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Phase Transitions in a Quenched Amorphous Ferromagnet

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Abstract Quenched thermodynamic states of an amorphous ferromagnet are studied. The magnet is a countable collection of point particles chaotically distributed over $\mathbb{R}^d$, $d \geq 2$, which is modeled by a Poisson random point field. Each particle bears a real-valued spin with symmetric a priori distribution; the spin-spin interaction is set to be pairwise and attractive. For every pair of particles, the interaction intensity is random with distribution dependent on the Euclidean distance between the particles. The intensities are independent of the Poisson field and also of each other for distinct pairs of particles. The main result of the paper is a statement that, with probability one, the mean magnetization per particle can be positive, and hence the magnet can be in a ferromagnetic phase, if the particle density and the interaction strength are large enough.

Keywords Phase transition · random graph · Poisson random point field · continuum percolation · Wells inequality

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1 Introduction

In this paper, we study thermodynamic states of the following model of an interacting particle system. A countable collection of point 'particles' is chaotically distributed over $\mathbb{R}^d$, $d \geq 2$. The corresponding mathematical model is a homogeneous Poisson random point field π_λ with intensity $\lambda > 0$. Each 'particle' represents a cluster of magnetically active physical particles, and hence is supposed to bear spin σ_x which takes any real value. We assume that $\sigma_x \in \mathbb{R}$ is characterized by a symmetric a priori distribution χ. The spin-spin interaction is supposed to be pair-wise and attractive. For the 'particles' located at x and y, it has the form $J_{xy}\sigma_x\sigma_y$ with intensity $J_{xy} = \phi(|x-y|)D_{xy}$, where a non-random (measurable) function ϕ takes values in $[\phi_*, \phi^*]$, $\phi_* > 0$, $\phi^* < \infty$. The random variables $\{D_{xy} : x, y \in \mathbb{R}^d, \ x \neq y\}$ take values 1 and 0 with probability $g(|x-y|)$ and $1 - g(|x-y|)$, respectively. They are mutually independent, and also independent of the underlying Poisson random field. We suppose that $g(r) \in [0, 1]$, and $g(r) = 0$ whenever $r > r_*$, where $r_* > 0$ is a fixed parameter of the model. The physical meaning of the factors D_{xy} is to take into account that some of the exchange interactions between the spins can be suppressed to zero by the random environment in which the system is placed. We call this model the *amorphous ferromagnet*, cf. [18, Section 11].

In view of the randomness mentioned above, the notion of the thermodynamic state of our model can be introduced in the following two ways. In the first one, the randomness is taken into account already at the level of local states defined on the space of joint configurations of particles, spins, and connection variables D. The global Gibbs measures constructed in this way are then the *annealed states*; they describe the equilibrium of the whole system. In the case of non-random $D_{xy} = 1$, $|x - y| \leq r_*$ and $D_{xy} = 0$, $|x - y| > r_*$, the mentioned configuration space would be the space of marked particle configurations $\hat{\gamma} = \{(x, \sigma_x) : x \in \gamma\}$, where γ is a locally finite subset of $\mathbb{R}^d$, see (1) below. The second approach, which we follow in this paper, is to construct thermodynamic states of the spin system alone for fixed *typical* configurations of the particles and the variables D. These are *quenched states.* The global observables characterizing such states are *self-averaging*, i.e., non-random. Note that studying quenched states is a more difficult problem as compared to that of annealed ones, in view of the present spatial irregularities which do not allow for applying here most of the methods effective for regular systems.

Actually, there exist only few publications on the mathematically rigorous theory of phase transitions in spin systems of general type living on non-crystalline (amorphous) substances, cf. [6–8, 20] where annealed states were considered. The reason for this is presumably that the methods for studying such phenomena, e.g., infrared estimates, are essentially based on the translation invariance (and other symmetries) of the underlying crystals. At the same time, for Ising spins $\sigma_x = \pm 1$, there exist methods applicable to the corresponding models on graphs, cf. [9, 10, 15]. The main idea of proving phase transitions in such models is to relate the appearance of multiple phases of the spin system with Bernoulli bond percolation on the underlying graph. On the other hand, an amorphous substance can be described in terms of random point fields, and thereby can also be considered as a random graph, in which one can observe a Bernoulli bond percolation, see [3, 16, 17]. The main aim of the present work is to combine the mentioned methods and prove that the mean magnetization in the model of an amorphous ferromagnet with spins $\sigma_x \in \mathbb{R}$ mentioned above can be positive almost surely, and hence the Gibbs states can be multiple, if the particle density and the interaction strength are large enough. The realization of the mentioned idea goes along the following line of arguments. First we establish the existence of the corresponding Gibbs states. For random graphs with unbounded vertex degrees, proving the existence of Gibbs states with properties suitable for physical applications is a nontrivial problem, especially if the single-spin distribution χ has noncompact support. In the latter case, there can exist states supported on configurations of spins with rapidly increasing $|\sigma_x|$ as $|x| \to +\infty$, whereas for typical configurations in a ferromagnetic phase, most of the spins take values close to same $s > 0$. Therefore, Gibbs measures of physical relevance ought to be supported on the configurations with tempered growth of $|\sigma_x|$. In this paper, the existence of such *tempered* Gibbs states is proven by means of a result of [11] and a property of the Poisson random field obtained in Proposition 1 below. Next, by means of the results of [3, 16, 17], we conclude that the underlying Poisson random field with the adjacency relation established by the function g almost surely has an infinite connected component, in which the Bernoulli bond percolation takes place if the intensity λ exceeds some threshold $\lambda_* \in (0, 1)$. By means of a result of [9], from this we deduce that the Ising model on such a graph can be in a ferromagnetic state. Then the generalization to all other types of the single-spin measures χ, including those corresponding to unbounded spins, is performed by means of the Wells inequality [21]. For the reader convenience, we present here a complete proof of the latter.

Finally, let us mention that, for our model with $\sigma_x \in \mathbb{R}$, the problem of uniqueness of Gibbs states – opposite to the one which we study here, remains open except for some special cases, see Remark 1 below. For the Ising model, the mentioned uniqueness can be established by showing that the Bernoulli site percolation on the underlying graph is absent for small enough values of the corresponding probability, see [5].

2 Quenched Gibbs states

2.1 The model

By Γ we denote the set of all locally finite configurations in $\mathbb{R}^d$, that is,

$$\Gamma = \{\gamma \subset \mathbb{R}^d : |\gamma \cap K| < \infty \text{ for any compact } K \subset \mathbb{R}^d \}, \tag{1}$$

where $|A|$ stands for the cardinality of a finite set A. This set is equipped with the vague topology being the weakest one in which the maps $\Gamma \ni \gamma \mapsto \sum_{x \in \gamma} f(x)$ are continuous for all continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ with compact support, see e.g., [1] for more detail. This allows for introducing the corresponding Borel σ-field $\mathcal{B}(\Gamma)$. The vague topology is metrizable in such a way that the corresponding metric space Γ is complete and separable. For $\lambda > 0$, by π_λ we denote the homogeneous Poisson measure on $(\Gamma, \mathcal{B}(\Gamma))$ with intensity λ. It is convenient for us to consider π_λ as the probability distribution of a point process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In that we assume the existence of a measurable map $\Omega \ni \omega \mapsto \gamma(\omega) \in \Gamma$ such that, for each $A \in \mathcal{B}(\Gamma)$, $\pi_\lambda(A) = \mathbb{P}(\gamma^{-1}(A))$.

For $r_* > 0$, let $g : \mathbb{R}_+ \to [0,1]$ be a non-increasing function with support in the interval $[0, r_*]$. We define a system of random variables $D := \{D_{xy} = D_{yx},\ x, y \in X,\ x \neq y\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that each D_{xy} takes values 1 and 0 with probability $g(|x-y|)$ and $1 - g(|x-y|)$, respectively. The function g is assumed to be such that

$$g_* := \int_{\mathbb{R}^d} g(|x|)dx > 0. \tag{2}$$

All D_{xy} are mutually independent and are also independent of the Poisson point process mentioned above. Now, for a fixed pair (γ, D), we consider a graph $\mathsf{G}(\gamma, D)$ with vertex set γ and edge set

$$\mathsf{E}(\gamma, D) = \{\{x, y\} \subset \gamma : D_{xy} = 1\}.$$

It is a random graph called the *random connection model with connection function g, driven by π_λ*, see [3,17]), and especially [16, pages 18–20], for a more detailed exposition of this model. It will serve us as the underlying set for spin configurations of the magnet we consider.

Let $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}_\mathrm{b}(\mathbb{R}^d)$ stand for the set of all Borel and all bounded Borel subsets of $\mathbb{R}^d$, respectively. For $\Lambda \in \mathcal{B}(\mathbb{R}^d)$ and $\gamma \in \Gamma$, by $\mathsf{E}_\Lambda(\gamma, D)$ we denote the set of edges of $\mathsf{G}(\gamma, D)$ both endpoints of which are in $\gamma_\Lambda := \gamma \cap \Lambda$. To each $x \in \gamma$, there is assigned a variable – spin $\sigma_x \in \mathbb{R}$. Then the configuration of spins corresponding to γ is $\sigma = (\sigma_x)_{x\in\gamma} \in \mathbb{R}^\gamma$. The set of all spin configurations $\mathbb{R}^\gamma$ is equipped with the product topology and with the corresponding Borel σ-field $\mathcal{B}(\mathbb{R}^\gamma)$. For $\Lambda \in \mathcal{B}(\mathbb{R}^d)$, by σ_Λ we denote the 'configuration in Λ', i.e., $\sigma_\Lambda := \{\sigma_x : x \in \gamma_\Lambda\}$. Given two configurations σ and $\bar{\sigma}$, by $\sigma_\Lambda \times \bar{\sigma}_{\Lambda^c}$ we mean the configuration such that its restriction to $x \in \gamma_\Lambda$ (respectively, to $x \in \gamma_{\Lambda^c}$) is σ_x (respectively, $\bar{\sigma}_x$); $\Lambda^c := \mathbb{R}^d \setminus \Lambda$. For $x \in \gamma$, by ∂x we denote the neighborhood of x in $\mathsf{G}(\gamma, D)$, that is, $\partial x := \{y \in \gamma : D_{xy} = 1\}$.

Let χ be a finite symmetric measure on $\mathbb{R}$. Our aim is to construct Gibbs measures on $\mathbb{R}^\gamma$ corresponding to the single-spin measures $\chi_x = \chi$ and to the following relative energy functionals

$$-E^\gamma_\Lambda(\sigma_\Lambda|\bar{\sigma}_{\Lambda^c}) = \sum_{\{x,y\}\in\mathsf{E}_\Lambda(\gamma,g)} J_{xy}\sigma_x\sigma_y + \sum_{x\in\gamma_\Lambda}\sum_{y\in\partial x\cap\gamma_{\Lambda^c}} J_{xy}\sigma_x\bar{\sigma}_y, \tag{3}$$

where the interaction intensities are

$$J_{xy} = \phi(|x-y|)D_{xy}, \tag{4}$$

and hence are random. In (4), $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-random measurable function such that, for some $\phi^* > \phi_* > 0$,

$$\phi(r) \in [\phi_*, \phi^*], \qquad \text{for all} \ \ r \in [0, r_*]. \tag{5}$$

For $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ and $\bar{\sigma} \in \mathbb{R}^\gamma$, we define

$$\Pi_\Lambda(A|\bar{\sigma}) = \frac{1}{Z_\Lambda(\bar{\sigma})} \int_{\mathbb{R}^{\gamma_\Lambda}} \mathbb{I}_A(\sigma_\Lambda \times \bar{\sigma}_{\Lambda^c}) \exp(-E_\Lambda^\gamma(\sigma_\Lambda|\bar{\sigma}_{\Lambda^c}))\chi_\Lambda(d\sigma_\Lambda), \quad (6)$$

where $\mathbb{I}_A$ is the indicator of $A \in \mathcal{B}(\mathbb{R}^\gamma)$, E_Λ^γ is as in (3), and

$$\chi_\Lambda(d\sigma_\Lambda) := \bigotimes_{x \in \gamma_\Lambda} \chi(d\sigma_x), \quad (7)$$

$$Z_\Lambda(\bar{\sigma}) := \int_{\mathbb{R}^\gamma} \exp(-E_\Lambda^\gamma(\sigma_\Lambda|\bar{\sigma}_{\Lambda^c}))\chi_\Lambda(d\sigma_\Lambda).$$

Thus, for each $A \in \mathcal{B}(\mathbb{R}^\gamma)$, $\Pi_\Lambda(A|\cdot)$ is $\mathcal{B}(\mathbb{R}^\gamma)$-measurable, and, for each $\bar{\sigma} \in \mathbb{R}^\gamma$, $\Pi_\Lambda(\cdot|\bar{\sigma})$ is a probability measure on $(\mathbb{R}^\gamma, \mathcal{B}(\mathbb{R}^\gamma))$. The collection of probability kernels $\{\Pi_\Lambda : \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)\}$ is called the *Gibbs specification* of the model we consider, see [4, Chapter 2]. It enjoys the consistency property

$$\int_{\mathbb{R}^\gamma} \Pi_{\Lambda_1}(A|\sigma)\Pi_{\Lambda_2}(d\sigma|\bar{\sigma}) = \Pi_{\Lambda_2}(A|\bar{\sigma}),$$

which holds for all $A \in \mathcal{B}(\mathbb{R}^\gamma)$, $\bar{\sigma} \in \mathbb{R}^\gamma$, and all $\Lambda_1, \Lambda_2 \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ such that $\Lambda_1 \subset \Lambda_2$.

Definition 1 A probability measure μ on $(\mathbb{R}^\gamma, \mathcal{B}(\mathbb{R}^\gamma))$ is said to be a *quenched Gibbs measure* of the model considered if it satisfies the Dobrushin-Lanford-Ruelle equation

$$\mu(A) = \int_{\mathbb{R}^\gamma} \Pi_\Lambda(A|\sigma)\mu(d\sigma), \qquad \text{for all} \quad A \in \mathcal{B}(\mathbb{R}^\gamma).$$

The set of all such measures is denoted by $\mathcal{G}(\gamma, D)$.

A priori it is not obvious whether $\mathcal{G}(\gamma, D)$ is nonempty. If this is the case, then $\mathcal{G}(\gamma, D)$ depends on ω through γ and D, not necessarily in a measurable way, cf. [12]. We leave aside here problems of this kind, which we are going to tackle in [2].

As mentioned above, the set $\mathcal{G}(\gamma, D)$ may contain elements which are not suitable for describing thermodynamic states of a ferromagnet since we have no a priori information concerning the support of the eventual $\mu \in \mathcal{G}(\gamma, D)$. At the same time, Gibbs measures of physical relevance ought to be supported on the so called *tempered* spin configurations, for which $|\sigma_x|$ increases 'not too fast' as $|x| \to +\infty$, cf. (12) below. In order to ensure the existence of such measures we first establish some properties of the graph $\mathsf{G}(\gamma, D)$.

2.2 Properties of the underlying graph

2.2.1 Estimating the degree growth

For $x \in \gamma$, let $n(x)$ be the number of neighbors of x, i.e., $n(x) := |\partial x|$. Clearly, $n(x)$ is almost surely finite since each γ is almost surely locally finite. Note, however, that $\sup_{x \in \gamma} n(x) = +\infty$, also almost surely.

For an $\alpha > 0$, we introduce the weight function

$$w_\alpha(x) = \exp(-\alpha|x|), \qquad x \in \mathbb{R}^d. \tag{8}$$

For $x \in \gamma$ and $\theta > 0$, we then consider, cf. Eqs. (4) and (5) in [11],

$$a(\alpha, \theta) := \sum_{\{x,y\}\in \mathsf{E}(\gamma,D)} [w_\alpha(x) + w_\alpha(y)][n(x)n(y)]^\theta,$$
$$b(\alpha) := \sum_{x\in\gamma} w_\alpha(x).$$

Proposition 1 *For each positive α and θ, both $a(\alpha, \theta)$ and $b(\alpha)$ are almost surely finite.*

Proof By the very definition of the Poisson measure π_λ, for each $n \in \mathbb{N}$ and any measurable and symmetric function $f : (\mathbb{R}^d)^n \to \mathbb{R}_+ := [0, +\infty)$, we have that

$$\int_\Gamma \left(\sum_{\{x_1,\ldots,x_n\}\subset\gamma} f(x_1, \ldots, x_n) \right) \pi_\lambda(d\gamma) \tag{9}$$
$$= \frac{\lambda^n}{n!} \int_{(\mathbb{R}^d)^n} f(x_1, \ldots, x_n) dx_1 \cdots dx_n$$

(the Mecke identity). Then

$$\mathbb{E}b(\alpha) = \lambda \int_{\mathbb{R}^d} w_\alpha(x)dx < \infty,$$

where $\mathbb{E}$ is the expectation with respect to $\mathbb{P}$. Hence $b(\alpha) < \infty$ almost surely. To complete the proof we write

$$a(\alpha, \theta) = 2\sum_{x\in\gamma} w_\alpha(x)m_\theta(x), \qquad m_\theta(x) := \sum_{y\in\partial x} [n(x)n(y)]^\theta. \tag{10}$$

Since $n(x)$ takes integer values only, we have

$$m_\theta(x) \le [n(x)]^{k+1} \max_{y\in\partial x} [n(y)]^k, \tag{11}$$

where k is the least integer such that $\theta \le k$. Let $\mathcal{I} : \mathbb{R}^d \to \{0, 1\}$ be the indicator of the ball $B_{2r_*} := \{x \in \mathbb{R}^d : |x| \le 2r_*\}$. Clearly,

$$\max_{y\in\{x\}\cup\partial x} n(y) \le \sum_{z\in\gamma} \mathcal{I}(z - x).$$

Applying this in (11) we get

$$m_\theta(x) \le \sum_{\{y_1,\ldots,y_{2k+1}\}\subset\gamma\setminus x} \prod_{j=1}^{2k+1} \mathcal{I}(y_j - x).$$

By this and (9), from (10) and (11) we then obtain

$$\begin{aligned}
\mathbb{E}a(\alpha,\theta) &\le 2\int_\Gamma \left(\sum_{x\in\gamma} w_\alpha(x) \sum_{\{y_1,\ldots,y_{2k+1}\}\subset\gamma\setminus x} \prod_{j=1}^{2k+1} \mathcal{I}(y_j - x)\right) \pi_\lambda(d\gamma) \\
&= 2\int_\Gamma \left(\sum_{\{y_1,\ldots,y_{2k+2}\}\subset\gamma} \sum_{j=1}^{2k+2} w_\alpha(y_j) \prod_{l=1, l\neq j}^{2k+2} \mathcal{I}(y_l - y_j)\right) \pi_\lambda(d\gamma) \\
&= \frac{\lambda^{2k+2}}{(2k+1)!} \int_{(\mathbb{R}^d)^{2k+2}} w_\alpha(y_1) \prod_{l=2}^{2k+2} \mathcal{I}(y_l - y_1) dy_1 \cdots dy_{2k+2} \\
&= \frac{\lambda^{2k+2}}{(2k+1)!} V_{2r_*}^{2k+1} \int_{\mathbb{R}^d} w_\alpha(y) dy < \infty,
\end{aligned}$$

which completes the proof. Here $V_{2r_*} = \int \mathcal{I}(x)dx$ is the volume of the ball B_{2r_*}.

2.2.2 Percolation in a typical graph

In Section 3 below, we explore the relationship between phase transitions in our model and two (related to each other) percolation problems on the graph $\mathsf{G}(\gamma, D)$.

The *continuum percolation* consists in the appearance of an infinite connected component of the random graph $\mathsf{G}(\gamma, D)$, see [16,17]. It is known that such a component exists with probability 1 (resp. 0) if the intensity λ of the Poisson measure π_λ satisfies inequality $\lambda > \lambda_g^*$ (resp. $\lambda < \lambda_g^*$), where $\lambda_g^* \in \mathbb{R}^+$ is a critical value, which depends on the connection function g. The threshold value satisfies $\lambda_g^* \ge g_*$, with g_* given in (2). Observe that there can only be a single infinite connected component, see [16, Theorem 6.3, page 172].

The *bond percolation problem* on $\mathsf{G}(\gamma, D)$ is posed as follows. Let $q \in (0,1)$ be fixed. Then each edge of $\mathsf{G}(\gamma, D)$ is marked independently open with probability q, and closed otherwise. Now we can form a new graph, by removing closed edges, and discuss the continuum percolation problem on it. To make this procedure consistent with the continuum percolation discussed above we introduce another system of random variables

$$\widehat{D} := \{\widehat{D}_{xy} = \widehat{D}_{yx},\ x, y \in \mathbb{R}^d,\ x \neq y,\ |x-y| \le r_*\}$$

on $(\Omega, \mathcal{F}, \mathbb{P})$ such that each $\widehat{D}_{xy}$ takes values 1 and 0 with probability q and $1-q$, respectively. All $\widehat{D}_{xy}$ are mutually independent, are independent of the connection variables D and of the Poisson point process. Let $\widehat{D}D$ denote the system of product random variables $\{\widehat{D}_{xy}D_{xy},\ x, y \in \mathbb{R}^d,\ x \neq y,\ |x-y| \le r_*\}$. We say that $\mathsf{G}(\gamma, D)$ admits Bernoulli bond percolation if the graph $\mathsf{G}(\gamma, \widehat{D}D)$ has an infinite connected component. It is clear that the probability that given $x, y \in \gamma$ are connected in $\mathsf{G}(\gamma, \widehat{D}D)$ is equal to $qg(|x-y|)$. That is, the bond percolation in $\mathsf{G}(\gamma, D)$ is equivalent to the continuum percolation with connection function qg. Hence, in accordance with the first part of this subsection, for a fixed intensity $\lambda > \lambda_g^*$, there exists a *critical probability* $q_* \in (0, 1)$ such that the graph $\mathsf{G}(\gamma, \widehat{D}D)$ contains an infinite connected component with probability 1 (resp. 0) if $q > q_*$ (resp. $q < q_*$). It is known that $q_* \geq 1/\lambda g_*$, cf. [3].

2.3 Existence of Gibbs states

As mentioned above, we aim at showing that the set of quenched Gibbs measures as in Definition 1 contains more than one element, cf. Theorem 2 below. However, for random graphs with unbounded vertex degrees, the existence of Gibbs states with properties suitable for physical applications is not immediate if the single-spin distribution χ has noncompact support. The usual way to exclude 'non-physical' Gibbs measures from the consideration is to take into account only *tempered* elements of $\mathcal{G}(\gamma, D)$ by prescribing a priori their support properties, see e.g., [14] or a more recent development in [13]. Thus, for $\alpha > 0$, we define

$$\Sigma(\alpha) := \left\{ \sigma \in \mathbb{R}^\gamma : \sum_{x \in \gamma} |\sigma_x|^2 w_\alpha(x) < \infty \right\}, \tag{12}$$

where w_α is as in (8). For each fixed γ, it is a Borel subset of $\mathbb{R}^\gamma$. The elements of $\Sigma(\alpha)$ are called tempered configurations. Then

$$\mathcal{G}_{\mathrm{t}}(\gamma, D) := \{\mu \in \mathcal{G}(\gamma, D) : \mu(\Sigma(\alpha)) = 1\} \tag{13}$$

is called the set of tempered quenched Gibbs measures. In view of Proposition 1, constant configurations $\sigma_x = s$, for all $x \in \gamma$, are tempered.

Now to ensure that the set $\mathcal{G}_{\mathrm{t}}(\gamma, D)$ as given in (13) is almost surely nonempty, we impose conditions on the single-spin measure. For positive u and $\varkappa$, we set

$$C_\pm(\varkappa) = \int_{\mathbb{R}} \exp\left(\pm\varkappa |t|^u\right) \chi(dt). \tag{14}$$

Theorem 1 *Let the single-spin measure χ be such that, for some $u > 2$ and each $\varkappa > 0$, the quantities in (14) obey $0 < C_-(\varkappa) \leq C_+(\varkappa) < \infty$. Then the set of Gibbs measures $\mathcal{G}_{\mathrm{t}}(\gamma, D)$ is almost surely nonempty. Moreover, for each positive ϑ and α, there exists an almost surely finite $C(\vartheta, \alpha) > 0$ such that, uniformly for all $\mu \in \mathcal{G}_{\mathrm{t}}(\gamma, D)$,*

$$\int_{\mathbb{R}^\gamma} \exp\left(\vartheta \sum_{x \in \gamma} |\sigma_x|^2 w_\alpha(x) \right) \mu(d\sigma) \leq C(\vartheta, \alpha). \tag{15}$$

Proof The results stated follow from Theorem 1 of [11] since all the conditions of that theorem are satisfied in view of (5), Proposition 1, and the assumed properties of χ.

Let us make some comments. First we mention that the existence of Gibbs measures follows from the relative weak compactness of the family $\{\Pi_\Lambda(\cdot|\bar{\sigma}) : \Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)\}$, for at least some $\bar{\sigma} \in \mathbb{R}^\gamma$. By Prokhorov's theorem, this fact is deduced from the tightness of the mentioned family. A typical choice of $\bar{\sigma}$, for which the tightness is proven, is $\bar{\sigma}_x = s \in \mathbb{R}$ for all $x \in \gamma$. Note that such $\bar{\sigma}$ is tempered, see (12). Then the accumulation points of the family $\{\Pi_\Lambda(\cdot|\bar{\sigma}) : \Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)\}$ are shown to obey the Dobrushin-Lanford-Ruelle equation and to satisfy the estimate in (15), in which the $C(\vartheta, \alpha)$ can be expressed explicitly in terms of the weights as in (8), cf. Proposition 1. The boundedness just mentioned guarantee that all $\Pi_\Lambda(\cdot|\bar{\sigma})$ are tempered measures. Let $\{\Lambda_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_{\rm b}(\mathbb{R}^d)$ be a *cofinal* sequence, which means that $\Lambda_n \subset \Lambda_{n+1}$, $n \in \mathbb{N}$, and each $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$ is contained in a certain Λ_n. The mentioned weak compactness of the family $\{\Pi_\Lambda(\cdot|\bar{\sigma}) : \Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)\}$ yields that, for each $s > 0$ and a cofinal sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$, the sequence $\{\Pi_{\Lambda_n}(\cdot|\bar{\sigma})\}_{n\in\mathbb{N}}$ with $\bar{\sigma}_x = s$ weakly converges to a certain element of $\mathcal{G}_{\rm t}(\gamma, D)$, which is independent of the choice of $\{\Lambda_n\}$, cf. [13, Theorem 3.8]. By similar arguments, one can show that $\mathcal{G}_{\rm t}(\gamma, D)$ is compact in the weak topology. For $a > 0$, by

$$\mu^{\pm a} \in \mathcal{G}_{\rm t}(\gamma, D) \tag{16}$$

we shall denote the limiting elements of $\mathcal{G}_{\rm t}(\gamma, D)$ for $s = \pm a$.

Now we turn to the single-spin measure. If χ has compact support, as was the case in [20], then clearly $C_+ < \infty$ and $C_-(\varkappa) > 0$. The most known example of such χ is

$$\chi(dt) = \delta_{-1}(dt) + \delta_{+1}(dt), \tag{17}$$

which corresponds to an Ising magnet. Here δ_s is the Dirac measure concentrated at $s \in \mathbb{R}$. Since this magnet will be used as a reference model, we reserve a special notation $\mathcal{G}^{\rm Ising}(\gamma, D)$ for the set of all corresponding Gibbs measures, which are automatically tempered. By $\nu^\pm \in \mathcal{G}^{\rm Ising}(\gamma, D)$, we denote the limiting Gibbs measures as in (16) with $a = 1$.

Another example is the measure which corresponds to 'unbounded' spins, cf. [11,13]. Here

$$\chi(dt) = \exp\left(-V(t)\right) dt, \tag{18}$$

where $V : \mathbb{R} \to \mathbb{R}$ is a measurable even function such that: (a) the set $\{t \in \mathbb{R} : V(t) < +\infty\}$ is of positive Lebesgue measure; (b) $V(t)$ increases at infinity as $|t|^{u+\epsilon}$ with some $\epsilon > 0$ and u being as in (14). This includes the case where V is a polynomial of even degree $d \geq 4$ with positive leading coefficient.

Remark 1 By means of the Brascamp-Lieb and other known inequalities, we can prove the following result. If $V(t)$ in (18) is convex and the energy functional instead of (3) has the form

$$-E^\gamma_\Lambda(\sigma_\Lambda|\bar{\sigma}_{\Lambda^c}) = -\sum_{\{x,y\}\in \mathsf{E}_\Lambda(\gamma,g)} J_{xy}(\sigma_x - \sigma_y)^2 - \sum_{x\in\gamma_\Lambda}\sum_{y\in\partial x\cap\gamma_{\Lambda^c}} J_{xy}(\sigma_x - \bar{\sigma}_y)^2,$$

then, for all values of the intensity λ of the underlying Poisson random field, the set $\mathcal{G}_{\rm t}(t, D)$ is almost surely a singleton, and hence no phase transition occurs.

Finally, we mention that the boundedness $|J_{xy}| \leq \phi^*$ assumed in (4) and (5) has been imposed for simplicity only. It can be relaxed by passing to 'tempered' interaction intensities as in [12].

3 The Phase Transition

3.1 The statement

Recall that by a phase transition in the considered quenched ferromagnet we mean the possibility that the set of tempered Gibbs measures $\mathcal{G}_{\rm t}(\gamma, D)$ almost surely contains at least two elements which are thermodynamic phases of the magnet. It is equivalent to the appearance of a nonzero magnetization in states $\mu^{\pm a} \in \mathcal{G}_{\rm t}(\gamma, D)$, cf. [4, Chapter 19].

Let us observe that there is no interaction between spins in different connected components of the underlying graph $\mathsf{G}(\gamma, D)$. Then for a phase transition to occur it is necessary that $\mathsf{G}(\gamma, D)$ almost surely possess an infinite connected component, that is, this graph admits a continuum percolation as described in subsection 2.2.2 above. This is the case if the intensity λ of the underlying Poisson random field obeys the bound $\lambda > \lambda_g^*$. For $\lambda < \lambda_g^*$, we have no infinite connected component in $\mathsf{G}(\gamma, D)$ and thus $|\mathcal{G}(\gamma, D)| = 1$ with probability 1. In order to obtain a sufficient condition for a phase transition to occur, we will explore the well-known relationship between the Bernoulli bond percolation on the fixed sample graph $\mathsf{G}(\gamma, D)$ and the existence of multiple Gibbs states in the corresponding Ising model, established in [9], as was discussed in subsection 2.2.2 above. Our goal is to prove the following result.

Theorem 2 *Let the measure χ be as in Theorem 1 and such that $\chi(\{0\}) < \chi(\mathbb{R})$. Assume also that the intensity λ of the underlying Poisson random field satisfies the condition $\lambda > \lambda_g^*$. Then there exists ϕ_* such that, for any ϕ satisfying (5), the set $\mathcal{G}_{\rm t}(\gamma, D)$ contains at least two elements with probability 1.*

The proof of this statement is based on the following result, cf. (16).

Lemma 1 *Let the conditions of Theorem 2 be satisfied. Then there exist $a > 0$ and ϕ_* such that, for ϕ satisfying (5), the following estimate holds with probability 1*

$$\int_{\mathbb{R}^\gamma} \sigma_o \mu^{+a}(d\sigma) > 0, \qquad (19)$$

for some $o \in \gamma$.

The proof of this lemma is given in the next subsection.

Proof of Theorem 2 By the symmetry of χ and of the interaction in (3), we have

$$\int_{\mathbb{R}^\gamma} \sigma_o \mu^{+a}(d\sigma) = -\int_{\mathbb{R}^\gamma} \sigma_o \mu^{-a}(d\sigma).$$

Then (19) yields $\mu^{+a} \neq \mu^{-a}$ and hence the multiplicity in question. Note that o in (19) belongs to the infinite connected component of $\mathsf{G}(\gamma, D)$, and the integral in (19) is the mean value of the spin at this vertex in state μ^{+a}.

3.2 Proof of Lemma 1

First, by means of percolation arguments of [9], we prove the lemma in the case of the Ising model. Then we extend the proof to the general case by comparison inequalities.

3.2.1 The case of the Ising model

Recall that the single-spin measure of the Ising model is given in (17), $\mathcal{G}^{\rm Ising}(\gamma, D)$ denotes the set of all corresponding Gibbs measures, and $\nu^+ \in \mathcal{G}^{\rm Ising}(\gamma, D))$ is the maximum Gibbs measure as in (16) with $a = 1$. The key fact proven in [9] which we are going to use is that, for an infinite graph $\mathcal{G}$, the Ising model living on $\mathcal{G}$ with constant intensities $J_{xy} = \phi_* > 0$ on the edges of $\mathcal{G}$, cf. (3), has at least two phases if and only if the graph admits the Bernoulli bond percolation with critical probability $q_* \in (0, 1)$ such that $\phi_* > (\log(1 + q_*) - \log(1 - q_*))/2$. In our case, the graph $\mathsf{G}(\gamma, D)$ with probability 1 admits this percolation and the threshold probability satisfies $q_* \geq 1/\lambda g_*$, as described in subsection 2.2.2. Then, for and some $o \in \gamma$, it follows that

$$\int_{\mathbb{R}^\gamma} \sigma_o \tilde{\nu}^+(d\sigma) > 0, \tag{20}$$

see [9, Theorem 2.1] and also the proof of Lemma 4.2 therein. Here $\tilde{\nu}^+$ is the corresponding Gibbs measure of the Ising model with $J_{xy} = \phi_* > 0$. By the standard GKS inequality, see, e.g., [9, Subsection 3.4], we have

$$\int_{\mathbb{R}^\gamma} \sigma_o \nu^+(d\sigma) \geq \int_{\mathbb{R}^\gamma} \sigma_o \tilde{\nu}^+(d\sigma),$$

which together with (20) yields the proof in this case.

3.2.2 The general case

Here we compare the integral in (19) calculated for the general model with the corresponding value for the Ising model with a rescaled interaction intensity. In view of this, we shall indicate the dependence on ϕ. That is, by $\mathcal{G}_{\rm t}(\gamma, D, \phi)$ we denote the set of Gibbs measures in the general case. The corresponding set for the Ising model is denoted by $\mathcal{G}^{\rm Ising}(\gamma, D, \phi)$. The proof of the lemma immediately follows from the Wells inequality used, e.g., in [19]. As the original publication [21] is hardly attainable, for the reader convenience we give a complete proof of this inequality here in the form suitable for our purposes. More general versions of this result can be proven in a similar way.

Proposition 2 (Wells inequality) *Let $a > 0$ be such that*

$$\chi([a\sqrt{2}, +\infty)) \geq \chi([0, a]). \tag{21}$$

Then, for each $x \in \gamma$ and for $\mu^{+a} \in \mathcal{G}(\gamma, D, \phi)$ and $\nu^{+} \in \mathcal{G}^{\mathrm{Ising}}(\gamma, D, a^2\phi)$, we have that

$$\int_{\mathbb{R}^\gamma} \sigma_x \mu^{+a}(d\sigma) \geq a \int_{\mathbb{R}^\gamma} \sigma_x \nu^{+}(d\sigma). \tag{22}$$

Proof For the general choice of χ, let Π_Λ^{+a} be defined as in (6) with $\bar{\sigma}_x = +a$ for all $x \in \gamma$. Since μ^{+a} is an accumulation point of the family $\{\Pi_\Lambda^{+a} : \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)\}$, one finds the sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$ such that the sequence $\{\Pi_{\Lambda_n}^{+a}\}_{n\in\mathbb{N}}$ converges weakly to μ^{+a}. Since the weak topology is introduced by bounded continuous functions, for unbounded spins the latter convergence does not imply the convergence of moments as in (22). The same estimate as in (15) can also be proven for all Π_Λ^{+a}, which together with (15) yields the convergence

$$\int_{\mathbb{R}^\gamma} \sigma_x \Pi_{\Lambda_n}^{+a}(d\sigma) \to \int_{\mathbb{R}^\gamma} \sigma_x \mu^{+a}(d\sigma), \qquad n \to +\infty.$$

Since the sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$ is exhausting it contains a subsequence, $\{\Lambda_{n_k}\}_{k\in\mathbb{N}}$, such that also

$$\int_{\mathbb{R}^\gamma} \sigma_x \Pi_{\Lambda_{n_k}}^{\mathrm{Ising}}(d\sigma) \to \int_{\mathbb{R}^\gamma} \sigma_x \nu^{+}(d\sigma), \qquad n \to +\infty,$$

where $\Pi_{\Lambda_{n_k}}^{\mathrm{Ising}}$ is the kernel (6) corresponding to the Ising single-spin measure (17), interaction intensities $a^2 J$, and the choice $\bar{\sigma}_x = +1$ for all $x \in \gamma$. Thus, the validity of (22) will follow if we prove that, for each Λ which contains x, the following holds

$$\int_{\mathbb{R}^\gamma} \sigma_x \Pi_\Lambda^{+a}(d\sigma) \geq a \int_{\mathbb{R}^\gamma} \sigma_x \Pi_\Lambda^{\mathrm{Ising}}(d\sigma). \tag{23}$$

Let $Z_\Lambda(a)$ and $Z_\Lambda^{\mathrm{Ising}}(1)$ be the corresponding normalizing factors defined in (7). Then by (6) we have, cf. (3),

$$\int_{\mathbb{R}^\gamma} \sigma_x \Pi_\Lambda^{+a}(d\sigma) - a \int_{\mathbb{R}^\gamma} \sigma_x \Pi_\Lambda^{\mathrm{Ising}}(d\sigma) = \Big(Z_\Lambda(a) Z_\Lambda^{\mathrm{Ising}}(1) \Big)^{-1} \tag{24}$$
$$\times \int_{\mathbb{R}^\gamma} \int_{\mathbb{R}^\gamma} (\sigma_x - a\tilde{\sigma}_x) \exp\Bigg\{ \sum_{\{x,y\}\in \mathsf{E}_\Lambda(\gamma,D)} J_{yz}[\sigma_y\sigma_z + a^2\tilde{\sigma}_y\tilde{\sigma}_z]$$
$$+ \sum_{y\in\Lambda} [\sigma_y + a\tilde{\sigma}_y] K_y \Bigg\} \bigotimes_{x\in\gamma_\Lambda} \Big(\chi(d\sigma_x) \otimes \chi^{\mathrm{Ising}}(d\tilde{\sigma}_x)\Big),$$

where $\chi^{\rm Ising}$ is given in (17) and $K_y = a\sum_{z\in\partial y\cap\Lambda^c} J_{yz}$ if $\partial y \cap \Lambda^c \neq \emptyset$, and $K_y = 0$ otherwise. Note that $K_y \geq 0$ in both cases. Then (23) will follow from the positivity of the integral on the right-hand side of (24). Now we rewrite the integrand in (24) in the variables $u_x^{\pm} := (\sigma_x \pm a\tilde{\sigma}_x)/\sqrt{2}$, and then expand the exponent and write the integral as the sum of the products over $x \in \gamma_\Lambda$ of 'one-site' integrals having the form

$$C_x \int_{\mathbb{R}^2} (u_x^+)^{m_x} (u_x^-)^{n_x} \chi(d\sigma_x) \otimes \chi^{\rm Ising}(d\tilde{\sigma}_x) \tag{25}$$
$$= C_x \int_{\mathbb{R}} [(\sigma_x + a)^{m_x}(\sigma_x - a)^{n_x} + (\sigma_x - a)^{m_x}(\sigma_x + a)^{n_x}]\, \chi(d\sigma_x), \quad C_x \geq 0.$$

Thus, to prove the statement we have to show that the right-hand side of (25) is nonnegative for all values of $m_x, n_x \in \mathbb{N}_0$. By the assumed symmetry of χ, the latter integral vanishes if m_x and n_x are of different parity. If both are even, then the positivity is immediate. Then it is left to consider the case of $m_x = 2k+1$ and $n_x = 2l+1$. By the symmetry of χ, we can take $k \geq l$. Thus, we have to prove the positivity of the following integral

$$\int_{\mathbb{R}} \left[(\sigma + a)^{2k+1}(\sigma - a)^{2l+1} + (\sigma - a)^{2k+1}(\sigma + a)^{2l+1}\right] \chi(d\sigma)$$
$$= 2\int_0^{+\infty} (\sigma^2 - a^2)^{2l+1} \left[(\sigma + a)^{k-l} + (\sigma + a)^{k-l}\right] \chi(d\sigma).$$

The function $\varphi(\sigma) := (\sigma + a)^{k-l} + (\sigma + a)^{k-l}$ is increasing on $[0, +\infty)$. The integral on the right-hand side of the latter equality can be written in the form

$$\int_0^{+\infty} (\sigma^2 - a^2)^{2l+1} \varphi(\sigma)\chi(d\sigma) = I_1(a) + I_2(a) + I_3(a), \tag{26}$$
$$I_1(a) := \int_0^{a} (\sigma^2 - a^2)^{2l+1} \varphi(\sigma)\chi(d\sigma) \geq -a^{4l+2}\varphi(a)\chi([0,a]),$$
$$I_2(a) := \int_a^{a\sqrt{2}} (\sigma^2 - a^2)^{2l+1} \varphi(\sigma)\chi(d\sigma) \geq 0,$$
$$I_3(a) := \int_{a\sqrt{2}}^{+\infty} (\sigma^2 - a^2)^{2l+1} \varphi(\sigma)\chi(d\sigma) \geq a^{4l+2}\varphi(a\sqrt{2})\chi([a\sqrt{2}, +\infty))$$

Then the property assumed in (21) yields that the sum on the right-hand side of (26) is nonnegative, which completes the proof.

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