**On the absolute continuity of the distributions of smooth functions on infinite-dimensional spaces with measures**

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**Abstract**

We consider sufficient conditions for the absolute continuity of the distributions of smooth functions on infinite-dimensional spaces with measures.

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In this note we are concerned with sufficient conditions for the absolute continuity of the distribution of a smooth function $f$ on an infinite-dimensional space $X$ equipped with a measure $\mu$. We shall assume that $X$ is a locally convex space and $\mu$ is a Radon probability measure on $X$ (see [2]). Various conditions of this sort are known for the most diverse classes of functions and measures, see [1], [3], [4], [5]. An important sufficient condition comes from the one-dimensional case where the following simple fact is known: if $\mu$ is an absolutely continuous measure and $f$ is an arbitrary function, then, letting $D$ be the set where $f$ has a nonzero derivative, we obtain that the restriction of $\mu$ to $D$ is taken by $f$ to an absolutely continuous measure, i.e., the measure $\mu|_D \circ f^{-1}$ is absolutely continuous. Throughout the image of $\mu$ under $f$ is denoted by $\mu \circ f^{-1}$ and is defined by $\mu \circ f^{-1}(B) = \mu(f^{-1}(B))$. It is known that in the considered case the set $D$ is always Lebesgue measurable and $f$ is measurable on $D$.

This fact suggests the following obvious infinite-dimensional (actually, dimension-free) extension. Suppose that $h \neq 0$ is a vector in $X$ such that $\mu$ admits absolutely continuous conditional measures $\mu^y$ on the straight lines $y + \mathbb{R}h$, where $y \in Y$ and $Y$ is a closed hyperplane complementing $\mathbb{R}h$. This means that

$$\mu(B) = \int_Y \mu^y(B)\, \mu_Y(dy),$$

where $\mu_Y$ is the image of $\mu$ under the natural projection on $Y$. Then, for any measurable function $f$, the image under $f$ of the restriction of $\mu$ to the set $D$ where the partial derivative $\partial_h f$ exists and does not vanish is absolutely continuous. The partial derivative is naturally defined as

$$\partial_h f(x) = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}.$$

The existence of absolutely continuous conditional measures is equivalent to the continuity of $\mu$ along $h$, i.e., the continuity of the functions $t \mapsto \mu(B + th)$ for all Borel sets $B$, which, in turn, is equivalent to the equality $\lim\limits_{t \to 0} \|\mu - \mu_{th}\| = 0$, where $\|\cdot\|$ is the variation norm and $\mu_h(B) = \mu(B - h)$.

As a corollary one obtains that $\mu \circ f^{-1}$ is absolutely continuous provided $f$ is a measurable function such that there is a countable collection of vectors $h_n$ along which $\mu$ is continuous and for almost every point $x$ there is $n$ such that $\partial_{h_n} f$ exists at $x$ and does not vanish.

There is also an multidimensional analog of this result for mappings $f = (f_1, \ldots, f_d)$ with values in $\mathbb{R}^d$: under the same assumptions about $\mu$, it is sufficient that $\mu$-almost everywhere one can find $h_{i_1}, \ldots, h_{i_d}$ such that the partial derivatives $\partial_{h_i} f_j$ exist and the matrix $(\partial_{h_i} f_j)_{i,j \le d}$ is nondegenerate. Apparently, one cannot expect efficient generalizations of these results. However, it could be of interest to have efficiently verified conditions that guarantee that the stated hypotheses are fulfilled. In that case, one could accept to deal with less general functions. In the one-dimensional case, the following simple observation may be helpful.

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**Lemma 1.** *If $f \in C^k(\mathbb{R})$, then the set*

$$\{x\colon\ f'(x)=0\}\cap\{x\colon\ f^{(k)}(x)\neq 0\}$$

*does not contain its limit points and is at most countable. Therefore, the image under $f$ of the restriction of Lebesgue measure to the set where some of the derivatives of $f$ does not vanish is absolutely continuous. In particular, if $f$ is infinitely differentiable and*

$$E=\{x\colon\ \text{there is } k \text{ such that } f^{(k)}(x)\neq 0\},$$

*then the image under $f$ of the restriction of Lebesgue measure to $E$ is absolutely continuous.*

*Proof.* Since we have the absolute continuity of the image under $f$ of the restriction of Lebesgue measure to the set where $f'$ does not vanish, it suffices to prove the first claim. This claim is, however, obvious: if $f'(x)=0$ and there is a nontrivial sequence $x_n\to x$ such that $f'(x_n)=0$ and $f^{(k)}(x_n)\neq 0$, then we can find another nontrivial sequence $z_n\to x$ such that $f''(z_n)=0$, hence $f''(x)=0$. Continuing in this way we conclude that $f^{(k)}(x)=0$, so $x$ does not belong to the indicated set. Since the set $\{x\colon\ f^{(k)}(x)\neq 0\}$ is open and consists of a finitely or countably many intervals, we obtain that the intersection of the closed set $\{x\colon\ f'(x)=0\}$ with any of these intervals is at most countable. □

Now an infinite-dimensional extension is immediate.

**Theorem 1.** *Let $\mu$ be a Radon probability measure on a locally convex space $X$ continuous along vectors from a countable set $S$ and let $f$ be a $\mu$-measurable function on $X$ such that all partial derivatives $\partial_{h_1}\cdots\partial_{h_n}f$ exist everywhere for all $h_1,\dots,h_n$ in the linear span of $S$. Let*

$$E=\{x\colon\ \exists\, h_1,\dots,h_n\in S \text{ with } \partial_{h_1}\cdots\partial_{h_n}f(x)\neq 0\}.$$

*Then the measure $\mu|_E\circ f^{-1}$ is absolutely continuous.*

*Proof.* Let us observe that every differential operator $\partial_{h_1}\cdots\partial_{h_n}$ can be written as a linear combination of differential operators of the form $\partial_v^k$, where $k\leq n$ and $v$ is a linear combination of $h_1,\dots,h_n$ with rational coefficients. Adding to the set $S$ all linear combinations of its elements with rational coefficients we may assume from the very beginning that $S$ is closed under taking such combinations. Therefore, it suffices to prove our claim for the set

$$E_0=\{x\colon\ \exists\, h\in S,\ n\in\mathbb{N} \text{ with } \partial_h^n f(x)\neq 0\}.$$

Now the lemma can be applied. Indeed, if $Z\subset\mathbb{R}$ is a set of Lebesgue measure zero, then in order to verify that $f^{-1}(Z)$ has $\mu$-measure zero, it suffices to show that the intersection of $f^{-1}(Z)$ with every set $E_{v,n}=\{x\colon\ \partial_h^n f(x)\neq 0\}$, where $v\in S$ and $n\in\mathbb{N}$ has measure zero. Therefore, it remains to observe that for every $y$ such that the conditional measure $\mu^y$ is absolutely continuous, the intersection $E_{v,n}\cap y+\mathbb{R}v$ has Lebesgue measure zero in the straight line $y+\mathbb{R}v$. □

We recall that a Radon probability measure $\mu$ on $X$ is called centered Gaussian (see [1], [3]) if every continuous linear functional on $X$ induces a centered Gaussian measure. The Cameron–Martin space $H$ of $\mu$ consists of all vectors $h$ such that the measures $\mu$ and $\mu_h$ are equivalent. It also coincides with the set of all vectors along which $\mu$ is continuous. It is known that $H$ is a separable Hilbert space with respect to the norm

$$|h|_H=\sup\{l(h)\colon\ l\in X^*,\ \|l\|_{L^2(\mu)}\leq 1\},$$

where $X^*$ is the space of all continuous linear functionals on $X$.

The Sobolev classes $W^{p,k}(\mu)$ are defined as the completions of the class of functions of the form

$$f(x)=f_0(l_1(x),\dots,l_n(x)),\quad f_0\in C_b^\infty(\mathbb{R}^n),\ l_1,\dots,l_n\in X^*,$$

with respect to the Sobolev norm $f\mapsto\|f\|_{p,k}$. The latter can be defined as follows. Let $\{e_n\}$ be an orthonormal basis in $H$. Then

$$\|f\|_{p,k}=\|f\|_{L^p(\mu)}+\biggl(\int_X\Bigl|\sum_{i_1,\dots,i_p}|\partial_{e_{i_1}}\cdots\partial_{e_{i_k}}f(x)|^2\Bigr|^{p/2}\mu(dx)\biggr)^{1/p}.$$

Every function $f \in W^{p,k}(\mu)$ has a Sobolev derivative $D_H^k f$ (defined by means of the integration by parts formula) with values in the space of $k$-linear Hilbert–Schmidt functions on $H$. See [1], [3] for more details. The class $W^{p,\infty}(\mu)$ is defined as the intersection of all $W^{p,k}(\mu)$, $k \in \mathbb{N}$. Let $W^\infty(\mu) = \bigcap_{p\in\mathbb{N}} W^{p,\infty}(\mu)$.

It is possible to introduce local Sobolev classes $W_{loc}^{p,k}(\mu)$ and their intersection. We say that $f$ belongs to $W_{loc}^{p,k}(\mu)$ if there exists a sequence of functions $\chi_n \in W^\infty(\mu)$ such that the sets $\{\chi_n = 1\}$ are increasing and their union has full measure. The sequence $\{\chi_n\}$ is called localizing.

**Remark 1.** For example, if $f$ is a Borel function on a sequentially complete space $X$ that is bounded on compact sets and has Fréchet derivatives $D_H^k f$ along $H$ of every order $k$ with values in the spaces $\mathcal{H}_k$ of $k$-linear Hilbert–Schmidt functions such that the corresponding Hilbert–Schmidt norms of the derivatives are also bounded on compact sets, then $f \in W_{loc}^\infty(\mu)$. Indeed, one can find a localizing sequence of functions $\chi_h \in W^\infty(\mu)$ such that $0 \le \chi_n \le 1$ and $\chi_n = 0$ outside of some compact absolutely convex set $K_n$ (their existence is shown below). Then $\chi_n f \in W^\infty(\mu)$, which follows by the known characterizations of Sobolev classes (see [1, Chapter 5] and [3, Chapter 8]). Let us explain how to construct $\chi_n$ (the details can be found in [1, Proposition 5.4.12 and Remark 5.4.13]). Since $X$ is sequentially complete, there exist metrizable absolutely convex compact sets $K_n$ such that their union has full measure (by Tsirelson's theorem, see [1, Theorem 3.4.1], for any $X$ one can find increasing metrizable compact sets with union of full measure, and in the case of a sequentially complete space the absolutely convex closed hull of a metrizable compact set is compact metrizable). By using the Minkowski functional of $K_n$ it is easy to construct a function $f_n \in W^{p,1}(\mu)$ such that $0 \le f_n \le 1$, $f$ is Lipschitzian along $H$, $f_n = 1$ on $K_n$, $f_n = 0$ outside of $2K_n$. Then the Ornstein–Uhlenbeck semigroup can be used for smoothing $f_n$ and obtaining the desired function.

**Corollary 1.** *Let $\mu$ be a centered Radon Gaussian measure on $X$, let $f \in W^{p,\infty}(\mu)$ and*

$$E := \bigcup_{n=1}^{\infty} \{x\colon\ D_H^n f(x) \ne 0\}.$$

*Then the measure $\mu|_E \circ f^{-1}$ is absolutely continuous. The same is true for the class $f \in W_{loc}^{p,\infty}(\mu)$.*

**Corollary 2.** *Let $\mu$ be a centered Radon Gaussian measure on $X$ and let $f$ be a Borel function on $X$ such that, for some orthonormal basis $\{e_n\}$ in $H$, the functions $t \mapsto f(x + te_n)$ are real analytic for almost each $x$. Then either $f$ has an absolutely continuous distribution or coincides $\mu$-almost everywhere with a constant.*

*Proof.* Suppose that $\mu \circ f^{-1}$ is not absolutely continuous. Then it follows that the intersection $Z$ of the sets $E_n = \{x\colon\ \partial_{e_n} f(x) = 0\}$ has positive measure. If we show that $\mu(Z) = 1$, then by the analyticity of $f$ along $e_n$ we obtain that, for every fixed $n$, the function $t \mapsto f(x + te_n)$ is constant for almost every $x$. By the zero-one law $f$ coincides with a constant almost everywhere (see [1, Corollary 3.2.11]). Therefore, it remains to show that $\mu(Z) = 1$. The function $t \mapsto \partial_{e_n} f(x + te_n)$ is real analytic for each $x$ such that $t \mapsto f(x + te_n)$ is real analytic. Hence the set $Z$ contains every straight line $x + \mathbb{R}e_n$ that intersects $Z$ by an uncountable set. Therefore, the set $Z$ coincides with $Z + \mathbb{R}e_n$ up to a measure zero set for each fixed $n$. Consequently, applying this to $n = 1, 2, \ldots$, we conclude that $Z$ coincides up to a measure zero set with $Z + L$, where $L$ is the linear span of $\{e_n\}$. Applying again the zero-one law, we obtain that $\mu(Z) = 1$, which completes our proof. □

Note that if $H$ is dense in $X$ and $f$ is continuous and equals a constant $\mu$-a.e., then it is constant. This follows by the known fact that $\mu$ is positive on all nonempty open sets if $H$ is dense.

Clearly, the same result is true for any measure $\mu$ with the following two properties: there is a sequence of vectors $e_n$ such that $\mu$ is quasi-invariant along $e_n$ (the shifts $\mu_{te_n}$ are equivalent) and any measurable function invariant under the shifts along the vectors $te_n$ is constant almost everywhere. It would be interesting to study this question for convex measures.

There is a natural multidimensional analog of the last corollary. Recall that if $f = (f_1, \dots, f_d)$, where $f_i \in W^{1,1}_{loc}(\mu)$, then the matrix with the entries $(D_H f_i, D_H f_j)_H$ is called the Malliavin matrix. Its nondegeneracy is sufficient for the absolute continuity of the measure $\mu\circ f^{-1}$ on $\mathbb{R}^d$ (see [3, Chapter 9]).

**Corollary 3.** *Let $f = (f_1, \dots, f_d)$, where each $f_i$ satisfies the same conditions as $f$ in the previous corollary and, in addition, $f \in W^{1,1}_{loc}(\mu)$. Then either the measure $\mu \circ f^{-1}$ is absolutely continuous on $\mathbb{R}^d$ or the Malliavin matrix for $f$ is degenerate almost everywhere.*

*Proof.* The determinant of the Malliavin matrix for $f$ is a Borel function satisfying the same assumptions as $f$ in the previous corollary. Hence its zero set has measure either 0 or 1. $\square$

Let $\mu$ be a Radon probability measure on a locally convex space $X$ infinitely differentiable along vectors from a densely embedded separable Hilbert space $H$ (see [3] for this concept). It is possible to introduce Sobolev spaces with respect to differentiable measures (see [3]). The same reasoning gives the following result.

**Corollary 4.** *Let $f \in W^\infty(\mu)$ and let $E = \bigcup_{n=1}^\infty \{x\colon\ D^n_H f(x) \neq 0\}$. Then the measure $\mu|_E \circ f^{-1}$ is absolutely continuous. The same is true for the local class $W^\infty_{loc}(\mu)$.*

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