On extensions of Sobolev functions on infinite-dimensional spaces

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Abstract

We study extensions of Sobolev and BV functions on infinite-dimensional domains. Some counter-examples and positive results are presented.

AMS Subject Classification: 28C20, 46G12, 60H07

Keywords: Sobolev class, extension, differentiable measure, Gaussian measure

Introduction and notation

In this paper we consider Sobolev and BV classes of functions on infinite-dimensional spaces with measures and the related classes on infinite-dimensional domains. We study extensions of functions in these classes to the whole space.

It is well-known that every function in the Sobolev class on a bounded convex set in $\mathbb{R}^n$ extends to a function on the whole space from the same Sobolev class (see [8, §4.4]). An analogous assertion is true also for weighted Sobolev classes with sufficiently regular weights, for example, Gaussian. It has been shown in the recent paper [10] that for a Gaussian measure on an infinite-dimensional space a weaker assertion is true: an extension exists for all functions in some everywhere dense set in the Gaussian Sobolev space (more precisely, in our notation, in the class $D^{2,1}(V,\gamma)$ defined through Sobolev derivatives) on a convex open set (or H-convex and H-open). We prove the existence of Sobolev functions on convex sets in an infinite-dimensional space equipped with a Gaussian measure without Sobolev extensions to the whole space. In the case of a Hilbert space such a set can be chosen convex and open. Next we introduce classes of functions of bounded variation on infinite-dimensional convex domains and prove that the functions in such classes admit BV extensions to the whole space in the case of domains of bounded perimeter.

We shall deal with the Gaussian measure γ on the countable product of the real lines $X=\mathbb{R}^\infty$ equal the countable product of the standard Gaussian measures on the real line. The objects and questions considered below are invariant with respect to measurable linear isomorphisms of locally convex spaces with centered Radon Gaussian measures (see [2]), hence by the Tsirelson isomorphism theorem everything proved below remains valid for those Radon Gaussian measures on locally convex spaces that are not concentrated on finite-dimensional spaces. Therefore, analogous results are true for the classical Wiener space or for any infinite-dimensional Gaussian measure on a Hilbert space.

We recall that the Cameron–Martin space of the measure γ is the Hilbert space $H=l^2$ with its usual norm $h\mapsto |h|_H=\Bigl(\sum_{i=1}^\infty h_i^2\Bigr)^{1/2}$, $h=(h_i)$; by definition the Cameron–Martin space consists of all vectors the shifts by which yield equivalent measures. Let $\{e_n\}$ be the standard orthonormal basis in H. Denote by FC_b^∞ the class of all functions on $\mathbb{R}^\infty$ of the form

$$f(x)=f_0(x_1,\dots,x_n),\quad f_0\in C_b^\infty(\mathbb{R}^n).$$

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The gradient of f along H is defined by the equality $\nabla f = (\partial_{x_n} f)_{n=1}^{\infty}$ and is a mapping with values in H.

Denote by $\|\cdot\|_p$ the standard norm in $L^p(\gamma)$; the same notation is used for the norm in the space $L^p(\gamma, H)$ of measurable mappings v with values in H such that $|v|_H \in L^p(\gamma)$. For $p \in [1, +\infty)$ the Sobolev class $W^{p,1}(\gamma)$ is defined as the completion of $\mathcal{F}C_b^{\infty}$ with respect to the Sobolev norm

$$\|f\|_{p,1} = \|f\|_p + \|\nabla f\|_p.$$

Each function f in $W^{p,1}(\gamma)$ has the Sobolev gradient $\nabla f \in L^p(\gamma, H)$ whose components $\partial_{x_n} f$ satisfy the identity

$$\int \varphi(x) \partial_{x_n} f(x)\, \gamma(dx) = -\int [f(x)\partial_{x_n}\varphi(x) - x_n f(x)\varphi(x)]\, \gamma(dx)$$

for all $\varphi \in \mathcal{F}C_b^{\infty}$. The given definition of the class $W^{p,1}(\gamma)$ is equivalent to the following one (see [2], [4]): $f \in W^{p,1}(\gamma)$ precisely when $f \in L^p(\gamma)$ and, for every fixed n, the function f has a version (i.e., an almost everywhere equal function) $\widetilde{f}$ such that the functions $t \mapsto \widetilde{f}(x + te_n)$, where $x \in X$, are absolutely continuous on the closed and the mapping ∇f whose component with the number n is $\partial_{x_n} \widetilde{f}$, belongs to $L^p(\gamma, H)$; here $\partial_{x_n} \widetilde{f}(x)$ is defined as the derivative at zero of the function $t \mapsto \widetilde{f}(x + te_n)$ (one can show that it exists γ-a.e.). Actually the definition of the Sobolev class does not depend on our choice of an orthonormal basis in the Cameron–Martin space.

In a particular way one introduces the space $BV(\gamma)$ of functions of bounded variation, containing $W^{1,1}(\gamma)$, see [9], [1]. It consists of functions $f \in L^1(\gamma)$ such that $x_n f \in L^1(\gamma)$ for all n and there is an H-valued measure Λf of bounded variation for which the scalar measures $(\Lambda f, e_n)_H$ satisfy the identity

$$\int \varphi(x)\, (\Lambda f, e_n)_H(dx) = -\int [f(x)\partial_{x_n}\varphi(x) - x_n f(x)\varphi(x)]\, \gamma(dx)$$

for all $\varphi \in \mathcal{F}C_b^{\infty}$. If $f \in W^{1,1}(\gamma)$, then $\Lambda f = \nabla f \cdot \gamma$. The space $BV(\gamma)$ is Banach with the norm $\|f\|_{1,1} = \|f\|_1 + \|\Lambda f\|$, where $\|\Lambda f\|$ is the variation of the vector measure Λf defined by $\|\Lambda f\| = \sup \sum_{i=1}^{\infty} |\Lambda f(B_i)|_H$, where sup is taken over all partitions of the space into disjoint Borel parts B_i. Let us note that there is $C > 0$ such that

$$\|\widehat{h} f\|_{L^1(\gamma)} \le C\|f\|_{1,1}, \quad f \in BV(\gamma),$$

where $\widehat{h}(x) = \sum_{n=1}^{\infty} h_n x_n$, $\sum_{n=1}^{\infty} h_n^2 \le 1$, and the series defining $\widehat{h}$ converges in $L^2(\gamma)$.

Suppose now that we are given a Borel or γ-measurable set $V \subset X$ of positive γ-measure such that its intersection with every straight line of the form $x + \mathbb{R}^1 e_n$ is a convex set $V_{x,n}$. If the sets $(V - x) \cap H$ are open in H for all $x \in V$, then V is called H-open (this property is equivalent to the fact that $V - x$ contains a ball from H for every $x \in V$, and is weaker than openness of V in X), and if all such sets are convex, then V is called H-convex. The latter property is weaker than the usual convexity.

There are several natural ways of introducing Sobolev classes on V. The first one is considering the class $W^{p,1}(V, \gamma)$ equal the completion of $\mathcal{F}C_b^{\infty}$ with respect to the Sobolev norm $\|\cdot\|_{p,1,V}$ with the order of integrability p, evaluated with respect to the restriction of γ to V. This class is contained in the class $D^{p,1}(V, \gamma)$ consisting of all functions f on V belonging to $L^p(V, \gamma)$ and having versions of the type indicated above, but with the difference that now the absolute continuity is required only on the closed intervals belonging to the sections $V_{x,n}$. The class $D^{p,1}(V, \gamma)$ is naturally equipped with the Sobolev norm $\|\cdot\|_{p,1,V}$ defined by the restriction of γ to V:

$$\|f\|_{p,1,V} = \Bigl(\int_V |f|^p\,d\gamma\Bigr)^{1/p} + \Bigl(\int_V |\nabla f|^p\,d\gamma\Bigr)^{1/p}.$$

In the paper [10] the class $D^{2,1}(V,\gamma)$ was used (denoted there by $W^{1,2}(V)$). In the finite-dimensional case for convex V both classes coincide, the infinite-dimensional situation is not clear, but for H-convex H-open sets one has $W^{2,1}(V,\gamma) = D^{2,1}(V,\gamma)$, which follows from [10]. It is readily verified that the spaces $W^{p,1}(V,\gamma)$ and $D^{p,1}(V,\gamma)$ with the Sobolev norm are Banach.

Note that analogous objects are defined in the same way also for every centered Radon Gaussian measure γ on a locally convex space X, just for $\mathcal{F}C_b^\infty$ one takes the class of functions of the form $f(x) = f_0(l_1(x),\dots,l_n(x))$, where $f_0 \in C_b^\infty(\mathbb{R}^n)$, the functions l_i are elements of the topological dual X^*. The Cameron–Martin space H is defined exactly in the same way and turns out to be a separable Hilbert space with the norm

$$|h|_H = \sup\{l(h)\colon\ l \in X^*, \|l\|_{L^2(\gamma)} \le 1\}.$$

A measurable linear isomorphism between $\mathbb{R}^\infty$ with the countable power of the standard Gaussian measure and such an abstract space (X,γ) is given by the formula $x \mapsto \sum_{n=1}^\infty x_n e_n$, where $\{e_n\}$ is an orthonormal basis in H.

1. Sobolev functions without extensions

Below we prove the existence of a function in $W^{p,1}(V,\gamma)$ that has no Sobolev extension to the whole space. Note that one can introduce more narrow Sobolev classes on V that admit extensions. For example, in the space $W^{p,1}(V,\gamma)$ one can take the closure $W_0^{p,1}(V,\gamma)$ of the set of functions from $W^{p,1}(\gamma)$ with compact support in V; the functions from $W_0^{p,1}(V,\gamma)$ extended by zero outside V belong to $W^{p,1}(\gamma)$. For certain very simple sets V (say, for half-spaces) it is easy to define explicitly an extension operator (see [2], [4]). It is not clear whether there are essentially infinite-dimensional domains V for which all Sobolev functions have extensions.

Lemma 1.1. *If the set V is such that for some $p > 1$ every function $f \in W^{p,1}(V,\gamma)$ has an extension $g \in W^{p,1}(\gamma)$, then there exists an extension $g_f \in W^{p,1}(\gamma)$ such that $\|g_f\|_{p,1} \le C\|f\|_{p,1,V}$ with some common constant C. If the condition is fulfilled for $p = 1$, then the conclusion is true with $g_f \in BV(\gamma)$.*

Proof. For every $f \in W^{p,1}(V,\gamma)$ we denote by $C(f)$ the infimum of numbers $C \ge 0$ for which f has an extension g with $\|g\|_{p,1} \le C\|f\|_{p,1,V}$. We observe that there exists an extension with $C = C(f)$. Indeed, take an extension g_n for which

$$\|g_n\|_{p,1} \le (C(f) + n^{-1})\|f\|_{p,1,V}.$$

If $p > 1$, then the space $W^{p,1}(\gamma)$ is reflexive, hence one can pass to a subsequence in $\{g_n\}$ with the arithmetic means convergent in $W^{p,1}(\gamma)$, which gives the desired extension.

If $p = 1$, then by the Komlos theorem (see [3]) one can find a subsequence with the arithmetic means convergent almost everywhere. This gives an extension of the class $BV(\gamma)$, since if functions f_j are uniformly bounded in $BV(\gamma)$ and converge a.e. to a function f, then $f \in BV(\gamma)$ (see [9]).

Thus, for every $f \in W^{p,1}(V,\gamma)$ there is the set $E(f)$ of all extensions with the minimal possible norm. It is clear that this set is convex and closed in $W^{p,1}(\gamma)$. If $p > 1$, then it consists of a single point because of the strict convexity of the norm in L^p. By Baire's theorem, there is a natural number M such that the set S_M of functions $f \in W^{p,1}(V,\gamma)$

with $C(f) \leq M$ has a nonempty interior in $W^{p,1}(V,\gamma)$. It is readily seen that increasing M one can find such a ball centered at the origin, whence the assertion of the lemma follows. $\square$

The main negative result of our paper is the following theorem.

Theorem 1.2. *The space $\mathbb{R}^\infty$ contains a convex Borel H-open set K of positive γ-measure with the following property: for every $p \in [1,+\infty)$ there is a function in the class $W^{p,1}(K,\gamma)$ having no extensions to a function of the class $W^{p,1}(\gamma)$. One can also find a convex compact set K with the same property.*

Proof. Let us fix a natural number m and consider the open rhomb K_m in the plane with vertices at the points $a = (m^2, 0)$, $b = (0, m)$, $c = -a$ and $d = -b$. The function $f_m(x) = \max(1-m^2|x-a|, 0)$, where $|z|$ is the usual norm on the plane, is Lipschitzian. Let us estimate its norm in the space $W^{p,1}(K_m,\gamma)$, where γ is the standard Gaussian measure on the plane with density ϱ. To this end we observe that whenever $|x-y| \leq \min(1, 1/|x|)$ the inequality

$$c_1 \leq \varrho(x)/\varrho(y) \leq c_2, \quad c_1 = e^{-1}, c_2 = e^{3/2} \tag{1.1}$$

holds. We have

$$|\nabla f_m(x)| = m^2 \quad \text{if } |x-a| < m^{-2}, \quad |\nabla f_m(x)| = 0 \quad \text{if } |x-a| > m^{-2}.$$

The domain in K_m, where f_m does not vanish, is the sector with vertex a which has the angle with tangent $1/m$. Hence by (1.1) there holds the inequality

$$\|f_m\|_{p,1,K_m} \leq c_3 \varrho(a)^{1/p} m^{2-5/p}, \tag{1.2}$$

where c_3 is some universal constant. Now observe that if a locally Sobolev function g is an extension of f_m to the plane (or in the case $p = 1$ the extension belongs to $BV(\gamma)$), then already for the open disc U with center a and radius $2m^{-2}$ there holds the estimate

$$\|g\|_{1,1,U} \geq c_4 m \|f\|_{1,1,K_m} \geq c_5 \varrho(a) m^{-2}, \tag{1.3}$$

where c_4 and c_5 are some universal constants. Indeed, by inequality (1.1) it suffices to obtain such an estimate for the usual Sobolev norms with respect to Lebesgue measure without Gaussian weight. It is known (see [8, §5.5]) that for every Sobolev function g on U one has the equality

$$\|\nabla g\|_{L^1(U)} = \int_{-\infty}^{+\infty} P(E_t)\, dt,$$

where $E_t = \{x \in U\colon\ g(x) > t\}$, $P(E_t)$ is the perimeter of the set E_t, which for almost all t equals $H_1(U \cap \partial E_t)$, where H_1 is the one-dimensional Hausdorff measure (the length) and ∂E_t is the boundary of the set E_t. An analogous equality is true for the function f on K_m. For a function $g \in BV(U)$ the indicated equality has the form

$$\|Dg\|(U) = \int_{-\infty}^{+\infty} P(E_t)\, dt,$$

where $\|Dg\|(U)$ is the value of the total variation of the vector measure Dg (the generalized gradient of g) on the set U. One can replace g by the function $\min(1, \max(g, 0))$ with values in $[0,1]$, which extends f and whose Sobolev norm does not exceed that of f. Hence we can consider only the values $t \in [0,1]$. Now it suffices to verify that for the sets $S_t = \{x \in K_m\colon\ f(x) > t\}$ we have the estimate

$$H_1(U \cap \partial E_t) \geq c m H_1(K_m \partial S_t)$$

with a universal constant c. The set $K_m \cap \partial S_t$ is an arc of the circle of radius $m^{-2}(1-t)$ centered at a. If the set ∂E_t oversteps the limits of U, then the length of $U \cap \partial E_t$ is not less than m^{-2} and the length of the arc $K_m \cap \partial S_t$ is not greater than cm^{-3}. If the set ∂E_t is entirely contained in U, then its length is also not less than $cmH_1(K_m \cap \partial S_t)$, since E_t contains the whole sector S_t.

From estimates (1.2) and (1.3) we obtain

$$\frac{\|g\|_{1,1,U}}{\|f_m\|_{p,1,K_m}} \geq c_6 \varrho(a)^{1-1/p} m^{5/p-4}$$

with a universal constant c_6. By Hölder's inequality we have

$$\frac{\|g\|_{p,1,U}}{\|g\|_{1,1,U}} \geq c_7 \varrho(a)^{1/p-1} m^{4/q}, \quad q = p/(p-1),$$

where c_7 is also some constant. Hence we finally obtain

$$\frac{\|g\|_{p,1,U}}{\|f_m\|_{p,1,K_m}} \geq cm^{1/p}$$

with some constant $c > 0$.

Now in the infinite-dimensional case we take for K the intersection of the product $\prod_{m=1}^{\infty} K_m$ with the liner subspace L consisting of all elements $x = (x_m)$ such that $\sum_{m=1}^{\infty} m^{-2}|x_m|^2 < \infty$ and having full measure; the equality $\gamma(L) = 1$ follows from the fact that the integral of x_m^2 equals 1. This subspace is Hilbert with respect to the norm $\|x\|_L = \Bigl(\sum_{m=1}^{\infty} m^{-2}|x_m|^2\Bigr)^{1/2}$.

In order to verify that K is H-open it suffices to show that K is open in the Hilbert space L, which is done as follows. Let $x = (x_m) \in K$. Suppose that for every n there is an element $h^n = (h_m^n) \in L$ for which $\|h^n\|_L < 1/n$ and $x + h^n \notin K$. Since $x \in L$, we have $|x_m| \leq m/4$ for all m, starting from some number m_1. Then $x_m + z \in K_m$ if $m \geq m_1$ and $|z| \leq m/4$, since K_m contains the ball of radius m. Hence $x_m + h_m^n \in K_m$ for all m and sufficiently large n, whence $x + h^n \in K$, which is a contradiction. It is clear that $\gamma(K) > 0$, since the Gaussian measures of the sets K_m are rapidly tending to 1.

Every function f_m constructed above on the plane generates a function (denoted by the same symbol) on the space $\mathbb{R}^\infty$, identified with the countable power of the plane, which acts by the formula $f_m(x) = f_m(x_m)$. The measure γ on $\mathbb{R}^\infty$ is also considered as the countable power of the standard Gaussian measure on the plane. By the lemma it suffices to show that $C(f_m) \geq cm^{1/p}$. It remains to observe that if a function $g \in W^{p,1}(\gamma)$ extends f_m, then for almost every fixed $y = (y_j)_{j \neq m}$ the function g_m of the remaining variable x_m gives an extension of the function $f_m(x_m)$ and hence one has the estimate

$$\int |\nabla g|^p \, d\gamma \geq c_0 m \int |\nabla f_m|^p \, d\gamma$$

with some constant c_0, which gives the required assertion. Note that our construction gives no explicit example of a function without extension, but for $p = 2$ one can explicitly indicate a function $f \in W^{2,1}(K, \gamma)$ without extensions to a function in $W^{2,1}(\gamma)$. To this end, it suffices to pick numbers $C_m > 0$ such that the series of the integrals of $C_m^2 |\nabla f_m|^2$ over K converges and the series of the integrals of $m^2 C_m^2 |\nabla f_m|^2$ over K diverges. □

Remark 1.3. Passing to the restriction of the measure γ to the Hilbert space L, we obtain a convex and open in L set K of positive measure, on which for every $p \in [1, +\infty)$ there is a function of the class $W^{p,1}(K, \gamma)$ without restrictions to a function in $W^{p,1}(\gamma)$. It is clear that the same example can be realized also on a larger weighted Hilbert space of sequences, in which K will be precompact. It is possible to combine H-openness of K with its relative compactness. In addition, if we take for γ the classical Wiener measure on $C[0,1]$ or $L^2[0,1]$, then the space L described above will coincide with $L^2[0,1]$, hence our convex set K will be open in the corresponding space.

For an arbitrary centered Radon Gaussian measure γ on a locally convex space X with the infinite-dimensional Cameron–Martin space H, the results proved above yield existence of an H-open convex Borel set V of positive γ-measure and, for every $p \in [1,+\infty)$, a function $f \in W^{p,1}(V,\gamma)$ without extensions to functions in the class $W^{p,1}(\gamma)$. It would be interesting to construct an example of a function in the intersection of all $W^{p,1}(V,\gamma)$ without extensions of the class $W^{1,1}(\gamma)$.

2. Classes BV on infinite-dimensional domains

In the paper [5] classes of functions of bounded variation on infinite-dimensional spaces with non-Gaussian differentiable measures have been considered. Earlier such classes in the case of Gaussian measures were studied in the papers [9], [10], [1]. As in [5], here we consider the two types of functions of bounded variation corresponding to vector measures of bounded variation or bounded semivariation arising as derivatives. However, some other nuances also appear due to boundary effects.

We recall that on a domain U in $\mathbb{R}^d$ with Lebesgue measure the class $BV(U)$ is defined as the set of all functions f integrable in U such that the generalized gradient Df is a vector measure on U of bounded variation. Here we identify almost everywhere equal functions; the class $BV(U)$ turns out to be a Banach space with norm $\|f\|_{BV} = \|f\|_{L^1(U)} + \mathrm{Var}\,(Df)$. On the real line with the standard Gaussian measure $\mu = \varrho dx$, $\varrho(x) = (2\pi)^{-1/2}\exp(-x^2/2)$, the class $BV(\mu)$ consists of all functions $f \in L^1(\mu)$ such that the function $xf(x)$ also belongs to $L^1(\mu)$ and the generalized derivative of the function $f\varrho$ is a bounded Borel measure on the real line. In this case one can introduce a bounded measure Λf on the real line for which the measure $(f\varrho)'$ is the sum of Λf and the measure with density $-xf(x)\varrho(x)$ with respect to Lebesgue measure. In [5], certain natural analogs of these classes for measures on infinite-dimensional spaces have been introduced. The principal feature of the infinite-dimensional case is connected with the fact that here one can consider vector measures of bounded variation as well as more general vector measures of bounded semivariation. Passing to domains we face an additional problem of defining generalized derivatives, since here domains are understood not in the topological sense any more. The latter is explained by that fact that many typical convex sets of positive measure have no inner points.

Let us describe our approach. We consider a Radon probability measure μ on a real locally convex space X with the topological dual X^*. We assume that X contains a continuously and densely embedded separable Hilbert space H. This embedding generates an embedding $X^* \to H$, since every functional $l \in X^*$ defines a vector $j_H(l) \in H$ by the formula $l(k) = (j_H(l), k)_H$, $k \in H$. The norm in H will be denoted by the symbol $|\cdot|_H$. Denote by $\mathcal{FC}^\infty$ the class of functions f on X of the form $f(x) = f_0(l_1(x), \ldots, l_n(x))$, where $f_0 \in C_b^\infty(\mathbb{R}^n)$, $l_i \in X^*$. If $X = \mathbb{R}^\infty$ (the countable product of real lines), then this class is just the union of all $C_b^\infty(\mathbb{R}^n)$, where any function on $\mathbb{R}^n$ is naturally considered as a function on the space $\mathbb{R}^\infty$.

Throughout we consider only Radon measures. A measure μ on X is called differentiable along a vector h in the sense of Skorohod if there exists a measure $d_h\mu$, called the Skorohod derivative of the measure μ along the vector h, such that

$$\lim_{t\to 0}\int_X \frac{f(x-th)-f(x)}{t}\,\mu(dx)=\int_X f(x)d_h\mu(dx)$$

for every bounded continuous function f on X. If the measure $d_h\mu$ is absolutely continuous with respect to the measure μ, then the measure μ is called Fomin differentiable along the vector h, the Radon–Nikodym density of the measure $d_h\mu$ with respect to μ is denoted by β_h^μ and called the logarithmic derivative of μ along h. The Skorohod differentiability of μ along h is equivalent to the identity

$$\int_X \partial_h f(x)\,\mu(dx)=-\int_X f(x)\,d_h\mu(dx),\quad f\in\mathcal{FC}^\infty,$$

where $\partial_h f(x)=\lim\limits_{t\to 0}(f(x+th)-f(x))/t$. On the real line the Fomin differentiability is equivalent to the membership of the density in the Sobolev class $W^{1,1}$ and the Skorohod differentiability is the boundedness of variation of the density. On differentiable measures, see [5].

For example, if μ is a Gaussian measure, then it is Fomin differentiable along all vectors in its Cameron–Martin space $H(\mu)$ (the set of all vectors the shifts along which give equivalent measures).

We recall some concepts related to conditional measures (see [3]). Suppose we are given a measure ν on X and a vector $h\in X$. Let us choose a closed hyperplane $Y\subset X$ for which $X=Y\oplus\mathbb{R}h$, and denote by $|\nu|_Y$ the projection of $|\nu|$ on Y under the natural projecting $\pi\colon X\to Y$. Then one can find measures $\nu^{y,h}$ on the straight lines $y+\mathbb{R}h$, $y\in Y$, called conditional measures, for which the equality $\nu=\nu^{y,h}|\nu|_Y(dy)$ holds in the sense of the identity

$$\nu(B)=\int_Y \nu^{y,h}(B)\,|\nu|_Y(dy)$$

for every Borel set B. In terms of the measure $|\nu|$ itself this can be written as

$$\nu(B)=\int_X \nu^{\pi x,h}(B)\,|\nu|(dx).$$

We can also assume that we are given conditional measures $\nu^{x,h}$ on the straight lines $x+\mathbb{R}h$ such that $\nu^{x,h}=\nu^{\pi x,h}$. If the measure $\nu^{x,h}$ has a density with respect to the natural Lebesgue measure on $x+\mathbb{R}h$ induced by the mapping $t\mapsto x+th$, then this density is called conditional and is denoted by $\varrho_\nu^{x,h}$. It is important to note that in place of the measure $|\nu|_Y$ on Y one can use any nonnegative measure σ with respect to which the measure $|\nu|_Y$ is absolutely continuous. If $|\nu|_Y=g\sigma$, then we obtain the representation $\nu=\nu^{y,h,\sigma}\,\sigma(dy)$, where $\nu^{y,h,\sigma}:=g(y)\nu^{y,h}$. Such a representation is called a disintegration of the measure. This may be convenient for a simultaneous disintegration of several measures. As above, if σ is given on X and $\nu\ll\sigma$, then we can write $\nu=\nu^{x,h,\sigma}\,\sigma(dx)$.

For a Gaussian measure, the conditional measures are Gaussian as well.

It is known (see [4]) that if μ is Skorohod or Fomin differentiable along h, then there exist conditional measures $\mu^{y,h}$ differentiable along h in the same sense and $d_h\mu=d_h\mu^{y,h}|\mu|_Y(dy)$; if in place of $|\mu|_Y$ we use a measure $\sigma\geq 0$ on Y with $|\mu|_Y\ll\sigma$, then we obtain $d_h\mu=d_h\mu^{y,h,\sigma}\,\sigma(dy)$. It follows from this that the projection of the measure $|d_h\mu|$ on Y is absolutely continuous with respect to the projection of the measure μ, although the measure $d_h\mu$ itself can be singular with respect to μ (it is easy to construct examples on the plane).

We assume throughout that we are given a probability measure μ Fomin differentiable along all vectors in H and that for every fixed $h \in H$ the continuous versions of the conditional densities on the straight lines $x + \mathbb{R}h$ are positive. Below these densities are denoted by $\varrho^{x,h}$ without indicating μ.

We recall the definitions of the variation and semivariation of vector measures (see [7]). A vector measure with values in H is an H-valued countably additive function η defined on a σ-algebra $\mathcal{A}$ of subsets of a space Ω. Such a measure automatically has finite semivariation defined by the formula

$$V(\eta) := \sup \Bigl|\sum_{i=1}^{n} \alpha_i \eta(X_i)\Bigr|_H,$$

where sup is taken over all finite partitions of Ω into disjoint parts $\Omega_i \in \mathcal{A}$ and all finite collections of real numbers α_i with $|\alpha_i| \le 1$. In other words, this is the supremum of the variations of the scalar measures $(\eta, h)_H$ over $h \in H$ with $|h|_H \le 1$. However, this does not mean that the variation of the vector measure η is finite, which is defined as

$$\mathrm{Var}(\eta) := \sup \sum_{i=1}^{n} |\eta(X_i)|_H,$$

where sup is taken over all finite partitions of Ω into disjoint parts $\Omega_i \in \mathcal{A}$. The variation of a measure η will be denoted by $\|\eta\|$ (but in [8] this notation is used for the semivariation). It is easy to give an example of a measure with values in an infinite-dimensional Hilbert space having bounded semivariation and infinite variation.

The class $SBV(\mu)$ introduced in [5] consists of all functions $f \in L^1(\mu)$ for which

$$\sup_{|h|\le 1} |f\beta_h|_{L^1(\mu)} < \infty$$

and there an exists an H-valued measure Λf of bounded semivariation such that the Skorohod derivative $d_h(f\mu)$ exists and equals $(\Lambda f, h)_H + f\beta_h\mu$. The subclass $BV(\mu) \subset SBV(\mu)$ consists of all functions f for which Λf has bounded variation.

For example, if f belongs to the Sobolev class $W^{1,1}(\mu)$, where μ is a Gaussian measure, then $f \in BV(\mu)$; here for the measure Λf we can take the measure with the vector density $D_H f$ with respect to μ. Even for a Gaussian measure μ (with an infinite-dimensional support) the class $SBV(\mu)$ is strictly larger than $BV(\mu)$.

It is important to note that the measure $(\Lambda f, h)_H$ can be singular with respect to μ (say, have atoms in the one-dimensional case), but it also admits a disintegration

$$(\Lambda f, h)_H = (\Lambda f, h)_H^{y,h,\mu_Y}\, \mu_Y(dy)$$

with some measures $(\Lambda f, h)_H^{y,h,\mu_Y}$ on the straight lines $y + \mathbb{R}h$, where $y \in Y$ and Y is a closed hyperplane complementing $\mathbb{R}h$. Indeed, we have

$$(\Lambda f, h)_H = d_h(f\mu) - f\beta_h\mu,$$

where the projections $|d_h(f\mu)|$ and $|f\beta_h|\mu$ on Y are absolutely continuous with respect to μ, since, as noted above, the projection of the measure $|d_h(f\mu)|$ is absolutely continuous with respect to the projection of $|f|\mu$.

For an interval J on the real line let $BV_{loc}(J)$ denote the class of all functions on J having bounded variation on every compact interval in J.

Lemma 2.1. *A function $f \in L^1(\mu)$ belongs to $SBV(\mu)$ precisely when there is an H-valued measure Λ of bounded semivariation such that for every $h \in H$ for μ-almost every*

x the function $t \mapsto f(x+th)$ belongs to $BV_{loc}(\mathbb{R})$ and its generalized derivative is

$$(\Lambda, h)_H^{x,h,\mu}/\varrho^{x,h}(t) + f(x+th)\partial_t\varrho^{x,h}(t)/\varrho^{x,h}(t).$$

A similar assertion is true for $BV(\mu)$, where the measure Λ must have bounded variation.

Proof. If the aforementioned condition is fulfilled, then the measure $(\Lambda f, h)_H + f\beta_h\mu$ serves as the Skorohod derivative of the measure $d_h(f\mu)$. With the help of conditional measures this is deduced from the fact that if on the real line a function f has a locally bounded variation and a function g is locally absolutely continuous, then $(fg)' = gf' + fg'$ in the sense of generalized functions. Conversely, if $f \in SBV(\mu)$, then almost all conditional measures for $f\mu$ on the straight lines $y + \mathbb{R}h$, where $y \in Y$, from the representation $f\mu = f\mu^{y,h}\,\mu_Y(dy)$ are Skorohod differentiable. Hence almost all functions $t \mapsto f(y+th)\varrho^{y,h}(t)$ have bounded variation, whence by the positivity of the conditional densities for μ we obtain that the functions $t \mapsto f(y+th)$ have locally bounded variation. In addition, the generalized derivative ψ of such a function satisfies the equality

$$\partial_t(f(y+th)\varrho^{y,h}(t)) = \varrho^{y,h}(t)\psi + f(y+th)\partial_t\varrho^{y,h}(t),$$

whence the required relationship for the derivative follows by the equality (in the sense of generalized functions)

$$\partial_t(f(y+th)\varrho^{y,h}(t)) = (\Lambda, h)_H^{y,h,\mu_Y} + f(y+th)\partial_t\varrho^{y,h}(t),$$

which, in turn, follows by the equality $d_h(f\mu) = (\Lambda f, h)_H + f\beta_h$ for the conditional measures taken with respect to the measure μ_Y on Y; here it is important to use a common measure on Y. The case of BV is similar. $\square$

Now let $U \subset X$ be a Borel set that is H-convex and H-open, that is, all sets $(U-x)\cap H$ are convex and open in H. For example, this can be a set that is convex and open in X. However, the ellipsoid $U = \left\{x\colon\ \sum_{n=1}^{\infty} n^{-2}x_n < 1\right\}$ is not open in $\mathbb{R}^\infty$, but is H-open.

For an H-convex and H-open set U the intersections

$$U_{x,h} := U \cap (x + \mathbb{R}h),$$

are open intervals on the straight lines $x + \mathbb{R}h$.

The symbol $L^p(U,\mu)$ will denote the space of equivalence classes of all μ-measurable functions f on U for which the functions $|f|^p$ are integrable with respect to the measure μ on U. Let $M(U,\mu)$ denote the class of all functions $f \in L^1(\mu, U)$ such that

$$\|f\|_M := \|f\|_{L^1(U,\mu)} + \sup_{|h|\le 1}\int_U |f(x)|\,|\beta_h(x)|\,\mu(dx) < \infty.$$

Definition 2.2. *We shall say that $f \in M(U,\mu)$ belongs to the class $SBV(U,\mu)$ if the function $t \mapsto f(x+th)\varrho^{x,h}(t)$ belongs to the class $BV_{loc}(U_{x,h})$ for every fixed $h \in H$ for almost all x and there exists an H-valued measure $\Lambda_U f$ on U of bounded semivariation such that, for every $h \in H$, the measure $(\Lambda_U f, h)_H$ admits the representation*

$$(\Lambda_U f, h)_H = (\Lambda_U f, h)_H^{x,h,\mu}\,\mu(dx),$$

where the measures $(\Lambda_U f, h)_H^{x,h,\mu}$ on the straight lines $x + \mathbb{R}h$ possess the property that

$$(\Lambda_U f, h)_H^{x,h} + f(x+th)\partial_t\varrho^{x,h}(t)$$

is the generalized derivative of the function $t \mapsto f(x+th)\varrho^{x,h}(t)$ on $U_{x,h}$.

The class $BV(U,\mu)$ consists of all $f \in SBV(U,\mu)$ such that the measure $\Lambda_U f$ has bounded variation.

An equivalent description of functions in $SBV(U,\mu)$ can be given in the form of integration by parts if in place of the class $\mathcal{FC}^\infty$ we use appropriate classes of test functions for every $h\in H$.

For any fixed $h\in H$ we choose a closed hyperplane Y complementing $\mathbb{R}h$ and consider the class $\mathcal{D}_h$ of all bounded functions φ on X with the following properties: φ is measurable with respect to all Borel measures, for each $y\in Y$ the function $t\mapsto\varphi(y+th)$ is infinitely differentiable and has compact support in the interval $J_{y,h}:=\{t\colon\ y+th\in U\}$, and the functions $\partial_h^n\varphi$ are bounded for all $n\ge 1$. Here $\partial_h^n\varphi(y+th)$ is the derivative of order n at the point t for the function $t\mapsto\varphi(y+th)$.

Note that $\psi\varphi\in\mathcal{D}_h$ for all $\varphi\in\mathcal{D}_h$ and $\psi\in\mathcal{FC}^\infty$.

Lemma 2.3. *A function $f\in M(U,\mu)$ belongs to $SBV(U,\mu)$ precisely when there exists an H-valued measure $\Lambda_U f$ on U of bounded semivariation such that, for every $h\in H$ and all $\varphi\in\mathcal{D}_h$, one has the equality*

$$\int_X \partial_h\varphi(x)f(x)\,\mu(dx)=-\int_X\varphi(x)\,(\Lambda_U f,h)_H(dx)-\int_X\varphi(x)f(x)\beta_h(x)\,\mu(dx).$$

A similar assertion is true for the class $BV(U,\mu)$.

Proof. If $f\in SBV(U,\mu)$, then the indicated equality follows from the definition and the integration by parts formula for conditional measures. Let us prove the converse assertion. Let us fix $k\in\mathbb{N}$. It is readily verified that the set Y_k of all points $y\in Y$ such that the length of the interval $J_{y,h}$ is not less than $8/k$ is measurable with respect to every Borel measure. In addition, it is not difficult to show that there exists a function $g_k\in\mathcal{D}_h$ measurable with respect to every Borel measure and possessing the following properties: $0\le g_k\le 1$, $g_k(y)=0$ if the length of $J_{y,h}$ is less than $8/k$, $g_k(y+th)=0$ if $t\notin J_{y,h}$ or if $t\in J_{y,h}$ and the distance from t to an endpoint of $J_{y,h}$ is less than $1/k$, $g_k(y+th)=1$ if $t\in J_{y,h}$ and the distance from t to an endpoint of $J_{y,h}$ is not less than $2/k$. It follows from our hypothesis that for all $\psi\in\mathcal{FC}^\infty$ we have the equality

$$\int_X\partial_h\psi(x)g_k(x)f(x)\,\mu(dx)=$$

$$=-\int_X\psi(x)g_k(x)\,(\Lambda_U f,h)_H(dx)-\int_X\psi(x)g_k(x)f(x)\beta_h(x)\,\mu(dx)-$$

$$-\int_X\psi(x)\partial_h g_k(x)f(x)\,\mu(dx).$$

Therefore, the measure $fg_k\mu$ is Skorohod differentiable and

$$d_h(fg_k\mu)=g_k(\Lambda_U f,h)_H+fg_k\beta_h\mu+\partial_h g_k f\mu.$$

Using the disintegration for $fg_k\mu$ and letting $k\to\infty$, we obtain the disintegration for $f\mu$ required by the definition. □

Theorem 2.4. *The set $SBV(U,\mu)$ is a Banach space with the norm*

$$\|f\|_{SBV}:=\|f\|_M+V(\Lambda_U f).$$

The set $BV(U,\mu)$ is a Banach space with the norm

$$\|f\|_{BV}:=\|f\|_M+\mathrm{Var}(\Lambda_U f).$$

Proof. Similarly to [5] we obtain that the space $M(U,\mu)$ is complete. Hence every Cauchy sequence $\{f_n\}$ in SBV converges in the M-norm to some function $f \in L^1(\mu, U)$. The sequence of measures $\Lambda_U f_n$ is Cauchy in semivariation, hence converges in the norm V to some measure ν of bounded semivariation. Applying Lemma 2 it is easy to show that $f \in SBV(U)$ and $\nu = \Lambda_U f$ is the corresponding H-valued measure. Since

$$\|f_n - f\|_M + V(\Lambda f_n - \Lambda f) \to 0,$$

it follows that f is a limit of $\{f_n\}$ in the norm of the space SBV. The proof of completeness of the space $BV(U,\mu)$ is similar. □

Theorem 2.5. *Suppose that $I_U \in SBV(\mu)$. Then for every function*

$$f \in SBV(U,\mu) \cap L^\infty(U,\mu)$$

its extension by zero outside of U gives a function in the class $SBV(\mu)$. Conversely, the restriction of every bounded function in $SBV(\mu)$ to U gives a function in $SBV(U,\mu)$. If $I_U \in BV(\mu)$, then analogous assertions are true for the class BV.

Proof. Let $f \in SBV(U,\mu) \cap L^\infty(U,\mu)$. We may assume that $|f| \le 1$. Let us fix $h \in H$. Then we can find a version of f whose restrictions to the straight lines $x+\mathbb{R}h$ have locally bounded variation. Let a_x be an endpoint of the interval $U_{x,h}$ (if it exists). Then the considered version of f has a limit at a_x (left or right, respectively), bounded by 1 in the absolute value. Defined by zero outside of U, the function f remains a function of locally bounded variation on all these straight lines, but at the endpoints a_x its generalized derivative may gain Dirac measures with coefficients bounded by 1 in the absolute value. However, such Dirac measures (with the coefficient 1 at the left end and the coefficient -1 at the right end) are already present at the derivative of the restriction of I_U. Thus, after adding these arising point measures to $(\Lambda_U f, h)_H^{x,h,\mu}$ we add to the measure $(\Lambda_U f, h)_H$ some measure with a semivariation not exceeding $\|(\Lambda_U f, h)_H\|$. Therefore, Lemma 1 gives the inclusion of the extension to $SBV(\mu)$. The converse assertion is proved similarly. In the case of $BV(\mu)$ one has also to use the fact that any H-valued measure of bounded variation is given by a Bochner integrable vector density with respect to a suitable scalar measure. □

Theorem 2.6. *The class $SBV(U,\mu)$ has the following property: if a sequence of functions f_n is norm bounded in it and converges almost everywhere to a function f, then f belongs to the same class, and the norm of f does not exceed the precise upper bound of the norms of the functions f_n.*

A proof can be obtained with the help of the same reasoning as in Lemma 2.3 and an analogous assertion for functions on the whole space proved in [5].

The simplest model example of the objects introduced is the standard Gaussian measure μ on the countable power of the real line $\mathbb{R}^\infty$ equal to the countable power of the standard Gaussian measure on the real line. Its Cameron–Martin space is the classical Hilbert space $H = l^2$. For every $h = (h_n) \in H$ the function β_h is the measurable linear functional $-\sum_{n=1}^\infty h_n x_n$, where the series converges almost everywhere and in $L^2(\mu)$. For U one can take any Borel convex set of positive measure for which the intersections $(U-x)\cap H$ are open in H. Note that even for a bounded convex set U the indicator function I_U does not always belong to $BV(\mu)$; explicit examples are given in [10]. On the other hand, the indicator of any open convex set in the Gaussian case belongs to $BV(\mu)$ (see [6]).

This work has been supported by the RFBR projects 13-01-00332, 11-01-90421-Ukr-f-a, 11-01-12104-ofi-m, and the SFB 701 at the university of Bielefeld.

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