NON-AUTONOMOUS MILD SOLUTIONS FOR SPDE WITH LEVY NOISE IN WEIGHTED L^p-SPACES

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ABSTRACT. In this paper we prove a comparison theorem for the stochastic differential equation

$$dX(t) = \quad (A(t)X(t) + F(t, X(t)))\,dt + \mathcal{M}_{\Sigma(t,X(t))}\,dW(t) + \int_{L^2} \mathcal{M}_{\Gamma(t,X(t))}x\,\tilde{N}(dt, dx), \quad t \in [0, T],$$

driven by a Wiener noise W and a Poisson noise $\tilde{N}$. The diffusion coefficients $\mathcal{M}_\Sigma$ and $\mathcal{M}_\Gamma$ are given by multiplication operators. The equation is considered in L^p- spaces with a finite weight measure μ_ρ over a (possibly unbounded) domain $\Theta \subset \mathbb{R}^d$. With the help of this comparison theorem we prove an existence result for the above equation in the case of non-Lipschitz drift F being a Nemytski-type operator of (at most) polynomial growth.

1. Introduction

In recent years there has been large interest in SDEs with general, not necessarily continuous, semimartingales as driving noises. This is reflected in a growing number of papers going beyond the well-known framework of SDEs with Wiener noise, e.g. by considering compensated Poisson random measures or Lévy processes as noise. Stochastic evolution equations in infinite dimensions are often used to describe complex models in natural sciences. Numerous examples of SDEs with Wiener noise in infinite dimensions can be found e.g. in the introductory chapter of the monograph by DaPrato and Zabczyk [11].

SDEs with compensated Poisson random measures or Lévy processes as driving random forces are candidates to model situations, where the system does not develop in a time-continuous way. The theory of SDEs with jumps in infinite-dimensional spaces plays a role in modelling critical phenomena. Among areas of application let us mention neurophysiology, environmental pollution and mathematical finance.

Consider $\mathbb{R}^d$, $d \in \mathbb{N}$, with Euclidean norm $|\cdot|$, Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ and Lebesgue measure dθ. Let us fix a (possibly unbounded) $\Theta \in \mathcal{B}(\mathbb{R}^d)$ and $\rho = 0$ (in the case of bounded Θ) resp. $\rho > d$ (for unbounded Θ). Denote by μ_ρ a finite measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ given by

$$\mu_\rho(d\theta) := (1 + |\theta|^2)^{-\frac{\rho}{2}}\,d\theta. \tag{1.1}$$

Given some $\nu \geq 1$, let $L^{2\nu}_\rho := L^{2\nu}_\rho(\Theta)$ be the Banach space of Borel-measurable, 2ν-integrable functions w.r.t. the measure μ_ρ on Θ. Such weighted spaces are of

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common use in the theory of (deterministic and stochastic) parabolic differential equations, see e.g. [12], [22] resp. [26].

In this paper, given some fixed $T > 0$, we study the following SDE in $L^{2\nu}_\rho$

$$(\mathbf{Eq.1}) \qquad \begin{aligned} dX(t) =& (A(t)X(t) + F(t, X(t)))dt + \mathcal{M}_{\Sigma(t,X(t))}dW(t) \\ &+ \int_{L2} \mathcal{M}_{\Gamma(t,\cdot,X(t))}x\,\tilde{N}(dt,dx), \quad t \in [\,0,T\,], \\ X(0) =& \xi. \end{aligned}$$

We will look for solutions to Eq. (1) in the Banach space $\mathcal{G}_\nu(T)$ of all predictable processes $[\,0,T\,] \ni t \longmapsto X(t) \in L^{2\nu}_\rho$ such that

$$||X||_{\mathcal{G}_\nu(T)} := \sup_{t\in[\,0,T\,]} \left(\mathbb{E}||X(t)||^{2\nu}_{L^{2\nu}_\rho}\right)^{\frac{1}{2\nu}} < \infty.$$

The precise setting will be given in Section 2 below. Here, we only point out, what kind of coefficients and noises are present in Eq.(1). For simplicity, we we shall supress explicit dependence on $\omega \in \Omega$ of all random elements, if no confusion can arise.

Everywhere below, we assume that:

- the family $(A(t))_{t\in[\,0,T\,]}$ generates a *strong evolution operator* $U = (U(t,s))_{0\le s\le t\le T}$ in L^2_ρ;
- F, Σ and Γ are time-dependent, random *Nemitskii-type nonlinear operators* defined through *predictable* functions f, σ, γ: $[\,0,T\,] \times \Omega \times \mathbb{R} \to \mathbb{R}$;
- $\mathcal{M}_\Sigma$ and $\mathcal{M}_\Gamma$ are the *multiplication operators* corresponding to Σ and Γ;
- $(W(t))_{t\in[\,0,T\,]}$ is a Q-Wiener process in L^2 with a trace class correlation operator $Q \ge 0$;
- $\tilde{N}$: $[\,0,T\,] \times \Omega \times L^2 \to \mathbb{R}$ is a compensated Poisson random measure with a *Lévy intensity measure* η on L^2.

All necessary technical assumptions on the growth of coefficients and regularity properties of noises will be given in Section 2. Note that by the Lévy-Itô decomposition it will be possible to have similar results also for stochastic evolution equations with Lèvy noise, see Remark 2.11 below.

The solutions to Eq.(1) will be understood in the mild sense (see Definition 2.6). In the particular case $\Gamma = 0$ such type of infinite dimensional equation was considered in [26]. Compared to [26], Eq.(1) has an additional *multiplicative* (i.e. solution-dependent) jump noise, which needs a careful analysis.

In this paper, we will extend the comparison method of [26] to the case of Poisson noise. Such extension is non-trivial, since now we are dealing with diffusion processes with jumps. To control the effect caused by a Poisson noise, we need additional assumptions (as compared to the Wiener case) on the jump diffusion terms (e.g., their monotonicity in **(Γ)** below).

Thus, in Section 2 (see Definition 2.6 there) we formulate a solution term, which differs from the one applied in [26] (cf. Definition 2.7, p. 55 there) in a more restrictive integrability condition and a weaker pathwise property (càdlàg paths vs. pathwise continuity).

Note that, given a strong evolution operator $(U(t,s))_{0\le s\le t\le T}$ generated by $(A(t))_{t\in[0,T]}$, we have to ensure the well-definedness of the so-called *Poisson stochastic convolution*

$$I_\Gamma^{\tilde N}(t) := \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\Gamma(s,X(s))}x\,\tilde N(ds,dx), \quad t\in[0,T], \tag{1.2}$$

in $L^{2\nu}_\rho$. This is done in Section 3 below. In particular, we prove a norm estimate in $L^{2\nu}_\rho$ (see (3.9) in Theorem 3.1).

The main results of this paper are:

- an existence and uniqueness result for Eq.(1) in the case of Lipschitz coefficients;
- a comparison theorem for solutions $X^{(i)}$, $i=1,2$, with initial conditions $\xi^{(i)}$ and drifts $F^{(i)}$, $i=1,2$, in the case of Lipschitz coefficients;
- an existence result for Eq.(1) in the case of polynomially growing drift F and the other coefficients being Lipschitzian.

The most delicate step will be to prove the comparison result, since, to the best of our knowledge, there is no infinite-dimensional comparison theorem for SDEs with jump noise applicable so far. In this paper, we solve this problem by applying related comparison results for SDEs with jumps which are known in finite dimensions, see e.g. [15], [34],[35] (Chapter 10 there)), [24], [30], [39]).

In order to apply finite-dimensional comparison results to prove an infinite-dimensional comparison result in the Lipschitz case, we have to define a family of finite-dimensional equations, whose solutions approximate the solution to Eq.(1). The key step is to consider the approximating equations in appropriate *Sobolev spaces*. The idea behind is that we want to apply the so-called *Sobolev embedding theorem*, which particularly implies the necessity to consider domains $\Theta\subset\mathbb{R}^d$ obeying the *weak cone property*.

In this paper, for given $m\ge 1$ and $p\ge 1$, we denote by $W^{m,p} := W^{m,p}(\Theta)$ the Sobolev space consisting of (equivalence classes of) functions $u\in L^p(\Theta) := L^p$, whose distributional derivatives $D^\alpha u$ of orders $|\alpha|\le m$ also belong to L^p and by $||\cdot||_{W^{m,p}}$ its norm given by

$$||u||_{W^{m,p}} = \left(\sum_{|\alpha|\le m} ||D^\alpha u||^p_{L^p}\right)^{\frac{1}{p}}.$$

Furthermore, fixing $p=2$ and $T>0$, we denote by $\mathcal{W}^2_m(T)$ the space of predictable processes $(Z(t))_{t\in[0,T]}$ s.t.

$$\sup_{t\in[0,T]} \mathbf{E}||Z(t)||^2_{W^{m,2}} < \infty$$

with norm $||\cdot||_{\mathcal{G}_\nu(T)}$ given by

$$||Z||_{\mathcal{G}_\nu(T)} := \sup_{t\in[0,T]} (\mathbf{E}||Z(t)||^{2\nu}_{L^{2\nu}_\rho})^{\frac{1}{2\nu}}.$$

By the by Sobolev embedding theorem (see e.g. [1]) the approximating equations are embedded into the space $C_b := C_b(\Theta)$ of continuous bounded functions u on Θ

equipped with norm $||\cdot||_{C_b}$ given by

$$||u||_{C_b} := \sup_{\theta\in\Theta} |u(\theta)| < \infty.$$

This gives us the possibility to evaluate the approximating equations pointwise at any $\theta \in \Theta$ and to apply one-dimensional comparison results for the evaluated equations.

The structure of the paper is as follows: In Section 2, we present the exact setting, the main results of this paper and some examples of SDEs, which are covered by this theory. Section 3 is devoted to the well-definedness of the Poisson stochastic convolution (1.2) in weighted L^2-spaces. In Section 4 we prove our main results. Section 5 is devoted to some illustrative examples.

2. Setting and main results

As usual, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a *right continuous* filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. In what follows, we fix a finite time horisont $T > 0$. Let $\mathcal{P}_T$ denote the σ-field of all *predictable* subsets of $[0, T] \times \Omega$.

2.1. **Assumptions on coefficients and noises.** For the nonlinear coefficients in Eq.(1) we suppose that F, Σ and Γ are *Nemytskii*-type (i.e., superposition) operators defined for *predictable*, i.e., $\mathcal{P}_T \times \mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable, functions

$$f,\ \sigma,\ \gamma\ : [0, T\,] \times \Omega \times \mathbb{R} \to \mathbb{R}$$

by setting, for any $\varphi : \mathbb{R} \to \mathbb{R}$,

$$(\mathbf{NEM}) \qquad F(t, \omega, \varphi)(\theta) := f(t, \omega, \varphi(\theta)), \quad \Sigma(t, \omega, \varphi)(\theta) := \sigma(t, \omega, \varphi(\theta)),$$
$$\Gamma(t, \omega, \varphi)(\theta) := \gamma(t, \omega, \varphi(\theta)), \quad \theta \in \mathbb{R}.$$

Multiplication operators corresponding to Σ and Γ will be denoted by $\mathcal{M}_\Sigma$ and $\mathcal{M}_\Gamma$, respectively.

Below we specify conditions on the generating functions f, σ and γ.

- (**F**) The mapping $f(t, \omega, \cdot) : \mathbb{R} \to \mathbb{R}$ is *continuous*, *one-sided linearly bounded* and *of at most polynomial growth*, uniformly with respect to $(t, \omega) \in [\,0, T\,] \times \Omega$. More precisely, there exist some $\nu \geq 1$ and $c_f(T) > 0$ such that
  - **(i)** $|f(t, \omega, y)| \leq c_f(T)(1 + |y|^\nu),\ \ y \in \mathbb{R}$;
  - **(ii)** $f(t, \omega, y) \leq \ \ c_f(T)(1 + y)$ if $y \geq 0$,
    $f(t, \omega, y) \geq -c_f(T)(1 - y)$ if $y \leq 0$.
- (**Σ**) The mapping $\sigma(t, \omega, \cdot) : \mathbb{R} \to \mathbb{R}$ is *Lipschitz continuous,* uniformly in the other variables: there exists some $c_\sigma(T) > 0$ such that
  - **(i)** $|\sigma(t, \omega, y_1) - \sigma(t, \omega, y_2)| \leq c_\sigma(T)|y_1 - y_2|,\ \ y_1, y_2 \in \mathbb{R}$;
  - **(ii)** $|\sigma(t, \omega, 0)| \leq c_\sigma(T)$.
- (**Γ**) The mapping $\gamma(t, \omega, \cdot) : \mathbb{R} \to \mathbb{R}$ is *Lipschitz continuous*, *bounded* and *monotonically increasing*, uniformly with respect to $(t, \omega) \in [\,0, T\,] \times \Omega$. There exists some $c_\gamma(T) > 0$ such that
  - **(i)** $|\gamma(t, \omega, y_1) - \gamma(t, \omega, y_2)| \leq c_\gamma(T)|y_1 - y_2|,\ \ y_1, y_2 \in \mathbb{R}$;
  - **(ii)** $|\gamma(t, \omega, y)| \leq c_\gamma(T),\ \ y \in \mathbb{R}$;
  - **(iii)** $\gamma(t, \omega, y_1) \leq \gamma(t, \omega, y_2)$ whenever $y_1 \leq y_2$.

Remark 2.1. **(i)** It is easy to see that (**F**) **(ii)** is equivalent (up to a positive constant $c_f(T)$) to claiming that

$$f(t,\omega,y)\cdot y \le c_f(T)(1+y^2), \quad y\in\mathbb{R} \tag{2.1}$$

In particular, the class of functions with one-sided linear growth includes all *quasi-dissipative* functions, i.e., those obeying

$$\left(f(t,\omega,y_1)-f(t,\omega,y_2)\right)(y_1-y_2) \le c_f(T)\left(y_1-y_2\right)^2, \quad y_1,y_2\in\mathbb{R} \tag{2.2}$$

A typical example of drifts f fulfilling the above conditions with $\nu := 2n+1$ is given by polynomials of the form

$$f(t,\omega,y) = \sum_{k=0}^{2n+1} b_k(t,\omega)y^k \tag{2.3}$$

with $\mathcal{P}_T$-measurable and bounded coefficients $b_{2n+1}(t,\omega) < 0$, $b_k(t,\omega)\in\mathbb{R}$, $0\le k\le 2n$, and $n\in\mathbb{N}\cup\{0\}$.
(ii) Condition (**Γ**) (**ii**) is imposed to ensure existence of càdlàg versions of the solution process to Eq. (1.1) (see Step 3 in the proof of Theorem 3.1 below), whereas (**Γ**) (**iii**) will play a crucial role in proving our main comparison result (cf. Theorem 2.8 and Section 4 below).
(iii) For the functions f and σ generating the drift and diffusion coefficients, it would suffices to assume only *progressive measurability.*

For the whole paper, let Θ be an open (possibly unbounded) subset of the Euclidean space $\left(\mathbb{R}^d, |\cdot|\right)$, $d\ge 1$. Furthermore, we assume that Θ obeys the *weak cone property*, i.e., there exists $\delta>0$ such that for each $\theta\in\Theta$ the Lebesgue measure of the set $\mathcal{K}_\theta$ is smaller than δ. Here, $\mathcal{K}_\theta := \{\xi\in B_1(\theta) \mid [\theta,\xi]\subset\Theta\}\in\mathcal{B}(\Theta)$ denotes the set of all points ξ in the open unit ball around θ such that the line segment $[\theta,\xi]$ joining θ to ξ lies entirely in Θ. This property is fulfilled by a wide class of domains in $\mathbb{R}^d$ including those with local Lipschitz or C^k-regular boundaries, in particular by any open ball or cube in $\mathbb{R}^d$ as well as by $\mathbb{R}^d$ itself (see e.g. Section 1 in [2] or Appendix A in [29]).

Remark 2.2. The weak cone property allows a well-developed theory of Sobolev spaces on the underlying domain Θ (see e.g. [**?**]). By Sobolev's embedding theorem we have a dence continuous embedding $W^{m,2}(\Theta)\underset{\to}{\subseteq} C_b(\Theta)$ for *any* $m>d/2$. *This* fact will be used in the proof of the comparison Theorem 2.2.3 in order to evaluate approximating equations for Eq. (1.1) pointwise at any $\theta\in\Theta$.

Remark 2.3. The $\mathcal{P}_T\times\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurability of the generating function $f:[0,T\,]\times\Omega\times\mathbb{R}\to\mathbb{R}$ and its continuity in the last variable allow for applying the well-known Krasnoselskii's theorem for the corresponding Nemytskii operator F (cf. e.g. [16], Section 3.4). So, if f satisfies the polynomial bound (**F**)(**ii**) for some $\nu\ge 1$ then the mapping $F:[0,T\,]\times\Omega\times L^{2\nu}_\rho\to L^2_\rho$ is well defined, $\mathcal{P}_T\times\mathcal{B}(L^{2\nu}_\rho)/\mathcal{B}(L^2_\rho)$ measurable and continuous in the last variable. Furthermore, one can standardly check by the monotone class theorem that, for any predictable $X:[0,T]\to L^{2\nu}_\rho$, the process $[0,T]\ni t\to F(t,X(t))\in L^2_\rho$ is again predictable.

Concerning the linear drift coefficients, we assume that $A := (A(t))_{t\in[0,T]}$ is the generator of a *strong evolution family* $U := (U(t,s))_{0\le s\le t\le T}$. More precisely, $U:\Delta(T)\to\mathcal{L}(L^2_\rho)$ is an operator-valued function defined on

$$\Delta(T) := \{(t,s)\mid t\ge s\}\subset[0,T]\times[0,T]$$

and such that

$$U(t,r)U(r,s) = U(t,s), \quad U(t,t) = \mathbf{I}, \quad 0 \le s \le r \le t \le T, \tag{2.4}$$

where $\mathbf{I}$ stands for the identity operator in L^2_ρ. Furthermore, the operator family U is *strongly continuous*, i.e., the mapping

$$U(\cdot,\cdot)\varphi : \Delta(T) \to L^2_\rho \text{ is continuous for any } \varphi \in L^2_\rho, \tag{2.5}$$

and hence U is *uniformly bounded*, i.e.,

$$||U||_{\mathcal{L}(L^2_\rho)} := \sup_{0\le s\le t\le T} ||U(t,s)||_{\mathcal{L}(L^2_\rho)} < \infty. \tag{2.6}$$

It is supposed that each $A(t) : \mathcal{D}(A(t)) \to L^2_\rho$ is a linear (possibly unbounded) operator closely defined on $\mathcal{D}(A(t)) \subset L^2_\rho$ and that their common domain

$$\mathcal{D}(A) := \bigcap_{t\in[\,0,T\,]} \mathcal{D}(A(t))$$

is dense in L^2_ρ. We say that $(A, \mathcal{D}(A))$ is a generator for U if for any $\varphi \in \mathcal{D}(A)$ and $t > s$

$$\frac{\partial}{\partial t}U(t,s)\varphi = A(t)U(t,s)\varphi, \quad \frac{\partial}{\partial s}U(t,s)\varphi = -U(t,s)A(s)\varphi. \tag{2.7}$$

For a general theory of evolution operators in Banach spaces see, e.g., [38] or [**?**].

More specific assumptions on $(A(t))_{t\in[0,T\,]}$ and on the associated evolution family U are as follows:

(**A1**) U is *positivity preserving*, i.e, $L^2_\rho \ni \varphi \ge 0$ implies $U(t,s)\varphi \ge 0$ for any $0 \le s \le t \le T$.

(**A2**) U is *pseudo contractive in* L^2_ρ, i.e., there exists a constant $\beta \ge 0$ such that

$$||U(t,s)||_{\mathcal{L}(L^2_\rho)} \le e^{\beta(t-s)}, \quad 0 \le s \le t \le T. \tag{2.8}$$

($\mathbf{A3}_\nu$) For a given $\nu \ge 1$, there exists a constant $c_\nu(T) > 0$ such that for any $\varphi \in L^{2\nu}_\rho$ and $0 \le s \le t \le T$

$$(U(t,s)|\varphi|)^\nu \le c_\nu(T)U(t,s)|\varphi|^\nu. \tag{2.9}$$

Combined with the positivity preserving, cf. (**A1**), this implies that U is bounded in $L^{2\nu}_\rho$ and

$$||U||_{\mathcal{L}(L^{2\nu}_\rho)} := \sup_{0\le s\le t\le T} ||U(t,s)||_{\mathcal{L}(L^{2\nu}_\rho)} \le \Big[c_\nu(T) \cdot ||U||_{\mathcal{L}(L^2_\rho)}\Big]^{1/\nu}. \tag{2.10}$$

Additionally we assume that U is *strongly continuous* in $L^{2\nu}_\rho$, i.e., the mapping

$$U(\cdot,\cdot)\varphi : \Delta(T) \to L^{2\nu}_\rho \text{ is continuous for any } \varphi \in L^{2\upsilon}_\rho. \tag{2.11}$$

($\mathbf{A4}_\nu$) For $\nu \ge 1$ from ($\mathbf{A3}_\nu$) and $0 \le s < t \le T$, the operator $U(t,s)$ extends to the domain

$$\mathfrak{M}_\nu := \{h \in L^\nu_\rho \mid h = \mathcal{M}_\varphi\psi, \ \ \varphi \in L^{2\nu}_\rho, \ \ \psi \in L^2\},$$

where

$$\mathcal{M}_\varphi : L^2 \ni \psi \to \mathcal{M}_\varphi\psi := \phi\psi \in L^\nu_\rho$$

denotes the multiplication operator corresponding to $\varphi \in L^{2\nu}_\rho$. Furthermore, there exist $\zeta \in [0,1)$ and $c_{\nu,\zeta}(T) > 0$ such that

$$||U(t,s)\mathcal{M}_\varphi \psi||^2_{L^{2\nu}_\rho} \le c_{\nu,\zeta}(T)(t-s)^{-\zeta}||\varphi||^2_{L^{2\nu}_\rho}||\psi||^2_{L^2}. \tag{2.12}$$

(**A4**) Assumption ($\mathbf{A4}_\nu$) holds with $\nu = 1$, $\zeta \in [0,1)$ and the corresponding constant $c_\zeta(T) > 0$, i.e.,

$$||U(t,s)\mathcal{M}_\varphi||^2_{\mathcal{L}(L^2,L^2_\rho)} \le c_{\nu,\zeta}(T)(t-s)^{-\zeta}||\varphi||^2_{L^2_\rho}, \quad \varphi \in L^2_\rho. \tag{2.13}$$

(**A5**) For a fixed $m > d/2$, there exists an approximating system of bounded operators $(A_N(t))_{\substack{t\in[0,T]\\ N\in\mathbb{N}}} \subset \mathcal{L}(L^2_\rho) \cap \mathcal{L}(W^{m,2})$ such that for any $N \in \mathbb{N}$

$$\begin{aligned} ||A_N||_{\mathcal{L}(L^2_\rho)} \quad &: \quad = \sup_{t\in[0,T]} ||A_N(t)||_{\mathcal{L}(L^2_\rho)} < \infty, \\ ||A_N||_{\mathcal{L}(W^{m,2})} \quad &: \quad = \sup_{t\in[0,T]} ||A_N(t)||_{\mathcal{L}(W^{m,2})} < \infty. \end{aligned} \tag{2.14}$$

Each $A_N :=(A_N(t))_{t\in[0,T]}$ generates a strong evolution family $U_N := (U_N(t,s))_{0\le s\le t\le T}$ in the both spaces L^2_ρ and $W^{m,2}$ that is positivity preserving. Furthermore, for any $\varphi \in L^2_\rho$

$$\sup_{0\le s\le t\le T} ||(U_N(t,s) - U(t,s))\varphi||^2_{L^2_\rho} \to 0, \quad N \to \infty. \tag{2.15}$$

In some cases it would be enough to assume a weaker version of ($\mathbf{A4}_\nu$) corresponding to the particular choise $\varphi \equiv 1$ in (2.12).

($\mathbf{A4}^*_\nu$) For $0 \le s < t \le T$, the operator $U(t,s)$ continuously maps L^2 into $L^{2\nu}_\rho$ and obeys the estimate with proper $\zeta \in [0,1)$ and $c_{\nu,\zeta}(T) > 0$

$$||U(t,s)||^2_{\mathcal{L}(L^2,L^{2\nu}_\rho)} \le c_{\nu,\zeta}(T)(t-s)^{-\zeta}. \tag{2.16}$$

Remark 2.4. ***(i)*** Clearly, the inequalities in (**A1**) and ($\mathbf{A3}_\nu$) are understood $d\theta$-almost surely. Assumptions (**A1**), ($\mathbf{A3}_\nu$) and (**A4**) have been introduced in the paper [26] dealing with SPDEs driven by a Wiener noise. The only reason to impose (**A2**) is to ensure existence of càdlàg versions of the Poisson convolution $I^{\tilde{N}}_\Gamma$ (cf. Section 3 below). Assumption ($\mathbf{A4}_\nu$) with $\nu \ge 1$ is already needed to prove unique solvability in $L^{2\nu}_\rho$ for Eq.(1) with Lipschitz coefficients, whereas the conditions in (**A5**) concerning $W^{m,2}$ will be relevant later when proving the comparison theorem in Section 4.

(ii) Observe that by the definition of $U_N := (U_N(t,s))_{0\le s\le t\le T}$ in (**A5**)

$$||U_N||_{\mathcal{L}(L^2_\rho)} \quad : \quad = \sup_{0\le s\le t\le T} ||U_N(t,s)||_{\mathcal{L}(L^2_\rho)} < \infty, \tag{2.17}$$

$$||U_N||_{\mathcal{L}(W^{m,2})} \quad : \quad = \sup_{0\le s\le t\le T} ||U_N(t,s)||_{\mathcal{L}(W^{m,2})} < \infty, \tag{2.18}$$

whereas (2.15) and (2.6) yield the uniform bound

$$\sup_{N\in\mathbb{N}} \sup_{0\le s\le t\le T} ||U_N(t,s)||_{\mathcal{L}(L^2_\rho)} < \infty. \tag{2.19}$$

(iii) The above conditions on $(A(t))_{t\in[0,T]}$ are satisfied by a large class of *elliptic differential operators* with smooth enough coefficients.(see Section 2.3 below).

We finish this subsection by specifying the properties of the noise terms in Eq.(1):

(W) $(W(t))_{t\in[0,T]}$ is a Q-Wiener process on $(\Omega, \mathcal{F}, P)$ taking values in L^2. Its correlation operator $Q \in \mathcal{L}_1^+(L^2)$ is of trace class with the eigenvectors $(e_n)_{n\in\mathbb{N}}$ building a complete orthonormal system in L^2 and with the corresponding nonnegative eigenvalues $(a_n)_{n\in\mathbb{N}}$ obeying

$$Qe_n = a_n e_n, \quad \mathrm{tr}Q = \sum_{n\in\mathbb{N}} a_n < \infty. \tag{2.20}$$

($\mathbf{E}_\infty$) The eigenvectors $(e_n)_{n\in\mathbb{N}}$ are equibounded in the sup-norm, i.e.,

$$\sup_{n\in\mathbb{N}} ||e_n||_{L^\infty} < \infty.$$

($\mathbf{N}_\nu$) $(\tilde{N}(t, dx))_{t\in[0,T],)}$ is a *compensated Poisson random measure on* $(L^2, \mathcal{B}(L^2))$, defined on the same stochastic basis and independent of $(W(t))_{t\in[0,T]}$. Furthermore, we assume that, for a given $\nu \geq 1$, the corresponding *Lévy intensity measure* η on $(L^2, \mathcal{B}(L^2))$ obeys the integrability property

$$\int_{L^2} ||x||_{L^2}^{2\nu}\,\eta(dx) + \int_{L^2} ||x||_{L^2}^{2}\,\eta(dx) < \infty.$$

As usual we set $\tilde{N}(t, \{0\}) = \eta(\{0\}) = 0$ for all $t \in [0,T]$.

To prove the comparison theorem for solutions of Eq. (1) with different drift coefficients, we should assume that the corresponding Poisson process has positive jumps.

(P) The intensity measure η is supported by the cone of *nonnegative* functions

$$L_+^2 := \left\{ x \in L^2 \mid x(\theta) \geq 0, \quad d\theta\text{-a. e.} \right\}$$

Remark 2.5. ***(i)*** Obviously, we have a coordinate representation convergent a.s. in L^2

$$W(t) = \sum_{n\in\mathbb{N}} \sqrt{a_n} \cdot w_n(t) e_n, \quad t \in [0,T],$$

where $(w_n(t) := (W(t), e_n)_{L^2})_{t\in[0,T]}$, $n \in \mathbb{N}$, is a family of independent real-valued Brownian motions. Assumption ($\mathbf{E}_\infty$) on the eigenvectors $(e_n)_{n\in\mathbb{N}}$ is satisfied in several important cases. A typical example is the Laplace operator with Dirichlet boundary conditions on a cube. Furthermore, by a general result (cf. e.g. p. 40 in [26]), the space $L^2(\Theta)$ always possesses a uniformly bounded othonormal system $(e_n)_{n\in\mathbb{N}}$, so that we can introduce a large class of correlation operators $Q \in \mathcal{L}(L^2)$ directly by (2.20).
(ii) The integrability condition ($\mathbf{N}_\nu$) is imposed to get the stochastic convolution integral $I_\Gamma^{\tilde{N}}$ well-defined, see Section 3 below.
(iii) Assumption (**P**) is related to the case of increasing jump coefficients γ, cf. ($\mathbf{\Gamma}$)(**iii**). Equivalently we can assume that γ is decaying and η is supported by L_-^2.

2.2. **Definition of solutions and main results.** As was mentioned in the Introduction, we look for mild solutions to Eq.(1) in the space $\mathcal{G}_\nu(T)$ of predictable processes $X : [0,T] \to L_\rho^{2\nu}(\Theta)$.

Definition 2.6. *Given an* $L_\rho^{2\nu}$*-valued,* $\mathcal{F}_0$*-measurable initial condition* ξ *such that* $E||\xi||_{L_\rho^{2\nu}}^{2\nu} < \infty$ *for some* $\nu \geq 1$*, a process* $X = (X(t))_{t\in[0,T]} \in \mathcal{G}_\nu(T)$ *is called a mild solution to Eq.(1) if*

- *the following identity in L^2_ρ holds $\mathbb{P}$-a. s for any $t \in [0,T]$*

$$X(t) = U(t,0)\xi + \int_0^t U(t,s)F(s,X(s))ds \tag{2.21}$$
$$+ \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X(s))}dW(s) + \int_0^t \int_{L2} U(t,s)\mathcal{M}_{\Gamma(s,X(s))}(x)\,\tilde{N}(ds,dx),$$

whereby all integrals on the right-hand side exist;
- $[0,T] \ni t \mapsto X(t) \in L^2_\rho$ *obeys a càdlàg version.*

Note that the càdlàg modification $\tilde{X}(t)$ of $X(t)$ satisfies $\mathbb{P}$-a.s. the following identity for all $t \in [0,T]$

$$\tilde{X}(t) = U(t,0)\xi + \int_0^t U(t,s)F(s,\tilde{X}(s))ds \tag{2.22}$$
$$+ \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,\tilde{X}(s))}dW(s) + \int_0^t \int_{L2} U(t,s)\mathcal{M}_{\Gamma(s,\tilde{X}(s-))}(x)\,\tilde{N}(ds,dx),$$

Below we present our main results in this setting:

1) an existence and uniqueness result for solutions of Eq.(1) with the Lipschitz coefficients F, Σ, Γ;
2) a comparison result for solutions $X^{(i)}$ to Eq.(1) with varying $F^{(i)}$, $i = 1,2$;
3) an existence theorem for Eq.(1) under the one-sided linear growth assumption $(\mathbf{F})$ on the drift term F.

As a preliminary step, we start with the existence and uniqueness result in the case of Lipschitz coefficients.

Theorem 2.7. ***(Lischitz case)*** *Let the initial condition $\xi \in L^{2\nu}_\rho$ be as in Definition 2.6, whereby we choose $\nu \in [1, \frac{1}{\zeta})$ with $\zeta \in [0,1)$ from $(\mathbf{A4}_\nu)$ and assume that $\tilde{N}$ satisfies $(\mathbf{N}_\nu)$. Let σ and γ are Lipschitz continuous and satisfy all conditions in Assumptions $(\mathbf{\Sigma})$ and $(\mathbf{\Gamma})$ respectively (except the monotonicity condition $(\mathbf{\Gamma})(\mathbf{iii})$). Furthermore, let the drift coefficient F is Lipschitz, i.e., it obeys $(\mathbf{\Sigma})$ with $\sigma := f$ and a proper $c_f(T) > 0$. Then, there exists a unique mild solution $X \in \mathcal{G}_\nu(T)$ to Eq.(1). The mapping $t \mapsto X(t)$ is continuous in $\mathbb{L}^{2\nu}(\Omega; L^{2\nu}_\rho)$. Furthermore, we have a moment estimate*

$$\sup_{t\in[0,T]} \mathbb{E}||X(t)||^{2\nu}_{L^{2\nu}_\rho} \le K(1 + \mathbb{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}) \tag{2.23}$$

with a certain constant $K > 0$ depending on T, ζ, ν, $c(T)$, $c_f(T)$, $c_\sigma(T)$ and $c_\gamma(T)$

Next, we state the comparison result in the Lipschitz case.

Theorem 2.8. ***(Comparison of solutions)*** *Suppose that Theorem 2.7 holds for $\nu = 1$ and assume additionally $(\mathbf{\Gamma})(\mathbf{iii})$ and $(\mathbf{P})$. Consider different initial conditions $\xi^{(1)}$, $\xi^{(2)} \in L^2_\rho$ and Lipschitz drift functions $f^{(1)}$, $f^{(2)}$. Then,*

$$\xi^{(1)} \le \xi^{(2)}, \quad \mathbb{P}\text{-a.s.},$$

and

$$f^{(1)}(t,y) \le f^{(2)}(t,y) \quad \textit{for all} \quad (t,y) \in [\,0,T\,] \times \mathbb{R}, \quad \mathbb{P}\text{-a.s.},$$

imply, for any $t \in [0,T]$,

$$X^{(1)}(t) \le X^{(2)}(t), \quad \mathbb{P}\text{-a.s.},$$

where $X^{(i)}$, $i = 1,2$, *are the corresponding solutions to Eq.(1).*

Finally, we formulate the most general existence result.

Theorem 2.9. ***(Existence in the general case)*** *Let* $(\mathbf{A1})$, $(\mathbf{A2})$, $(\mathbf{A5})$ *hold as well as* $(\mathbf{A3}_\nu)$ *and* $(\mathbf{A4}_\nu)$ *with some* $\zeta \in [\,0,1\,)$. *Let* Σ, Γ *and* F *fulfill respectively Assumptions* $(\mathbf{\Sigma})$, $(\mathbf{\Gamma})$ *and* $(\mathbf{F})$ *with some* $\nu \in [\,1,\frac{1}{\zeta}\,)$. *Suppose that the initial condition* $\xi \in L^{2\nu}_\rho$ *is as in Definition 2.6 with the same* ν. *Finally, assume the Lévy measure* η *corresponding to* $\tilde{N}$ *obeys the integrability property* $(\mathbf{N}_{\nu'})$ *with* $\nu' := \nu^2$ *and the support property* $(\mathbf{P})$. *Then, there exists a mild solution* $X \in \mathcal{G}_\nu(T)$ *to Eq.(1) such that* $t \mapsto X(t)$ *is continuous in* $\mathbb{L}^2(\Omega; L^2_\rho)$.

These theorems will be proven in Sections 4.1–4.3 respectively.

Remark 2.10. One can drop Assumption $(\mathbf{A4}_\nu)$ by assuming that $\tilde{N}(t,dx)$ is concentrated on the Sobolev space $W^{m,2}$ with some $m > d/2$ and satisfies the integrability condition $(\mathbf{N}_\nu)$ with $||\cdot||_{W^{m,2}}$ standing for $||\cdot||_{L^2}$, see Remark **??** below.

We finish this subsection by the following remark concerning a similar SDE driven by a Lévy noise

$$\begin{aligned} dX(t) &= (A(t)X(t) + F(t,\cdot,X(t)))dt \\ &+ \mathcal{M}_{\Sigma(t,\cdot,X(t))} dL(t), \qquad t \in [0,T], \\ X(0) &= \xi. \end{aligned} \tag{2.24}$$

Remark 2.11. Let $(L(t))_{t \in [\,0,T\,]}$ be an L^2-valued Lévy process (for its definition see e.g. the monograph [31]). Then, as was shown e.g. in [4], [6], [7], it obeys the Lévy-Itô decomposition

$$L(t) = mt + W(t) + \int\limits_{\{||x||_{L^2}<1\}} x\,\tilde{N}(t,dx) + \int\limits_{\{||x||_{L^2}\ge 1\}} x\,N(t,dx), \quad t \in [0,T], \tag{2.25}$$

with a drift vector $m \in L^2$, a Q-Wiener process W with $Q \in \mathcal{L}_1^+(L^2)$ and a Poisson random measure $N(t,dx)$ on $(L^2, \mathcal{B}(L^2 \backslash \{0\}))$, whereby W is independent of $N(\cdot, A)$ for all $A \in \mathcal{B}(L^2 \backslash \{0\})$. By setting $\eta(A) := \mathbb{E}\,[N(1,A)]$ and $\eta(\{0\}) := 0$, we define the associated Levy intensity measure η on $(L^2, \mathcal{B}(L^2))$ with the property $\int\limits_{L^2} \left(|x||^2_{L^2} \wedge 1\right) \eta(dx) < \infty$. If η additionally fulfills $(\mathbf{N}_\nu)$ with $\nu = 1$, the above decomposition simplifies as

$$L(t) = \tilde{m}t + W(t) + \int\limits_{L^2} x\,\tilde{N}(t,dx), \quad t \in [0,T], \tag{2.26}$$

with

$$\tilde{m} := m + \int\limits_{\{||x||_{L^2}\ge 1\}} x\,\eta(dx) \in L^2.$$

Furthermore, by Proposition 6.9 in [31], $(\mathbf{N}_\nu)$ with $\nu \geq 1$ *implies that* $\mathbb{E}||L(t)||^{2\nu} < \infty$ for all $t \in [0, T]$. Thus, Eq.(2) can be seen as a particular case of the initial Eq. (1). However, Assumption $(\mathbf{E}_\infty)$ on the eigenvector basis $(e_n)_{n\in\mathbb{N}}$ in (2.20) may fail in general. Nevertheless, this problem can be overcome by imposing an additional constraint on the parameters ς and ν.

3. Well-definedness of Poisson stochastic convolutions

In Section 3.1 we briefly introduce stochastic integration with respect to compensated Poisson random measures $\tilde{N}(ds, dx)$ in general Hilbert spaces, while in Section 3.2 we study in more detail the well-definedness of the Poisson stochastic convolution

$$I_\Gamma^{\tilde{N}}(Z)(t) := \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\Gamma(s,Z(s))}(x)\, \tilde{N}(ds,dx), \quad t \in [0,T]. \tag{3.1}$$

and its properties in the weighted spaces $L^{2\nu}_\rho$, $\nu \geq 1$. A main technical problem here is that $\mathcal{M}_{\Gamma(s,Z(s))}$ cannot be defined as a bounded operator sending L^2 into $L^{2\nu}_\rho$, $\nu > 1$. So, the desired properties of the stochastic convolution (3.1) should be achieved by additional regularity assumptions on $U = (U(t,s))_{0\leq s\leq t\leq T}$ like that in $(\mathbf{A4}_\nu)$.

3.1. **Poisson stochastic integrals in L^2.** Here we briefly summarize the definition and properties of the stochastic integral w.r.t. the compensated Poisson measure $\tilde{N}(dt, dx)$ for vector-valued integrands $\Phi : \Omega \times [0, T] \times L^2 \to L^2_\rho$.

The Poisson integral

$$I_\Phi^{\tilde{N}}(t) := \int_0^t \int_{L^2} \Phi(s,x)\, \tilde{N}(ds,dx) \tag{3.2}$$

$$= \int_0^T \int_{L^2} 1_{(0,t]}(s)\Phi(s,x)\, \tilde{N}(ds,dx) \in L^2_\rho, \quad t \in [0,T], \tag{3.3}$$

is defined by the Itô-isometry

$$\mathbb{E}\left\| \int_0^t \int_{L^2} \Phi(s,x)\, \tilde{N}(ds,dx) \right\|^2_{L^2_\rho} = \mathbb{E} \int_0^t \int_{L^2} ||\Phi(s,x)||^2_{L^2_\rho}\, \eta(dx)ds \tag{3.4}$$

extended (from simple processes) to all predictable processes $\Phi \in \mathbb{L}^2_T(L^2/L^2_\rho)$, where

$$\mathbb{L}^2_T(L^2/L^2_\rho) := \mathbb{L}^2([0,T] \times \Omega \times L^2, \mathcal{P}_T \otimes \mathcal{B}(L^2), dt \otimes \mathbb{P} \otimes \eta\ ;\ L^2_\rho). \tag{3.5}$$

The process $\left(I_\Phi^{\tilde{N}}(t)\right)_{t\in[0,T]}$ is a square-integrable càdlàg $\mathcal{F}_t$-martingale in L^2_ρ, with mean zero and predictable quadratic variation (i.e., Meyer process)

$$\langle I_\Phi^{\tilde{N}} \rangle_t = \int_0^t \int_{L^2} ||\Phi(s,x)||^2_{L^2_\rho}\, \eta(dx)ds.$$

All necessary details about such a construction in general Hilbert spaces can be found e.g. in [6], [36].

Maximal $\mathbb{L}^p$-estimates for the Poisson integral (3.2) can be obtained by means of an infinite-dimensional version of *Bichteler-Jacod inequality,* see Lemma 3.1 in [27]. More precisely, one has for $p \geq 2$

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left\|\int_0^t\int_{L^2}\Phi(s,x)\,\tilde{N}(ds,dx)\right\|_{L^2_\rho}\right)^p \tag{3.6}$$
$$\leq \quad C\cdot\mathbb{E}\int_0^T\left[\int_{L^2}||\Phi(s,x)||^p_{L^2_\rho}\,\eta(dx)+\left(\int_G||\Phi(s,x)||^2_{L^2_\rho}\,\eta(dx)\right)^{\frac{p}{2}}\right]ds<\infty,$$

whereby $(p,T)\to C:=C(p,T)\in\mathbb{R}_+$ is continuous. Estimate (3.6) holds for any $\Phi\in\mathbb{L}^2_T(L^2/L^2_\rho)$ such that the expectation on the right-hand side is finite. A lower bound for the left-hand side in (3.6) was established in the recent work [13]

To finish this subsection let us check the well-definedness of the Poisson stochastic convolution (3.1) in L^2_ρ. For a fixed $Z\in\mathcal{G}_\nu(T)$ and $t\in[0,T]$, let us define

$$\Phi:[\,0,T\,]\times\Omega\times L^2\to L^2_\rho, \tag{3.7}$$
$$\Phi(s,x):=1_{(0,t]}(s)U(t,s)M_{\Gamma(s,Z(s))}x,\quad (s,x)\in[\,0,T\,]\times L^2.$$

It is easy to see (cf. Lemma 4.5 in [19]) that the $\mathcal{P}_T\otimes\mathcal{B}(L^2)/\mathcal{B}(L^2_\rho)$-measurability required for Φ would be a sequel of the same measurability property valid for

$$[\,0,T\,]\times\Omega\times L^2\ni(s,\omega,x)\mapsto M_{\Gamma(s,Z(s))}x\in L^2_\rho, \tag{3.8}$$

Note that the multiplication operator

$$x\to M_{\Gamma(s,Z(s))}x=\gamma(s,Z(s))x$$

is continuous in L^2 for any fixed (s,ω), since by $(\mathbf{\Gamma})(\mathbf{ii})$ the generating function γ is assumed to be bounded. On the other hand, for any fixed $x\in L^2$, the process

$$(s,\omega)\to\gamma(s,Z(s,\omega))x\in L^2_\rho$$

is $\mathcal{P}_T$*-measurable.* Indeed, by Remark 2.3 $s\to\gamma(s,Z(s))y\in L^2_\rho$ is predictable for each $y\in L^\infty$. Approximating $x\in L^2$ by $\{y_n\}_{n\geq1}\subset L^\infty$ and using boundedness of σ, we get the required predictability of $s\to\gamma(s,Z(s))x$ as a pointwise limit of $\gamma(s,Z(s))y_n$ in L^2_ρ. Thus, by Theorem 6.1 from [17], there exists a $\mathcal{P}_T\otimes\mathcal{B}(L^2)/\mathcal{B}(L^2_\rho)$-measurable realization of (3.8) and hence of the integrand function (3.7), which we again denote by Φ. By $(\mathbf{\Gamma})(\mathbf{ii})$**,** $(\mathbf{N}_\nu)$ and (2.6) we have

$$\mathbb{E}\int_0^T\int_{L^2}||\Phi(s,x)||^2_{L^2_\rho}\,\eta(dx)ds\leq Tc^2_\gamma(T)||U||_{\mathcal{L}(L^2_\rho)}\int_{L^2}||x||^2_{L^2}\,\eta(dx)<\infty,$$

which shows that $\Phi \in \mathbb{L}^2_T(L^2/L^2_\rho)$. This allows us to define for each fixed $t \in [0,T]$ a random variable

$$I^{\tilde N}_\Gamma(Z)(t) := \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\Gamma(s,Z(s))}(x)\,\tilde N(ds,dx) = \int_0^T \int_{L^2} \Phi(s,x)\,\tilde N(ds,dx) \in L^2_\rho \quad (\mathbb{P}\text{-}a.s.).$$

3.2. **Poisson stochastic convolution in $L^{2\nu}_\rho$.** Next we prove even more that, for any $Z \in \mathcal{G}_\nu(T)$, the Poisson convolution $I^{\tilde N}_\Gamma(Z)$ obeys a predictable modification in the weighted spaces $L^{2\nu}_\rho \subset L^2$.

Theorem 3.1. *Let us fix some $\nu \geq 1$ and let η corresponding to $\tilde N$ obeys (N_ν) with the same ν. For $Z \in \mathcal{G}_\nu(T)$, the process $I^{\tilde N}_\Gamma(Z)$ is well-defined in $\mathcal{G}_\nu(T)$. In particular, we have the estimate*

$$\begin{aligned}(3.9)\qquad &\sup_{t\in[0,T]} E||I^{\tilde N}_\Gamma(Z)(t)||^{2\nu}_{L^{2\nu}_\rho} \\ &\leq C(\nu,T)\left\{\mathbb{E}\int_0^t\int_{L^2} ||U(t,s)\mathcal{M}_{\Gamma(s,Z(s))}x||^{2\nu}_{L^{2\nu}_\rho}\,\eta(dx) \right.\\ &\left. +\mathbb{E}\int_0^t\left(\int_{L^2}||U(t,s)\mathcal{M}_{\Gamma(s,Z(s))}x||^2_{L^{2\nu}_\rho}\eta(dx)\right)^\nu ds)\right\} \\ &\leq C_\gamma\left[\int_{L^2}||x||^{2\nu}_{L^{2\nu}}\,\eta(dx)+\left(\int_{L^2}||x||^2_{L^{2\nu}}\,\eta(dx)\right)^\nu\right] < \infty\end{aligned}$$

with a positive constant C_γ depending on ν, ζ, T, $||U||_{\mathcal{L}(L^2_\rho)}$ and on the uniform bound $c_\gamma(T)$ on γ. Furthermore, $[\,0,T\,] \ni t \mapsto I^{\tilde N}_\Gamma(Z)(t)$ is continuous in $L^{2\nu}(\Omega;L^{2\nu}_\rho)$ and obeys a càdlàg modification in L^2_ρ.

Proof: Let us fix an arbitrary $t \in [\,0,T\,]$. To shorten notation, let us write $I_\Gamma{:=}I^{\tilde N}_\Gamma(Z)$.The key issues in the proof are the following:

Step 1 *Finding a measurable representative $\tilde\Phi$ of the mapping (3.7), i.e. a $\mathcal{P}_T \otimes \mathcal{B}(L^2)\otimes\mathcal{B}(\Theta)/\mathcal{B}(\mathbb{R})$-measurable function*

$$\tilde\Phi : \Omega\times[\,0,T\,]\times L^2\times\Theta\to\mathbb{R}$$

such that

$$(3.10)\qquad \mathbb{E}\int_0^T\int_{L^2} ||\Phi(s,x)-\tilde\Phi(s,x,\cdot)||^2_{L^2_\rho}\eta(dx)\,ds = 0$$

Step 2 *Proving that for each $t \in [0,T]$*

$$I_\Gamma(t) = \tilde{I}(t) := \int_0^t \int_L \tilde{\Phi}(s,x,\cdot)\, \tilde{N}(ds,dx) \in L^{2\nu}_\rho, \quad \mathbb{P}\text{-a.s.},$$

whereby $\tilde{I}(t) \in \mathcal{G}_\nu(T)$ and satisfies the Bichteler-Jacod inequality (3.9).

Step 3 *Showing that $t \mapsto I_\Gamma(t)$ is continuous in $\mathbb{L}^{2\nu}(\Omega; L^{2\nu}_\rho)$ and obeys a càdlàg modification in L^2_ρ.*

Step 1: Let us denote by $(\delta_k)_{k\in\mathbb{N}} \subset L^1(\mathbb{R}^d)$ a (general-) *Dirac sequence* (for its definition and properties see e.g. Chapter 2 in [5]) constructed from a function $\varphi \in C_0^\infty(\mathbb{R}^d)$ by

$$\delta_k(\theta) := \left(\frac{1}{k}\right)^{-d} \varphi(k\theta), \quad \theta \in \mathbb{R}^d. \tag{3.11}$$

Then we can approximate $\Phi(s,x) \in L^2_\rho$ by standard convolutions

$$\Phi_m(s,x) := \mu_\rho^{-1}\, \mathrm{conv}(\delta_m\,,\, \mu_\rho\, \Phi(s,x)) \in L^2_\rho \bigcap C(\mathbb{R}^d) \tag{3.12}$$

in such a way that, for all $(s,\omega,x) \in [\,0,T\,] \times \Omega \times L^2$,

$$\lim_{m\to\infty} ||\Phi_m(s,x) - \Phi(s,x)||^2_{L^2_\rho} = 0 \tag{3.13}$$

and

$$\sup_{m\in\mathbb{N}} ||\Phi_m(s,x)||_{L^2_\rho} \le ||\Phi(s,x)||_{L^2_\rho}. \tag{3.14}$$

Since $\Phi_m(s,x) \in C(\mathbb{R}^d)$, we can evaluate $\Phi_m(s,x,\theta)$ for any $\theta \in \Theta$. Furthermore, Theorem 6.1 from [17] guarantees that there exists a $\mathcal{P}_T \otimes \mathcal{B}(L^2) \otimes \mathcal{B}(\Theta)/\mathcal{B}(\mathbb{R})$-measurable realization of $(s,\omega,x,\theta) \mapsto \Phi_m(s,\omega,x,\theta)$. Sufficient conditions to apply this theorem are:

(i) continuity of the mapping

$$\Theta \ni \theta \mapsto \Phi_m(s,\omega,x,\theta) \in \mathbb{R}$$

for almost any $(s,\omega,x) \in [\,0,T\,] \times \Omega \times L^2$. (Note that this property holds by (3.12);

(ii) $\mathcal{P}_T \otimes \mathcal{B}(L^2)$-measurability of

$$[\,0,T\,] \times \Omega \times L^2 \ni (s,\omega,x) \mapsto \Phi_m(s,\omega,x,\theta) \in \mathbb{R}$$

for any fixed $\theta \in \Theta$.

The latter readily follows by Fubini's theorem from the definition (3.12) saying that

$$\Phi_m(s,\omega,x,\theta) = \mu_\rho^{-\frac{1}{2}}(\theta) < \delta_m(\cdot - \theta), \mu_\rho^{\frac{1}{2}} F(s,x) >_{L^2}.$$

Next, by (3.13), (3.14) and Lebesgue's theorem, we observe that the corresponding sequence of measurable realizations Φ_m, $m \in \mathbb{N}$, is a Cauchy sequence in the Hilbert space

$$\mathbb{L}^2 := \mathbb{L}^2([0,T] \times \Omega \times L^2 \times \Theta,\ \mathcal{P}_T \otimes \mathcal{B}(L^2) \otimes \mathcal{B}(\Theta),\, dt \otimes \mathbb{P} \otimes \eta \otimes \mu_\rho;\ L^2_\rho).$$

Indeed, for all $n, m \in \mathbb{N}$

$$\mathbb{E}\int_0^T\int_{L^2}\int_{\Theta} |\Phi_m(s,x,\theta) - \Phi_n(s,x,\theta)|^2 \mu_\rho(d\theta)\,\eta(dx)\,ds$$
$$= \mathbb{E}\int_0^T\int_{L^2} ||\Phi_m(s,x) - \Phi_n(s,x)||^2_{L^2_\rho}\,\eta(dx)\,ds$$

with the uniform bound

$$\sup_{m\in\mathbb{N}} ||\Phi_m(s,x)||_{L^2_\rho} \le ||\Phi(s,x)||_{L^2_\rho}, \quad ||\Phi||_{L^2_\rho} \in \mathbb{L}^2([0,T]\times\Omega\times L^2).$$

Thus, there exists a limit function $\tilde\Phi \in \mathbb{L}^2$ such that

$$\lim_{m\to\infty} \mathbb{E}\int_0^T\int_{L^2}\int_{\Theta} |\tilde\Phi(s,x,\theta) - \Phi_m(s,x,\theta)|^2\,\mu_\rho(d\theta)\,\eta(dx)\,ds \tag{3.15}$$
$$= \lim_{m\to\infty} \mathbb{E}\int_0^T\int_{L^2} ||\tilde\Phi(s,x) - \Phi_m(s,x)||^2_{L^2_\rho}\,\eta(dx)\,ds = 0.$$

Combining (3.13)–(3.15), we conclude that

$$\mathbb{E}\int_0^T\int_{L^2} ||\Phi(s,x) - \tilde\Phi(s,x)||^2_{L^2_\rho}\,\eta(dx), ds = 0,$$

i.e. $\tilde\Phi(s,x,\theta)$ is a measurable realization of $\Phi(s,x)$ that obeys (3.10). △

Step 2: This step involves the following two claims:

Claim 1: *For each $t \in [0,T]$, consider the stochastic integral*

$$\tilde I(t,\theta) := \int_0^t\int_{L^2} \tilde\Phi(s,x,\theta)\,\tilde N(ds,dx) \in \mathbb{R}, \tag{3.16}$$

depending on the parameter $\theta \in \Theta$. Then:

(i) *$\tilde I(t,\theta)$ allows an $\mathcal{F}_t \otimes \mathcal{B}(\Theta)$-measurable modification (which will be denoted by $\tilde I$ again);*

(ii) *$\tilde I(t,\cdot)$ coincides $\mathbb{P}$-a.s. with the L^2_ρ-valued random variable*

$$I_{\tilde\Phi}(t) := \int_0^t\int_{L^2} \tilde\Phi(s,x,\cdot)\,\tilde N(ds,dx),$$

i.e.,

$$\mathbb{E}||\tilde I(t) - I_{\tilde\Phi}(t)||^2_{L^2_\rho} = \mathbb{E}\int_{\Theta} |\tilde I(t,\theta) - (I_{\tilde\Phi}(t))(\theta)|^2\,\mu_\rho(d\theta) = 0. \tag{3.17}$$

Claim 2: *$\tilde{I}(t,\theta)$, as a function of $\theta \in \Theta$, belongs $\mathbb{P}$-a.s. to $L^{2\nu}_\rho$*, whereby $\sup_{t\in[0,T]} E||\tilde{I}(t)||^{2\nu}_{L^{2\nu}_\rho}$ *is finite and can be estimated by the constant on the right-hand side in (3.9).*

Claim 1 (i) readily follows from a general measurability result for Poisson stochastic integrals depending on parameters (which in an explicit form can be found e.g. in [10], A.1.(b)) . To prove Claim 1 (ii), let us consider cylinder functions $F \in \mathbb{L}^2(\Omega; L^2_\rho)$ of the form

$$F(\omega,\theta) = F_1(\omega)\left(\sum_{j=1}^{J} d_j \mathbf{1}_{B_j}(\theta)\right)$$

with $F_1 \in \mathbb{L}^2(\Omega)$, $d_j \in \mathbb{R}$ and disjoint $B_j \in \mathcal{B}(\Theta)$ for $1 \le j \le J \in \mathbb{N}$. By the stochastic Fubini theorem for Poisson integrals (see again [10], A.1.(b)) one can check that

$$\begin{aligned}
\langle I_{\tilde{\Phi}}(t), F\rangle_{L^2(\Omega;L^2_\rho)} &= \mathbb{E}\int_\Theta F(\omega,\theta)\left(\int_0^t\int_{L^2}\tilde{\Phi}(s,x,\cdot)\,\tilde{N}(ds,dx)\right)(\theta)\mu_\rho(d\theta) \\
&= \mathbb{E}\int_0^t\int_{L^2}\int_\Theta F_1(\omega)F_2(\theta)\tilde{\Phi}(s,x,\theta)\mu_\rho(d\theta)\tilde{N}(ds,dx) \\
&= \mathbb{E}\int_\Theta F(\omega,\theta)\tilde{I}(t,\theta)\,\mu_\rho(d\theta) = \langle \tilde{I}(t), F\rangle_{L^2(\Omega;L^2_\rho)}.
\end{aligned}$$

Since such F constitute a total set in $\mathbb{L}^2(\Omega; L^2_\rho)$, this proves (3.17). △

To prove Claim 2 we make use of a scalar variant of the Bichteler-Jacod inequality, cf. (3.6),

$$\begin{aligned}
&\mathbb{E}\left(\int_0^t\int_{L^2}\tilde{\Phi}(s,x,\theta)\,\tilde{N}(ds,dx)\right)^{2\nu}\mu_\rho(d\theta) \qquad (3.18)\\
&\le C(\nu,T)\left\{\mathbb{E}\int_\Theta\int_0^t\int_{L^2}|\tilde{\Phi}(s,x,\theta)|^{2\nu}\,\eta(dx)\,ds\mu_\rho(d\theta)\right.\\
&\left.+\int_\Theta\int_0^t\mathbb{E}\left(\int_{L^2}|\tilde{\Phi}(s,x,\theta)|^2\eta(dx)\right)^{\nu} ds\mu_\rho(d\theta)\right\}.
\end{aligned}$$

Let us estimate the two integrals in the right hand side of (3.18), which we shall call $\mathcal{I}_1(t)$ and $\mathcal{I}_1(t)$. By the measurabilty property of $\tilde{\Phi}$ and Fubini's theorem we get

$$\begin{aligned}
\mathcal{I}_1(t) &= \mathbb{E}\int_0^t\int_{L^2}||U(t,s)\mathcal{M}_{\Gamma(s,Z(s))}x||^{2\nu}_{L^{2\nu}_\rho}\,\eta(dx)\,ds \qquad (3.19)\\
&\le [c_\gamma(T)]^{2\nu}[c_{\nu,\zeta}(T)]^{\nu}\int_0^t(t-s)^{-\varsigma\nu}ds\int_{L^2}||x||^{2\nu}_{L^{2\nu}}\eta(dx) < \infty,
\end{aligned}$$

where in the last line we have used $(\mathbf{\Gamma})(\mathbf{ii})$, $(\mathbf{A4}^*)$ and $(\mathbf{N}_\nu)$.

Applying the same arguments and additionally Minkowski's inequality (Theorem 2.4 in [25]), for the second integral on the right-hand side of (3.18) we get

$$\begin{aligned}
\mathcal{I}_2(t) &\le \mathbb{E}\int_0^T \left(\int_{L^2} \left(\int_\Theta |\tilde{\Phi}(s,x,\theta)|^{2\nu} \mu_\rho(d\theta) \right)^{1/\nu} \eta(dx) \right)^\nu ds \\
&= \mathbb{E}\int_0^t \left(\int_{L^2} ||U(t,s)\mathcal{M}_{\Gamma(s,Z(s))}x||^2_{L^{2\nu}_\rho} \eta(dx) \right)^\nu ds \\
&\le [c_\gamma(T)]^{2\nu}[c_{\nu,\zeta}(T)]^\nu \int_0^t (t-s)^{-\varsigma\nu} ds \left(\int_{L^2} ||x||^2_{L^{2\nu}} \eta(dx) \right)^\nu < \infty.
\end{aligned} \tag{3.20}$$

Thus, combining (3.19) and 3.20 we get the upper bound required in (3.9), which completes the proof of Claim 2. $\triangle$

Step $\dot{\mathbf{3}}$ Let us first show the continuity of $t \mapsto I_\Gamma(t)$ in $\mathbb{L}^{2\nu}(\Omega; L^{2\nu}_\rho)$. To this end, we extend a method of proving mean-square continuity, which was used e.g. in [19], to the case of non-Hilbert-Schmidt operator valued coefficients $\mathcal{M}_{\Gamma(s,Z(s))}$ and two-parameter evolution operators $U(t,s)$.

For $\alpha > 1$, consider the process

$$\chi^\alpha(t) := \int_0^{\frac{t}{\alpha}} \int_{L^2} U(t,s)\mathcal{M}_{\Gamma(s,Z(s))}x\, \tilde{N}(ds,dx) \in L^{2\nu}_\rho, \quad t \in [\,0,T\,]. \tag{3.21}$$

We claim that $t \mapsto \chi^\alpha(t)$ is continuous in $\mathbb{L}^{2\nu}(\Omega; L^{2\nu}_\rho)$. Indeed, similarly to (3.18)–(3.20) we have for $0 \le r \le t \le T$

$$\begin{aligned}
&\mathbb{E}||\chi^\alpha(t) - \chi^\alpha(r)||^{2\nu}_{L^{2\nu}_\rho} \\
&\le C(\nu,T) \Bigg\{ \mathbb{E}\int_0^{\frac{r}{\alpha}} \Bigg[\int_{L^2} ||[U(t,\alpha s) - U(r,\alpha s)]U(\alpha s,s)\mathcal{M}_{\Gamma(s,Z(s))}x||^{2\nu}_{L^{2u}_\rho} \eta(dx) \\
&\quad + \left(\int_{L^2} ||[U(t,\alpha s) - U(r,\alpha s)]U(\alpha s,s)\mathcal{M}_{\Gamma(s,Z(s))}x||^2_{L^{2\nu}_\rho} \eta(dx) \right)^\nu \Bigg] ds \\
&\quad + \mathbb{E}\int_{\frac{r}{\alpha}}^{\frac{t}{\alpha}} \Bigg[\int_{L^2} ||[U(t,s)\mathcal{M}_{\Gamma(s,Z(s))}x||^{2\nu}_{L^{2u}_\rho} \eta(dx) \\
&\quad + \left(\int_{L^2} ||[U(t,s)\mathcal{M}_{\Gamma(s,Z(s))}x||^2_{L^{2\nu}_\rho} \eta(dx) \right)^\nu \Bigg] ds \Bigg\}.
\end{aligned} \tag{3.22}$$

By $(\mathbf{A4}^*_\nu)$ we see that $U(\alpha s,s)\mathcal{M}_{\Gamma(s,Z(s))}x \in L^{2\nu}_\rho$ for any $x \in L^2$. Therefore, by the continuity assumption $(\mathbf{A3}_\nu)$ on U we have, for any $x \in L^2$ and $s \in [\,0,T\,]$,

$$||1_{[\,0,\frac{r}{\alpha}\,]}(s)[U(t,\alpha s) - U(r,\alpha s)]U(\alpha s,s)\mathcal{M}_{\Gamma(s,Z(s))}(x)||_{L^{2\nu}_\rho} \to 0$$

as $r \uparrow t$ resp. $t \downarrow r$. Furthermore, by $(\mathbf{\Gamma})(\mathbf{i})/(\mathbf{ii})$, $(\mathbf{A3})$, $(\mathbf{A4}^*_\nu)$, $(\mathbf{N}_\nu)$ and Hölder's inequality we have the following estimate

$$\mathbb{E}\int_0^{\frac{r}{\alpha}}\int_{L^2} ||[U(t,\alpha s)-U(r,\alpha s)]U(\alpha s,s)\mathcal{M}_{\Gamma(s,Z(s))}x||^{2\nu}_{L^{2\nu}_\rho}\,\eta(dx)$$
$$\leq C_1(\gamma,T)\int_0^{\frac{r}{\alpha}}[(\alpha-1)s]^{-\varsigma}\,ds\left(\int_{L^2}||x||^{2\nu}_{L^{2\nu}}\,\eta(dx)\right)\cdot\left(\sup_{t\in[0,T]}\mathbb{E}||\Gamma(t,Z(t))||^{2\nu}_{L^{2\nu}_\rho}\right)<\infty.$$

Thus, we can apply Lebesgue's dominated convergence theorem to show that the first integral on the right-hand side of (3.22) vanishes for $r \uparrow t$ resp. $t \downarrow r$. The proof for the second integral runs completely analogous. Thus, it remains to consider the third and the fourth integral on the right-hand side of (3.22. Concerning the third integral we have

$$\mathbb{E}\int_{\frac{r}{\alpha}}^{\frac{t}{\alpha}}\int_{L^2} ||U(t,s)\mathcal{M}_{\Gamma(s,Z(s))}x||^{2\nu}_{L^{2\nu}_\rho}\,\eta(dx)\,ds$$
$$\leq C_2(\gamma,T)\int_{\frac{r}{\alpha}}^{\frac{t}{\alpha}}(t-s)^{-\varsigma\nu}\,ds\left(\int_{L^2}||x||^{2\nu}_{L^{2\nu}}\,\eta(dx)\right)\cdot\left(\sup_{t\in[0,T]}\mathbb{E}||\Gamma(t,Z(t))||^{2\nu}_{L^{2\nu}_\rho}\right),$$

which vanishes as $r \uparrow t$ resp. $t \downarrow r$. By similar arguments also the last integral in (3.22) tends to 0.

Finally, we observe that for any $\alpha > 1$

$$\sup_{t\in[0,T]}\mathbb{E}||I_\Gamma(t)-\chi^\alpha(t)||^{2\nu}_{L^{2\nu}_\rho}=\sup_{t\in[0,T]}\int_{\frac{t}{\alpha}}^{t}\int_{L^2}\mathbb{E}||U(t,s)\mathcal{M}_{\Gamma(s,Z(s))}x||^{2\nu}_{L^{2\nu}_\rho}\,\eta(dx)\,ds$$
$$\leq C_3(\gamma,T)\int_0^{T(1-\frac{1}{\alpha})}s^{-\varsigma\nu}\,ds\left(\int_{L^2}||x||^{2\nu}_{L^{2\nu}}\,\eta(dx)\right)\left(\sup_{t\in[0,T]}\mathbb{E}||\Gamma(t,Z(t))||^{2\nu}_{L^{2\nu}_\rho}\right)$$

with some positive constant $C_3(\gamma,T)$. Since the right-hand side tends to 0 as $\alpha \downarrow 1$, the process I_Γ is $\mathbb{L}^{2\nu}(\Omega;L^{2\nu}_\rho)$-continuous as a uniform limit in $C([0,T],\mathbb{L}^{2\nu}(\Omega;L^{2\nu}_\rho))$ of I^α as $\alpha \downarrow 1$. $\triangle$

Concerning the càdlàg property note that by the uniform boundedness of γ, the multiplication operators $\mathcal{M}_{\Gamma(t,X(t))}$ are bounded both in L^2 and L^2_ρ with

$$\sup_{t\in[0,T]}\mathbf{E}||\mathcal{M}_{\Gamma(t,X(t))}||_{\mathcal{L}}\leq c_\gamma(T)<\infty.$$

Hence, by setting

$$M(t):=\int_{L^2}\mathcal{M}_{\Gamma(s,X(s))}x\,\tilde{N}(t,dx),\quad t\in[0,T],$$

we get a square integrable càdlàg $(\mathcal{F}_t)$-martingale in L^2, which gives us the identity for all $t \in [0,T]$, $\mathbb{P}$-a.s.,

$$I_\Gamma^{\tilde N}(t) = \int_0^t U(t,s)\, dM(s) \tag{3.23}$$

(see e.g. Proposition 3.10 in [19]). Now, by the pseudo-contractivity of U in L^2_ρ (cf. Assumption (**A2**)), we readily get the càdlàg property of $I_\Gamma^{\tilde N}$ in L^2_ρ by a general result from Corollary 2.1 in [20] resp. Remark 1.2 1. in [21], which finishes Step 3. $\triangle$

It remains to show that $I_\Gamma^{\tilde N} \in \mathcal{G}_\nu(T)$, which in particular requires existence of a predictable modification of $(I_\Gamma(t))_{t\in[0,T]}$. By Claim 1(i) in Step 2 we know that, for each $t \in [0,T]$, there is an $\mathcal{F}_t \otimes \mathcal{B}(\Theta)$-measurable version of the random variable $\tilde I(t,\theta)$, cf. (3.16). On the other hand, the continuity of $t \mapsto I_\Gamma(t)$ in $\mathbb{L}^2(\Omega; L^2_\rho)$ just proved implies that $[0,T] \ni t \mapsto \tilde I(t,\omega,\theta) \in \mathbb{R}$ is mean-square continuous (and hence continuous in probability) with respect to $\mathbb{P}\otimes\mu_\rho$. Let us recall a general fact (see e.g. Proposition 3.6 (ii) in [11]) that any adapted and stochastically continuous process in a separable Banach space obeys a predictable modification. In our setting, this means existence of the $\mathcal{P}_T \otimes \mathcal{B}(\Theta)$-measurable modification of the Poisson convolution I_Γ. The norm $||I_\Gamma|||_{\mathcal{G}_\nu(T)} := \sup_{t\in[0,T]} (\mathbb{E}||I_\Gamma(t)||^{2\nu}_{L^{2\nu}_\rho})^{\frac{1}{2\nu}} < \infty$ was estimated in Step 2. $\blacksquare$.

4. Proofs of the main results

In this section, we prove the main results of this paper, Theorems 2.7–2.9.

4.1. **Proof of Theorem 2.7.** The existence result in this theorem can be shown by the Picard iteration method. Note that in the case of Wiener noise, i.e. when $\Gamma = 0$ in Eq.(1), such procedure has been applied e.g. in the proof of Theorem 3.2.1 in [26]. Since the proof is rather standard, we mainly outline the issues related with the presence of a jump noise.

Given an initial condition $\xi \in L^{2\nu}_\rho$, we define a sequence of processes $(X_n)_{n\in\mathbb{N}}$ by

$$\begin{aligned} X_n(t) := \quad & X_0(t) + \int_0^t U(t,s)F(s,X_{n-1}(s))\, ds \\ & + \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X_{n-1}(s))}\, dW(s) \\ & + \int_0^t \int_{L2} U(t,s)\mathcal{M}_{\Gamma(s,X_{n-1}(s))}(x)\, \tilde N(ds,dx), \quad t\in[0,T], \quad n\geq 1, \end{aligned}$$

where

$$X_0(t) := U(t,0)\xi, \quad t\in[0,T].$$

It is an immediate consequence of Theorem 3.1 and the results known in the Wiener case (see e.g. [26]) that each X_n again belongs to $\mathcal{G}_\nu(T)$. Obviously, for any $n \in \mathbb{N}$

and $t \in [0,T]$

$$\mathbf{E}||X_{n+1}(t)-X_n(t)||^{2\nu}_{L^{2\nu}_\rho} \le c(\nu)\left\{\mathbb{E}\left|\left|\int_0^t U(t,s)[F(s,X_n(s))-F(s,X_{n-1}(s))]\,ds)\right|\right|^{2\nu}_{L^{2\nu}_\rho}\right.$$
$$+\mathbb{E}\left|\left|\int_0^t U(t,s)[\mathcal{M}_{\Sigma(s,X_n(s))}-\mathcal{M}_{\Sigma(s,X_{n-1}(s))}]\,dW(s)\right|\right|^{2\nu}_{L^{2\nu}_\rho}$$
$$\left.+\mathbb{E}\left|\left|\int_0^t U(t,s)[\mathcal{M}_{\Gamma(s,X_n(s))}-\mathcal{M}_{\Gamma(s,X_{n-1}(s))}]x\,\tilde{N}(ds,dx)\right|\right|^{2\nu}_{L^{2\nu}_\rho}\right\}.$$

Now, we are arive at the point where we need $(\mathbf{A4}_\nu)$, since to estimate $X_{n+1}-X_n$ in terms of X_n-X_{n-1} we can only use the Lipschitz property but not the boundedness of γ. By means of the Burkholder-Davis-Gundy inequality for Wiener integrals (cf. e.g. Lemma 7.2 in [11]) and the Bichteler-Jacod inequality (cf. Eqs. (3.18)-(3.20)) we get that

$$\mathbf{E}||X_{n+1}(t)-X_n(t)||^{2\nu}_{L^{2\nu}_\rho} \le C\int_0^t (t-s)^{-\zeta\nu}\mathbf{E}||X_n(s)-X_{n-1}(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds, \tag{4.1}$$

where C is a certain constant that depends on ζ, ν, T and the Lipschitz constants from $(\mathbf{F})$, $(\mathbf{\Sigma})$ and $(\mathbf{\Gamma})$. Next, we use the Gronwall-Bellman lemma (see Lemma 3.2.4 in [26]), whose formulation is given below for the reader's convenience.

Lemma 4.1. *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions $g_n\colon \mathbb{R}_+\to\mathbb{R}_+$ obeying for any $t\in[0,T]$ and $n\in\mathbb{N}$*

$$g_n(t) \le q + b\int_0^t (t-s)^{-\delta} g_{n-1}(s)\,ds,$$

with some $\delta\in[0,1)$, $b>0$ and $q\ge 0$. Then,

$$g_n(t) \le q\sum_{k=0}^{n-1} q_k\,t^{k(1-\delta)} + q_n\,t^{n(1-\delta)}\cdot \sup_{s\in[0,T]} g_0(s)$$

with

$$q_0=1,\quad q_1=\frac{b}{1-\delta},\quad q_k=\frac{c^k(b,\delta)}{\Gamma(k(1-\delta)+1)},\ k\ge 2,$$

where Γ is the Euler Gamma-function. Furthermore, $\sum_{k=0}^{\infty} q_k\,T^{k(1-\delta)}<\infty$ and in particular $\lim_{k\to\infty} q_k = 0$.

Applying Lemma 4.1 to (4.1) we get that for all $n\in\mathbb{N}$

$$\sup_{t\in[0,T]} \mathbb{E}||X_{n+1}(t)-X_n(t)||^{2\nu}_{L^{2\nu}_\rho}$$
$$\le CT^{n(1-\zeta\nu)}\sup_{t\in[0,T]}\mathbf{E}||X_1(t)-X_0(t)||^{2\nu}_{L^{2\nu}_\rho}\ ,$$

whereas by Hölder's inequality

$$
\begin{aligned}
\sup_{t\in[\,0,T\,]} \mathbf{E}||X_1(t)-X_0(t)||^{2\nu}_{L^{2\nu}_\rho} \quad \le \quad & c(\nu)\Bigg\{ \sup_{t\in[\,0,T\,]} \mathbb{E}\Bigg|\Bigg|\int_0^t U(t,s)F(s,U(s,0)\xi\, ds)\Bigg|\Bigg|^{2\nu}_{L^{2\nu}_\rho} \\
(4.2) \qquad & + \sup_{t\in[\,0,T\,]} \mathbb{E}\Bigg|\Bigg|\int_0^t U(t,s)\mathcal{M}_{\Sigma(s,U(s,0)\xi)}]dW(s)\Bigg|\Bigg|^{2\nu}_{L^{2\nu}_\rho} \\
& + \sup_{t\in[\,0,T\,]} \mathbb{E}\Bigg|\Bigg|\int_0^t U(t,s)[\mathcal{M}_{\Gamma(s,U(s,0)\xi)}x\,\tilde{N}(ds,dx)\Bigg|\Bigg|^{2\nu}_{L^{2\nu}_\rho}\Bigg\} \\
\le \quad & C(1+||\xi||^{2\nu}_{L^{2\nu}_\rho}).
\end{aligned}
$$

Finally, this leads to the following bound

$$||X_{n+1}-X_n||^{2\nu}_{\mathcal{G}_\nu(T)} \le T^{n(1-\zeta\nu)}C(\xi) < \infty.$$

Since $0<\nu<\frac{1}{\zeta}$, the right-hand side vanishes as $n\to\infty$, which shows that $(X_n)_{n\in\mathbb{N}}$ is a Cauchy sequence the Banach space $\mathcal{G}_\nu(T)$ and hence gives us the existence of a mild solution $X\in\mathcal{G}_\nu(T)$ to Eq.(1) on the whole intervall $[\,0,T\,]$. Assuming that X and Y are two different solutions to Eq.(1) with the same initial condition ξ, we get in a similar way

$$\mathbf{E}||X(t)-Y(t)||^{2\nu}_{L^{2\nu}_\rho} \le C\int_0^t (t-s)^{-\zeta\nu}\mathbf{E}||X(s)-Y(s)||^{2\nu}_{L^{2\nu}_\rho}\, ds.$$

The Gronwall lemma then yields for each $t\in[\,0,T\,]$ that $||X(t)-Y(t)||^{2\nu}_{L^{2\nu}_\rho}=0$, $\mathbb{P}$-a.s., which proves the uniqueness in $\mathcal{G}_\nu(T)$.

Recall that by Definition 2.2.1 $[\,0,T\,]\ni t\mapsto X(t)\in L^2_\rho$ should obey a càdlàg modification. While the càdlàg property of the Poisson stochastic convolution (3.1) was shown in Theorem 3.1, with the help of Remark 1.2 1 in [23] we even get a continuous modification of the Wiener stochastic convolution

$$(4.3) \qquad [\,0,T\,]\ni t\mapsto \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X(s))}\, dW(s)$$

being considered as a process in L^2_ρ. Furthermore, the above mapping is continuous in $L^{2\nu}(\Omega;L^{2\nu}_\rho)$ by the arguments similar to those used in the Poisson case (see the proof of Theorem 3.1 and also Proposition 3.3.5 from [29]). Under the polynomial growth assumptions imposed on f (cf. Remark 2.3), it is straighforward to prove that the Bochner integral

$$(4.4) \qquad [\,0,T\,]\ni t\mapsto \int_0^t U(t,s)F(s,X(s))\, ds$$

is pathwise continuous in L^2_ρ as well as continuous in $L^{2\nu}(\Omega;L^{2\nu}_\rho)$. Thus, X constructed above is reallly a solution to Eq.(1) in the sense of Definiton 2.6.

It remains to show the moment bound (2.23). To this end, we note that for any $t \in [0,T]$

$$
\begin{aligned}
\mathbf{E}||X(t)||^{2\nu}_{L^{2\nu}_\rho} \quad &\le \quad c(\nu)\Bigg\{\mathbb{E}\,||U(t,0)\xi||^{2\nu}_{L^{2\nu}_\rho} + \mathbb{E}\left|\left|\int_0^t U(t,s)F(s,X(s))ds)\right|\right|^{2\nu}_{L^{2\nu}_\rho} \\
&+\mathbb{E}\left|\left|\int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X(s))}\, dW(s)\right|\right|^{2\nu}_{L^{2\nu}_\rho} \\
&+\mathbb{E}\left|\left|\int_0^t U(t,s)\mathcal{M}_{\Gamma(s,X(s))}x\,\tilde{N}(ds,dx)\right|\right|^{2\nu}_{L^{2\nu}_\rho}\Bigg\} \\
&\le \quad C\left[1+\mathbb{E}||\xi||^{2\nu}_{L^{2\nu}_\rho} + \int_0^t (t-s)^{-\zeta\nu}\mathbb{E}||X(s)||^{2\nu}_{L^{2\nu}_\rho}\, ds.\right],
\end{aligned}
$$

where again $C > 0$ is some constant whose explicit value is not relevant. Herefrom, by Lemma 4.1 we get the required estimate (2.23). ■

Remark 4.2. It is also possible to drop the most restrictive condition ($\mathbf{A4}_\nu$) (and hence ($\mathbf{A4}^*_\nu$) by assuming that the Lévy measure η is concentrated on the Sobolev space $W^{m,2}$ with some $m > d/2$ and obeys the integrability condition

$$
\int_{W^{m,2}} ||x||^{2\nu}_{W^{m,2}}\,\eta(dx) + \int_{W^{m,2}} ||x||^{2}_{W^{m,2}}\,\eta(dx) < \infty. \tag{4.5}
$$

Indeed, $W^{m,2}_{\mathrm{b}} \underset{\rightarrow}{\subset} C_{\mathrm{b}}$ by Sobolev's embedding theorem (see e.g., Theorem 1 in Section 2 of [2]) and hence there exists some $c_m > 0$ such that for all $x \in W^{m,2}$

$$
\sup_{\theta\in\Theta}|x(\theta)| =: ||x||^{2\nu}_{C_{\mathrm{b}}} \le c_m||x||^{2\nu}_{W^{m,2}}.
$$

Then, our basic estimate (4.1) change for $Z_1, Z_2 \in \mathcal{G}_\nu(T)$ as follows

$$
\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}||I^{\tilde{N}}_\Gamma(Z_1-Z_2)(t)||^{2\nu}_{L^{2\nu}_\rho} \\
(4.6)\le \quad &C(\nu,T)c_\gamma(T)||U||^{2\nu}_{\mathcal{L}(L^{2\nu}_\rho)}\left[\int_{W^{m,2}} ||x||^{2\nu}_{L^\infty}\,\eta(dx) + \left(\int_{W^{m,2}} ||x||^{2}_{L^\infty}\,\eta(dx)\right)^\nu\right].
\end{aligned}
$$

Here, in the last step we have used ($\mathbf{A3}_\nu$) and the Lipschitz property of γ. Similar modifications are also required in the Bichteler-Jacod inequality (3.9). This guarantees that all claims proven above for remain true.

4.2. **Proof of Theorem 2.8.** In general, we will follow a scheme developed in [26] to prove comparison theorems for SPDEs with Wiener noises. A new issue is to take care of an additional jump term coming from Poisson integration. To control this term we will construct approximations of Eq.(1) by similar equations but with more regular coefficients and noises taking values in Sobolev spaces $W^{m,2}$. This has the following two reasons. First, we would like to keep a Hilbert space setting in order to develop a proper stochastic analysis and define Wiener and Poisson stochastic integrals via Itô's isometry (which is not evident in general Banach spaces). Second, we need to evaluate the equations pointwise at any $\theta \in \Theta$, which then allows us to apply finite-dimensional comparison results for jump diffusions obtained e.g. in [24], [30], [39]. Thus, a natural choise for our purposes is the Sobolev spaces $W^{m,2}$ with $m > \frac{d}{2}$, so that $W^{m,2}$ is densely continuously embedded in C_b.After establishing the comparison result for the regularized equations that are unique solvable in $W^{m,2}$ and then taking limits in L^2_ρ, we readily get a similar result for the initial equation.

Below we desribe the approximation scheme. For any fixed $i = 1, 2$ and $N, M, L, J \in \mathbb{N}$, we consider the following equations

$$\begin{aligned} dX_J^{(i)}(t) &= (A(t)X_J^{(i)}(t) + F_J^{(i)}(t, X_J^{(i)}(t)))dt \\ &\quad + \mathcal{M}_{\Sigma_J(t,X_J(t))}\, dW(t) + \int_{L^2} \mathcal{M}_{\Gamma_J(t,X_J^{(i)}(t))} I_J(x)\, \tilde{N}(dt, dx), \\ & X_J^{(i)}(0) = \xi_J^{(i)}; \end{aligned} \tag{4.7}$$

$$\begin{aligned} dX_{L,J}^{(i)}(t) &= (A(t)X_{L,J}^{(i)}(t) + F_J^{(i)}(t, X_{L,J}^{(i)}(t)))dt \\ &\quad + \mathcal{M}_{\Sigma_J(t,X_{M,J}^{(i)}(t))}\, dW_L(t) + \int_{L^2} \mathcal{M}_{\Gamma_J(t,X_{L,J}^{(i)}(t))} I_J(x)\, \tilde{N}(dt, dx), \\ & X_{L,J}^{(i)}(0) = \xi_J^{(i)}; \end{aligned} \tag{4.8}$$

$$\begin{aligned} dX_{M,L,J}^{(i)}(t) &= (A(t)X_{M,L,J}^{(i)}(t) + F_J^{(i)}(t, X_{M,M,J}^{(i)}(t)))dt \\ &\quad + \mathcal{M}_{\Sigma_J(t,X_{M,L,J}(t))}\, dW_{M,L}(t) + \int_{L^2} \mathcal{M}_{\Gamma_J(t,X_{M,L,J}^{(i)}(t))} I_J(x)\, \tilde{N}(dt, dx), \\ & X_{M,L,J}^{(i)}(0) = \xi_J^{(i)}; \end{aligned} \tag{4.9}$$

$$\begin{aligned} dX_{N,M,L,J}^{(i)}(t) &= (A_N(t)X_{N,M,L,J}^{(i)}(t) + F_J^{(i)}(t, X_{N,M,L,J}^{(i)}(t)))dt \\ &\quad + \mathcal{M}_{\Sigma_J(t,X_{N,M,L,J}(t))}\, dW_{M,L}(t) + \int_{L^2} \mathcal{M}_{\Gamma_J(t,X_{N,M,L,J}^{(i)}(t))} I_J(x)\, \tilde{N}(dt, dx), \\ & X_{N,M,L,J}^{(i)}(0) = \xi_J^{(i)}. \end{aligned} \tag{4.10}$$

Analogously to the proof of Theorem 3.3.1 in [26], $(A_N(t))_{t\in[0,T]} \subset \mathcal{L}(L^2_\rho) \cap \mathcal{L}(W^{m,2})$ is an element of the approximating family corresponding to $(A(t))_{t\in[0,T]}$, cf. (**A5**), and W_L is a finite-dimensional Wiener process defined by

$$W_L(t) := \sum_{n=1}^{L} \sqrt{a_n} e_n w_n(t), \quad w_n(t) := \langle W(t), e_n \rangle, \quad t \in [0,T]. \tag{4.11}$$

Compared to [26], we also need smooth approximations for the coefficient functions $f^{(i)}, \sigma, \gamma$, basis vectors e_n and initial conditions $\xi^{(i)}$. To this end, we shall use standard convolution operators in $\mathbb{R}^d$. So, for $\xi^{(i)} \in L^2_\rho(\Theta)$ (trivially extended outside Θ) we set

$$\xi_J^{(i)} := (1 + |\cdot|^2)^{\frac{\varrho}{4}} \cdot conv\left(\chi_J(|\cdot|)(1 + |\cdot|^2)^{-\frac{\varrho}{4}} \xi^{(i)}\ ,\ \delta_J\right), \quad J \in \mathbb{N}, \tag{4.12}$$

where $(\delta_J)_{J\in\mathbb{N}}$ is a Dirac sequence on $\mathbb{R}^d$ defined in (3.11) and $(\chi_J)_{J\in\mathbb{N}}$ is a sequence of cut-off functions $\chi_J : \mathbb{R}_+ \to [0,1]$, $\chi_J \in C_b^\infty(\mathbb{R}_+)$, with the properties $\chi_J(r) = 1$ for $r \in [0, J]$, $\chi_J(r) = 0$ for $r \geq J+1$ and $\chi_{J+1}(r) = \chi_J(r-1)$ for $r \geq 1$. Then, each $\xi_J^{(i)}$ is an $\mathcal{F}_0$-measurable random variable in $W^{m,2}$ such that $\mathbb{E}||\xi_J^{(i)}||^2_{W^{m,2}} < \infty$. Obviously, $\mathbb{E}||\xi^{(i)} - \xi_J^{(i)}||^2_{L^2_\rho} \to 0$ as $J \to \infty$ and $\xi^{(1)} \leq \xi^{(2)}$ implies $\xi_J^{(1)} \leq \xi_J^{(2)}$.

We next approximate each basis vector $e_n \in L^2$ by a sequence

$$e_{n,M} := conv\,(\chi_M(|\cdot|)e_n\ ,\ \delta_M) \in C_0^\infty(\mathbb{R}^d), \quad M \in \mathbb{N}, \tag{4.13}$$

and respectively define

$$W_{M,L}(t) := \sum_{n=1}^{L} \sqrt{a_n}\, e_{n,M} w_n(t), \quad t \in [0,T]. \tag{4.14}$$

Similarly, the mappings

$$L^2 \ni \psi \mapsto I_J(\psi) := conv(\psi, \delta_J), \quad J \in \mathbb{N}, \tag{4.15}$$

form an approximation of the identity function $I(\psi) := \psi$ in L^2.

Finally, using cut-off and convolution operators in $\mathbb{R}^d$, we approximate $f^{(i)}$, σ and γ by smooth functions with the following properties:

- The k-th derivatives of $f_J^{(i)}$, σ_J and γ_J w.r.t. $y \in \mathbb{R}$ are bounded and continuous for $k = 0, 1, ..., m+1$ and $i = 1, 2$;
- $f_J^{(1)}(t,y) \leq f_J^{(2)}(t,y)$ for all $(t,y) \in [0,T] \times \mathbb{R}$, $\mathbb{P}$-a.s.;
- $f_J^{(i)}(t,\omega,y) \to f^{(i)}(t,\omega,y)$ as $J \to \infty$ for any $(t,\omega,y) \in [0,T] \times \Omega \times \mathbb{R}$ and $i = 1, 2$.
  The same holds for σ_J and γ_J.
- The functions $f_J^{(i)}$, σ_J and γ_J are Lipschitz continuous and locally bounded at 0 (like as in $(\mathbf{\Sigma})$(**i**),(**ii**)), uniformly in $J \in \mathbb{N}$.

A crucial moment here is that the convolution operator is contraction not only in L^2 but also in the Lipschitz norm (see e.g. [37]), which guarantees that all $f_J^{(i)}$, $J \in \mathbb{N}$, are Lipschitzian with the same constant. Furthermore, the convolution operator also preserves the monotonicity property, which implies that $f_J^{(1)} \leq f_J^{(2)}$ and $\xi_J^{(1)} \leq \xi_J^{(2)}$.

Theorem 2.7 gives us existence of the unique mild solutions to Eqs. (4.7)–(4.10) in the space L^2_ρ. Furthermore, since all generating functions wetre chosen to have bounded smooth derivatives up to order $m+1$, the associated Nemitskii operators are *Lipschitz continuous* in $W^{m,2}$. So, we also have the unique *strong* (=mild) solution $X^{(i)}_{N,M,L,J} \in \mathcal{W}^2_m(T)$ to (4.10), where $\mathcal{W}^2_m(T)$ is the Banach space of all predictable (up to a stochastic modification) $W^{m,2}$-valued processes $X = (X(t))_{t\in[0,T]}$ such that

$$\sup_{t\in[0,T]} \mathbf{E}||X(t)||^2_{W^{m,2}} < \infty. \tag{4.16}$$

The solution process $X^{(i)}_{N,M,L,J} \in \mathcal{W}^2_m(T)$ obeys a càdlàg modification in $W^{m,2}$. Furthermore, all solutions to (4.7)– (4.10) have càdlàg versions in L^2_ρ.

The proof of the comparison theorem can be divided into two consecutive steps:

Step 1 *Given any $N, M, L, J \in \mathbb{N}$, for the càdlàg mild solutions $Y^{(i)} := X^{(i)}_{N,M,L,J}$, $i = 1, 2$, we show that*

$$Y^{(1)}(t) \le Y^{(2)}(t), \quad \text{for all } t \in [\,0, T\,], \quad \mathbb{P}\text{-a.s.}.$$

Step 2 *We check the following convergence for the solutions to Eq.(1) and (4.7)–(4.10)*

$$\lim_{N\to\infty} \mathbb{E}||X^{(i)}_{N,M,L,J}(t) - X^{(i)}_{M,L,J}(t)||^2_{L^2_\rho} = 0, \qquad \lim_{M\to\infty} \mathbb{E}||X^{(i)}_{M,L,J}(t) - X^{(i)}_{L,J}(t)||^2_{L^2_\rho} = 0,$$
$$\lim_{L\to\infty} \mathbb{E}||X^{(i)}_{L,J}(t) - X^{(i)}_{J}(t)||^2_{L^2_\rho} = 0, \qquad \lim_{J\to\infty} \mathbb{E}||X^{(i)}_{J}(t) - X^{(i)}(t)||^2_{L^2_\rho} = 0.$$

Proof of Step 1: Adapting the time-discretization scheme proposed in [26], we fix $j \in \mathbb{N}$ and take a partition $t_k := kT/j$, $0 \le k \le j$, of the interval $[0, T]$. Consider the following equations

$$Z^{(i)}_{0,j}(t) := \qquad \xi^{(i)}_J + \int_0^t \mathcal{M}_{\Sigma_J(s, Z^{(i)}_{0,j}(s))}\, dW_{M,L}(s) + \int_0^t \int_{L2} \mathcal{M}_{\Gamma_J(s, Z^{(i)}_{0,j}(s))} I_J(x)\, \tilde{N}(ds, dx),$$

$$V^{(i)}_{0,j}(t) := \qquad Z^{(i)}_{0,j}(t_1) + \int_0^t (A_N(s) V^{(i)}_{0,j}(s) + F^{(i)}_J(s, V^{(i)}_{0,j}(s)))\, ds$$

for $t \in [\,0, t_1\,]$ and

$$Z^{(i)}_{k,j}(t) := \qquad V^{(i)}_{k-1,j}(t_k) + \int_{t_k}^t \mathcal{M}_{\Sigma_J(s, Z^{(i)}_{k,j}(s))}\, dW_{M,L}(s) + \int_{t_k}^t \int_{L2} \mathcal{M}_{\Gamma_J(s, Z^{(i)}_{k,j}(s))} I_J(x)\, \tilde{N}(ds, dx),$$

$$V^{(i)}_{k,j}(t) := \qquad Z^{(i)}_{k,j}(t_{k+1}) + \int_{t_k}^t (A_N(s) V^{(i)}_{k,j}(s) + F^{(i)}_J(s, V^{(i)}_{k,j}(s)))\, ds$$

for $t \in [t_k, t_{k+1}]$ and $k = 1, 2, ..., j-1$. Due to the Lipschitz continuity of the coefficients, the above equations obey unique strong solutions $V^{(i)}_{k,j}$, $Z^{(i)}_{k,j} \in \mathcal{W}^2_m([\,t_k, t_{k+1}\,])$. Next, we define processes $V^{(i)}_j$, $Z^{(i)}_j \in \mathcal{W}^2_m(T)$ by setting

$$
\begin{aligned}
(4.17) \qquad Z^{(i)}_j(t) &:= \quad Z^{(i)}_{k,j}(t), \quad t \in [\,t_k, t_{k+1}\,), \quad k = 0, 1, 2, ..., j-1, \\
V^{(i)}_j(0) &:= \quad \xi^{(i)}_J, \\
V^{(i)}_j(t) &:= \quad V^{(i)}_{k,j}(t), \quad t \in (t_k, t_{k+1}], \quad k = 0, 1, ..., j-1, \\
Z^{(i)}_j(T) &:= \quad V^{(i)}_j(T).
\end{aligned}
$$

Because of the continuous embedding $W^{m,2} \underset{\rightarrow}{\subseteq} C_b$, we can evaluate $V^{(i)}_j(t, \theta)$ resp. $Z^{(i)}_j(t, \theta)$ for any $t \in [\,0, T\,]$ and $\theta \in \Theta$.

Merely speaking, the idea of such discretization is to separate stochastic and deterministic (i.e., operator) terms in Eq. (1), since their analysis require quite different methods. Below we will check that this approximation preserves the odering and that $Z^{(i)}_j(t)$ and $V^{(i)}_j(t)$ both converge to $Y^{(i)} := X^{(i)}_{N,M,L,J}$ as $j \to \infty$.

Claim 1: For V_j, Z_j defined as above we have, $\mathbb{P}$-a.s., for any $\theta \in \Theta$ and $t \in [0,T]$.

$$\text{(i)} \qquad V_j^{(1)}(t,\theta) \le V_j^{(2)}(t,\theta), \tag{4.18}$$
$$\text{(ii)} \qquad Z_j^{(1)}(t,\theta) \le Z_j^{(2)}(t,\theta).$$

Proof: Let us start with the intervall $[0,t_1]$. By construction, for all $t \in [0,t_1)$

$$Z_j^{(i)}(t) = Z_{0,j}^{(i)}(t) = \xi_J^{(i)} + \int_0^t \mathcal{M}_{\Sigma_J(s,Z_j^{(i)}(s))}\, dW_{M,L}(s) + \int_0^t \int_{L^2} \mathcal{M}_{\Gamma_J(s,Z_j^{(i)}(s))} I_J(x)\, \tilde{N}(ds,dx). \tag{4.19}$$

To estimate the values of $Z_j^{(i)}(t,\theta)$ pointwise, we consider the pairing of $Z_j^{(i)}(t) \in W^{m,2}$ with δ_θ, which is the δ-function at a fixed θ. As δ_θ is a linear bounded functional on $W^{m,2}$, we get

$$\begin{aligned} Z_j^{(i)}(t,\theta) &= \langle Z_j^{(i)}(t), \delta_\theta\rangle_{L^2} \\ &= \xi_J^{(i)}(\theta) + \sum_{n=1}^{L} \sqrt{a_n} \int_0^t \langle \mathcal{M}_{\Sigma_J(s,Z_j^{(i)}(s))} e_{n,M}\ ,\ \delta_\theta\rangle_{L^2}\, dw_n(s) \\ &\quad + \int_0^t \int_{L^2} \langle \mathcal{M}_{\Gamma_J(s,Z_j^{(i)}(s))} I_J(x), \delta_\theta\rangle_{L^2}\, \tilde{N}(ds,dx). \end{aligned} \tag{4.20}$$

Obviously, for any $\theta \in \Theta$ we have $\mathbb{P}$-a.s.

$$\int_0^t \langle \mathcal{M}_{\Sigma_J(s,Z_j^{(i)}(s))} e_{n,M}\ ,\ \delta_\theta\rangle_{L^2}\, dw_n(s) = \int_0^t \sigma_J(s, Z_j^{(i)}(s,\theta)) e_{n,M}(\theta)\, dw_n(s).$$

The integral w.r.t. $\tilde{N}$ in (4.20) can be rewritten as

$$\int_0^t \int_{L^2} [\gamma_J(s, Z_j^{(i)}(s)) I_J(x)](\theta)\, \tilde{N}(ds,dx). \tag{4.21}$$

Note that

$$I_J(x)(\theta) := \int_{\mathbb{R}^d} x(y)\delta_J(\theta - y)\, dy = \langle x\ ,\ \delta_{\theta,J}\rangle_{L^2},$$

where $0 \le \delta_{\theta,J} \in C_0^\infty(\mathbb{R}^d)$ is defined by $\delta_{\theta,J}(y) := \delta_J(\theta - y)$, $y \in \mathbb{R}^d$, whereby $(\delta_J)_{J\in\mathbb{N}}$ is a Dirac sequence on $\mathbb{R}^d$, cf. (3.11). Thus, we have the identity

$$\int_0^t \int_{L^2} \langle \mathcal{M}_{\Gamma_J(s,Z_j^{(i)}(s))} I_J(x), \delta_\theta\rangle_{L^2}\, \tilde{N}(ds,dx) = \int_0^t \int_{\mathbb{R}} \gamma_J(s, Z_j^{(i)}(s,\theta)) u\, \tilde{N}_\theta(ds,du), \tag{4.22}$$

where $\tilde{N}_\theta(s, du)$ is the projection of $\tilde{N}(s, dx)$ on $L^2_\theta \cong \mathbb{R}$, which is a one-dimensional subspace in L^2 defined by

$$L^2_\theta := \{\langle x, \delta_{J,\theta}\rangle_{L^2} \cdot \delta_{J,\theta} \,|\, x \in L^2(\Theta)\}.$$

Obviously, $\tilde{N}_\theta(s, du)$ is a Poisson random measure on $[0, T] \times \mathbb{R}$ with the Lévy intensity measure η_θ being the corresponding projection of η. Since by Assumption (**P**) the measure η is supported on $L^2_{\geq 0}$, the measure η_θ is respectively supported on $\mathbb{R}_+$. Here, we took into account that $< x, \delta_{J,\theta} >_{L^2} \geq 0$ for any $x \in L^2_{\geq 0}$.

Now, by the Lipschitz properties of σ and γ and the monotonicity of γ, we can apply the comparison results for càdlàg solutions of finite dimensional SDEs from [24], [30], [39]. According to its definition in (4.17), $Z^{(i)}_j$ obeys a càdlàg version in $W^{m,2}$. Then by (4.20)–(4.22), $Z^{(i)}_j(t, \theta) =< Z^{(i)}_j(t), \delta_\theta >_{L^2} \in \mathbb{R}$ is a càdlàg solution to the equation

$$\begin{aligned} Z^{(i)}_j(t,\theta) \quad &= \quad \xi^{(i)}_J(\theta) + \sum_{n=1}^{L} \sqrt{a_n} \int_0^t \sigma_J(s, Z^{(i)}_j(s,\theta)) e_{n,M}(\theta)\, dw_n(s) \\ &\quad + \int_0^t \int_{\mathbb{R}} \gamma_J(s, Z^{(i)}_j(s-,\theta)) u\, \tilde{N}_\theta(ds, du). \end{aligned} \tag{4.23}$$

Thus, for each fixed $\theta \in \Theta$, we have $Z^{(1)}_j(t,\theta) \leq Z^{(2)}_j(t,\theta)$, $\mathbb{P}$-a.s., for all $t \in [\,0, t_1\,)$. Since $\theta \mapsto Z^{(i)}_j(t,\theta)$ is a continuous function, there exists a subset Ω_t of full $\mathbb{P}$-measure such that

$$Z^{(1)}_j(t,\omega,\theta) \leq Z^{(2)}_j(t,\omega,\theta)$$

for all $\theta \in \Theta$ and $\omega \in \Omega_t$. By considering a càdlàg version of $Z^{(i)}_j(t) \in W^{m,2}$, we conclude that the above inequality holds, for all $t \in [\,0, t_1)$ and $\theta \in \Theta$, on a universal subset Ω_0 of full $\mathbb{P}$-measure. The same reasoning applied to a cadlag version of $Z^{(i)}_{0,j}(t) \in W^{m,2}$ shows that for all $\theta \in \Theta$ and $\omega \in \Omega_{t_1}$ with $\mathbb{P}(\Omega_{t_1}) = 1$

$$Z^{(1)}_{0,j}(t_1,\omega,\theta) \leq Z^{(2)}_{0,j}(t_1,\omega,\theta).$$

Now we consider the terms $V^{(1)}_j(t), V^{(2)}_j(t) \in W^{m,2} \subset L^2_\rho$, $t \in [0, t_1]$. Obviously, $V^{(1)}_j(0) \leq V^{(2)}_j(0)$, $\mathbb{P}-$a.s.. To this end, we define linear operators $B_J(t) : L^2_\rho \to L^2_\rho$ by

$$B_J(t)\varphi := \frac{F^{(2)}_J(t, V^{(2)}(t)) - F^{(2)}_J(s, V^{(1)}(t))}{V^{(2)}_j(t) - V^{(1)}_j(t)}\,\varphi, \quad \varphi \in L^2_\rho,$$

in the case of $V^{(2)}_j(t) \neq V^{(1)}_j(t)$ and

$$B_J(t)\varphi := c_f(T)\varphi, \quad \varphi \in L^2_\rho,$$

otherwise, where $c_f(T) > 0$ is a Lipschitz constant common for all $F^{(i)}_J$, $J \in \mathbb{N}$, $i = 1, 2$. Now we can apply general arguments from the perturbation theory of evolution operators (see pp. 64-65 in [26]) to get

$$V^{(1)}_j(t) \leq V^{(2)}_j(t) \;\text{ in }\; W^{m,2}(\Theta) \;\text{ for all } t \in [\,0, t_1\,], \;\; \mathbb{P}-\text{a.s.}.$$

By the continuous embedding property $W^{m,2} \underset{\to}{\subseteq} C_b$, this gives us the pointwise inequality (4.18) (ii) on $[\,0, t_1\,]$.

By repeating the previous procedure on all intervalls $[t_k, t_{k+1}]$, $k = 1, 2, ..., j-1$, and we get (4.18) on the whole $[\,0, T\,]$. $\triangle$

Claim 2: *For* $i = 1, 2$

$$\lim_{j\to\infty} Z_j^{(i)} = Y^{(i)}$$

in $\mathcal{H}^2(T) := \mathcal{G}_\nu(T) = \mathcal{W}^{m,2}$ *for* $\nu = 1$ *and* $m = 0$, *that is*

$$\lim_{j\to\infty} \sup_{t\in[\,0,T\,]} \mathbb{E}||Z_j^{(i)}(t) - Y^{(i)}(t)||^2_{L^2_\rho} = 0. \tag{4.24}$$

Hence, inequality (4.18) (cf. Claim 1) implies that $Y^{(1)}(t) \le Y^{(2)}(t)$, $\mathbb{P}$-a.s., *for all* $t \in [\,0, T\,]$.

Proof: Actually, in (4.24) we have the pointwise estimate $Y^{(1)}(t,\theta) \le Y^{(2)}(t,\theta)$, $\mathbb{P}$-a.s., for all $t \in [\,0,T\,]$ and $\theta \in \Theta$. This will follow for the càdlàg solutions $Y^{(1)}$, $Y^{(2)}$ due to the continuous embedding $W^{m,2} \underset{\to}{\subseteq} C_b$.

Analogously to the proof of Theorem 3.3.1 in [26], we express $Z_j^{(i)}(t) - Y^{(i)}(t)$ in terms of $V_j^{(i)}(t) - Y^{(i)}(t)$. Since γ is Lipschitz continuous in $\mathbb{R}$ and unifomly bounded, one can check by Gronwall's lemma that

$$\mathbb{E}||V_j^{(i)}(t) - Y^{(i)}(t)||^2_{L^2_\rho} \le C(T)\cdot \beta_j(t_{k+1}) \tag{4.25}$$

for any $t \in [\,t_k, t_{k+1}\,)$, $k \in \{0, 1, \dots, j-1\}$ and some $C(T) > 0$ depending on M, $N, L \in \mathbb{N}$ and $c_f(T)$, $c_\sigma(T)$, $c_\gamma(T)$. Here,

$$\beta_j(t_{k+1}) := \int_0^{t_{k+1}} \mathbb{E}||Z_j^{(i)}(t) - Y^{(i)}(t)||^2_{L^2_\rho} ds + \frac{1}{j}\left(\sup_{t\in[\,0,T\,]} \mathbb{E}||Y^{(i)}(t)||^2_{L^2_\rho} + 1\right). \tag{4.26}$$

On the other hand,

$$\begin{aligned} &\mathbb{E}||Z_j^{(i)}(t) - Y^{(i)}(t)||^2_{L^2_\rho} \\ \le\ & C\Big\{\mathbb{E}||V_j^{(i)}(t) - Y^{(i)}(t)||^2_{L^2_\rho} \\ &+\mathbb{E}\left\|\int_{t_k}^{t} \left(\mathcal{M}_{\Sigma_J(s,Z_j^{(i)}(s))} - \mathcal{M}_{\Sigma_J(s,Y^{(i)}(s))}\right) dW_{M,L}(s)\right\|^2_{L^2_\rho} \\ &+\mathbb{E}\left\|\int_{t_k}^{t}\int_{L^2} \left(\mathcal{M}_{\Gamma_J(s,Z_j^{(i)}(s))} - \mathcal{M}_{\Gamma_J(s,Y^{(i)}(s))}\right) I_J(x)\,\tilde{N}(ds,dx)\right\|^2_{L^2_\rho} \\ &+ \mathbb{E}\int_{t_k}^{t} \left(||A_N||^2_{\mathcal{L}(L^2_\rho)}||Y^{(i)}(s)||^2_{L^2_\rho} + ||F^{(i)}(s, Y^{(i)}(s)||_{L^2_\rho}\right) ds\Big\}. \end{aligned} \tag{4.27}$$

Except the term with Poisson integral, all other terms in (4.27) can be estimated similarly to [26]. Using that all γ_J are uniformly Lipschitz and that $||I_J(x)||^2_{L^2} \le$

$||x||^2_{L^2}$, by Itô's isometry we get that

$$\begin{aligned} &\mathbb{E}\left\|\int_{t_k}^{t}\int_{L^2}\mathcal{M}_{\Gamma_J(s,Z_j^{(i)}(s))-\Gamma_J(s,Y^{(i)}(s))}I_J(x)\,\tilde{N}(ds,dx)\right\|^2_{L^2_\rho} \\ \leq\;& c_\gamma^2(T)\left(\int_{L^2}||x||^{2\nu}_{L^2}\,\eta(dx)\right)\int_{t_k}^{t}\mathbb{E}||Z_j^{(i)}(s)-Y^{(i)}(s)||^2_{L^2_\rho}\,ds. \end{aligned}$$

Summing up in (4.27), we get with some constant $C(T,M,N,L)>0$

$$\mathbb{E}||Z_j^{(i)}(t)-Y^{(i)}(t)||^2_{L^2_\rho}\leq C(T,M,N,L)\cdot\left(\frac{1}{j}+\int_0^t\mathbf{E}||Z_j^{(i)}(s)-Y^{(i)}(s)||^2_{L^2_\rho}\,ds\right).$$

Gronwall's lemma finally implies

$$\lim_{j\to\infty}\mathbb{E}||Z_j^{(i)}(t)-Y^{(i)}(t)||^2_{L^2_\rho}=0,$$

which was needed to prove Claim 2. $\triangle$

Thus, we are finished with Step 1.

Proof of Step 2: We only sketch how to check the J-convergence, while the M-, N- and L-convergence can be established in the spirit of the proof of Theorem 3.3.1 in [26]. So, for a fixed $J\in\mathbb{N}$ we have

$$X_J^{(i)}-X^{(i)}=a_{\xi,J}+a_{F,J}+b_{F,J}+a_{\Sigma,J}+b_{\Sigma,J}+a_{\Gamma,J}+b_{\Gamma,J}\ ,$$

with the corresponding terms defined for $t\in[0,T]$ by

$$\begin{aligned} a_{\xi,J}(t)\ &:=U(t,0)[\xi_J^{(i)}-\xi^{(i)}],\\ a_{F,J}(t)\ &:=\int_0^t U(t,s)[F_J^{(i)}(s,X^{(i)}(s))-F^{(i)}(s,X^{(i)}(s))]\,ds,\\ b_{F,J}(t)\ &:=\int_0^t U(t,s)[F_J^{(i)}(s,X_J^{(i)}(s))-F_J^{(i)}(s,X^{(i)}(s))]\,ds,\\ a_{\Sigma,J}(t)\ &:=\int_0^t U(t,s)[\mathcal{M}_{\Sigma_J(s,X^{(i)}(s))}-\mathcal{M}_{\Sigma(s,X^{(i)}(s))}]\,dW(s),\\ b_{\Sigma,J}(t)\ &:=\int_0^t U(t,s)[\mathcal{M}_{\Sigma_J(s,X_J^{(i)}(s))}-\mathcal{M}_{\Sigma_J(s,X^{(i)}(s))}]\,dW(s), \end{aligned}$$

and

$$\begin{aligned} a_{\Gamma,J}(t)\ &:=\int_0^t\int_{L^2}U(t,s)[\mathcal{M}_{\Gamma_J(s,X^{(i)}(s))}I_J(x)-\mathcal{M}_{\Gamma(s,X^{(i)}(s))}x]\,\tilde{N}(ds,dx),\\ b_{\Gamma,J}(t)\ &:=\int_0^t\int_{L^2}U(t,s)[\mathcal{M}_{\Gamma_J(s,X_J^{(i)}(s))}-\mathcal{M}_{\Gamma_J(s,X^{(i)}(s))}]I_J(x)\,\tilde{N}(ds,dx). \end{aligned}$$

Below we prove that, for any $t \in [0,T]$, the a_J-terms tend to 0 in $\mathbb{L}^2(\Omega; L^2_\rho)$ as $J \to \infty$ and that

$$\mathbb{E}||b_{F,J}(t)||^2_{L^2_\rho} \leq C_F \int_0^t \mathbb{E}||X_J^{(i)} - X^{(i)}||^2_{L^2_\rho}\, ds, \tag{4.28}$$

$$\mathbb{E}||b_{\Sigma,J}(t)||^2_{L^2_\rho} \leq C_\Sigma \int_0^t \mathbb{E}||X_J^{(i)} - X^{(i)}||^2_{L^2_\rho}\, ds, \tag{4.29}$$

$$\mathbb{E}||b_{\Gamma,J}(t)||^2_{L^2_\rho} \leq C_\Gamma \int_0^t (t-s)^{-\varsigma}\mathbb{E}||X_J^{(i)} - X^{(i)}||^2_{L^2_\rho}\, ds \tag{4.30}$$

with positive constants C_F, C_Σ and C_Γ that are uniform for all $J \in \mathbb{N}$. Indeed, since

- $I_J(x) \underset{J\to\infty}{\longrightarrow} x$ in L^2 and $||I_J(x)||_{L^2} \leq ||x||_{L^2}$,
- $f_J \to f$ as $J \to \infty$, $(f_J)_{J\in\mathbb{N}}$ uniformly Lipschitz in J,
- $\sigma_J \to \sigma$ as $J \to \infty$, $(\sigma_J)_{J\in\mathbb{N}}$ unifomly Lipschitz in J,

we immediately get the required property for $a_{\xi,J}$, $a_{F,J}$ and $a_{\Sigma,J}$. Using (**A4**) we obtain

$$\begin{aligned}
\mathbb{E}||a_J(\Gamma)||^2_{L^2_\rho} &= \int_0^t \int_{L^2} \mathbb{E}\left\|\mathcal{M}_{\Gamma_J(s,X^{(i)}(s))} I_J(x) - \mathcal{M}_{\Gamma(s,X^{(i)}(s))} x\right\|^2_{L^2_\rho} \eta(dx)\, ds \\
&\leq 2\Bigg\{ \int_0^t \int_{L^2} \mathbb{E}\left\|U(t,s)[\mathcal{M}_{\Gamma_J(s,X^{(i)}(s))}(I_J(x) - x)]\right\|^2_{L^2_\rho} \eta(dx)\, ds \\
&\quad + \int_0^t \int_{L^2} \mathbb{E}\left\|U(t,s)[\mathcal{M}_{\Gamma_J(s,X^{(i)}(s))} - \mathcal{M}_{\Gamma(s,X^{(i)}(s))}]x\right\|^2_{L^2_\rho} \eta(dx)\, ds \Bigg\} \\
&\leq 2\Bigg\{ T C_\gamma^2(T) ||U||^2_{\mathcal{L}(L^2_\rho)} \left(\int_{L^2} ||I_J(x) - x||^2_{L^2}\, \eta(dx)\right) \\
&\quad + c_{1,\varsigma} \left(\int_{L^2} ||x||^2_{L^2}\, \eta(dx)\right) \int_0^t (t-s)^{-\varsigma} \mathbb{E}\left\|\gamma_J(s, X^{(i)}(s)) - \gamma(s, X^{(i)}(s))\right\|^2_{L^2_\rho} ds \Bigg\},
\end{aligned}$$

which by Lebesgue's dominated convergence theorem tends to 0 as $J \to \infty$.

Estimates (4.28) and (4.29) for $b_{F,J}$ and $b_{\Sigma,J}$ follow by standard properties of Bochner and Wiener integrals. Furthermore, (**A4**) and ($\mathbf{N}_\nu$) with $\nu = 1$ lead to

$$\begin{aligned}
&\mathbb{E}||b_{\Gamma,J}(t)||^2_{L^2_\rho} \\
&\leq c_{1,\varsigma} \left(\int_{L^2} ||I_J(x)||^2_{L^2}\, \eta(dx)\right) \int_0^t (t-s)^{-\varsigma} \mathbb{E}\left\|\gamma_J(s, X_J^{(i)}(s)) - \gamma_J(s, X^{(i)}(s))\right\|^2_{L^2_\rho} ds \\
&\leq c_{1,\varsigma} c_\gamma^2(T) \left(\int_{L^2} ||x||^2_{L^2}\, \eta(dx)\right) \int_0^t (t-s)^{-\varsigma} \mathbb{E}\left\|X_J^{(i)}(s)) - X^{(i)}(s)\right\|^2_{L^2_\rho} ds,
\end{aligned}$$

which holds uniformly in $J \in \mathbb{N}$. The Gronwall-Bellman lemma now gives the final result. ■

4.3. **Proof of Theorem 2.9.** The proof of the existence result is strongly based on the comparison result established in the previous theorem. To this end, we adapt a standard scheme used for similar SPDEs in weighted $\mathbb{L}^p$-spaces driven by a Wiener noise in [23] and [26]. For convenience, we devide the proof into several steps.

Step 1: For $N, M \in \mathbb{N}$ define

$$f_N(t,y) \ : \ = f(t,y) \vee (-N), \tag{4.31}$$

$$f_{N,M}(t,y) \ : \ = \inf_{u\in\mathbf{R}} \{ f_N(t,u) + M|u-y| \}, \quad (t,y) \in [0,T]\times\mathbb{R}, \tag{4.32}$$

such that

$$\begin{aligned} &f_{N,M} \uparrow f_N \quad \text{as} \quad M\to\infty, \quad f_N \downarrow f \quad \text{as} \quad N\to\infty, \\ &|f_{N,M}(t,y) - f_{N,M}(t,y)| \le M|u-v|, \quad u,v\in\mathbb{R}. \end{aligned} \tag{4.33}$$

Hence, for each N, $M \in \mathbb{N}$ there exists a unique solution $X_{N,M} \in \mathcal{G}_v(T)$ to Eq.(1) with F being replaced by the Nemytskii operator $F_{N,M}$ generated by function $f_{N,M}$. Furthermore, by the monotonicity property of $f_{N,M}$ and Theorem 2.7, for any $t \in [\,0,T\,]$ we have $\mathbb{P}-$a.s.

$$X_{N,M}(t) \le X_{N,M+1}(t). \tag{4.34}$$

Similarly to the proof of Theorem 3.4.1 in [26], we intend to prove an M-independent estimate for the $X_{N,M}$. To this end, let us also consider

- $\bar{X}_{0,M}$ – the unique solution to Eq.(1) with initial condition $\xi^+ := \xi \vee 0$ and drift $F_{0,M}$,
- $\underline{X}_{N,M}$ – the unique solution to Eq.(1) with initial condition $\xi^- := \xi \wedge 0$ and drift $F^-_{N,M} := F_{N,M} \wedge 0$,
- V – the unique solution to Eq.(1) with initial condition $\xi = 0$ and drift $F = 0$.

Theorem 2.7 tells us that $\mathbb{P}$-a.s.

$$\begin{aligned} \underline{X}_{N,M}(t) \le X_{N,M}(t) \le \bar{X}_{0,M}(t), \\ \underline{X}_{N,M}(t) \le V(t) \le \bar{X}_{0,M}(t), \end{aligned} \tag{4.35}$$

for each $t \in [\,0,T\,]$ and $N, M \in \mathbb{N}$. We first prove that

$$\sup_{\substack{t\in[\,0,T\,]\\ M\in\mathbb{N}}} \mathbb{E}||\bar{X}_{0,M}(t)||^{2\nu}_{L^{2\nu}_\rho} < \infty. \tag{4.36}$$

Indeed,

$$\mathbf{E}||\bar{X}_{0,M}(t)||^{2\nu}_{L^{2\nu}_\rho} \le c(\nu)\left\{ \bar{I}^{(1)}(t) + \bar{I}^{(2)}_M(t) + \bar{I}^{(3)}_M(t) + I^{(4)}_M(t) \right\}$$

with

$$\bar I^{(1)}(t) := \quad \mathbb{E}||U(t,0)\xi^+||^{2\nu}_{L^{2\nu}_\rho},$$

$$\bar I^{(2)}_M(t) := \quad \mathbb{E}\left|\left|\int_0^t U(t,s)F_{0,M}(s,\bar X_{0,M}(s))\,ds\right|\right|^{2\nu}_{L^{2\nu}_\rho},$$

$$\bar I^{(3)}_M(t) := \quad \mathbb{E}\left|\left|\int_0^t U(t,s)\mathcal{M}_{\Sigma(s,\bar X_{0,M}(s))}\,dW(s)\right|\right|^{2\nu}_{L^{2\nu}_\rho},$$

$$\bar I^{(4)}_M(t) := \quad \mathbb{E}\left|\left|\int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\Gamma(s,\bar X_{0,M}(s))}(x)\,\tilde N(ds,dx)\right|\right|^{2\nu}_{L^{2\nu}_\rho}.$$

Here, $\bar I^{(2)}_M(t)$, $\bar I^{(2)}_M(t)$ and $\bar I^{(3)}_M(t)$ are the same as on page 73 in [26]. A crucial moment is the estimate

$$\begin{aligned}\bar I^{(2)}_M(t) \quad : \quad &= \mathbb{E}\left|\left|\int_0^t U(t,s)F_{0,M}(s,\bar X_{0,M}(s))\,ds\right|\right|^{2\nu}_{L^{2\nu}_\rho}\\ &\le \quad c(\nu,T,c_f(T))\left[1+\int_0^t \mathbb{E}||\bar X_{0,M}(s)||^{2\nu}_{L^{2\nu}_\rho}ds+\int_0^T \mathbb{E}||V(s)||^{2\nu^2}_{L^{2\nu}_\rho}ds\right],\end{aligned}$$

where $V\in\mathcal{G}_\upsilon(T)$ is the solution to

$$V(t)=\int_0^t U(t,s)\mathcal{M}_{\Sigma(s,V(s))}dW(s)+\int_0^t\int_{L2} U(t,s)\mathcal{M}_{\Gamma(s,V(s))}(x)\,\tilde N(ds,dx).$$

To derive it one uses the one-sided growth condition on f. Next, similarly to the proof of (4.1) one can show that $\sup_{t\in[0,T]}\mathbb{E}||V(t)||^{2\nu^2}_{L^{2\nu}_\rho}ds<\infty$ provided $(\mathbf{N}_{\eta'})$ holds with $\eta'=\nu^2$. Furthermore, $\sup_{M\in\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}||\bar I^{(4)}_M(t)||^{2\nu}_{L^{2\nu}_\rho}ds<\infty$ due to the boundedness assumption imposed on of γ. By the Gronwall-Belmann lemma we finally arrive at (4.36).

In a similar way one can derive an M-independent estimate for $\underline{X}_{N,M}$, which then by (4.35) implies

$$\sup_{M\in\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}||X_{N,M}(t)||^{2\nu}_{L^{2\nu}_\rho}ds\le C_\nu(T,N)<\infty. \tag{4.37}$$

Step 2: For $N\in\mathbb{N}$ we define processes $(X_N)_{t\in[0,T]}$ by

$$X_N(t):=Z_N(t)+X_{N,1}(t),\quad t\in[0,T], \tag{4.38}$$

with

$$0\le Z_N(t):=\sup_{M\in\mathbb{N}}Z_{N,M}(t),\quad t\in[0,T],\quad N\in\mathbb{N},$$

and

$$Z_{N,M}(t):=X_{N,M}(t)-X_{N,1}(t),\quad t\in[0,T],\quad N,M\in\mathbb{N}.$$

Note that by (4.34)

$$0\le Z_{N,M}(t)\le Z_{N,M+1}(t).$$

Actually, for each $t \in [0,T]$ and $N \in \mathbb{N}$, the random variable $Z_N(t)$ is uniquely defined up to a $\mathbb{P} \otimes \mu_\rho$-zero set in $\Omega \times \Theta$ (which depends on the $\mathcal{B}(\Omega) \otimes \mathcal{B}(\Theta)$ representations chosen for $Z_{N,M}(t)$). Furthermore, by (4.37) and B. Levi's monotone convergence theorem

$$\sup_{t\in[0,T]} \mathbb{E}||Z_N(t)||^{2\nu}_{L^{2\nu}_\rho} ds = \sup_{M\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}||Z_{N,M}(t)||^{2\nu}_{L^{2\nu}_\rho} ds < \infty$$

and hence

$$\sup_{t\in[0,T]} \mathbb{E}||X_N(t)||^{2\nu}_{L^{2\nu}_\rho} ds \leq \bar{C}_\nu(T,N) < \infty. \tag{4.39}$$

By construction $t \to X_N(t) \in L^{2\nu}_\rho$ obeys a predictable modification. Thus, $(X_N(t))_{t\in[0,T]}$ is a process in $\mathcal{G}_v(T)$. Furthermore, by Lebegue's dominated convergence theorem we have

(4.40)

$$\lim_{M\to\infty} \int_0^T \mathbb{E}||X_{N,M}(t) - X_N(t)||^{2\nu}_{L^{2\nu}_\rho} \, dt = \lim_{M\to\infty} \int_0^T \mathbb{E}||Z_{N,M}(t) - Z_N(t)||^{2\nu}_{L^{2\nu}_\rho} \, dt = 0.$$

Analogously we define the processes $\underline{X}_N$, $\bar{X} \in \mathcal{G}_v(T)$ such that

$$\underline{X}_N(t) \leq X_N(t) \leq \bar{X}(t) \tag{4.41}$$

and

(4.42)

$$\lim_{M\to\infty} \int_0^T \mathbb{E}||\underline{X}_{N,M}(t) - \underline{X}_N(t)||^{2\nu}_{L^{2\nu}_\rho} \, dt = 0, \quad \lim_{M\to\infty} \int_0^T \mathbb{E}||\bar{X}_{0,M}(t) - \bar{X}(t)||^{2\nu}_{L^{2\nu}_\rho} \, dt = 0.$$

Step 3: We next show that the processes X_N solves Eq.(1) with F being replaced by F_N. Let us denote by $\mathcal{E}(X_N)$ resp. $\mathcal{E}(X_{N,M})$ the right-hand side of Eq.(1) after substituting there X_N resp. $X_{N,M}$. Then,

$$\begin{aligned}
(4.43)\quad & \mathbb{E}||X_N(t) - \mathcal{E}(X_N)(t)||^2_{L^2_\rho} \\
= \quad & \mathbb{E}||X_N(t) - \mathcal{E}(X_N)(t) + X_{N,M}(t) - \mathcal{E}(X_{N,M})(t)||^2_{L^2_\rho} \\
= \quad & \mathbb{E}\Bigg\| X_N(t) - U(t,0)\xi + \int_0^t U(t,s)F_N(s,X_N(s))ds \\
& - \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X(s))}dW(s) + \int_0^t \int_{L2} U(t,s)\mathcal{M}_{\Gamma(s,X(s))}(x)\,\tilde{N}(ds,dx) \Bigg\|^2_{L^2_\rho} \\
\leq \quad & C\left\{ I^{(1)}_{N,M}(t) + I^{(2)}_{N,M}(t) + I^{(3)}_{N,M}(t) + I^{(4)}_{N,M}(t) \right\},
\end{aligned}$$

where we set

$$
\begin{aligned}
I^{(1)}_{N,M}(t) :=& \quad \mathbb{E}||X_N(t) - X_{N,M}(t)||^2_{L^2_\rho},\\
I^{(2)}_{N,M}(t) :=& \quad \mathbb{E}\left|\left|\int_0^t U(t,s)[F_N(s,X_N(s)) - F_{N,M}(s,X_{N,M}(s))]\,ds\right|\right|^2_{L^2_\rho},\\
I^{(3)}_{N,M}(t) :=& \quad \mathbb{E}\left|\left|\int_0^t U(t,s)[\mathcal{M}_{\Sigma(s,X_N(s))} - \mathcal{M}_{\Sigma(s,X_{N,M}(s))}]\,dW(s)\right|\right|^2_{L^2_\rho},\\
I^{(4)}_{N,M}(t) :=& \quad \mathbb{E}\left|\left|\int_0^t \int_{L^2} U(t,s)[\mathcal{M}_{\Gamma(s,X_N(s))} - \mathcal{M}_{\Gamma(s,X_{N,M}(s))}]x\,\tilde{N}(ds,dx)\right|\right|^2_{L^2_\rho}.
\end{aligned}
$$

With the help of (4.39) and (4.40) it follows that $\lim\limits_{M\to\infty} I^{(i)}_{N,M}(t) = 0$ for $i = 1,2,3$ analogously to pp. 76–77 in [26]. Finally, we deal with $I^{(4)}_{N,M}(t)$ and show that by Hölder's inequalty

$$
\begin{aligned}
I^{(4)}_{N,M}(t) &\leq C_{\nu,\varsigma}(T)\left(\int_0^t (t-s)^{-\varsigma}\mathbb{E}||X_{N,M}(s) - X_N(s)||^2_{L^2_\rho}\,ds\right)\\
&\leq C_{\nu,\varsigma}(T)\left(\int_0^t s^{-\varsigma\frac{\nu}{\nu-1}}\,ds\right)^{1-\frac{1}{\nu}}\left(\int_0^t \mathbb{E}||X_{N,M}(s) - X_N(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds\right)^{\frac{1}{\nu}},
\end{aligned}
$$

which tends to 0 for $M\to\infty$ by (4.40). The above estimates show that X_N solves obeys Eq.(1) and that

$$
\lim_{M\to\infty}\sup_{t\in[0,T]} \mathbb{E}||X_{N,M}(t) - X_N(t)||^{2\nu}_{L^{2\nu}_\rho} = 0.
$$

Along the same lines one proves that $\underline{X}_N \in \mathcal{G}_\upsilon(T)$ solves Eq.(1) with ξ^- and F^-_N $\in \mathcal{G}_\upsilon(T)$ respectively with $\xi^+ := \xi \vee 0$ and F^+.Similar arguments

Step 4: In this final step, we shall check that

$$
X(t) := \inf_{N\in\mathbb{N}} X_N(t) \tag{4.44}
$$

is a solution to Eq.(1). First we obtain an N-indpendent estimate of the $\mathcal{G}_\nu(T)$-norms of X_N, $N\in\mathbb{N}$. Similarly to proving the M-independent estimate in Step 1, we have

$$
\mathbb{E}||\underline{X}_N(t)||^{2\nu}_{L^{2\nu}_\rho} \leq c(\nu)\left\{\underline{I}^{(1)}(t) + \underline{I}^{(2)}_N(t) + \underline{I}^{(3)}_N(t) + \underline{I}^{(4)}_N(t)\right\},
$$

where

$$\begin{aligned}
\underline{I}^{(1)}(t) :=& \quad \mathbb{E}||U(t,0)\xi^-||^{2\nu}_{L^{2\nu}_\rho},\\
\underline{I}^{(2)}_N(t) :=& \quad \mathbb{E}\left|\left|\int_0^t U(t,s)F^-_N(s,\underline{X}_N(s))\,ds\right|\right|^{2\nu}_{L^{2\nu}_\rho},\\
\underline{I}^{(3)}_N(t) :=& \quad \mathbb{E}\left|\left|\int_0^t U(t,s)\mathcal{M}_{\Sigma(s,\underline{X}_N(s))}\,dW(s)\right|\right|^{2\nu}_{L^{2\nu}_\rho},\\
I^{(4)}_M(t) :=& \quad \mathbb{E}\left|\left|\int_0^t\int_{L2} U(t,s)\mathcal{M}_{\Gamma(s,\underline{X}_N(s))}(x)\,\tilde{N}(ds,dx)\right|\right|^{2\nu}_{L^{2\nu}_\rho}.
\end{aligned}$$

The terms $\underline{I}^{(i)}(t)$ for $i=1,2,3$ are dealt with on p. 79 in [26]. Concerning $\underline{I}^{(4)}_N$ note that by (3.9)

$$\mathbb{E}||\underline{I}^{(4)}(t)||^{2\nu}_{L^{2\nu}_\rho} \le C_\gamma\left[\int_{L2}||x||^{2\nu}_{L^{2\nu}}\,\eta(dx) + \left(\int_{L2}||x||^2_{L^{2\nu}}\,\eta(dx)\right)^\nu\right].$$

Summing up we get for each $t\in[0,T]$

$$\mathbb{E}||\underline{X}_N(t)||^{2\nu}_{L^{2\nu}_\rho} \le c(\nu,T,c_f(T))\left[1+\mathbb{E}||\xi||^{2\nu}_{L^{2\nu}_\rho} + \int_0^t \mathbb{E}||\underline{X}_N(s)||^{2\nu}_{L^{2\nu}_\rho}ds\right],$$

which by Gronwalls lemma gives us the N-independent estimate for $\underline{X}_N$. Together with (4.41) this yields

$$\sup_{t\in[0,T]}\mathbb{E}||X_N(t)||^{2\nu}_{L^{2\nu}_\rho}ds \le C_\nu(T)<\infty. \tag{4.45}$$

Finally, we check that $X(t):=\inf_{N\in\mathbb{N}}X_N(t)$, $t\in[0,T]$, defines a solution to Eq.(1) in the sense of Definition 2.2.1. By (4.45) we have

$$\begin{aligned}
&\lim_{N\to\infty}\mathbb{E}||X_N(t)-X(t)||^{2\nu}_{L^{2\nu}_\rho}=0, \quad t\in[0,T],\\
&\lim_{N\to\infty}\mathbb{E}\int_0^T||X_N(t)-X(t)||^{2\nu}_{L^{2\nu}_\rho}\,dt=0.
\end{aligned} \tag{4.46}$$

In full analogy to the previous step, cf. (4.43), we estimate the following terms

$$I_N^{(1)} := \quad \mathbb{E}||X_N(t) - X(t)||^2_{L^2_\rho},$$
$$I_N^{(2)} :=$$
$$I_N^{(3)} := \quad \mathbb{E}\left|\left|\int_0^t U(t,s)[\mathcal{M}_{\Sigma(s,X_N(s))} - \mathcal{M}_{\Sigma(s,X(s))}\, dW(s)\right|\right|^2_{L^2_\rho},$$
$$I_N^{(4)} := \quad \mathbf{E}\left|\left|\int_0^t \int_{L2} U(t,s)[\mathcal{M}_{\Gamma(s,X_N(s))} - \mathcal{M}_{\Gamma(s,X(s))}\, \tilde{N}(ds,dx)\right|\right|^2_{L^2_\rho}$$

and by Gronwall's lemma show that they vanishes as $N \to \infty$.Thus, X solves Eq. (1). The required mean-square continuity properties of $t \mapsto X(t)$ follow from the similar properties of the Wiener and Poisson convolution integrals.. ■

4.4. **Examples of SPDEs.** Here we present an example of SPDEs to which the above results are applicable. So, let $\Theta \subset \mathbb{R}^d$ obey the weak cone property and fix $m \in \mathbb{N}$ s.t. $m > d/2$.

Let us consider a second order elliptic partial differential operator A such that for $\varphi \in \mathcal{D}(A) := W^{m,2}$

$$A\varphi(\theta) := \sum_{1\le i,j\le d} a_{i,j}(\theta)\frac{\partial^2}{\partial\theta_i\partial\theta_j}\varphi(\theta) + b_i(\theta)\frac{\partial}{\partial\theta_i}\varphi(\theta) + c(\theta)\varphi(\theta), \;\; \theta \in \mathbb{R}^d, \tag{4.47}$$

where $a_{i,j}$, b_i, $c \in C_b^\infty$ (= the set of bounded, infinitely differentiable functions on Θ). Furthermore, suppose that $c \ge 0$ and that $a_{i,j}$ fulfill the *standard ellipticity condition*, i.e. there is some $\delta > 0$ such that for all $\theta, \xi \in \mathbb{R}^d$

$$\sum_{i,j=1}^d a_{ij}(\theta)\xi_i\xi_j \ge \delta|\xi|^2. \tag{4.48}$$

Proposition 4.3. *Operator A defined by (4.47), (4.48) obeys conditions* $(\mathbf{A1}) - (\mathbf{A3})$ *and* $(\mathbf{A5})$ *from Section 2.1. Furthermore,* $(\mathbf{A4}_\nu)$ *is fulfilled in the case* $d = 1$.

Proof: By Section B.2 in [31], there exists a continuous Green function G: $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ corresponding to the parabolic operator $\frac{\partial}{\partial t} + A$ such that defining U by $U(0) = \mathbf{I}$ and

$$(U(t)\varphi)(\theta) := \int_\Theta G(t,\theta,\xi)\varphi(\xi)\, d\xi, \;\; \theta \in \mathbb{R}^d, \;\; \varphi \in L^2_\rho, \;\; t > 0, \tag{4.49}$$

gives us a $\mathcal{C}_0$-semigroup in L^2_ρ (cf. Theorem B.9 in [31]) obeying (cf. Theorem B.7 in [31])

$$\sup_{t\in[0,T]} ||U(t)||_{\mathcal{L}(L^2_\rho)} < \infty.$$

It is a well-known fact (see e.g. Lemma 2.1 in [14]) that $G(t,\theta,\xi)$ obeys a sub-Gaussian growth, i.e. there are positive constants c_1, c_2 such that

$$0 < G(t,\theta,\xi) \le c_1 t^{-\frac{d}{2}} \exp\left(-c_2\frac{|\theta-\xi|^2}{t}\right), \quad (t,\theta,\xi) \in (0,T] \times \mathbb{R}^d \times \mathbb{R}^d. \tag{4.50}$$

Furthermore, by (4.48) we have for $0 < t \le T$

(4.51)
$$\int\limits_{\Theta} G(t,s,\theta,\xi)(1+|\theta-\xi|^2)^{\frac{\rho}{2}}\,d\xi + \int\limits_{\Theta} G(t,s,\theta,\xi)(1+|\xi-\theta|^2)^{\frac{\rho}{2}}\,d\theta \le c(T) < \infty.$$

As was shown in Example 2.5 in [26] with the help of (4.51) one gets (**A3**). By (4.48), A obeys the assumptions of Proposition 2.7 in [8] and thus the $\mathcal{C}_0$-semigroup in L^2_ρ generated by A and having the representation (4.49) is positive, i.e. (**A1**) is fulfilled. Furthermore, since $c \ge 0$, U is contractive in L^2 by Proposition 2.7 from [8], so A also obeys (**A2**).

Thus, for the general claim it remains to consider (**A5**)**.** Concerning the approximation property in $W^{m,2}$, we note the following. By the definition of the semigroup U (see (2.11)) and the properties of the derivatives of a convolution, we immediately get $U(t)\varphi \in W^{m,2}$ for any $\varphi \in L^2$. Thus, setting $\mathcal{D}(A_N) = W^{m,2}$ for any $N \in \mathbb{N}$ and $A_N := N \cdot (U(1/N) - \mathbf{I})$, $N \in \mathbb{N}$, we get a family $(A_N)_{N\in\mathbb{N}}$ of linear bounded operators on $W^{m,2}$. Then, the corresponding evolution family in L^2_ρ is given by

$$U_N(t) := \exp(tA_n) = \exp(tN \cdot U(1/N)) \exp(-tN).$$

Furthermore, for this evolution family we have

$$\sup_{t\in[0,T]} ||U_N(t,s)\varphi||_{W^{2,2}} \le c_N(T)||\varphi||_{W^{2,2}}.$$

Thus, the approximation property in $W^{m,2}$ is fulfilled.

Finally, let us show ($\mathbf{A4}_\nu$) in the special case $d = 1$. In this case (4.50) becomes

$$0 < G(t,\theta,\xi) \le c_1 t^{-\frac{1}{2}} \exp\left(-c_2\frac{|\theta-\xi|^2}{t}\right), \quad (t,\theta,\xi) \in (0,T] \times \mathbb{R} \times \mathbb{R}, \tag{4.52}$$

which yields that

$$\begin{aligned} &\int\limits_{\Theta} G(t,s,\theta,\xi)\alpha^\rho(\theta-\xi)\,d\xi \\ \le\ & c(\rho)\left[\int\limits_{\mathbb{R}} t^{-1}\exp\left(-\frac{|\xi|^2}{t}\right)d\xi + \int\limits_{\Theta} t^{-1}|\xi|^{\rho}\exp\left(-\frac{|\xi|^2}{t}\right)d\xi\right] \le c(\rho,T)t^{-\frac{1}{2}} < \infty, \quad t > 0. \end{aligned}$$

Similarly to the above considerations, by the symmetry of $\alpha^\rho(\theta-\xi)$, we also get

$$\int\limits_{\Theta} G^2(t,\theta,\xi)\alpha^\rho(\theta-\xi)\,d\theta \le c(\rho,T)t^{-\frac{1}{2}} < \infty, \quad t > 0.$$

Setting $\zeta = \frac{1}{2}$, by (4.52) we get the following chain of estimates

$$\begin{aligned}
\int_\Theta (U(t,s)\mathcal{M}_\varphi(\psi))^{2\nu}(\theta)\,\mu_\rho(d\theta) &= \int_\Theta \left(\int_\Theta G(t,s,\theta,\xi)\varphi(\xi)\psi(\xi)\,d\xi\right)^{2\nu} \mu_\rho(d\theta) \\
&\leq ||\psi||_{L^2}^{2\nu} \int_\Theta \left[\int_\Theta G(t,s,\theta,\xi)^{\frac{2(\nu-1)}{\nu}} G(t,s,\theta,\xi)^{\frac{2}{\nu}} \varphi^2(\xi)\,d\xi\right]^\nu \mu_\rho(d\theta \\
&\leq ||\psi||_{L^2}^{2\nu} \int_\Theta \left[\left(\int_\Theta G^2(t,s,\theta,\xi)\,d\xi\right)^{\frac{\nu-1}{\nu}} \left(\int_\Theta G^2(t,s,\theta,\xi)\varphi^{2\nu}(\xi)\,d\xi\right)^{\frac{1}{\nu}}\right]^\nu \mu_\rho(d\Theta) \\
&\leq c(\rho,\nu,c(T))||\psi||_{L^2}^{2\nu}(t-s)^{-(\nu-1)\zeta} \int_\Theta \left[\int_\Theta G^2(t,s,\theta,\xi)\alpha^\rho(\theta-\xi)d\theta\right] \varphi^{2\nu}(\xi)\,\mu_\rho(d\xi) \\
&\leq c(\rho,\nu,c(T))(t-s)^{-\zeta\nu}||\psi||_{L^2}^{2\nu}||\varphi||_{L^{2\nu}_\rho}^{2\nu}.
\end{aligned}$$

Thus, $(\mathbf{A4}_\nu)$ holds, which finishes the proof. ■

Remark 4.4. Setting $a_{i,i} \equiv 1$, $a_{i,j} \equiv 0$, $i \neq j$, $b_i \equiv 0$ and $c_i \equiv 0$, $1 \leq i \leq d$ in (4.47) gives us $A = \Delta$ such that by Proposition 4.3 the Laplace operator obeys $(\mathbf{A1}) - (\mathbf{A3})$ and $(\mathbf{A5})$**.** In the special case $d = 1$, $(\mathbf{A4}_\nu)$ is fulfilled with $\zeta = \frac{1}{2}$.

References

[1] R. Adams, J. Fournier, *Sobolev Spaces*, Academic Press, 1991.
[2] R.Adams, J. Fournier, Cone conditions and properties of Sobolev spaces, *J. Math. Analysis Appl.* **61** (1977), 713–734.
[3] S. Albeverio, V. Mandrekar, B. Rüdiger, Existence of mild solutions for stochastic differential equations and semilinear equations with non-Gaussian Lévy noise, *Stochastic Processes Appl.* **119** (2009), 835–863.
[4] S. Albeverio, B. Rüdiger, Stochastic integrals in the Lévy-Itô decomposition theorem on separable Banach spaces, *Stochastic Anal. Appl.*, **23(2)**(2005), 217–253.
[5] H. W.Alt, *Lineare Funktionalanalysis*, Springer, 1999.
[6] D. Applebaum, *Lévy Processes and Stochastic Calculus*, Cambridge Studies in Advanced Mathematics, 2004.
[7] D. Applebaum, Martingale-valued measures,Ornstein-Uhlenbeck processes with jumps and operator self-decomposability in Hilbert space, *Lect. Notes Math.* **1874**, Springer, Berlin, 2006.
[8] W. Arendt, C.Batty, D.Robinson, Positive semigroups generated by elliptic operators on Lie groups, *J. Operator Theory* **23** (1990), 369–407.
[9] J. Bertoin, *Lévy Processes*, Cambridge University Press, 1996.
[10] T. Björk, G. DiMasi, Y. Kabanov, W. Ruggaldier, Towards a general theory of bond markets, *Finance and Stochastics* **1** (1997), pp.141–174.
[11] G. Da Prato, J.Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.
[12] G. DaPrato, J. Zabczyk, Convergence to equilibrium for classical and quantum spin systems, *Probab. Theory Related Fields* **103** (1995), 529–552.
[13] S. Dirksen, Noncommutative and vector-valued Rosenthal inequalities, Technische Universiteit Delft, PhD Thesis, 2011.
[14] T. Funaki, Regularity properties for stochastic partial differential equations of parabolic type, *Osaka J. Math.* **28** (1991), 495–516.
[15] L. I. Galchuk, A comparison theorem for stochastic equations with integrals with respect to martingales and random measures, *Theory Probab. Appl.* **27(3)** (1982), 450–460.

[16] L. Gasínski, N. S. Papageorgiou, *Nonlinear Analysis*, Chapman & Hall, 2005.
[17] C. J. Himmelberg, Measurable relations, *Fundamenta Mathematica* **87** (1976), 53–72.
[18] N. Ikeda, S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North-Holland / Kodansha Ltd., 1981.
[19] C. Knoche, Mild Solutions of SPDEs driven by Poisson noise in Infinite Dimensions and their dependence on initial conditions, PhD thesis, BiBoS preprint server E05-10-194, 2005.
[20] P. Kotelenez, A submartingale type inequality with applications to stochastic partial differential equations, *Stochastics* **8** (1982), pp.139–151.
[21] P. Kotelenez, A stopped Doob inequality for stochastic convolution integrals and stochastic evolution equations, *Stochastic Analysis Appl.* **2(3)** (1984), 245–265.
[22] P. Kotelenez, Comparison methods for a class of function valued stochastic partial differential equations, *Probab. Theory Related Fields* **93(1)** (1992), 1–19.
[23] P. Kotelenez, Existence, uniqueness and smoothness for a class of function valued stochastic partial differential equations, *Stochastics Stoch. Reports* **41(3)** (1992), 177-199.
[24] V. Krasin, A. Melnikov, On comparison theorem and its applications to finance in optimality and risk, *Modern Trends in Math. Finance*, 171–181, 2009.
[25] E. Lieb, M. Loss, *Analysis. Second Edition*, Graduate Studies in Mathematics **14**, 2001.
[26] R. Manthey , T. Zausinger, Stochastic evolution equations in $L^{2\nu}_\rho$, *Stochastics Stoch. Reports* **66** (1999), 37–85.
[27] C. Marinelli, C. Prévôt, M. Röckner, Regular dependence on initial data for stochastic evolution equations with multiplicative Poisson noise, arXiv-Server 0808.1509v1, 2008.
[28] C. Marinelli, M. Röckner, Well-posedness and asymptotic behaviour for stochastic reaction-diffusion equations with multiplicative Poisson noise, *Electronic J. Probab.* **15**, 1528–1555, 2010.
[29] S. Michel, Stochastic evolution equations in weighted L^p spaces with jump noise, PhD Thesis, University of Bielefeld, Publication Server http://pub.uni-bielefeld.de/publication/2492386, 2012.
[30] S. Peng, X. Zhu, Necessary and sufficient condition for comparison theorem of 1-dimensional stochastic differential equations, *Stoch. Processes Appl.* **116(3)** (2006), 370–380.
[31] S. Peszat, J. Zabczyk, *Stochastic Differential Equations with Levy Noise*, Cambridge University Press, 2007.
[32] C. Prévôt, M. Röckner, *A Concise Course on Stochastic Partial Differential Equations*, Lecture Notes Math. **1905**, Springer, 2006.
[33] P. Protter, D. Talay, The Euler scheme for Lévy driven stochastic differential equations, *Annals Probab.* **25(1)** (1997), 393-423.
[34] S. Rong, On strong solution, uniqueness, stability and comparison theorems for a stochastic system with Poisson jumps, *Lecture Notes Control Information Theory* **75** (1985), 352–381.
[35] S. Rong, *Theory of Stochastic Differential Equations With Jumps and Applications*, Springer, 2005
[36] B. Rüdiger, G. Ziglio, Ito formula for stochastic integrals w.r.t. compensated Poisson random measures on separable Banach spaces, *Stochastics Stoch. Reports* **78(6)** (2006), 377–410.
[37] G. M. Troianello, *Elliptic Differential Equations and Obstacle Problems*, Plenum Press, 1987.
[38] S. L. Yadava, Stochastic equations in locally convex spaces, *Proceedings Math. Sciences*, **95(2)** (1986), 79–94.
[39] X. Zhu, On the comparison theorem for multidimensional SDEs with jumps, arXiv-server 1006.1454v1, 2010.

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