Nonlinear parabolic equations for probability measures

O.A. Manita, S.V. Shaposhnikov

Department of Mechanics and Mathematics, Moscow State University,

119991 Moscow, Russia, starticle@mail.ru

Abstract

A new existence result is established for weak parabolic equations for probability measures. Sufficient conditions for the existence of local and global-in-time probability solutions of the Cauchy problem for such equations are given. Some conditions under which global-in-time solutions do not exist are indicated.

AMS Subject Classification: 35K10, 35K55, 35K65.

1. Introduction and main definitions

In this paper we study the well-posedness of the following Cauchy problem for probability measures:

$$\partial_t \mu_t = \partial_{x_i x_j}(a^{ij}(x,t,\mu)\mu_t) - \partial_{x_i}(b^i(x,t,\mu)\mu_t), \quad \mu_0 = \nu, \tag{1}$$

where ν is probability measure on $\mathbb{R}^d$.

First and second order nonlinear parabolic equations for probability measures belong to the most frequently used equations in physics and statistical mechanics. These equations have been an objet of intensive studies, especially over the past decade (for example, transport equations in [1, 2, 23]). However, they deal with coefficients like convolutions (see the Vlasov equations [16, 19, 20] or more general equations in [11, 12]), and only few results concerning the general case are known (see [9, 18, 22]). It should be noted that in the cases of degenerate and non-smooth coefficients the study of such equations in the space of finite measures rather than in function spaces seems more appropriate and allows one to extend finite-dimentional results to differential equations on infinite-dimentional spaces. Surveys of the recent studies on linear parabolic equations for measures are given in [8, 6, 10, 27].

Let us consider the operator L_μ defined by

$$L_\mu u = a^{ij}(x,t,\mu)\partial_{x_i x_j} u + b^i(x,t,\mu)\partial_{x_i} u,$$

where summation is taken over all repeated indices.

Let $\mathcal{M}(\mathbb{R}^d \times [0,\tau])$ be the linear space of finite Borel measures on $\mathbb{R}^d \times [0,\tau]$. Given a family of Borel measures $(\mu_t)_{t\in[0,\tau]}$ on $\mathbb{R}^d$, we associate to it a measure $\mu \in \mathcal{M}(\mathbb{R}^d \times [0,\tau])$ (in this case we write $\mu = (\mu_t)_{[0,\tau]}$) if the mapping $t \mapsto \mu_t(B)$ is Borel measurable on $[0,\tau]$ for each Borel set B and for each function $u \in C_0^\infty(\mathbb{R}^d \times (0,\tau))$ the following identity holds:

$$\int_{\mathbb{R}^d\times[0,\tau]} u(x,t)\, d\mu = \int_0^\tau \int_{\mathbb{R}^d} u(x,t)\, d\mu_t\, dt.$$

It is obvious that the last identity extends to all functions of the form fu where u is the same as before and f is integrable with respect to the measure μ on each compact set in $\mathbb{R}^d \times (0,\tau)$.

A Borel measure σ on $\mathbb{R}^d$ is called a probability measure if $\sigma \ge 0$ and $\sigma(\mathbb{R}^d) = 1$.

We shall say that $\mu = (\mu_t)_{t\in[0,\tau]}$ satisfies the Cauchy problem (1) if μ_t are probability measures and

- for all $1 \le i, j \le d$ we have Borel mappings

$$(x,t) \mapsto a^{ij}(x,t,\mu), \quad (x,t) \mapsto b^i(x,t,\mu)$$

  are defined on $\mathbb{R}^d \times [0,\tau]$ and $a^{ij}, b^i \in L^1(U \times [0,\tau], d\mu)$ for each closed ball $U \subset \mathbb{R}^d$,
- for all $t \in [0,\tau]$ and all $\varphi \in C_0^\infty(\mathbb{R}^d)$ the following identity holds:

$$\int \varphi d\mu_t - \int \varphi d\nu = \int_0^t \int L_\mu \varphi d\mu_s ds. \tag{2}$$

Typical coefficients of parabolic equations for measures contain expressions like

$$\int K(x,y,t)\, d\mu_t \quad \text{or} \quad \int_0^t \int K(x,y,s)\, d\mu_s\, ds,$$

where the kernel K grows at infinity.

We recall that $\mathcal{M}(\mathbb{R}^d \times [0,\tau])$ is a normed space with respect to the Kantorovich–Rubinshtein norm

$$\|\mu\| = \sup\Bigl\{ \int f\, d\mu \,:\, f \in \mathrm{Lip}_1, \quad |f| \le 1 \Bigr\},$$

where Lip_1 is the class of Lipschitzian functions with constant 1. Moreover, the topology generated by this norm on the space of nonnegative measures coincides with the topology of weak convergence (see [5, Theorem 8.3.1]). We also recall that a sequence of finite measures μ_n on $\mathbb{R}^d$ or on $\mathbb{R}^d \times [0,\tau]$ converges weakly to a measure μ if for each continuous bounded function f one has

$$\lim_{n\to\infty} \int f\, d\mu_n = \int f\, d\mu.$$

Further, a sequence of measures μ_n is called a Cauchy sequence if, for every bounded continuous function f, the sequence $\int f\, d\mu_n$ is a Cauchy sequence. We recall that every Cauchy sequence of Borel measures is weakly convergent (see [5, Theorem 8.7.1]).

Our approach to the proof of the existence result for probability measures is based on the Schauder fixed-point theorem: *Let Q be a continuous mapping of a convex compact set K in a normed space into itself. Then it has a fixed point, that is there exists $x \in K$ such that $Q(x) = x$* (see [17]).

Let $C([0,\tau])$ and $C^+([0,\tau])$ denote the spaces of continuous and nonnegative continuous functions on $[0,\tau]$, respectively.

Let τ_0 be a fixed positive number and let V be a nonnegative function. For each function $\alpha \in C^+([0,\tau_0])$ and each $\tau \in (0,\tau_0]$ let $M_{\tau,\alpha}(V)$ denote the set of nonnegative measures $\mu = (\mu_t)_{t\in[0,\tau]}$ in $\mathcal{M}(\mathbb{R}^d \times [0,\tau])$ such that for all $t \in [0,\tau]$ the following estimate holds:

$$\int V(x)\, d\mu_t \le \alpha(t).$$

We observe that α belongs to $C^+([0,\tau_0])$, but not to $C^+([0,\tau])$ because this condition enables us to choose α and τ independently.

Let us introduce the following conditions on the coefficients a^{ij} and b^i.

(H1) Suppose that there is a function $V \in C^2(\mathbb{R}^d)$ such that

$$V(x) > 0, \qquad \lim_{|x|\to+\infty} V(x) = +\infty,$$

two mappings Λ_1 and Λ_2 of the spaces $C^+([0,\tau_0])$ into $C^+([0,\tau_0])$ such that for all $\tau\in(0,\tau_0]$ and every $\alpha\in C^+([0,\tau_0])$ the functions a^{ij} and b^i are defined on $M_{\tau,\alpha}=M_{\tau,\alpha}(V)$ and for all $\mu\in M_{\tau,\alpha}$ and all $(x,t)\in\mathbb{R}^d\times[0,\tau]$ one has

$$L_\mu V(x,t)\le \Lambda_1[\alpha](t)+\Lambda_2[\alpha](t)V(x).$$

We shall call such a function V a Lyapunov function for L_μ. It will be important below that V is strictly positive.

Example 1.1. Condition (H1) is fulfilled, for example, if $V(x)=1+|x|^2$ and

$$L_\mu u=(x,\nabla u)G\Bigl(\int |y|^2d\mu_t\Bigr),$$

where G is a increasing nonnegative continuous function on $[0,+\infty)$. In this case

$$L_\mu V=2|x|^2G\Bigl(\int |y|^2d\mu_t\Bigr)\le 2G(\alpha(t))V(x)$$

and $L_\mu V\le\Lambda_1+\Lambda_2V$, $\Lambda_1[\alpha]\equiv0$ and $\Lambda_2[\alpha]=2G(\alpha)$.

We also note that typical examples of Λ_1 and Λ_2 are mappings of the type $\alpha(t)\mapsto G(\alpha(t))$ or

$$\alpha(t)\mapsto\int_0^t G(\alpha(s))\,ds.$$

As it has been mentioned, in typical cases the coefficients are the convolutions with growing functions, so we need that our coefficients be continuous with respect to some stronger (than weak) convergence.

We shall say that a sequence of measures $\mu^n=(\mu^n_t)_{t\in[0,\tau]}$ in $M_{\tau,\alpha}$ is V-convergent (or V-converge) to a measure $\mu=(\mu_t)_{t\in[0,\tau]}$ in $M_{\tau,\alpha}$ if for all $t\in[0,\tau]$ one has

$$\lim_{n\to\infty}\int F(x)d\mu^n_t=\int F(x)d\mu_t$$

for every continuous function F such that $\lim\limits_{|x|\to\infty}F(x)/V(x)=0$.

(H2) For all $\tau\in(0,\tau_0]$, $\alpha\in C^+([0,\tau_0])$, $\sigma\in M_{\tau,\alpha}$, and $x\in\mathbb{R}^d$ the mappings

$$t\mapsto a^{ij}(x,t,\sigma)\quad\text{or}\quad t\mapsto b^i(x,t,\sigma)$$

are Borel measurable on $[0,\tau]$ and for each closed ball $U\subset\mathbb{R}^d$ the mappings

$$x\mapsto b^i(x,t,\sigma)\quad\text{or}\quad x\mapsto a^{ij}(x,t,\sigma)$$

are bounded on U uniformly in $\sigma\in M_{\tau,\alpha}$ and $t\in[0,\tau]$ and are equicontinuous on U uniformly in $\sigma\in M_{\tau,\alpha}$ and $t\in[0,\tau]$. Moreover, if a sequence $\mu^n\in M_{\tau,\alpha}$ is V-convergent to $\mu\in M_{\tau,\alpha}$, then for all $(x,t)\in\mathbb{R}^d\times[0,\tau]$ one has

$$\lim_{n\to\infty}a^{ij}(x,t,\mu^n)=a^{ij}(x,t,\mu),\qquad \lim_{n\to\infty}b^i(x,t,\mu^n)=b^i(x,t,\mu).$$

Remark 1.1. (i) The continuity of the mappings $x\mapsto a^{ij}(x,t,\mu)$ and $x\mapsto b^i(x,t,\mu)$ and the measurability of $t\mapsto a^{ij}(x,t,\mu)$ and $t\mapsto b^i(x,t,\mu)$ ensure that the mappings $(x,t)\mapsto a^{ij}(x,t,\mu)$ and $(x,t)\mapsto b^i(x,t,\mu)$ are measurable with respect to the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d\times[0,\tau])$ (see [5, Lemma 6.4.6, Exercise 6.10.39]).

(ii) For each $t \in [0,\tau]$ the sequences $a^{ij}(x,t,\mu^n)$ and $b^i(x,t,\mu^n)$ are uniformly in x convergent on all balls $U \subset \mathbb{R}^d$. This follows from their pointwise convergence, the uniform boundedness and equicontinuity.

Note that we do not assume that the coefficients are continuous in t.

Example 1.2. Condition (H2) is fulfilled, for example, for

$$b(x,t,\mu) = \int K(x,y)d\mu_t$$

with a Lyapunov function V and a continuous vector field K on $\mathbb{R}^d \times [0,\tau]$ which satisfy the following estimate:

$$|K(x,y)| \leq C_1(x) + C_2(x)V^{1-\gamma}(y),$$

where $\gamma \in (0,1)$ and $C_1(x)$, $C_2(x)$ are bounded functions.

Finally, we need one more assumption.

(H3) For each $\tau \in (0,\tau_0]$, each function $\alpha \in C^+([0,\tau_0])$ and each measure $\sigma \in M_{\tau,\alpha}$, the matrix $A(x,t,\sigma) = (a^{ij}(x,t,\sigma))_{1\leq i,j\leq d}$ is symmetric and nonnegative definite, that is $a^{ij} = a^{ji}$ and for all $\xi \in \mathbb{R}^d$ one has $(A(x,t,\sigma)\xi,\xi) \geq 0$.

Remark 1.2. One can see from the proof of the main theorem that it is sufficient that the conditions (H1), (H2) and (H3) be fulfilled not on every set $M_{\tau,\alpha}$, but on some of them with certain fixed τ and α, which are completely determined by the initial data in the Cauchy problem and the mappings Λ_1 and Λ_2.

The main result of this paper is the following theorem.

Theorem 1.1. *Let the coefficients a^{ij} and b^i have properties* (H1), (H2), (H3) *above. Let also the initial data ν be a probability measure on $\mathbb{R}^d$ and $V \in L^1(\nu)$. Then the following assertions are valid*:

(i) *There exists $\tau \in (0,\tau_0]$ such that the Cauchy problem* (1) *has a solution on the interval* $[0,\tau]$.

(ii) *If Λ_1 and Λ_2 are constant, then the Cauchy problem* (1) *has a solution on the whole interval* $[0,\tau_0]$.

(iii) *If $\Lambda_1[\alpha] = 0$ and $\Lambda_2[\alpha](t) = G(\alpha(t))$, where G is a strictly increasing continuous positive function on $[0,+\infty)$, then the Cauchy problem* (1) *has a solution on each interval $[0,\tau]$, where $\tau \in (0,\tau_0]$, $\tau < T$ and*

$$T = \int_{u_0}^{+\infty} \frac{1}{uG(u)}\,du, \quad u_0 = \int V(x)\,d\nu.$$

(iv) *If $\Lambda_1[\alpha](t) = G(\alpha(t))$ and $\Lambda_2[\alpha] = 0$, where G is a strictly increasing continuous positive function on $[0,+\infty)$, then the Cauchy problem* (1) *has a solution on each interval $[0,\tau]$, where $\tau \in (0,\tau_0]$, $\tau < T$ and*

$$T = \int_{u_0}^{+\infty} \frac{1}{G(u)}\,du, \quad u_0 = \int V(x)\,d\nu.$$

Moreover, in all these cases for any solution $(\mu_t)_{t\in[0,\tau]}$ one has μ_t are probability measures and

$$\sup_{t\in[0,\tau]} \int V(x)d\mu_t < \infty.$$

It is natural to call a solution of the Cauchy problem (1) on the whole interval $[0, \tau_0]$ a global solution and a solution on an interval $[0, \tau]$, where $\tau < \tau_0$, a local solution. In these terms, in (i) we claim the existence of a local solution and in (ii) the existence of a global one.

The assertion in (ii) was established in a special case in [9]. The assertion in (iii) with $T = +\infty$ was established in [18], where the well-posedness of the appropriate martingale problem was proved (applications of probabilistic methods to the study of nonlinear parabolic equations are also considered in [3]). The main difference between our results and the known ones is that, on the one hand, we do not restrict the growth of the coefficients of L_μ, only the existence of a Lyapunov function is needed, and, on the other hand, we do not use the exact form of the coefficients. Thus we cover typical cases considered in [16], [19], [20], [11], [12], [15], and [22]. Moreover, we do not assume that the coefficients belong to a Sobolev class or the class of functions with bounded total variation, only their continuity is assumed. Also we investigate the existence time for the solution and give the exact estimates of the moment after which the solution does not exist. We note that in [16] the Vlasov equation is considered in the space of finite measures with the Kantorovich norm and the existence is established by using the contraction mapping theorem. In our more general case this theorem cannot be used, because, for example, generally speaking, a solution is not unique. In [11], [12], [21], and [22], nonlinear transport equations with coefficients of the convolution form are also studied in the space with the Kantorovich norm and the solutions are regarded as geodesics. However, this approach is based on the specific form of the coefficients. We also note that the stationary equation was studied in [29].

This paper consists of 3 sections. The first one is the introduction, the formulation of the problem and the main result. In the second section we prove Theorem 1.1 and give some examples of its application. The proof of Theorem 1.1 is divided into several steps.

1. From the Schauder fixed-point theorem and the linear theory of parabolic equations for measures we obtain the assertion of Theorem 1.1 in the case of a non-degenerate and sufficiently smooth matrix A .

2. By using the method of "vanishing viscosity" we obtain the assertion of Theorem 1.1 in the case of a degenerate and sufficiently smooth matrix A.

3. Finally, we proceed to the general case.

The main reason for such division (with respect to the smoothness of A) is that we need the uniqueness of solution to an appropriate linear equation in order to apply the Schauder theorem, and this is possible only with additional restrictions on the smoothness of A.

Finally, the last section is concerned with the lack of global solutions.

2. Proof of the theorem 1.1

We shall deal first with the case of a nondegenerate matrix A. This assumption enables us to apply the linear theory with its broad conditions on the coefficients and it provides the well-posedness of the Cauchy problem for the linear equation.

2.1. **Nondegenerate case.** In this section and in the following one we suppose that in place of (H3) the following stronger condition is fulfilled:

(H3') condition (H3) is fulfilled and for every $\tau \in (0, \tau_0)$, each function $\alpha \in C([0, \tau_0])$, each closed ball $U \subset \mathbb{R}^d$ and each $\sigma \in M_{\tau,\alpha}$, there exists a number $\lambda = \lambda(\sigma, U) > 0$ such that for all $x, y \in U$, $t \in [0, \tau]$ one has $\det A(x, t, \sigma) \neq 0$ and the following estimate holds:

$$|A(x, t, \sigma) - A(y, t, \sigma)| \leq \lambda |x - y|.$$

Moreover, we suppose that for each measure $\sigma \in M_{\tau,\alpha}$ there exist numbers C_1 and C_2 such that

$$|\sqrt{A(x, t, \sigma)} \nabla V(x)| \leq C_1 + C_2 V(x).$$

Let $\sigma \in M_{\tau,\alpha}$. Consider the following Cauchy problem for the linear equation:

$$\partial_t \mu_t = \partial_{x_i x_j} (a^{ij}(x, t, \sigma) \mu_t) - \partial_{x_i} (b^i(x, t, \sigma) \mu_t), \quad \mu_0 = \nu.$$

It is proved in [6] (see Theorem 3.1) that (H1), (H2) and (H3') are sufficient for the existence of a probability solution $\mu = (\mu_t)_{t \in [0,\tau]}$. We also observe that the existence result in the linear case follows from the well-posedness of the appropriate martingale problem (see [28]). It is proved in [27, Theorem 2.3] (see also [10]) that Conditions (H1), (H2) and (H3') are sufficient for the uniqueness of a probability solution. A survey of the principle results concerning the local and global properties of solutions μ can be found in [8].

Hence the mapping $Q\colon\ M_{\tau,\alpha} \mapsto \mathcal{M}(\mathbb{R}^d \times [0, \tau])$

$$Q(\sigma) = \chi \Longleftrightarrow \partial_t \chi_t = \partial_{x_i x_j} (a^{ij}(x, t, \sigma) \chi_t) - \partial_{x_i} (b^i(x, t, \sigma) \chi_t), \quad \chi_0 = \nu$$

is well defined.

It is obvious that μ is a solution of (1) if and only if μ is a fixed point of Q. Thus, it is natural to prove the existence result employing the Schauder fixed-point theorem.

We observe that the uniqueness of a solution to the Cauchy problem for the linear equation will be also used when we prove the continuity of Q.

By definition, a measure $\mu = (\mu_t)_{t \in [0,\tau]} \in M_{\tau,\alpha}$ belongs to the set $N_{\tau,\alpha}$ if and only if for each $\varphi \in C_0^\infty(\mathbb{R}^d)$ and all $t, s \in [0, \tau]$ the following estimate holds:

$$\left| \int \varphi d\mu_t - \int \varphi d\mu_s \right| \leq \Lambda(\tau, \alpha, \varphi) |t - s|, \tag{3}$$

where

$$\Lambda(\tau, \alpha, \varphi) = \sup \bigl\{ |L_\mu \varphi(x, t)| \colon\ (x, t) \in \mathbb{R}^d \times [0, \tau],\ \mu \in M_{\tau,\alpha} \bigr\}$$

does not depend on $\mu \in N_{\tau,\alpha}$. Note that sup in this definition is finite due to (H2).

In order to use the Schauder theorem we have to find a convex compact set K in $\mathcal{M}(\mathbb{R}^d \times [0, \tau])$ such that $Q(K) \subset K$. With an appropriate choice of τ and α the set $N_{\tau,\alpha}$ will be the required compact.

Lemma 2.1. *Every sequence of measures $\mu^n = (\mu_t^n)_{t \in [0,\tau]}$ in $N_{\tau,\alpha}$ has a subsequence $\{\mu^{n_l}\}$ such that $\{\mu^{n_l}\}$ converges weakly to $\mu \in N_{\tau,\alpha}$ and $\{\mu_t^{n_l}\}$ converges weakly to μ_t for all $t \in [0, \tau]$.*

Proof. Taking into account the definition of $N_{\tau,\alpha}$ and Chebyshev's inequality, we obtain that the set $\{\mu_t^n\}$ is uniformly tight for each t. Let $T = \{t_1, t_2, \ldots\}$ be a countable dense set in $[0, \tau]$. Prohorov's theorem yields that for each j in $\mu_{t_j}^n$ there is a weakly convergent subsequence. Using the diagonal method, we find a subsequence $\mu_t^{n_l}$ which converges weakly for each $t \in T$.

Let us show that it is a Cauchy sequence for all $t \in [0,\tau]$. Consider $t \in [0,\tau]$, $s \in T$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$. One has

$$\left|\int \varphi d\mu_t^{n_p} - \int \varphi d\mu_t^{n_k}\right| \le \left|\int \varphi d\mu_t^{n_p} - \int \varphi d\mu_s^{n_p}\right| + \left|\int \varphi d\mu_s^{n_p} - \int \varphi d\mu_s^{n_k}\right| + \\ + \left|\int \varphi d\mu_s^{n_k} - \int \varphi d\mu_t^{n_k}\right| \le 2\Lambda(\tau,\alpha,\varphi)\cdot|t-s| + \left|\int \varphi d\mu_s^{n_p} - \int \varphi d\mu_s^{n_k}\right|.$$

Choosing s close to t, we can make the first summand less than $\varepsilon/2$ for every given $\varepsilon > 0$. As the sequence $\{\mu_s^{n_l}\}$ converges, it is a Cauchy sequence, hence there is a number N such that for all $p, k > N$ the second summand is less than $\varepsilon/2$. So for each function $\varphi \in C_0^\infty(\mathbb{R}^d)$ the sequence $\displaystyle\int \varphi(x)\, d\mu_t^{n_l}$ is a Cauchy sequence. Using the uniform tightness of $\mu_t^{n_l}$ for each $\varepsilon > 0$ we find a ball U such that $\mu_t^{n_l}(\mathbb{R}^d\backslash U) < \varepsilon$ for all l. Let f be a continuous bounded function with $|f(x)| \le M$. Let $\psi \in C_0^\infty(\mathbb{R}^d)$ be such that $|f(x) - \psi(x)| < \varepsilon$ for every $x \in U$ and $|\psi(x)| \le M+1$. Then

$$\left|\int f d\mu_t^{n_p} - \int f d\mu_t^{n_k}\right| \le \int_U |f-\psi|\, d\mu_t^{n_p} + \int_U |f-\psi|\, d\mu_t^{n_k} + \\ + \left|\int \psi d\mu_t^{n_p} - \int \psi d\mu_t^{n_k}\right| + \int_{\mathbb{R}^d\backslash U} (|f| + |\psi|)\, d\mu_t^{n_p} + \int_{\mathbb{R}^d\backslash U} (|f| + |\psi|)\, d\mu_t^{n_k},$$

which is obviously bounded by

$$4\varepsilon(M+1) + \left|\int \psi d\mu_t^{n_p} - \int \psi d\mu_t^{n_k}\right|.$$

The last summand can be made as small as required by letting $p \to \infty$ and $k \to \infty$. So we obtain that the sequence $\{\mu_t^{n_l}\}$ is a Cauchy sequence for each $t \in [0,\tau]$ and thus it converges weakly to some measure μ_t for each $t \in [0,\tau]$. Hence for every continuous bounded function f the mapping $t \mapsto \displaystyle\int f\, d\mu_t$ is Borel measurable on $[0,\tau]$ as a limit of measurable functions. Consider the class Φ of bounded Borel functions ϕ on $\mathbb{R}^d$ for which the mapping $t \mapsto \displaystyle\int \phi\, d\mu_t$ is Borel measurable on $[0,\tau]$. The set Φ contains the algebra of continuous bounded functions on $\mathbb{R}^d$ and is closed with respect to uniform and monotone limits. By the monotone class theorem (see [5, Theorem 2.19.9]) the set Φ contains all bounded Borel functions on $\mathbb{R}^d$. In particular, the mapping $t \to \mu_t(B)$ is Borel measurable on $[0,\tau]$ for each Borel set B.

Consider $\mu = (\mu_t)_{t\in[0,\tau]}$. Let us show that $\{\mu^{n_l}\}$ converges weakly to μ. Let h be a continuous bounded function on $\mathbb{R}^d \times [0,\tau]$. As shown above, for each fixed t one has

$$\lim_{l\to\infty} \int h(x,t) d\mu_t^{n_l} = \int h(x,t) d\mu_t.$$

The function h is bounded and each $\mu_t^{n_l}$ is a probability measure, hence the quantities $\displaystyle\int h(x,t) d\mu_t^{n_l}$ are uniformly (with respect to t and n_l) bounded. The Lebesgue dominated convergence theorem yields that

$$\lim_{l\to\infty} \int_0^\tau \int h(x,t) d\mu_t^{n_l} dt = \int_0^\tau \int h(x,t) d\mu_t dt.$$

This gives the weak convergence of μ^{n_l} to μ.

Finally, we show that $\mu \in N_{\tau,\alpha}$. By the weak convergence of $\{\mu_t^{n_l}\}$ to μ_t one has

$$\int \min\{V(x), N\}\, d\mu_t \le \alpha(t)$$

for each integer N. Letting $N \to \infty$ and using Fatou's lemma, we obtain that $\mu \in M_{\tau,\alpha}$. Letting $l \to \infty$ in the estimate

$$\left|\int \varphi\, d\mu_t^{n_l} - \int \varphi\, d\mu_s^{n_l}\right| \le \Lambda(\tau,\alpha,\varphi)|t-s|$$

we get

$$\left|\int \varphi\, d\mu_t - \int \varphi\, d\mu_s\right| \le \Lambda(\tau,\alpha,\varphi)|t-s|.$$

This proves the lemma. □

Corollary 2.1. *The set $N_{\tau,\alpha}$ is a convex compact set in $\mathcal{M}(\mathbb{R}^d \times [0,\tau])$.*

Lemma 2.2. *If a sequence of measures $\mu^n \in N_{\tau,\alpha}$ converges weakly, then it V-converges.*

Proof. Let the sequence μ^n be weakly convergent to μ. Then for all $t \in [0,\tau]$ the sequence of measures μ_t^n converges weakly to μ_t. Indeed, using Lemma 2.1, in each subsequence of indices n_l we can find a subsequence $\{n_{l_k}\}$ such that $\{\mu_t^{n_{l_k}}\}$ weakly converges to μ_t. Hence the whole sequence $\{\mu_t^n\}$ converges weakly to μ_t. Let F be a continuous function on $\mathbb{R}^d$. Consider the function $g(x) = F(x)(1+V(x))^{-1}$, for which $\lim\limits_{|x|\to\infty} g(x) = 0$. Then, for each $\varepsilon > 0$, there exists a function $\psi \in C_0^\infty(\mathbb{R}^d)$ such that $|g(x) - \psi(x)| < \varepsilon$ for all $x \in \mathbb{R}^d$. Therefore,

$$\left|\int F\, d\mu_t^n - \int F\, d\mu_t\right| = \left|\int g(1+V)\, d\mu_t^n - \int g(1+V)\, d\mu_t\right| \le$$

$$\le \left|\int \psi(1+V)\, d\mu_t^n - \int \psi(1+V)\, d\mu_t\right| + \varepsilon(2+2\alpha(t)).$$

Finally, we observe that the following identity holds due to the weak convergence of μ_t^n:

$$\lim_{n\to\infty}\left|\int \psi(1+V)\, d\mu_t^n - \int \psi(1+V)\, d\mu_t\right| = 0.$$

This completes the proof. □

Corollary 2.2. *Suppose that for some $\tau \in (0,\tau_0]$ and $\alpha \in C^+([0,\tau_0])$ one has $Q(N_{\tau,\alpha}) \subseteq N_{\tau,\alpha}$. Then Q is a continuous mapping of $N_{\tau,\alpha}$ to $N_{\tau,\alpha}$.*

Proof. Suppose that $\mu^n, \mu \in N_{\tau,\alpha}$ and the sequence $\{\mu^n\}$ converges weakly to μ. Set $\chi^n = Q(\mu^n)$ by definition. To prove convergence of $\{\chi^n\}$ it is sufficient to prove that in every subsequence in $\{\chi^n\}$ there is a subsequence convergent to one and the same measure χ and that it satisfies the equality $Q(\mu) = \chi$. Without loss of generality we prove this for $\{\chi^n\}$.

Using Lemma 2.1 we find a subsequence of indices n_k such that $\{\chi^{n_k}\}$ converges weakly to some measure $\chi \in N_{\tau,\alpha}$. Moreover, $\{\chi_t^{n_k}\}$ converges weakly to χ_t for each $t \in [0,\tau]$. Combining convergence of μ^{n_k} to μ and Lemma 2.2, we obtain V-convergence of μ^{n_k} to μ.

Due to (H2) for each $t \in [0,\tau]$ the sequences of functions $x \mapsto a^{ij}(x,t,\mu^{n_k})$ and $x \mapsto b^i(x,t,\mu^{n_k})$ are equicontinuous and uniformly bounded and converge pointwise to $a^{ij}(x,t,\mu)$ and $b^i(x,t,\mu)$, respectively, on each ball $U \subset \mathbb{R}^d$. Hence the sequences $a^{ij}(x,t,\mu^{n_k})$ and $b^i(x,t,\mu^{n_k})$ converge uniformly on $U \subset \mathbb{R}^d$ to $a^{ij}(x,t,\mu)$ and $b^i(x,t,\mu)$, respectively.

Let us show that $Q(\mu)=\chi$. Let $\varphi\in C_0^\infty(\mathbb{R}^d)$ and let U be a ball containing the support of φ. Since $Q(\mu^{n_k})=\chi^{n_k}$, one has

$$\int_U \varphi d\chi_t^{n_k}-\int_U\varphi d\nu=\int_0^t\int_U L_{\mu^{n_k}}\varphi d\chi_s^{n_k}ds.$$

Note that

$$\int_U L_{\mu^{n_k}}\varphi d\chi_s^{n_k}=\int_U(L_{\mu^{n_k}}\varphi-L_\mu\varphi)d\chi_s^{n_k}+\int_U L_\mu\varphi d\chi_s^{n_k}ds.$$

Letting $k\to\infty$ we see that the first summand tends to zero due to the uniform convergence of $a^{ij}(x,s,\mu^{n_k})$ and $b^i(x,s,\mu^{n_k})$ on U, and the second one tends to $\displaystyle\int_U L_\mu\varphi d\chi_s$ due to the weak convergence of $\{\chi_s^{n_k}\}$ to χ_s. We recall that $|L_{\mu^{n_k}}\varphi|\le\Lambda(\tau,\alpha,\varphi)$. Hence the quantities $\displaystyle\int_U L_{\mu^{n_k}}\varphi d\chi_s^{n_k}$ are uniformly bounded. Using Lebesgue's dominated convergence theorem we obtain

$$\lim_{k\to\infty}\int_0^t\int_U L_{\mu^{n_k}}\varphi d\chi_s^{n_k}ds=\int_0^t\int_U L_\mu\varphi d\chi_s.$$

The sequence $\{\chi_t^{n_k}\}$ converges weakly to χ_t for each t, so one has $\displaystyle\int\varphi d\chi_t^{n_k}\to\int\varphi d\chi_t$. Letting $k\to\infty$ we obtain

$$\int_U\varphi d\chi_t-\int_U\varphi d\nu=\int_0^t\int_U L_\mu\varphi d\chi ds.$$

Since φ was arbitrary, we conclude that $Q(\mu)=\chi$. Finally, we observe that under our assumptions χ is uniquely defined. □

To complete the proof of Theorem 1.1 we have to find a number $\tau\in(0,\tau_0]$ and a function $\alpha\in C^+([0,\tau])$ such that $Q(N_{\tau,\alpha})\subseteq N_{\tau,\alpha}$. We emphasize that we need only the inclusion $Q(M_{\tau,\alpha})\subseteq M_{\tau,\alpha}$, because Condition (3) holds automatically for any χ such that $\chi=Q(\mu)$.

Lemma 2.3. *Let $\mu\in N_{\tau,\alpha}$ and $\chi=Q(\mu)$ on $[0,\tau]$. Then for all $t\in[0,\tau]$ the following estimate holds*:

$$\int V(x)\,d\chi_t\le S[\alpha](t)+R[\alpha](t)\int V(x)\,d\nu,$$

where

$$R[\alpha](t)=\exp\Bigl(\int_0^t\Lambda_2[\alpha](s)\,ds\Bigr),\quad S[\alpha](t)=R[\alpha](t)\int_0^t\frac{\Lambda_1[\alpha](s)}{R[\alpha](s)}\,ds.$$

The mappings Λ_1 and Λ_2 are defined in Condition (H1).

Applying Lemma 2.3 we can find τ and α in each of cases (i)–(iv) in Theorem 1.1. In the next corollary we find τ and α in case(i).

Corollary 2.3. *There exist $\tau\in(0,\tau_0]$ and a constant function $\alpha(t)\equiv\alpha>0$ such that $Q(N_{\tau,\alpha})\subseteq N_{\tau,\alpha}$.*

Proof. Due to Lemma 2.3 for $\chi=Q(\mu)$, where $\mu\in N_{\tau,\alpha}$, one has

$$\int V\,d\chi_t\le S[\alpha](t)+R[\alpha](t)\int V(x)\,d\nu.$$

Let

$$\alpha=2\int V\,d\nu+1$$

and note that the functions $S[\alpha]$ and $R[\alpha]$ do not depend on τ, because Λ_1 and Λ_2 do not depend on τ. Moreover, $\lim_{t\to 0} S[\alpha](t) = 0$ and $\lim_{t\to 0} R[\alpha](t) = 1$. Choosing τ such that $S[\alpha](t) < 1$ and $R[\alpha](t) < 2$ for all $t \in [0, \tau]$, we arrive at the estimate

$$\int V\, d\chi_t \le \alpha$$

for all $t \in [0, \tau]$. □

Let us find τ and α in case (ii) of Theorem 1.1.

Corollary 2.4. *If the mappings Λ_1 and Λ_2 are constant, then for every τ_0 there is a constant function $\alpha(t) \equiv \alpha$ on $[0, \tau_0]$ such that $Q(N_{\tau_0,\alpha}) \subseteq N_{\tau_0,\alpha}$.*

Proof. Let

$$\alpha = \max_{t\in[0,\tau_0]} \Bigl(S(t) + R(t) \int V(x)\, d\nu\Bigr),$$

where S and R are the functions from Lemma 2.3. Note that S and R do not depend on α, because they are functions of Λ_1 and Λ_2, which do not depend on α, that is, $\Lambda_1[\alpha] = \Lambda_1[0]$ and $\Lambda_2[\alpha] = \Lambda_2[0]$. Now it is obvious that

$$\int V\, d\chi_t \le \alpha$$

for all $t \in [0, \tau_0]$. □

Let us find τ and α in case (iii) of Theorem 1.1.

Corollary 2.5. *Suppose that $\Lambda_1[\alpha] = 0$ and $\Lambda_2[\alpha](t) = G(\alpha(t))$, where G is a strictly increasing continuous positive function on $[0, +\infty)$. Let*

$$T = \int_{u_0}^{+\infty} \frac{du}{uG(u)}, \quad u_0 = \int V(x)\, d\nu.$$

Then for each $\tau \in (0, \tau_0]$ with $\tau < T$ there exists a function $\alpha \in C^+([0, \tau_0])$ such that $Q(N_{\tau,\alpha}) \subseteq N_{\tau,\alpha}$.

Proof. Applying Lemma 2.3 for $\chi = Q(\mu)$, we obtain

$$\int V\, d\chi_t \le \int V\, d\nu \exp\Bigl(\int_0^t G(\alpha(s))\, ds\Bigr).$$

Let $\tau \in (0, \tau_0]$, $\tau < T$, and let the function α on $[0, \tau]$ be defined by the following identity:

$$t = \int_{\alpha(0)}^{\alpha(t)} \frac{du}{uG(u)}, \quad \alpha(0) = \int V(x)\, d\nu.$$

If $\tau < \tau_0$, then we set $\alpha(t) = \alpha(\tau)$ for all $t > \tau$. Then the function α is continuously differentiable and strictly increasing on $[0, \tau]$ and continuous on $[0, \tau_0]$. Moreover, $\alpha' = \alpha G(\alpha)$. Hence for all $t \in [0, \tau]$ we have

$$\int V\, d\nu \exp\Bigl(\int_0^t G(\alpha(s))\, ds\Bigr) = \alpha(t)\alpha^{-1}(0) \int V\, d\nu = \alpha(t).$$

Thus, for this function α the inclusion $\mu \in N_{\tau,\alpha}$ yields that $\chi \in N_{\tau,\alpha}$. □

Let us find τ and α in case (iv) of Theorem 1.1.

Corollary 2.6. *Suppose that* $\Lambda_1[\alpha](t) = G(\alpha(t))$ *and* $\Lambda_2[\alpha] = 0$*, where* G *is a strictly increasing continuous positive function on* $[0, +\infty)$*. Let*

$$T = \int_{u_0}^{+\infty} \frac{du}{G(u)}, \quad u_0 = \int V(x)\, d\nu.$$

Then for each $\tau \in (0, \tau_0]$ *with* $\tau < T$ *there exists a function* $\alpha \in C^+([0, \tau_0])$ *such that* $Q(N_{\tau,\alpha}) \subseteq N_{\tau,\alpha}$*.*

Proof. Applying Lemma 2.3 to $\chi = Q(\mu)$, where $\mu \in N_{\tau,\alpha}$, we obtain

$$\int V\, d\chi_t \leq \int V\, d\nu + \int_0^t G(\alpha(s))\, ds.$$

Let $\tau \in (0, \tau_0]$, $\tau < T$, and let the function α on $[0, \tau]$ be defined by the following identity:

$$t = \int_{\alpha(0)}^{\alpha(t)} \frac{1}{G(u)}\, du, \quad \alpha(0) = \int V\, d\nu.$$

If $\tau < \tau_0$, then we set $\alpha(t) = \alpha(\tau)$ for all $t > \tau$. Then the function α is continuously differentiable and strictly increasing on $[0, \tau]$ and continuous on $[0, \tau_0]$. Moreover, $\alpha' = G(\alpha)$. Hence for all $t \in [0, \tau]$ we have

$$\int V\, d\nu + \int_0^t G(\alpha(s))\, ds = \alpha(t) + \int V\, d\nu - \alpha(0) \leq \alpha(t).$$

Thus, for this function α the inclusion $\mu \in N_{\tau,\alpha}$ yields that $\chi \in N_{\tau,\alpha}$. $\square$

Remark 2.1. Let us observe that in Corollaries 2.3, 2.4, 2.5, and 2.6 our choice of τ and α has been completely determined by the mappings Λ_1 and Λ_2. Thus, one can assume that Conditions (H1), (H2) and (H3') are fulfilled on some fixed (and completely determined by Λ_1 and Λ_2) set $M_{\tau,\alpha}$.

Proof of Theorem 1.1 in the non-degenerate case. Using Corollaries 2.3, 2.4, 2.5, and 2.6, for each case (i)–(iv) we find appropriate α and τ. According to Corollary 2.1, the set $N_{\tau,\alpha}$ is a convex compact set in $\mathcal{M}(\mathbb{R}^d \times [0, \tau])$, hence, due to Corollary 2.2, the mapping $Q\colon N_{\tau,\alpha} \to N_{\tau,\alpha}$ is continuous. Taking into account the Schauder theorem, we obtain that there exists a measure $\mu \in N_{\tau,\alpha}$ that is a fixed point of Q. Hence it is a solution of the Cauchy problem (1). $\square$

2.2. **Degenerate case.** The proof of the main theorem in the degenerate case is based on the well-known method of "vanishing viscosity" (see, e.g., [15, 26]). We assume again that all the hypotheses of (H3') hold except for the nondegeneracy of the matrix A, that is, we do not need $\det A$ to be positive.

Let $\varrho \in C^\infty(\mathbb{R}^d)$ with $\varrho > 0$ be such that for all $x \in \mathbb{R}^d$ the following estimate holds:

$$\varrho(x)(|\Delta V(x)| + |\nabla V(x)|^2) \leq \min\{V(x), 1\}.$$

Such a function ϱ exists, because one has $\min V > 0$ due to (H1). For each $\varepsilon > 0$ we consider the following Cauchy problem:

$$\partial_t \mu_t = \varepsilon\varrho(x)\Delta\mu_t + \partial_{x_i x_j}(a^{ij}(x, t, \mu)\mu_t) - \partial_{x_i}(b^i(x, t, \mu)\mu_t), \quad \mu_0 = \nu. \tag{4}$$

Let us show that all the assertions in Theorem 1.1 in the non-degenerate case are fulfilled. First we check (H1). Let

$$L_{\mu,\varepsilon} := \varepsilon\varrho\Delta + L_\mu.$$

Let $\varepsilon \in (0, \varepsilon_0)$. Then one has

$$L_{\mu,\varepsilon}V(x,t) = \varepsilon\varrho(x)\Delta V(x) + L_\mu V(x,t) \le \varepsilon_0 \min\{V(x), 1\} + \Lambda_1[\alpha](t) + \Lambda_2[\alpha](t)V(x).$$

Thus, (H1) is fulfilled with $\widehat{\Lambda}_1[\alpha](t) = \Lambda_1[\alpha](t) + \varepsilon_0$ instead of $\Lambda_1[\alpha]$. If one has $\Lambda_1 = 0$ (like in assertion (iii) in Theorem 1.1), then we replace Λ_2 with $\widehat{\Lambda}_2[\alpha](t) = \Lambda_2[\alpha](t) + \varepsilon_0$. We recall that in (iii) and (iv) we take $\tau < T$, where

$$T = \int_{u_0}^{+\infty} \frac{du}{uG(u)} \quad \text{and} \quad T = \int_{u_0}^{+\infty} \frac{du}{G(u)},$$

respectively. We replace Λ_1 with $\Lambda_1 + \varepsilon_0$ or Λ_2 with $\Lambda_2 + \varepsilon_0$, so we have a new moment T_{ε_0} with the function $G + \varepsilon_0$ in place of T with the function G. Note that $T_{\varepsilon_0} < T$ and $T_{\varepsilon_0} \to T$ if $\varepsilon_0 \to 0$. Let $\varepsilon_0 > 0$ be such that $\tau < T_{\varepsilon_0} < T$. It is obvious that (H2) and (H3') are also fulfilled. Hence, for each assertion (i)–(iv) in Theorem 1.1, taking into account Corollaries 2.3, 2.4, 2.5, and 2.6, one has appropriate τ and α. We recall that τ and α depend only on Λ_1 and Λ_2 and hence they do not depend on ε. Let $\varepsilon = 1/n < \varepsilon_0$. Using the non-degenerate case, we obtain that for each n there exists a measure $\mu^n \in N_{\tau,\alpha}$ that is a solution to (4).

Applying Lemma 2.1, we obtain a subsequence of indices n_k such that $\{\mu^{n_k}\}$ converges weakly to $\mu \in N_{\tau,\alpha}$. Using Lemma 2.2 we obtain that this sequence V-converges to the measure μ. Taking into account (H2), we observe that for each $t \in [0, \tau]$ the sequences of functions $a^{ij}(x, t, \mu^{n_k})$ and $b^i(x, t, \mu^{n_k})$ converge uniformly on every ball $U \subset \mathbb{R}^d$. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ and let U be a ball containing the support of φ. The following identity holds:

$$\int_U \varphi d\mu_t^{n_k} - \int_U \varphi d\nu = \int_0^t \int_U [n_k^{-1}\varrho\Delta\varphi + L_{\mu^{n_k}}\varphi] d\mu_s^{n_k} ds. \tag{5}$$

Taking into account the uniform convergence of $a^{ij}(x, s, \mu^{n_k})$, $b^i(x, s, \mu^{n_k})$ and the weak convergence of $\mu_s^{n_k}$ for each $s \in [0, t]$, one has

$$\lim_{k\to\infty} \int_U L_{\mu^{n_k}}\varphi \, d\mu_s^{n_k} = \int_U L_\mu\varphi \, d\mu_s.$$

Since $|L_{\mu^{n_k}}\varphi| \le \Lambda(\tau, \alpha, \varphi)$, applying Lebesgue's dominated convergence theorem we obtain

$$\lim_{k\to\infty} \int_0^t \int_U L_{\mu^{n_k}}\varphi \, d\mu_s^{n_k} \, ds = \int_0^t \int_U L_\mu\varphi \, d\mu_s \, ds.$$

It is obvious that

$$\lim_{k\to\infty} \int_0^t \int_U n_k^{-1} \varrho\Delta\varphi \, d\mu_s \, ds = 0.$$

Letting $k \to \infty$ in (5), we arrive at the equality

$$\int \varphi d\mu_t - \int \varphi d\nu = \int_0^t \int L_\mu\varphi d\mu_s ds.$$

Thus, μ is the required solution of (1).

2.3. **The case of a non-smooth matrix** A. Now suppose that Conditions (H1), (H2), and (H3) are fulfilled.

Let us begin with an important special case. We suppose that there exists a number $R > 0$ such that $A(x, t, \sigma) \equiv 0$ for all x if $|x| > R$. Then, taking into account (H2), for each $\tau \in (0, \tau_0]$ and $\alpha \in C^+([0, \tau_0])$ there exists a number $C(\alpha, \tau) > 0$ such that $|a^{ij}(x, t, \sigma)| \le C(\alpha, \tau)$ for all $x \in \mathbb{R}^d$, $t \in [0, \tau]$ and $\sigma \in M_{\tau,\alpha}$. Moreover, the mappings $x \mapsto a^{ij}(x, t, \sigma)$ are continuous on $\mathbb{R}^d$ uniformly in $\sigma \in M_{\tau,\alpha}$ and $t \in [0, \tau]$.

Let $\omega \in C^\infty(\mathbb{R}^d)$, $\omega \ge 0$, $\omega(x) = 0$ if $|x| > 1$ and $\|\omega\|_{L^1(\mathbb{R}^d)} = 1$. Whenever $0 < \delta < 1$, we set $\omega_\delta(x) = \delta^{-d}\omega(x/\delta)$ and $A_\delta = (a^{ij}_\delta)$, where

$$a^{ij}_\delta(x, t, \sigma) = a^{ij} * \omega_\delta(x, t, \sigma) = \int a^{ij}(y, t, \sigma)\omega_\delta(x - y)\, dy.$$

Note that $A_\delta(x, t, \sigma) \equiv 0$ if $|x| > R + 1$. It is obvious that (H2) is fulfilled for A_δ and for every measure σ the mapping $A_\delta(x, t, \sigma)$ is Lipschitzian uniformly in t. Moreover, the mapping $x \mapsto A(x, t, \sigma)$ is equicontinuous in σ and t, hence for each $\varepsilon > 0$ there exists $\delta \in (0, 1)$ such that for every $x \in \mathbb{R}^d$, $t \in [0, \tau]$ and $\sigma \in M_{\tau,\alpha}$ the following inequalities hold:

$$|A(x, t, \sigma) - A_\delta(x, t, \sigma)| \le \int |A(x, t, \sigma) - A(x + y, t, \sigma)|\omega_\delta(y)\, dy < \varepsilon.$$

Therefore, for each τ and α one can find a sequence $\delta_n > 0$ such that for every $\sigma \in M_{\tau,\alpha}$ the following estimate holds:

$$\max_{(x,t)\in\mathbb{R}^d\times[0,\tau]} |A(x, t, \sigma) - A_n(x, t, \sigma)| < \frac{1}{n}, \quad A_n = A_{\delta_n}.$$

Let us consider a new operator

$$L_{n,\mu}u := a^{ij}_n \partial_{x_i}\partial_{x_j}u + b^i\partial_{x_i}u.$$

Let $\varepsilon_0 > 0$. Let $\widehat{\Lambda}_1[\alpha](t) = \Lambda_1[\alpha](t) + \varepsilon_0$ and $\widehat{\Lambda}_2[\alpha] = \Lambda_2[\alpha]$. If one has $\Lambda_1 = 0$ (like in assertion (iii) in Theorem 1.1), the we replace Λ_2 with $\widehat{\Lambda}_2[\alpha](t) = \Lambda_2[\alpha](t) + \varepsilon_0$. We recall that in (iii) and (iv) we take $\tau < T$, where

$$T = \int_{u_0}^{+\infty} \frac{du}{uG(u)} \quad \text{and} \quad T = \int_{u_0}^{+\infty} \frac{du}{G(u)},$$

respectively. We replace Λ_1 with $\Lambda_1 + \varepsilon_0$ or Λ_2 with $\Lambda_2 + \varepsilon_0$, so we have a new moment T_{ε_0} with the function $G + \varepsilon_0$ in place of T with the function G. Note that $T_{\varepsilon_0} < T$ and $T_{\varepsilon_0} \to T$ if $\varepsilon_0 \to 0$. Let $\varepsilon_0 > 0$ be such that $\tau < T_{\varepsilon_0} < T$. Hence for every assertion (i)–(iv) in Theorem 1.1, taking into account Corollaries 2.3, 2.4, 2.5, and 2.6, one has appropriate τ and α. According to Remark 2.1, it suffices to have Conditions (H1), (H2) and (H3') only on this set $M_{\tau,\alpha}$.

There is an index N such that for every $n > N$ and every measure $\mu \in M_{\tau,\alpha}$ one has

$$|L_{n,\mu}V(x, t) - L_\mu V(x, t)| = |(a^{ij}_n(x, t) - a^{ij}(x, t))\partial_{x_i}\partial_{x_j}V(x)| \le \varepsilon_0 \min\{V(x), 1\}.$$

We recall that Condition (H1) holds for L_μ . Let $\mu \in M_{\tau,\alpha}$ and $n > N$. Then

$$L_{n,\mu}V = L_\mu V + \big(L_{n,\mu}V - L_\mu V\big) \le \Lambda_1 + \Lambda_2 V + \varepsilon_0 \min\{V, 1\} \le \widehat{\Lambda}_1 + \widehat{\Lambda}_2 V.$$

Hence Condition (H1) is fulfilled on $M_{\tau,\alpha}$ for $L_{n,\mu}$ if $n > N$. Moreover, τ and α are such that $Q(N_{\tau,\alpha}) \subseteq N_{\tau,\alpha}$. We recall that $A_n = 0$ if $|x| > R+1$. Hence the condition $|\sqrt{A_n}\nabla V| \le C_1+C_2V$ is fulfilled for some constants C_1 and C_2. Therefore, Conditions (H1), (H2), and (H3') hold and for each $n > N$ there exists a solution μ^n to the Cauchy problem (1) with the operator $L_{n,\mu}$ and the measures μ^n belong to $N_{\tau,\alpha}$. Also, as before, one can find a subsequence $\{\mu^{n_k}\}$ that is weakly and V-convergent to some measure $\mu \in N_{\tau,\alpha}$. Let us show that for every function $\varphi \in C_0^\infty(\mathbb{R}^d)$ one has

$$\lim_{k\to\infty} \int_0^t \int L_{n_k,\mu^{n_k}} \varphi \, d\mu_s^{n_k} \, ds = \int_0^t \int L_\mu \varphi \, d\mu_s \, ds. \tag{6}$$

Indeed, we have

$$\int_0^t \int L_{n_k,\mu^{n_k}} \varphi \, d\mu_s^{n_k} \, ds \quad = \quad \int_0^t \int (L_{n_k,\mu^{n_k}} \varphi \quad - \quad L_{\mu^{n_k}} \varphi) \, d\mu_s^{n_k} \, ds \ + \ \int_0^t \int L_{\mu^{n_k}} \varphi \, d\mu_s^{n_k} \, ds.$$

The first summand can be estimated as follows:

$$\left| \int_0^t \int L_{n_k,\mu^{n_k}} \varphi - L_{\mu^{n_k}} \varphi \, d\mu_s^{n_k} \, ds \right| \le n_k^{-1} t \max_x |\partial_{x_i} \partial_{x_j} \varphi(x)|,$$

Hence it tends to zero. Repeating the reasoning from Subsection 2.2, we obtain

$$\lim_{k\to\infty} \int_0^t \int L_{\mu^{n_k}} \varphi \, d\mu_s^{n_k} \, ds = \int_0^t \int L_\mu \varphi \, d\mu_s \, ds.$$

This proves (6). For each k and every function $\varphi \in C_0^\infty(\mathbb{R}^d)$ one has

$$\int \varphi d\mu_t^{n_k} - \int \varphi d\nu = \int_0^t \int L_{n_k,\mu^{n_k}} \varphi \, d\mu_s^{n_k} ds.$$

Letting $k \to \infty$, we arrive at the equality

$$\int \varphi d\mu_t - \int \varphi d\nu = \int_0^t \int L_\mu \varphi d\mu_s ds.$$

Hence μ is the required solution of (1).

Finally, we proceed to the general case and reduce it to the already studied case. Let $\psi \in C^\infty(\mathbb{R}^d)$, $0 \le \psi \le 1$, $\psi(x) = 1$ if $|x| < 1$ and $\psi(x) = 0$ if $|x| > 2$. Set $\psi_n(x) = \psi(x/n)$ and $L_{n,\mu} := \psi_n L_\mu$. One can easily see that the matrix A_n of $L_{n,\mu}$ vanishes outside of the ball $|x| < n$. It is obvious that Conditions (H2) and (H3) are fulfilled for $L_{n,\mu}$. In addition, one has

$$L_{n,\mu} V = \psi_n L_\mu V \le \psi_n \Lambda_1 + \psi_n \Lambda_2 V \le \Lambda_1 + \Lambda_2 V,$$

so (H1) is fulfilled. Let μ^n be a solution to the Cauchy problem with the operator $L_{n,\mu}$. One can assume again that μ^n belong to one and the same set $N_{\tau,\alpha}$. Repeating the same reasoning with weak- and V-converging subsequence $\{\mu^{n_k}\}$, letting $k \to \infty$ and taking into account that $L_{n,\mu}\varphi(x) = L_\mu \varphi(x)$ if $|x| < n$, we complete the proof of Theorem 1.1.

2.4. **Examples.** Consider the following Cauchy problem:

$$\partial_t \mu_t = \mathrm{div}\Big(\mu_t \int \nabla W(x-y) \, d\mu_t\Big), \quad \mu_0 = \nu. \tag{7}$$

Proposition 2.1. *Let $W(x) = K(|x|)$, where $K \in C^2([0, +\infty))$, $K(0) = 0$, $K'(0) = 0$, $K'(u) > 0$, $K''(u) > 0$ for $u > 0$ and there exist constants $C_1 > 0$, $C_2 > 0$ such that for all $u, v > 0$ one has*

$$K'(u+v) \le C_1 K(u) + C_2 K(v).$$

Suppose also that $K^m(|x|) \in L^1(\nu)$ for some $m > 1$. Then there exists a positive number τ such that the Cauchy problem (7) *has a probability solution $\mu = (\mu_t)_{t\in[0,\tau]}$ on $[0, \tau]$. Moreover, one has*

$$\sup_{t\in[0,\tau]} \int K^m(|x|)\, d\mu_t < \infty.$$

Proof. It follows from our assumptions that $a^{ij} = 0$ and

$$b(x, t, \mu) = -\int \frac{K'(|x-y|)(x-y)}{|x-y|}\, d\mu_t.$$

Let $V(x) = K^m(|x|) + 1$. Note that

$$K'(|x-y|) \le C_1 K(|x|) + C_2 K(|y|) \le C_1 K(|x|) + C_2 V^{1-\gamma}(y),$$

where $\gamma = (m-1)/m$. Hence Condition (H2) is fulfilled and, as before, we only have to ensure Condition (H1). Let $f(u) = K'(u)$. One has

$$(b(x, t, \mu), x) = -\int \frac{f(|x-y|)(x-y, x)}{|x-y|}\, d\mu_t \le$$

$$\le -\int f(|x-y|)|x-y|\, d\mu_t + \int f(|x-y|)|y|\, d\mu_t.$$

We recall that for $c, d > 0$ the following Young inequality holds:

$$cd \le \int_0^c f^{-1}(s)\, ds + \int_0^d f(t)\, dt,$$

where f^{-1} is the inverse function to f. Applying this inequality to $f(|x-y|)|y|$ and using that

$$\int_0^{f(|x-y|)} f^{-1}(u)\, du \le |x-y| f(|x-y|),$$

one obtains

$$(b(x, t, \mu), x) \le \int \Bigl(\int_0^{|y|} f(u)\, du\Bigr)\, d\mu_t.$$

Notice that $\nabla V(x) = mK^{m-1}(|x|) f(|x|) x/|x|$. Hence for some $c_1 > 0$ and $c_2 > 0$ one has

$$L_\mu V(x, t) \le \bigl(c_1 + c_2 V(x)\bigr)\Bigl(\int V(y)\, d\mu_t\Bigr)^{1/m} = \bigl(c_1 + c_2 V(x)\bigr)\alpha^{1/m}(t).$$

This yields (H1). Finally, we apply assertion (i) in Theorem 1.1. $\square$

Let $\tau_0 > 0$ and let $a(x, t)$ be a nonnegative continuous function on $\mathbb{R}^d \times [0, \tau_0]$ and $b(x, t)$ be a continuous vector field on $\mathbb{R}^d \times [0, \tau_0]$. Let F be a continuous nonnegative function on $\mathbb{R}^d$ and let G be a strictly increasing continuous positive function on $[0, +\infty)$, $G(0) > 1$. Let us set

$$b(x, t, \mu) = b(x, t) G\Bigl(\int F(y)\, d\mu_t\Bigr).$$

Consider the following Cauchy problem:

$$\partial_t \mu_t = a(x, t)\Delta\mu_t + \mathrm{div}\Bigl(\mu_t b(x, t, \mu)\Bigr), \quad \mu_0 = \nu, \tag{8}$$

Proposition 2.2. *Suppose that there exists a function $V \in C^2(\mathbb{R}^d)$ such that*

$$V \geq 1, \qquad \lim_{|x|\to+\infty} V(x) = +\infty, \qquad \lim_{|x|\to+\infty} F(x)/V(x) = 0$$

and for some constants C_1 and C_2 and all $(x,t) \in \mathbb{R}^d \times [0,\tau_0]$ one has

$$a(x,t)|\Delta V(x)| + (b(x,t), \nabla V(x)) \leq C_1 + C_2 V(x).$$

Let $V \in L^1(\nu)$. Then there exists $\tau \in (0,\tau_0]$ such that on $[0,\tau]$ the Cauchy problem (8) has a probability solution $\mu = (\mu_t)_{t\in[0,\tau]}$. Moreover, one has

$$\sup_{t\in[0,\tau]} \int V(x)\, d\mu_t < \infty.$$

Furthermore, if one has

$$\int_{u_0}^{+\infty} \frac{du}{uG(u)} = +\infty, \quad u_0 = \int V\, d\nu.$$

then a solution exists on the whole interval $[0,\tau_0]$.

Proof. Let $\mu \in M_{\tau,\alpha}$. Note that

$$\begin{aligned} L_\mu V(x,t) &= a(x,t)\Delta V + (b(x,t), \nabla V(x))G(\alpha(t)) = \\ &= a(x,t)\Delta V + (b(x,t), \nabla V(x)) + (b(x,t), \nabla V(x))(G(\alpha(t)) - 1) \leq (C_1 + C_2 V)G(\alpha(t)). \end{aligned}$$

The required assertion follows immediately from Theorem 1.1. □

For example, let $a = 1$ and $V(x) = \exp(M|x|^r)$, where $M > 0$ and $r \geq 2$. To ensure the hypotheses of Proposition 2.2 we need the following two estimates:

$$\exp(M|x|^r) \in L^1(\nu), \quad |F(x)| \leq \exp(M'|x|^r), \quad M' < M$$

and, for some c_1 and $c_2 > drM$ and all $(x,t) \in \mathbb{R}^d \times [0,\tau_0]$,

$$(b(x,t), x) \leq c_1 - c_2|x|^r.$$

3. The absence of global solutions

In this section we obtain sufficient conditions for the absence of global solutions. Let us observe that there is an extensive literature concerned with the so-called "blow-up" of solutions to such equations, see [24, 25, 4, 12, 21].

We recall that if one has $\Lambda_1[\alpha] = 0$ and $\Lambda_2[\alpha](t) = G(\alpha(t))$ in (H1), then, under Conditions (H2) and (H3), assertion (iii) in Theorem 1.1 gives the existence of solutions on each interval $[0,\tau]$, where

$$\tau < \int_{u_0}^{+\infty} \frac{du}{uG(u)}.$$

A similar assertion is given in (iv). Our next theorem shows that, in a sense, such estimates for the existence time are exact.

Theorem 3.1. *Let $V \in C^2(\mathbb{R}^d)$, $V \geq 0$, $\lim_{|x|\to\infty} V(x) = +\infty$. Let G be a continuous positive increasing function on $[0,+\infty)$. Suppose that the coefficients of the operator*

$$L_\mu = a^{ij}(x,t,\mu)\partial_{x_i}\partial_{x_j} + b^i(x,t,\mu)\partial_{x_i}$$

are defined on every set $M_{\tau,\alpha}(V)$ and for all $\mu \in M_{\tau,\alpha}(V)$ and all $(x,t) \in \mathbb{R}^d \times [0,\tau]$ one has

(i) $L_\mu V(x,t) \geq G\Big(\int V(x)d\mu_t\Big)V(x)$

or

(ii) $L_\mu V(x,t) \geq G\Big(\int V(x)d\mu_t\Big)$.

Suppose that $|\sqrt{A(x,t,\mu)}\nabla V(x)|^2 \leq C_1 + C_2V(x)$ *for some* $C_1 > 0$ *and* $C_2 > 0$. *Suppose that* $u_0 = \displaystyle\int V\,d\nu > 0$ *and in case* (i) *one has*

$$T = \int_{u_0}^{\infty} \frac{du}{uG(u)} < +\infty,$$

and in case (ii) *one has*

$$T = \int_{u_0}^{\infty} \frac{du}{G(u)} < +\infty.$$

Then the Cauchy problem (1) *has no probability solution* $\mu = (\mu_t)_{t\in[0,T]}$ *on* $[0,\tau]$ *with* $\tau \geq T$ *and*

$$\sup_{t\in[0,T]} \int V(x)\,d\mu_t < \infty.$$

Proof. We first consider (i). Let $\tau > 0$. If $\mu = (\mu_t)_{t\in[0,\tau]}$ is a solution to the Cauchy problem, then for every $\varphi \in C_0^\infty(\mathbb{R}^d)$ one has

$$\int \varphi d\mu_t - \int \varphi d\nu = \int_0^t \int L_\mu \varphi d\mu_s ds.$$

Let $\zeta_m \in C^\infty[0,+\infty)$ be such that $0 \leq \zeta_m^{'}(z) \leq 1$, $\zeta_m^{''} \leq 0$, $\zeta_m(z) = z$ for $z \leq m-1$, and $\zeta_m(z) = m$ for $z > m$. Set $\varphi(x) = \zeta_m(V(x)) - m$. We note that

$$L_\mu \zeta_m(V) = \zeta_m^{'}(V)L_\mu V + \zeta_m^{''}(V)(A\nabla V, \nabla V).$$

Hence

$$\int \zeta_m(V)d\mu_t = \int \zeta_m(V(x))d\nu + \int_0^t \int \zeta_m^{'}(V)L_\mu V d\mu_s ds + \int_0^t \int \zeta_m^{''}(V)|\sqrt{A}\nabla V|^2 d\mu_s ds,$$

which has a lower bound

$$\int \zeta_m(V)d\nu + \int_0^t \int \zeta_m^{'}(V)VG\Big(\int V\,d\mu_s\Big)d\mu_s ds - \int_0^t \int_{|V|\geq m} |\zeta_m^{''}(V)|(C_1 + C_2V)d\mu_s ds.$$

Letting $m \to \infty$ and taking into account that $V \in L^1(\mu)$, we obtain that

$$\int V d\mu_t \geq \int V d\nu + \int_0^t G\Big(\int V\,d\mu_s\Big)\int V d\mu_s ds.$$

Let

$$H(u) = G(u)u, \quad g(t) := \int V\,d\nu + \int_0^t H\Big(\int V\,d\mu_s\Big)ds.$$

Then the inequality can be rewritten as $H^{-1}(g^{'}(t)) \geq g(t)$ or, in other words, $g^{'}(t) \geq H(g(t))$. Dividing it by $H(g) = gG(g)$ and integrating over $[0,t]$, we arrive at the following inequality for all t:

$$t \leq \int_{g(0)}^{g(t)} \frac{du}{uG(u)} \leq \int_{g(0)}^{+\infty} \frac{du}{uG(u)} = T.$$

We recall that $g(0) = \int V(x)\,d\nu$. Let $\tau \geq T$. Then for $t = T$ we have $g(t) = +\infty$ and this contradicts the estimate $\sup\limits_{t\in[0,\tau]} \int V(x)\,d\mu_t < \infty$. The proof in case (ii) is analogous. $\square$

Example 3.1. Consider the Cauchy problem (1) with

$$a^{ij} = 0, \quad b(x,t,\mu) = x\int |y|^2 d\mu_t.$$

Suppose that $|x|^p \in L^1(\nu)$ for some $p > 2$. Let $m \in (2,p)$. Then

$$L_\mu |x|^m = (b, \nabla |x|^m) = m|x|^m \int |y|^2\,d\mu_t.$$

Thus, for $V(x) = \delta + |x|^m$ with $\delta > 0$ one has

$$L_\mu V \leq mV\Bigl(\int V\,d\mu_t\Bigr)^{2/m}$$

and hence, according to assertion (iii) of Theorem 1.1, there exists a solution $(\mu)_{[0,\tau]}$ on $[0,\tau]$ whenever

$$\tau < \int_{u_0}^{+\infty} \frac{du}{mu^{1+2/m}} = \frac{1}{2u_0^{2/m}}, \quad u_0 = \delta + \int |x|^m\,d\nu.$$

Taking into account that m is an arbitrary number in $(2,p)$ and δ is an arbitrary positive number, letting $m \to 2$ and $\delta \to 0$, we obtain that a solution exists on $[0,\tau]$ for each moment $\tau < \Bigl(2\int |x|^2\,d\nu\Bigr)^{-1}$. If $\int |x|^2\,d\nu > 0$ (that is, ν is not Dirac's measure δ_0 at the origin), then the hypotheses in assertion (i) in Theorem 3.1 with the function $V(x) = |x|^2$ are fulfilled. Hence there is no solution on $[0,\tau]$ for $\tau \geq \Bigl(2\int |x|^2\,d\nu\Bigr)^{-1}$. It can be easily proved that $\mu_t \equiv \delta_0$ is a solution of the Cauchy problem with the initial measure $\nu = \delta_0$ on each $[0,\tau]$.

This work has been supported by the RFBR projects 10-01-00518-a, 11-01-00348-a, 11-01-12018-ofi-m-2011, 12-01-92103-JFa, the Russian Federation President grant MK-3674.2011.1 and the SFB 701 at Bielefeld University.

The authors are grateful to V.I. Bogachev for useful discussions and valuable remarks.

References

[1] Ambrosio L. Transport equation and Cauchy problem for BV vector fields. Invent. Math., 2004, v. 158(2). P. 227–260.

[2] Ambrosio L. Transport equation and Cauchy problem for non-smooth vector fields. Lecture Notes in Mathematics "Calculus of Variations and Non-Linear Partial Differential Equations" (CIME Series, Cetraro, 2005), 1927, B.Dacorogno, P.Marcellini eds., 2008. P. 2–41.

[3] Belopolskaya Ya. A Probabilistic Approach to a Solution of Nonlinear Parabolic Equations. Theory Probab. Appl., 49(4). P. 589–611.

[4] Bertozzi A.L., Carrillo J.A., Laurent T. Blow-up in multidimensional aggregation equations with mildly singular interaction kernels. Nonlinearity, 2009, v. 22. P. 683–710.

[5] Bogachev V. Measure theory. Springer-Verlag, Berlin, 2007.

[6] Bogachev V.I., Da Prato G., Roeckner M. On parabolic equations for measures. Commun. Partial Differ. Equations, 2008, v. 33. P. 397–418.

[7] Bogachev V.I., Da Prato G., Roeckner M., Shaposhnikov S.V. Nonlinear evolution equations for measures on infinite dimensional spaces. Stochastic Partial Differential Equations and Applications. Quaderni di Matematica, 2010, v. 25, P. 51–64.

[8] Bogachev V., Krylov N., Roeckner M. Elliptic and parabolic equations for measures. Russian Math. Surveys, 2009, 64:6 P. 973–1078.

[9] Bogachev V., Rockner M., Shaposhnikov S. Nonlinear evolution and transport equations for measures. Journal of Math. Sciences. 2011. V. 179. 1. P. 747.

[10] Bogachev V.I., Roeckner M., Shaposhnikov S.V. On uniqueness problems related to the Fokker-Planck-Kolmogorov equations for measures. J. Math. Sci. (New York), 2011, v. 179, N 1. P. 7–47.

[11] Carrillo J.A., McCann R.J., Villani C. Kinetic equilibration rates for granular media and related equations: entropy, dissipation and mass transportation estimates. Rev. Math. Iberoamericana, 2003, v. 19. P. 971–1018.

[12] Carrillo J.A., Difrancesco M., Figalli A., Laurent T., Slepcev D. Global-in-time weak measure solutions and finite-time aggregation for non-local interaction equations. Duke Math. J., 2011, v. 156, N 2. P. 229–271.

[13] Daletskii Yu. Infinite-dimensional elliptic operators and parabolic equations connected with them. Russian Math. Surveys, 1967, 22:4 P. 1–53.

[14] Daletskii Yu., Fomin S. Measures and differential equations in infinite-dimensional space. Springer, 1991.

[15] DiPerna R.J., Lions P.L. Ordinary differential equations, transport theory and Sobolev spaces. Invent. Math., 1989, v. 98. P. 511–547.

[16] Dobrushin R. Vlasov equations. Functional Analysis and Its Applications, 1979, 13:2 P. 115–123.

[17] Dunford N., Schwartz Ja. Linear Operators, General Theory, Wiley Classics Library, 2009 .

[18] Funaki T. A certain class of duffusion processes associated with nonlinear parabolic equations. Z. Wahrscheinlichkeitstheorie verw. Geb., 1984, B. 67. P. 331–348.

[19] Kozlov V. The generalized Vlasov kinetic equation, Russian Math. Surveys, 63:4 (2008) P. 691–726.

[20] Kozlov V. The Vlasov kinetic equation, dynamics of continuum and turbulence.Nelin. Dinam., 6:3 (2010) P. 489–512.

[21] Li H., Toscani G. Long-time asymptotics of kinetic models of granular flows. Arch. Rational Mech. Anal., 2004, v. 172. P. 407–428.

[22] Lorenz T. Mutational Analysis. A Joint Framework for Cauchy Problems in and Beyond Vector Spaces. Springer Verlag, Lecture Notes in Mathematics, 2010, v. 1996.

[23] Maniglia S. Probabilistic representation and uniqueness results for measure-valued solutions of transport equation. J. Math. Pures Appl., 2007, v. 87. P. 601–626.

[24] Mitidieri E., Pohozaev S. Apriori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities. Proc. Steklov Inst. Math., 2001, 234 P. 1–362.

[25] Mitidieri E., Pohozaev S. Liouville theorems for some classes of nonlinear nonlocal problems. Proc. Steklov Math. Inst., 2005, 248 P. 164–184.

[26] Oleinik O., Radkevic E. Second order equations with nonnegative characteristic form., Mathematical analysis, 1969, P. 7–252.

[27] Shaposhnikov S. On Uniqueness of a Probability Solution to the Cauchy Problem for the Fokker-Planck-Kolmogorov Equation. Theory Probab. Appl., 56(1) P. 96–115.

[28] Stroock, D.W., Varadhan, S.R.S. Multidimensional diffusion processes. Springer-Verlag, Berlin – New York, 1979.

[29] Tonoyan L. Nonlinear elliptic equations for measures. Dokl. Russian Acad. Sci. 2011, V. 439:2 P. 174–177.