c>0$ is so small that $cN_{k+1}W_{k+1}<1/2$. We may assume that $\delta\leq 1$. We have

$$\int_0^{T_0}\int_{\{\Theta\geq n_1\}} G\,d\mu_{t,n}\,dt=\int_0^{T_0}\int_{\{\Theta\geq n_1\}}(\Theta^{-1}+\delta(\Theta))V^k\Theta\,d\mu_{t,n}\,dt\leq c\varepsilon\int_0^{T_0}\int_{H_n}V^k\Theta\,d\mu_{t,n}\,dt\leq\varepsilon/2.$$

For any $N\geq n_1$ and $t<T_0$ we have

$$\int_{\{G\geq N,\Theta\leq n_1\}} G\,d\mu_{t,n}\leq(1+n_1)\int_{\{V^k\geq N/(1+n_1)\}}V^k\,d\mu_{t,n}\leq N^{-1}(1+n_1)^2N_kW_k,$$

which can be made smaller than $\varepsilon/2$ uniformly in $t<T_0$ for all N sufficiently large.

Thus, it remains to justify the desired convergence in the case of f_N, which will be now denoted by f. We recall that the restriction of such a function f to every set $\{\Theta\leq R\}\times[0,T_0]$ is continuous in the first variable. Dividing by N we assume that $|f|\leq 1$. If f were continuous in x on the whole space, this would follow at once from the weak convergence of $\mu_{t,n}$ for every fixed t. Our situation reduces to this one in the standard way: given $\varepsilon>0$, we find R so large that the set $\{\Theta\leq R\}\times[0,T_0]$ has measure less than ε with respect to all measures $\mu_{t,n}\,dt$ and $\mu_t\,dt$. By our assumption the set $\Omega=\{\Theta\leq R\}$ is compact in $\mathbb{R}^\infty$. The mapping $t\mapsto f(\,\cdot\,,t)$ from $[0,T_0]$ to $C(\Omega)$ is Borel measurable. By Dugundji's theorem (see [13, Chapter III, Section 7]), there is a linear extension operator $E\colon C(\Omega)\to C_b(\mathbb{R}^\infty)$ such that $E\varphi(x)=\varphi(x)$ for all $\varphi\in C(\Omega)$, $x\in\Omega$ and $\|E\varphi\|_\infty=\|\varphi\|_\infty$. Letting $g(x,t)=Ef(\,\cdot\,,t)(x)$, we obtain a Borel function (since it is Borel measurable in t and continuous in x, see [2, Lemma 6.4.6]) such that $|g|\leq 1$ and $g(t,x)=f(t,x)$ for all $x\in\Omega$. The integral of g with respect to $\mu_{t,n}\,dt$ converges to the integral of g with respect to $\mu_t\,dt$ and the integrals of $|f-g|$ with respect to these measures do not exceed ε. Therefore, the measure $\mu=\mu_t\,dt$ satisfies our parabolic equation with initial distribution ν. □

The condition that $V\geq 1$ is taken just for simplicity of estimates: it can be replaced by $V\geq 0$ if we add constants in the right sides of (3.1) and (3.2).

In typical examples V and Θ are quadratic functions (with added constants). For example, we shall use $V(x)=\sum_{i=1}^\infty\beta_i x_i^2+1$ and $\Theta(x)=\sum_{i=1}^\infty\alpha_i x_i^2$. There is also a version of this theorem applicable to exponents of quadratic functions (the first inequality in (3.1) is not suitable for such functions).

Theorem 3.2. *Suppose that in Theorem* 3.1 *condition* (3.1) *is replaced by*

$$LV(x,t)\leq V(x)-V(x)\Theta(x)\tag{3.7}$$

and (3.2) *is replaced by*

$$|a^{ij}(x,t)|+|B^i(x,t)|\leq C_i(1+\delta(V(x)\Theta(x))V(x)\Theta(x)),\ (x,t)\in\mathbb{R}^\infty\times[0,T_0].\tag{3.8}$$

Then, for every Borel probability measure μ_0 on $\mathbb{R}^\infty$ with $W_1:=\sup_n\|V\circ P_n\|_{L^1(\mu_0)}<\infty$ the Cauchy problem (1.1) *with initial distribution μ_0 has a solution of the form $\mu=\mu_t\,dt$ with Borel probability measures μ_t on $\mathbb{R}^\infty$ such that for $t\in[0,T_0]$*

$$\int_{\mathbb{R}^\infty}V\,d\mu_t+\int_0^t\int_{\mathbb{R}^\infty}V\Theta\,d\mu_s\,ds\leq 4W_1.\tag{3.9}$$

Proof. The reasoning is much the same as in the previous theorem, but we use only one Lyapunov function V and use (3.7) in place of (3.4) to obtain the estimate

$$\int_{H_n}V(x)\,\mu_{t,n}(dx)+\int_0^t\int_{H_n}V(x)\Theta(x)\,\mu_{s,n}(dx)\,ds\leq(e+1)\int_{H_n}V(x)\,\mu_{0,n}(dx)\leq 4W_1.$$

Another place where some difference arises is the estimate of the integral of $fI_{|f|>N}$, where $|f|$ is estimated by $C(1+\delta(V\Theta)V\Theta)$, but this is easily done by using the previous inequality and the condition that $\delta(s)\to 0$ as $s\to\infty$. □

Let us apply the last theorem to the Fokker–Planck–Kolmogorov equation associated with the stochastic Burgers type equations (see Example 2.10).

Example 3.3. ("Stochastic Burgers equation") Let us return to the situation of Example 2.10. Let u be from the linear span of $\{e_k\}$. Note that

$$\langle B(u),u\rangle_2=-\|u\|^2_{H_0^1}.$$

Let $V(u)=\exp\bigl(\delta\|u\|_2^2\bigr)$. We have

$$LV(u)\leq 2\delta\bigl(\mathrm{tr}S+2\delta\langle Su,u\rangle_2-\|u\|^2_{H_0^1}\bigr)V(u).$$

Taking $\delta < \varepsilon_0/4$ we obtain

$$LV(u) \le (1-\Theta(u))V(u), \quad \Theta(u) = 1 - 2\delta\,\mathrm{tr}S + \delta\|u\|^2_{H^1_0}.$$

In addition, $|B^k(u)| \le C(k) + C(k)\|u\|_2^2$. According to Theorem 3.2 for every initial condition ν with $\exp(\delta\|u\|_2^2) \in L^1(\nu)$ there exists a probability solution μ of the Cauchy problem $\partial_t\mu = L^*\mu$, $\mu|_{t=0} = \nu$ such that

$$\int_0^{T_0}\int_{L^2((0,1))} \|u\|^2_{H^1_0} \exp\bigl(\delta\|u\|_2^2\bigr)\,\mu_t(du)\,dt < \infty.$$

According to Example 2.10 this μ is the unique probability solution with this property.

Example 3.4. Let us return to the situation of Example 2.11(i). Assume that $a^{ij} = 0$ if $i \ne j$ and that $\sum_i a^{ii} < \infty$. Let u be from the linear span of $\{e_k\}$. Set

$$V(u) = (1 + \|u\|^{2m+2}_{2m+2})\exp\bigl(\delta\|u\|_2^2\bigr)$$

Note that for some positive constants C_1, C_2 and C_3 we have

$$L(1 + \|u\|^{2m+2}_{2m+2}) \le C_1 - C_2\|u^{m+1}\|^2_{H^1_0} - C_3\|u\|^{4m+2}_{4m+2}.$$

Using the calculations from the previous example we obtain

$$LV(u) \le (1-\Theta(u))V(u), \quad \Theta(u) = \widetilde{C}_1 + \delta\widetilde{C}_2\|u\|^2_{H^1_0} + \bigl(\widetilde{C}_3\|u^{m+1}\|^2_{H^1_0} + \widetilde{C}_4\|u\|^{4m+2}_{4m+2}\bigr)\bigl(1 + \|u\|^{2m+2}_{2m+2}\bigr)^{-1}$$

for some positive constants $\widetilde{C}_1$, $\widetilde{C}_2$, $\widetilde{C}_3$ and $\widetilde{C}_4$. According to Theorem 3.2 for every initial condition ν with

$$(1 + \|u\|^{2m+2}_{2m+2})\exp(\delta\|u\|_2^2) \in L^1(\nu)$$

there exists a probability solution μ of the Cauchy problem $\partial_t\mu = L^*\mu$, $\mu|_{t=0} = \nu$ such that

$$\int_0^{T_0}\int_{L^2((0,1))} \bigl(\|u\|^{4m+2}_{4m+2} + \|u^2\|_{H^1_0}\bigr)\exp\bigl(\delta\|u\|_2^2\bigr)\,\mu_t(du)\,dt < \infty.$$

According to Example 2.11(i), this μ is the unique probability solution with this property.

In the same way applying Theorem 3.2 with

$$V(u) = (1 + \|u\|_2^2 + \|u\|^{2l+2}_{2l+2})\exp\bigl(\delta\|u\|^{2m-2}_{2m-2}\bigr)$$

one can obtain existence of a probability solution in the situation of Example 2.11(ii). Thus, there is a unique probability solution μ such that

$$\int_0^{T_0}\int_{L^2((0,1))} \bigl(\|u\|^{4l+2}_{4l+2} + \|u\|^2_{H^1_0} + \|u^{m-1}\|^2_{H^1_0}\bigr)\exp\bigl(\delta\|u\|^{2m-2}_{2m-2}\bigr)\,\mu_t(du)\,dt < \infty.$$

We note only that $\|u^m\|_{H^1_0} \le \|u\|^2_{H^1_0} + \|u^{m-1}\|^2_{H^1_0}$, since $m \ge 2$ and $\|u\|_\infty \le \|u\|_{H^1_0}$. This partially generalizes a result in [30].

Let us apply the existence theorems to the Fokker–Planck–Kolmogorov equation associated with the stochastic Navier–Stokes equation in any dimension (a special case has been considered in Example 2.12).

Example 3.5. The stochastic equation of Navier–Stokes type is considered in the space V_2 of $\mathbb{R}^d$-valued mappings $u = (u^1, \dots, u^d)$ such that $u^j \in H_0^{2,1}(D)$ and $\operatorname{div} u = 0$, where $D \subset \mathbb{R}^d$ is a bounded domain with smooth boundary. The space V_2 is equipped with its natural Hilbert norm $\|u\|_{V_2}$ defined by

$$\|u\|^2_{V_2} := \sum_{j=1}^d \|\nabla_z u^j\|_2^2.$$

Let H be the closure of V_2 in $L^2(D,\mathbb{R}^d)$ and let P_H denote the orthogonal projection on H in $L^2(D,\mathbb{R}^d)$. The stochastic Navier–Stokes equation is formally written as

$$du(z,t) = \sqrt{2}dW(z,t) + P_H\Bigl[\Delta_z u(z,t) - \sum_{j=1}^d u^j(z,t)\partial_{z_j}u(z,t) + F(z,u(z,t),t)\Bigr]dt,$$

where W is a Wiener process of the form $W(z,t) = \sum_{n=1}^\infty \sqrt{\alpha_n}w_n(t)\eta_n(z)$, where

$$\alpha_n \ge 0, \quad \sum_{n=1}^\infty \alpha_n < \infty,$$

w_n are independent Wiener processes, and $\{\eta_n\}$ is an orthonormal basis in H, and $F\colon D\times\mathbb{R}^d\times(0,T_0) \to \mathbb{R}^d$ is a bounded continuous mapping. No interpretation of this equation is needed for the sequel, it should be regarded only as a heuristic expression leading to a specific form of the corresponding elliptic operator.

The case $F = 0$ is the classical stochastic Navier–Stokes equation. Note that the action of P_H in the right-hand side is defined in the natural way: $P_H \Delta_z u(z,t) := P_H \Delta_z u(\,\cdot\,,t)(z)$ and similarly for the other terms. Since the Laplacian Δ is not defined on all of V_2, this equation requires some interpretation. Our approach suggests the following procedure. It is known (see [26]) that there exists an orthonormal basis $\{\eta_n\}$ in H formed by eigenfunctions of Δ with eigenvalues $-\lambda_n^2$ such that $\eta_n \in V_2$. Employing the fact that $\langle P_H w, \eta_n\rangle_2 = \langle w, \eta_n\rangle_2$ for any $w \in L^2(D, \mathbb{R}^d)$, we introduce the "coordinate" functions

$$\begin{aligned} B^n(u,t) &= \langle u, \Delta\eta_n\rangle_2 - \sum_{j=1}^{d} \langle P_H(u^j \partial_{z_j} u), \eta_n\rangle_2 + \langle P_H F(\,\cdot\,, u(\,\cdot\,,t),t), \eta_n\rangle_2 \\ &= \langle u, \Delta\eta_n\rangle_2 - \sum_{j=1}^{d} \langle \partial_{z_j} u, u^j \eta_n\rangle_2 + \langle F(\,\cdot\,, u(\,\cdot\,,t),t), \eta_n\rangle_2. \end{aligned}$$

These functions are defined by the last line on all of V_2. They are continuous on balls in V_2 with respect to the topology of $L^2(D, \mathbb{R}^d)$, which follows by the compactness of the embedding of $H^{2,1}(D) \to L^2(D)$. Choosing a Wiener process of the above form, we arrive at the operator

$$L\varphi(u,t) = \sum_{n=1}^{\infty} \alpha_n \partial_{\eta_n}^2 \varphi(u,t) + \sum_{n=1}^{\infty} B^n(u,t) \partial_{\eta_n} \varphi(u,t).$$

Since for every u from the linear span of $\{\eta_n\}$ one has

$$\sum_{n=1}^{\infty} \sum_{j=1}^{d} \langle u, \eta_n\rangle_2 \langle \partial_{z_j} u, u^j \eta_n\rangle_2 = \sum_{j=1}^{d} \langle u, u^j \partial_{z_j} u\rangle_2 = -\frac{1}{2} \int_D |u(z)|^2 \operatorname{div} u(z)\, dz = 0$$

and $\langle \Delta u, u\rangle_2 = -\|u\|_{V_2}^2$, we have the estimate

$$\sum_{n=1}^{N} \langle u, \eta_n\rangle_2 B^n(u,t) \le C_1 - C_1 \|u\|_{V_2}^2$$

for all u in the linear span of $\eta_1, \ldots, \eta_N$, where C_1 is a constant independent of N. Clearly, we have also $|B^n(u,t)| \le C_2(n) + C_2(n)\|u\|_2^2$.

Therefore, by Theorem 3.1 applied with $\Theta(u) = C_1\|u\|_{V_2}^2$ and $V(u) = \|u\|_2^2 + 1$ (the above estimates along with convergence of the series of α_n mean that we have (3.1)) there is a probability measure $\mu = \mu_t dt$ on $V_2 \times [0, T_0)$, such that $\mu_t(H) = 1$ for all t and $\mu_t(V_2) = 1$ for almost all t, and solving the Cauchy problem (1.1) with any initial distribution μ_0 for which $\|u\|_2^k \in L^1(\mu_0)$ for all k.

It should be also noted that Flandoli and Gatarek [20] proved (under the stated assumptions) the existence of a solution to the martingale problem associated with the operator L such that this solution possesses all moments in H. One can show that the measure generated by this solution satisfies the Fokker–Planck–Kolmogorov equation in our sense.

Let us consider the 2d-Navier–Stokes equation, i.e., $d = 2$ and $F = 0$. Recall that for every u from the linear span of $\{\eta_n\}$ one has

$$\sum_{n=1}^{\infty} \sum_{j=1}^{2} \langle u, \Delta\eta_n\rangle_2 \langle \partial_{z_j} u, u^j \eta_n\rangle_2 = 0.$$

Set $V(u) = \exp(\delta\|u\|_{V_2}^2)$. Let u be from the linear span of $\{\eta_n\}$. We have

$$LV(u) = 2\delta\Bigl(\sum_n \alpha_n \lambda_n^2 + 2\delta \sum_n \alpha_n \lambda_n^4 u_n^2 - \sum_n \lambda_n^4 u_n^2\Bigr) V(u).$$

Assume that $\sum_{n=1}^{\infty} \alpha_n \lambda_n^2 < \infty$. Hence for sufficiently small $\delta > 0$

$$LV(u) \le (1 - \Theta(u))V(u), \quad \Theta(u) = 1 - \delta \sum_{n=1}^{\infty} \alpha_n \lambda_n^2 + \delta \|\Delta u\|_2^2,$$

where $\Theta(u) = +\infty$ if $u \notin H^{2,2}(D)$. According to Theorem 3.2 for every initial condition ν with $\exp(\delta\|u\|_{V_2}^2) \in L^1(\nu)$ there exists a probability solution μ of the Cauchy problem $\partial_t \mu = L^*\mu$, $\mu|_{t=0} = \nu$ such that

$$\int_0^{T_0} \int_H \bigl(1 + \|\Delta u\|_2^2\bigr) e^{\delta\|u\|_{V_2}^2}\, \mu_t(du)\, dt < \infty.$$

According to Example 2.12 this μ is the unique probability solution with this property.

Finally, we formulate one more existence and uniqueness result which is a combination of Theorem 3.1 and Theorem 2.3.

Corollary 3.6. *Let $a^{ij}=0$ if $i\neq j$ and $a^{ii}=\alpha_i>0$. Suppose that the hypotheses of Theorem* 3.1 *are fulfilled with certain functions V and Θ. If there exists a Borel mapping $F=(F_n)\colon \mathbb{R}^\infty\times[0,T_0]\to\mathbb{R}^\infty$ and numbers $p>0$, $C>0$ such that $\|F(x,t)\|^2_{l^2_\alpha}\le CV(x)^p\Theta(x)$ and for each natural number n the difference $B^n(x,t)-F^n(x,t)$ depends only on t and $x_1,x_2,\ldots,x_n$, then for every initial condition ν with $V\in L^k(\nu)$ for every $k\ge 1$ the class $\mathcal{P}_\nu$ consists of exactly one element.*

Example 3.7. Let $a^{ij}=0$ if $i\neq j$ and $a^{ii}=\alpha_i>0$. Suppose that

$$B^n(x,t)=-\beta_n x_n+F^n(x,t),\quad \text{where } \beta_n>0.$$

Let $\gamma_n\in(0,+\infty)$ be such that

$$\sum_{n=1}^\infty \alpha_n\gamma_n<\infty.$$

Let

$$V(x)=1+\sum_{n=1}^\infty \gamma_n x_n^2,\quad \Theta(x)=\sum_{n=1}^\infty \beta_n\gamma_n x_n^2.$$

Let c_{00} denote the subspace of all vectors $x\in\mathbb{R}^\infty$ with at most finitely many nonzero coordinates.

Suppose that a Borel mapping $F(\,\cdot\,,\cdot\,)\colon \mathbb{R}^\infty\times[0,T_0]\to\mathbb{R}^\infty$ satisfies the following conditions: for each t it is continuous in x on every set $\{\Theta\le R\}$ and there are numbers $\varepsilon\in(0,1)$, $C_1>0$, $C_2>0$, and $p>0$ such that for all $t\in(0,T)$ and $x\in c_{00}$ one has

$$\sum_{n=1}^\infty \gamma_n F^n(t,x)x_n\le\varepsilon\Theta(x)+C_1V(x),\quad \sum_{n=1}^\infty \alpha_n^{-1}|F^n(t,x)|^2\le C_2\left(1+\Theta(x)\right)V(x)^p,$$

Then, for every initial condition ν with $V\in L^k(\nu)$ for every $k\ge1$, the class $\mathcal{P}_\nu$ consists of exactly one element.

Remark 3.8. As already noted, if the infinite-dimensional stochastic differential equation (SDE) associated to our Fokker–Planck–Kolmogorov equation has a solution in the sense of Stroock–Varadhan, then one gets a solution to the FPK-equation (but not vice versa). In contrast to that, uniqueness of solutions to the martingale problem does not imply the uniqueness of solutions to the FPK-equation, here the converse is true. Therefore, the existence parts in our Examples 3.3 – 3.5 can partly also be derived by probabilistic methods. It should also be pointed out that in these examples we always assume that (a^{ij}) is trace class. For existence results by probabilistic means in case of Example 3.3 and the first part of Example 3.4 without this condition we refer to [23] and its recent improvement [32]. Furthermore, we believe that by a similar method as in [15] one can also prove uniqueness for the FPK-equation in the Burgers case (see Example 3.3) without the trace class condition. Finally, we point out that here we consider the Burgers case only on the bounded domain $D=(0,1)\subset\mathbb{R}$. If $D=\mathbb{R}$ existence, however, also holds. This follows from the probabilistic results in [24].

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