VECTOR ANALYSIS FOR DIRICHLET FORMS AND QUASILINEAR PDE AND SPDE ON FRACTALS

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Abstract. Starting with a regular symmetric Dirichlet form on a locally compact second countable Hausdorff space X, our paper studies elements of vector analysis, L_p-spaces of vector fields and related Sobolev spaces. These tools are then employed to obtain existence and uniqueness results for some quasilinear elliptic PDE and SPDE in variational form on X by standard methods. For many of our results locality is not assumed, but most interesting applications involve local regular Dirichlet forms on fractal spaces such as nested fractals and Sierpinski carpets.

Contents

1. Introduction and setup

This paper is concerned with some elements of vector analysis on locally compact spaces X that carry a regular Dirichlet form. We start from the notion of 1-forms based on energy as recently introduced by Cipriani and Sauvageot in [11, 12] and further studied in [30]. A priori this concept is of global, non-local nature, and the space $\mathcal{H}$ of 1-forms defined in [11, 12] is a Hilbert space which in classical smooth cases agrees with the Hilbert space of L_2-differential 1-forms. It is shown below that for local Dirichlet forms this approach may

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Date: October 1, 2012.

2010 *Mathematics Subject Classification.* 60J25, 60J35, 28A80, 81Q35, 30D30, 31E05, 35P20, 47A75, 81Q10.

Key words and phrases. Regular symmetric Dirichlet form, vector stochastic calculus, p-energy and p-Laplacian, nonlinear and quasilinear elliptic SPDE. .

[1]Research supported in part by NSF grant DMS-0505622 and by the Alexander von Humboldt Foundation Feodor (Lynen Research Fellowship Program).

[2]Research was supported in part by the German Science Foundation (DFG) through CRC 701.

[3]Research supported in part by NSF grant DMS-0505622.

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be seen as an extension of closely related and preceding constructions of Eberle, [15], based on abstract (local) differential operators. Within the framework of [11, 12] we propose to study some basic notions of vector analysis such as vector fields, gradient and divergence operators. Furthermore, a direct integral representation of $\mathcal{H}$ allows to define L_p-spaces over measurable fields of Hilbert spaces (fibers) $(\mathcal{H}_x)_{x\in X}$ such that the space $\mathcal{H}$ of 1-forms in the sense of [11, 12] appears for $p = 2$. Related Sobolev spaces of functions and vector fields come up naturally after that. We show that these tools can be applied to quasilinear elliptic PDE on X in divergence and non-divergence form and to SPDE on X in variational form such as, for instance, the stochastic p-Laplace equation. The proposed notions of vector analysis allows to obtain existence and uniqueness results by classical fixed point and monotonicity arguments. We finally discuss a probabilistic counterpart of $\mathcal{H}$ which goes back to Nakao, [44]. This allows to give probabilistic interpretations of our vector analysis in terms of additive functionals. The mentioned direct integral representation of $\mathcal{H}$ nicely connects to (analytic and probabilistic) perturbation results for Dirichlet forms, e.g. [18], what permits to define analogs of non-divergence form operators in our context.

The main motivation for the present study comes from the analysis on fractals, cf. [1, 33, 53]. For certain classes of fractal sets the existence of a Laplace operator has been proved, see [2, 3, 20, 33, 40, 46, 51] and the references therein for some examples. Linear elliptic and parabolic PDE on fractals can then be treated by standard methods, [16]. Semilinear equations have been studied in [17]. There are also methods that apply to fully nonlinear problems, see for instance [4, 49] for porous medium equations. However, to our knowledge quasilinear equations of type

$$div(a(\nabla u)) = f$$

or

$$\Delta u + b(\nabla u) = f$$

with generally nonlinear coefficients a and b have not been considered so far, as an appropriate notion of gradient ∇ on fractals had not yet been sufficiently developed. The present paper addresses these problems. It also establishes a basis for further studies of first order differential operators on fractals, which have never been carried out before. Examples of such operators and related equations include for instance Dirac operators, magnetic Schrödinger operators or the Navier-Stokes equations on fractals, investigated in the companion papers [28] and [27], respectively. A short survey is given in [29].

In [11], and later in [12] and [30], a Hilbert space $\mathcal{H}$ of 1-forms and a related analog ∂ of the exterior derivation (in L_2-sense) had been introduced by means of tensor products and energy norms, see Section 2 below for precise definitions. Its norm is most conveniently expressed in terms of energy measures in the sense of LeJan and Fukushima, [19, 39]. Without too much effort a related notion of weighted energy measures for 1-forms can be introduced, what yields a coherent picture (especially in probabilistic terms) and is useful for some applications in [27] and [28].

The energy measure of a bounded energy finite function may be absolutely continuous with respect to the given reference measure or not. In Eberle [15, Section 3.2 and Appendix D] it is shown how to construct derivation operators if the energy measures are absolutely continuous for all functions from a dense algebra contained in the domain of the generator. On fractal spaces energy measures are typically singular with respect to the self-similar Hausdorff measure on the base space, cf. [5, 22, 23, 38]. However, the construction in [15, Theorem 3.11] is still possible if we choose a finite or countable pool of functions admitting energy densities and being energy dense in the space of bounded energy finite functions. Switching to a suitable measure m if needed (a so-called *energy dominant measure* [25] or, more specifically, a *Kusuoka measure* $\widetilde{m}$, see [35, 38, 58]), this can be realized for any regular Dirichlet form.

Following [15] we therefore obtain a measurable field of Hilbert spaces, [14, 57]. Rewriting the construction using some simple manipulations it can be shown that, roughly speaking, the resulting direct integral is a Hilbert space isomorphic to the space of 1-forms $\mathcal{H}$. Moreover, the direct integral of Eberle's fiberwise operators coincides with the derivation ∂ in the sense of Cipriani and Sauvageot in the case of local commutative Dirichlet forms. Apart from minor modifications this material is not new in substance. However, the direct connection between these two constructions had not been well established before. Even more importantly, our reasoning provides a constructive fiberwise interpretation for $\mathcal{H}$ that carries over from [15]. Our results imply that the construction in [11, 12] could be viewed as an extension of that in [15, Theorem 3.11], now based on a regular Dirichlet form instead of an abstract differential operator.

By the self-duality of $\mathcal{H}$ we regard its elements also as vector fields and ∂ a gradient operator. As a first new result, a corresponding divergence operator is defined as the adjoint of ∂. Note that although Eberle considers the adjoint of the derivation operator, [15, Chapter 3 b), Section 1], in his case it is part of the basic hypotheses and the discussion there aims at constructing Sobolev spaces of functions rather than at investigating spaces of vector fields.

Under the additional assumption that the given Dirichlet form is local and either transient or induced by a regular resistance form, its restriction to a suitable core $\mathcal{C}$ is also closable with respect to some energy dominant measure $\widetilde{m}$. This follows by arguments from [50] in the first case and by results of [36] in the second.

Using the above mentioned fiberwise interpretation, it is straightforward to define L_p-spaces of vector fields. Based on the previous closability result we then introduce Sobolev spaces of functions that make the derivation a closed operator for any $p \geq 2$, provided there exists a core $\mathcal{C}_p$ of functions having $p/2$-integrable energy densities which is dense in L_p and moreover such that $\mathcal{C}_p \otimes \mathcal{C}_p$ is dense in the corresponding L_p-space of vector fields. These assumptions are clearly satisfied in the classical smooth context. To verify them for non-classical examples we propose to investigate abstract continuous coordinates with respect to a measure. Harmonic coordinates in the sense of [32, 35, 58] constitute a prototype example. Any symmetric regular Dirichlet form admits such continuous coordinates with respect to the aforementioned energy dominant measure $\widetilde{m}$. Therefore we observe that if the original Dirichlet form is local and transient or induced by a local resistance form, the Sobolev spaces are well defined and the derivations are closed operators.

The applications to PDE and SPDE follow standard patterns that become applicable thanks to the definitions and results described above.

The space $\mathcal{H}$ is isometrically isomorphic to the Hilbert space $\mathring{\mathcal{M}}$ of martingale additive functionals of finite energy as studied for instance in [19, 44]. Given our setup, this isomorphism is almost immediate. Weighted energy measures of 1-forms and energy measures of martingale additive functionals correspond to each other, and gradients can be understood in terms of the martingale part in the Fukushima decomposition. The divergence may be expressed in terms of Nakao's divergence functional.

Our basic setup is as follows: X is assumed to be a locally compact and second countable Hausdorff space; $\mathcal{M}(X)$ denotes the space of (signed) Radon measures on X and $\mathcal{M}_+(X)$

the cone consisting of its non-negative elements; a measure $\mu \in \mathcal{M}_+(X)$ is an *admissible reference measure* on X if each open set $U \subset X$ has positive measure $\mu(U) > 0$. In the sequel we assume that μ is an admissible reference measure on X and, furthermore, we assume that $(\mathcal{E}, \mathcal{F})$ is a regular symmetric Dirichlet (energy) form on $L_2(X, \mu)$, cf. [19]. More exactly, we begin our arguments with an admissible reference measure μ, and later switch to an energy dominant measure m if necessary, see Lemma 2.2 below.

Set $\mathcal{C} := C_0(X) \cap \mathcal{F}$. By regularity the space $\mathcal{C}$ is dense in $\mathcal{F}$. It is an algebra endowed with the norm $\|f\|_{\mathcal{C}} := \mathcal{E}(f)^{1/2} + \sup_X |f|$ we have in particular

$$\mathcal{E}(fg)^{1/2} \leq \|f\|_{\mathcal{C}} \, \|g\|_{\mathcal{C}} \, , \tag{1}$$

as a consequence of the Markov property, see for instance [6]. Here we use the notation $\mathcal{E}(f) := \mathcal{E}(f, f)$, and we will do similarly for any other bilinear expression. Since $C_0(X) \subset L_2(X, \mu)$ for any Radon measure μ, we have

$$\mathcal{C} = \{f \in C_0(X) : \mathcal{E}(f) < \infty\} \, .$$

For any $g, h \in \mathcal{C}$ there exists a unique signed finite measure $\Gamma(g, h) \in \mathcal{M}(X)$ such that for any $f \in \mathcal{C}$,

$$2 \int_X f \, d\Gamma(g, h) = \mathcal{E}(fg, h) + \mathcal{E}(fh, g) - \mathcal{E}(gh, f) \ . \tag{2}$$

Obviously $\Gamma : \mathcal{C} \times \mathcal{C} \to \mathcal{M}(X)$ is a well defined symmetric bilinear mapping, and for any $g \in \mathcal{C}$, $\Gamma(g) \in \mathcal{M}_+(X)$. Note that in particular $\mathcal{E}(g, h) = \Gamma(g, h)(X)$. Therefore $\Gamma(g)$ is called the *energy measure of* g, cf. [19, 39]. Using the estimate

$$\left| \left(\int_X f d\Gamma(g) \right)^{1/2} - \left(\int_X f d\Gamma(h) \right)^{1/2} \right| \leq 2 \sup_{x \in X} |f(x)| \mathcal{E}(g - h)^{1/2}, \tag{3}$$

[19, p. 111] we can define (finite) energy measures $\Gamma(g, h) \in \mathcal{M}(X)$ for arbitrary $g, h \in \mathcal{F}$ and even for arbitrary $g, h \in \mathcal{F}_e$, where $(\mathcal{F}_e, \mathcal{E})$ denotes the extended Dirichlet space with respect to μ, that is the collection of μ-measurable μ-a.e. finite functions g on X for which there exists a $\mathcal{E}$-Cauchy sequence $(g_n)_n \subset \mathcal{F}$ such that $\lim_n g_n = g$ μ-a.e. The form $\mathcal{E}$ extends to $\mathcal{F}_e$ by $\mathcal{E}(g) := \lim_n \mathcal{E}(g_n)$, the limit being independent of the choice of $(g_n)_n$. See [19].

In the next section the definition on the space $\mathcal{H}$ of 1-forms is given and the concept of energy measure is extended to 1-forms. A fiberwise perspective is investigated and $\mathcal{H}$ is shown to coincide with the direct integral considered in [15, Appendix D]. Section 3 introduces gradient and divergence, equipped with suitable domains, and Section 4 presents some applications to quasilinear PDE. In Section 5 we discuss the question of closability when changing from the original to the energy dominant measure. Sobolev spaces and abstract continuous coordinates are introduced in Sections 6 and 7, respectively, while Section 8 contains some further applications, now to SPDE in the variational framework. A brief discussion of p-energies is provided in Section 9. We conclude the paper with some remarks on stochastic calculus in Section 10. To keep notation short, sequences or families indexed by the naturals (or pairs of naturals) will be written with index set suppressed, e.g. $(a_n)_n$ stands for $(a_n)_{n \in \mathbb{N}}$. Similarly, $\lim_n a_n$ abbreviates $\lim_{n \to \infty} a_n$.

2. The space $\mathcal{H}$ and weighted energy measures

By $\mathcal{B}_b(X)$ we denote the space of bounded Borel functions on X. Consider $\mathcal{C}\otimes\mathcal{B}_b(X)$, endowed with the symmetric bilinear form

$$\langle a\otimes b, c\otimes d\rangle_{\mathcal{H}} := \int_X bd\, d\Gamma(a,c), \tag{4}$$

$a\otimes b, c\otimes d\in\mathcal{C}\otimes\mathcal{B}_b(X)$, and let $\|\cdot\|_{\mathcal{H}}$ denote the associated seminorm on $\mathcal{C}\otimes\mathcal{B}_b(X)$. We write

$$ker\ \|\cdot\|_{\mathcal{H}} := \left\{\sum_i a_i\otimes b_i\in\mathcal{C}\otimes\mathcal{B}_b(X) : \left\|\sum_i a_i\otimes b_i\right\|_{\mathcal{H}} = 0\right\}$$

(with finite linear combinations). The completion of $\mathcal{C}\otimes\mathcal{B}_b(X)/ker\ \|\cdot\|_{\mathcal{H}}$ with respect to $\|\cdot\|_{\mathcal{H}}$ is denoted by $\mathcal{H}$. Following [11, 12] we refer to $\mathcal{H}$ as the *space of differential 1-forms on X*. Obviously it is a Hilbert space. Unlike for later constructions we agree to use the same notation $a\otimes b$ for a simple tensor from $\mathcal{C}\otimes\mathcal{B}_b(X)$ and for its equivalence class in $\mathcal{H}$.

Remark 2.1. The space $\mathcal{C}\otimes\mathcal{C}$ is dense in $\mathcal{H}$, and therefore $\mathcal{H}$ can be constructed from $\mathcal{C}\otimes\mathcal{C}$ in an analogous manner.

The space $\mathcal{H}$ becomes a bimodule if we declare the algebras $\mathcal{C}$ and $\mathcal{B}_b(X)$ to act on it in the following manner: For $a\otimes b\in\mathcal{C}\otimes\mathcal{B}_b(X)$, $c\in\mathcal{C}$ and $d\in\mathcal{B}_b(X)$ set

$$c(a\otimes b) := (ca)\otimes b - c\otimes(ab) \tag{5}$$

and

$$(a\otimes b)d := a\otimes(bd). \tag{6}$$

As shown in [11] and [30], (5) and (6) extend to well defined left and right actions of the algebras $\mathcal{C}$ and $\mathcal{B}_b(X)$, respectively. In particular, we have

$$\|c(a\otimes b)\|_{\mathcal{H}} \le \sup_X |c|\,\|a\otimes b\|_{\mathcal{H}} \quad\text{and}\quad \|(a\otimes b)d\|_{\mathcal{H}} \le \sup_X |d|\,\|a\otimes b\|_{\mathcal{H}}.$$

If $(\mathcal{E},\mathcal{F})$ is local, then the Leibniz rule for energy measures [19, Lemma 3.2.5] together with (4) implies that left and right multiplication agree for any $c\in\mathcal{C}$, and by approximation they are seen to agree for all $c\in\mathcal{B}_b(X)$. See [26] or [30] for further details.

We continue the preceding ideas and develop a *global perspective*. The following results apply even if the energy measures are possibly not absolutely continuous with respect to the reference measure μ. From Γ an $\mathcal{M}(X)$-valued bilinear mapping on $\mathcal{H}$ can be constructed. For simple tensors $a\otimes b, c\otimes d\in\mathcal{H}$ set

$$\Gamma_{\mathcal{H}}(a\otimes b, c\otimes d) := bd\,\Gamma(a,c)\ , \tag{7}$$

seen as an $\mathcal{M}(X)$-equality.

Lemma 2.1. *(7) extends to a well defined and uniquely determined symmetric bilinear mapping $\Gamma_{\mathcal{H}}:\mathcal{H}\times\mathcal{H}\to\mathcal{M}(X)$ such that for any $\omega\in\mathcal{H}$, $\Gamma_{\mathcal{H}}(\omega)\in\mathcal{M}_+(X)$. For any $\omega,\eta\in\mathcal{H}$ we have $\Gamma_{\mathcal{H}}(\omega,\eta)(X) = \langle\omega,\eta\rangle_{\mathcal{H}}$.*

The measure $\Gamma_{\mathcal{H}}(\omega,\eta)$ may be interpreted as a *weighted energy measure*.

Proof. First note that for any finite linear combination $\sum_i a_i \otimes b_i \in \mathcal{C} \otimes \mathcal{B}_b(X)$ for which

$$\left\| \sum_i a_i \otimes b_i \right\|_{\mathcal{H}}^2 = \sum_i \sum_j b_i b_j \Gamma(a_i)(X) = 0,$$

we have

$$\sum_i \sum_j b_i b_j \Gamma(a_i) \equiv 0, \tag{8}$$

seen as equality in $\mathcal{M}(X)$. In fact, if $c_1, ..., c_m, d_1, ..., d_n$ are such that

$$0 = \left\| \sum_{i=1}^m \sum_{j=1}^n c_i \otimes d_j \right\|_{\mathcal{H}}^2 = \int_X \left(\sum_{j=1}^n d_j \right)^2 d\Gamma \left(\sum_{i=1}^m c_i \right),$$

then $\left(\sum_{j=1}^n d_j\right)^2 \Gamma\left(\sum_{i=1}^m c_i\right) \equiv 0$ in $\mathcal{M}(X)$ because $\Gamma\left(\sum_{i=1}^m c_i\right)$ is a non-negative measure. The case $c_i = a_i$ and $d_j = b_i$ if $j = i$ and 0 otherwise yields (8). Now consider finite linear combinations $\sum_i f_i \otimes g_i \in \mathcal{H}$. For each i let $\widetilde{f}_i \otimes \widetilde{g}_i \in \mathcal{C} \otimes \mathcal{B}_b(X)$ be a representant of $f_i \otimes g_i$ and set

$$\Gamma_{\mathcal{H}} \left(\sum_i f_i \otimes g_i \right) := \sum_i \sum_j \widetilde{g}_i \widetilde{g}_j \Gamma(\widetilde{f}_i) \geq 0. \tag{9}$$

By the previous arguments (9) is a well defined element of $\mathcal{M}(X)$. Given a general 1-form $\omega \in \mathcal{H}$, let $(\omega_k)_k$ be a sequence of finite linear combinations

$$\omega_k = \sum_{i=1}^{n_k} f_i^{(k)} \otimes g_i^{(k)} \in \mathcal{H}$$

approximating ω in $\mathcal{H}$. For a non-negative function $\varphi \in \mathcal{B}_b(X)$ obviously $\sqrt{\varphi} \in \mathcal{B}_b(X)$ and by (6),

$$\begin{aligned} \lim_k \int_X \varphi \, d\Gamma_{\mathcal{H}}(\omega_k) &= \lim_k \sum_i \sum_j \int_X \varphi \, g_i^{(k)} g_j^{(k)} d\Gamma(f_i^{(k)}) \\ &= \lim_k \|\omega_k \sqrt{\varphi}\|_{\mathcal{H}}^2 \\ &= \|\omega \sqrt{\varphi}\|_{\mathcal{H}}^2 . \end{aligned}$$

Set

$$\Gamma_{\mathcal{H}}(\omega)(\varphi) := \lim_k \int_X \varphi \, d\Gamma_{\mathcal{H}}(\omega_k). \tag{10}$$

For arbitrary $\varphi \in \mathcal{B}_b(X)$ consider the standard decomposition $\varphi = \varphi_+ - \varphi_-$ with $\varphi_+ = \max(\varphi, 0)$, $\varphi_- = \max(-\varphi, 0)$ and define a linear functional on $\mathcal{B}_b(X)$ by

$$\Gamma_{\mathcal{H}}(\omega)(\varphi) := \lim_k \int_X \varphi \, d\Gamma_{\mathcal{H}}(\omega_k) = \lim_k \int_X \varphi_+ \, d\Gamma_{\mathcal{H}}(\omega_k) - \lim_k \int_X \varphi_- \, d\Gamma_{\mathcal{H}}(\omega_k). \tag{11}$$

As this equals $\|\omega\sqrt{\varphi_+}\|_{\mathcal{H}}^2 - \|\omega\sqrt{\varphi_-}\|_{\mathcal{H}}^2$, we have

$$|\Gamma_{\mathcal{H}}(\omega)(\varphi)| \leq 2 \sup_x |\varphi(x)| \, \|\omega\|_{\mathcal{H}}^2 . \tag{12}$$

(11) and (12) hold in particular for any $\varphi \in C_0(X)$, (10) is non-negative if $\varphi \geq 0$. Hence by the Riesz representation theorem there exists a unique non-negative Radon measure $\Gamma_{\mathcal{H}}(\omega)$ on X such that

$$\int_X \varphi \, d\Gamma_{\mathcal{H}}(\omega) = \Gamma_{\mathcal{H}}(\omega)(\varphi) \quad \text{for all } \varphi \in C_0(X).$$

By (12) and denseness this extends to all $\varphi \in C_b(X)$, and $\Gamma_{\mathcal{H}}(\omega)$ is seen to be the weak limit of the measures $\Gamma_{\mathcal{H}}(\omega_k)$. Finally, a corresponding bilinear mapping $\Gamma_{\mathcal{H}}$ can be defined via polarization, and the last statement of the lemma follows easily from (9) and (10). □

To the support of the measure $\Gamma_{\mathcal{H}}(\omega)$ we refer as the *support of the* 1*-form* $\omega \in \mathcal{H}$.

Corollary 2.1.

(i) *If $\omega \in \mathcal{H}$ is such that $\|\omega\|_{\mathcal{H}} = 0$, then $\Gamma_{\mathcal{H}}(\omega) = 0$ in $\mathcal{M}(X)$.*

(ii) *For any $\omega, \eta \in \mathcal{H}$ and any Borel set A,*

$$|\Gamma_{\mathcal{H}}(\omega, \eta)|(A) \leq \Gamma_{\mathcal{H}}(\omega)(A)^{1/2}\Gamma_{\mathcal{H}}(\eta)(A)^{1/2}$$

for any Borel set $A \in \mathcal{B}(X)$. In particular, $\Gamma_{\mathcal{H}}(\omega, \eta) = 0$ in $\mathcal{M}(X)$ if ω and η have disjoint supports.

Proof. (i) is a consequence of (12). The first statement in (ii) follows by a standard argument, see e.g. [43, Proposition 3.3]: By Lemma 2.1,

$$0 \leq \Gamma_{\mathcal{H}}(\omega - \lambda\eta) = \Gamma_{\mathcal{H}}(\omega) - 2\lambda\Gamma_{\mathcal{H}}(\omega, \eta) + \lambda^2\Gamma_{\mathcal{H}}(\eta).$$

For any relatively compact Borel set A and any $\lambda > 0$,

$$|\Gamma_{\mathcal{H}}(\omega, \eta)|(A) \leq \frac{1}{2}\left(\lambda^{-1}\Gamma_{\mathcal{H}}(\omega)(A) + \lambda\Gamma_{\mathcal{H}}(\eta)(A)\right).$$

If, without loss of generality, $\Gamma_{\mathcal{H}}(\eta) = 0$, then we can let λ go to zero to see the left hand side is zero. If both $\Gamma_{\mathcal{H}}(\omega)$ and $\Gamma_{\mathcal{H}}(\eta)$ are nonzero, consider

$$\lambda = \frac{\Gamma_{\mathcal{H}}(\omega)(A)^{1/2}}{\Gamma_{\mathcal{H}}(\eta)(A)^{1/2}}$$

to arrive at the desired inequality. By the regularity properties of the measures it extends to arbitrary Borel sets. The last statement in (ii) is a simple consequence. □

Remark 2.2. In the present paper the weighted energy measures $\Gamma_{\mathcal{H}}(\omega, \eta)$ will not play a predominant role. However, they are substantially used in [27] and [28], and we feel that for systematic reasons they should be discussed here.

The above picture can be complemented by a *fiberwise perspective*. The following fact is well known, see for instance [25, Lemmas 2.2-2.4]. For the convenience of the reader we briefly sketch it.

Lemma 2.2. *Given a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L_2(X, \mu)$, it is always possible to construct an admissible reference measure $\widetilde{m}$ such that for all $f \in \mathcal{C}$, the measure $\Gamma(f)$ is absolutely continuous with respect to $\widetilde{m}$ and the density $\frac{d\Gamma(f)}{d\widetilde{m}}$ is in $L_1(X, \widetilde{m})$. Moreover, $\widetilde{m}$ may chosen to be finite.*

As usual we write $\mathcal{E}_1(f) := \mathcal{E}(f) + \|f\|^2_{L_2(X,\mu)}$, $f \in \mathcal{F}$.

Proof. As $(\mathcal{F}, \mathcal{E}_1)$ is a separable Hilbert space, it possesses a countable dense subset $\{e_n\}_n$ (in practice we may for instance take a countable orthonormal basis and its finite linear combinations with rational coeffcients). For fixed n, let $(\varphi_{n,k})_k$ be a sequence of functions from $\mathcal{C}$ such that

$$\mathcal{E}_1(e_n - \varphi_{n,k})^{1/2} \leq 2^{-k}, \quad k \in \mathbb{N}.$$

Then $\{\varphi_{n,k}\}_{k,n}$ is a countable family of functions from $\mathcal{C}$ and also dense in $\mathcal{F}$ with respect to $\mathcal{E}_1$. Let $\{\psi_n\}_n$ be an enumeration of this family. We may assume that each ψ_n has positive energy. Set

$$f_n := \frac{\psi_n}{\mathcal{E}(\psi_n)^{1/2}}. \tag{13}$$

For each $n \in \mathbb{N}$, $\Gamma(f_n)$ is a probability measure. Now put

$$\widetilde{m} := \sum_{n=0}^{\infty} 2^{-n} \Gamma(f_n). \tag{14}$$

This series obviously converges set-wise, and proceeding as in the proof of Lemma 2.1 it is also seen to converge in the weak topology. For any $f \in \mathcal{C}$ there is some approximating sequence $(f_{n_j})_j$ and by construction each $\Gamma(f_{n_j})$ is absolutely continuous with respect to $\widetilde{m}$. If $B \in \mathcal{B}(X)$ is such that $\widetilde{m}(B) = 0$, then $\Gamma(f_{n_j})(B) = 0$ for all j and since

$$|\Gamma(f)(B)^{1/2} - \Gamma(f_{n_j})(B)^{1/2}| \leq \Gamma(f - f_{n_j})(B)^{1/2} \leq \mathcal{E}(f - f_{n_j})^{1/2},$$

by (3), we have $\Gamma(f)(B) = 0$, too.

Indeed $\widetilde{m}$ is an admissible reference measure: assume there were a non-empty open set $U \subset X$ such that $\Gamma(f_n)(U) = 0$ for all n. Let $\varphi \in \mathcal{C}$ be nontrivial and supported in U. As φ can be approximated in the $\mathcal{E}_1$-norm by a sequence $(f_{n_j})_j$, this would entail

$$\mathcal{E}(\psi, \varphi) = 0 \text{ for all } \psi \in \mathcal{C} \text{ supported in } U, \tag{15}$$

recall for instance Corollary 2.1 (ii). But then $\mathcal{E}(\varphi) = 0$, a contradiction to the previous assumptions on φ. Consequently $\Gamma(f_n)(U) > 0$ for some n. □

Let us return to the fixed regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L_2(X, \mu)$ as used in (2) and (4). From now on we assume the following:

Assumption 2.1. m is an admissible reference measure such that for any $f \in \mathcal{C}$, the measure $\Gamma(f)$ is absolutely continuous with respect to m.

Note that in this case $\Gamma(f) = \frac{d\Gamma(f)}{dm}$ is in $L_1(X, m)$ for any $f \in \mathcal{C}$. If all energy measures $\Gamma(f)$, $f \in \mathcal{C}$, are absolutely continuous with respect to μ, we may use $m := \mu$. If not, we switch to the measure $m := \widetilde{m}$ from Lemma 2.2. As this is sufficient for later purposes, the above assumption is no additional restriction.

Remark 2.3. If $(\mathcal{E}, \mathcal{F})$ is local and transient or if it is induced by a local regular resistance form then $(\mathcal{E}, \mathcal{C})$ can be shown to be closable in $L_2(X, \widetilde{m})$. This will be discussed in Section 6. In the present and the next two sections closability is not needed.

We recall a construction from [15]. Let $\mathcal{A}_0 = \{f_n\}_n$ be a countable collection of functions which is $\mathcal{E}$-dense in $\mathcal{C}$, i.e. such that for any $f \in \mathcal{C}$ there exists a sequence $(f_{n_j})_j \subset \mathcal{A}_0$

with $\lim_j \mathcal{E}(f - f_{n_j}) = 0$. For any finite linear combination $u = \sum_{i=1}^N \lambda_i f_i$ and any Borel set $A \subset X$ we have

$$0 \le \Gamma(u)(A) = \sum_{i=1}^N \sum_{j=1}^N \lambda_i \lambda_j \int_A \Gamma_x(f_i, f_j) m(dx)$$
$$= \int_A \overline{\lambda}^T (\Gamma_x(f_i, f_j))_{i,j=1,...,N} \overline{\lambda}\, m(dx),$$

where $\overline{\lambda} = (\lambda_1, ..., \lambda_N) \in \mathbb{R}^N$ and $\overline{\lambda}^T$ is its transpose. We can therefore choose Borel versions $x \mapsto \Gamma_x(f_i, f_j)$ of the classes $\Gamma(f_i, f_j) \in L_1(X, m)$ such that for all $N \in \mathbb{N}$ and all $x \in X$, the matrix $(\Gamma_x(f_i, f_j))_{i,j=1,...,N}$ is symmetric and nonnegative definite over $\mathbb{Q}^N$. For two finite linear combinations $u = \sum_i \lambda_i f_i$ and $v = \sum_j \mu_j f_j$ from $\mathcal{A} := \text{span}(\mathcal{A}_0)$ set

$$\Gamma_x(u, v) := \sum_i \sum_j \lambda_i \mu_j \Gamma_x(f_i, f_j).$$

Then for all $x \in X$, Γ_x clearly is a non-negative definite bilinear form on $\mathcal{A}$. Consider the factor $\mathcal{A}/ker\,\Gamma_x$, where $ker\,\Gamma_x := \{f \in \mathcal{A} : \Gamma_x(f) = 0\}$ and let $d_x f$ denote the equivalence class of $f \in \mathcal{A}$. Define

$$(d_x f, d_x g)_{\mathcal{B}_x} = \Gamma_x(f, g) \tag{16}$$

for all $f, g \in \mathcal{A}$ and let $\mathcal{B}_x$ denote the completion of $\mathcal{A}/ker\,\Gamma_x$ in $(\cdot,\cdot)_{\mathcal{B}_x}$, clearly a Hilbert space.

For convenience we recall the following definitions: A collection $(H_x)_{x\in X}$ of Hilbert spaces $(H_x, (\cdot,\cdot)_{H_x})$ together with a subspace M of $\prod_{x\in X} H_x$ is called a *measurable field of Hilbert spaces* if

(i) an element $\xi \in \prod_{x\in X} H_x$ is in M if and only if $x \mapsto (\xi, \eta)_{H_x}$ is measurable for any $\eta \in M$,
(ii) there exists a countable set $\{\xi_i : i \in \mathbb{N}\} \subset M$ such that for all $x \in X$ the span of $\{\xi_i(x) : i \in \mathbb{N}\}$ is dense in H_x.

The elements of M are usually referred to as *measurable sections*. Two measurable fields of Hilbert spaces $(H_x)_{x\in X}$ and $(\widetilde{H}_x)_{x\in X}$ are *essentially isometric* if there are a null set $N \subset X$ and a collection $(\Phi_x)_{x\in X\setminus\mathcal{N}}$ of isometries $\Phi_x : H_x \to \widetilde{H}_x$ such that $\xi \in \prod_{x\in X} H_x$ is a member of M if and only if $x \mapsto \Phi_x(\xi(x)) \in \widetilde{M}$. If N may be chosen to be empty, we say that $(H_x)_{x\in X}$ and $(\widetilde{H}_x)_{x\in X}$ are *isometric*.

Remark 2.4. Orthonormalizing the ξ_i from (ii) in the respective spaces one obtains the following useful fact: There is a countable set $\{\eta_i : i \in \mathbb{N}\} \subset M$ such that for any x with H_x infinite-dimensional, it provides a orthonormal basis and for any x with $dim\,H_x = d(x)$, $\eta_1(x), ..., \eta_{d(x)}(x)$ is an orthonormal basis and $\eta_i(x) = 0$, $i > d(x)$. For a proof see [14, Proposition II.4.1] or [57, Lemma 8.12]. Note that every $\eta_i(x)$ is a finite linear combination of elements $\xi_j(x)$. $\{\eta_i : i \in \mathbb{N}\} \subset M$ is then referred to as a *measurable field of orthogonal bases*.

Lemma 2.3.

(i) *The collection $(\mathcal{B}_x)_{x\in X}$ is a measurable field of Hilbert spaces.*
(ii) *Different choices of versions above lead to essentially isometric fields of Hilbert spaces.*

Proof. Let $\mathcal{M}$ be the subspace of all $\xi \in \prod_{x\in X} \mathcal{B}_x$ such that $x \mapsto (\xi(x), d_x f_n)_{\mathcal{B}_x}$ is measurable for any n. Obviously all $x \mapsto d_x f$, $f \in \mathcal{A}$, are in $\mathcal{M}$. For general $\xi \in \prod_{x\in X} \mathcal{B}_x$ and each $x \in X$ there is a sequence $(g_k) \subset \mathcal{A}$ such that

$$\lim_k \|\xi(x) - d_x g_k\|_{\mathcal{B}_x} = 0.$$

Hence a section ξ is in $\mathcal{M}$ if and only if $x \mapsto (\xi(x), d_x f_n)_{\mathcal{B}_x}$ are measurable for all $n \in \mathbb{N}$. This shows (i).

To see (ii), assume $x \mapsto \widetilde{\Gamma}_x(f_i, f_j)$ are further versions of $\Gamma(f_i, f_j) \in L_1(X, m)$ so that the previous agreements are valid and denote the similarly constructed spaces by $\widetilde{\mathcal{B}}_x$. Then there exists a null set $\mathcal{N}$ such that

$$(\widetilde{d}_x f_i, \widetilde{d}_x f_j)_{\widetilde{\mathcal{B}}_x} = (d_x f_i, d_x f_j)_{\mathcal{B}_x}$$

for all $i, j \in \mathbb{N}$ and $x \in X \setminus \mathcal{N}$. By the denseness of $\mathcal{A}/ker\,\Gamma_x$ in $\mathcal{B}_x$ and $\mathcal{A}/ker\,\widetilde{\Gamma_x}$ in $\widetilde{\mathcal{B}}_x$ we obtain a unique isometry Φ_x from $\mathcal{B}_x$ onto $\widetilde{\mathcal{B}}_x$ for any $x \in X \setminus \mathcal{N}$. If now $\xi \in \mathcal{M}$ then

$$(\Phi_x(\xi(x)), \widetilde{d}_x f_n)_{\widetilde{\mathcal{B}}_x} = (\xi(x), d_x f_n)_{\mathcal{B}_x}$$

for $x \in X \setminus \mathcal{N}$ and all $n \in \mathbb{N}$, and the right-hand side is a measurable function of x. Therefore $\Phi_x(\xi(x))$ is a measurable section. Similarly for the converse direction. □

This construction may be rephrased as follows. For any point $x \in X$ and arbitrary simple tensors $a \otimes b, c \otimes d \in \mathcal{A} \otimes \mathcal{B}_b(X)$ put

$$\Gamma_{\mathcal{H},x}(a \otimes b, c \otimes d) := b(x)d(x)\Gamma_x(a, c). \tag{17}$$

As a consequence of the above choice of versions every $\Gamma_{\mathcal{H},x}$, $x \in X$, defines a non-negative definite bilinear form on $\mathcal{A} \otimes \mathcal{B}_b(X)$. Set

$$ker\,\Gamma_{\mathcal{H},x} := \left\{ \sum_i a_i \otimes b_i \in \mathcal{A} \otimes \mathcal{B}_b(X) : \Gamma_{\mathcal{H},x}(\sum_i a_i \otimes b_i) = 0 \right\}$$

and let $\mathcal{H}_x$ be the Hilbert space obtained as the completion of $\mathcal{A} \otimes \mathcal{B}_b(X)/ker\,\Gamma_{\mathcal{H},x}$ with respect to scalar product determined by

$$\langle [a \otimes b]_x, [c \otimes d]_x \rangle_{\mathcal{H}_x} = \Gamma_{\mathcal{H},x}(a \otimes b, c \otimes d),$$

where $[a \otimes b]_x \in \mathcal{A} \otimes \mathcal{B}_b(X)/ker\,\Gamma_{\mathcal{H},x}$ denotes the equivalence class of $a \otimes b$. Note that

$$[a \otimes b]_x = [a \otimes b(x)]_x = b(x)[a \otimes \mathbf{1}]_x \text{ for any } x \in X \tag{18}$$

and any $a \otimes b \in \mathcal{A} \otimes \mathcal{B}_b(X)$, because $\Gamma_{\mathcal{H},x}(a \otimes (b - b(x)) = 0$ by (17).

Lemma 2.4. *The collection $(\mathcal{H}_x)_{x\in X}$ is a measurable field of Hilbert spaces on X. The measurable fields $(\mathcal{H}_x)_{x\in X}$ and $(\mathcal{B}_x)_{x\in X}$ are isometric.*

Proof. The first assertion may be seen as in the previous lemma. For any $x \in X$ define a bilinear mapping $\Psi_x : \mathcal{A}/ker\,\Gamma_x \to \mathcal{H}_x$ by

$$\Psi_x(d_x a) := [a \otimes \mathbf{1}]_x, \quad a \in \mathcal{A}. \tag{19}$$

Since

$$\|\Psi_x(d_x a)\|^2_{\mathcal{H}_x} = \|[a \otimes \mathbf{1}]_x\|^2_{\mathcal{H}_x} = \Gamma_{\mathcal{H},x}(a \otimes \mathbf{1}) = \Gamma_x(a) = \|d_x a\|^2_{\mathcal{B}_x} \tag{20}$$

and $d_x\widetilde{a} = d_x a$ if and only if $\Gamma_x(\widetilde{a} - a) = 0$, Ψ_x is well defined. By (20) and denseness it extends to a uniquely determined isometry from $\mathcal{B}_x$ into $\mathcal{H}_x$. Ψ_x is also surjective: For any

$[a \otimes b]_x \in \mathcal{A} \otimes \mathcal{B}_b(X)/ker\, \Gamma_{\mathcal{H},x}$ consider $b(x)d_x a$. Then by linearity and (18), $\Psi_x(b(x)d_x a) = b(x)[a \otimes \mathbf{1}]_x = [a \otimes b]_x$. On the other hand, $\mathcal{A} \otimes \mathcal{B}_b(X)/ker\, \Gamma_{\mathcal{H},x}$ is dense in $\mathcal{H}_x$. $\square$

Lemma 2.5. *The space $\mathcal{A} \otimes \mathcal{B}_b(X)$ is dense in $\mathcal{H}$.*

Proof. By construction, any simple tensor $a \otimes b \in \mathcal{C} \otimes \mathcal{B}_b(X)$ can be approximated by elements of $\mathcal{A} \otimes \mathcal{B}_b(X)$. $\square$

Recall that given a measurable field of Hilbert spaces $(H_x)_{x \in X}$, a measurable section ξ is called *square-integrable* if

$$\int_X \|\xi(x)\|^2_{H_x}\, m(dx) < \infty. \tag{21}$$

The set of all square-integrable sections together with the scalar product induced by (21) is called the *direct integral* of $(H_x)_{x \in X}$ and denoted by $\int_X^{\oplus} H_x m(dx)$.

Remark 2.5. If $\{\eta_i : i \in \mathbb{N}\}$ is a measurable field of orthonormal bases according to Remark 2.4 and $\omega \in H = \int_X^{\oplus} H_x m(dx)$, then the sections ω_n, given by

$$\omega_n(x) = \sum_{i=0}^{n} (\omega(x), \eta_i(x))_{H_x} \eta_i(x)$$

approximate ω in H. A proof is given in [14, Proposition II.1.6].

Given $a \otimes b \in \mathcal{A} \otimes \mathcal{B}_b$ with corresponding classes $[a \otimes b]_x \in \mathcal{H}_x$, the symbol $[a \otimes b]$ denotes the measurable section $x \mapsto [a \otimes b]_x$. Similarly for more general measurable sections ω.

Theorem 2.1. *The Hilbert spaces $\mathcal{H}$ and $\int_X^{\oplus} \mathcal{H}_x m(dx)$ are isometrically isomorphic. In particular, for all $\omega, \eta \in \mathcal{H}$,*

$$\langle \omega, \eta \rangle_{\mathcal{H}} = \int_X^{\oplus} \langle \omega, \eta \rangle_{\mathcal{H}_x} m(dx).$$

Consequently also $\mathcal{H}$ and $\int_X^{\oplus} \mathcal{B}_x m(dx)$ are isometrically isomorphic. In particular, up to an isomorphism, the definition of 1-forms in [15, Chapter 3 b) and Appendix D] arises as a special case of that in [11, 12].

Proof. For simple tensors $a \otimes b \in \mathcal{A} \otimes \mathcal{B}_b(X)$ set $\chi(a \otimes b) := [a \otimes b]$ and extend linearly to a mapping $\chi : \mathcal{A} \otimes \mathcal{B}_b(X) \to \int_X^{\otimes} (\omega, \eta)_{\mathcal{H}_x} m(dx)$. Since

$$\int_X \|[a \otimes b]_x\|_{\mathcal{H}_x} m(dx) = \int_X b(x)^2 \|[a \otimes \mathbf{1}]_x\|_{\mathcal{H}_x} m(dx) = \int_X b(x)^2 \Gamma_x(a) m(dx) = \|a \otimes b\|^2_{\mathcal{H}},$$

By denseness χ extends to an isometry from $\mathcal{H}$ into $\int_X^{\otimes} \mathcal{H}_x m(dx)$. To conclude surjectivity we make use of a totality argument from [15, Theorem 7.3.11]. Suppose $\omega \in \int_X^{\oplus} \mathcal{H}_x m(dx)$ is such that

$$0 = \langle \omega, [a \otimes b] \rangle_{\mathcal{H}} = \int_X b(x) \langle \omega(x), [a \otimes \mathbf{1}]_x \rangle_{\mathcal{H}_x} m(dx)$$

for all $a \otimes b \in \mathcal{A} \otimes \mathcal{B}_b(X)$. Then in particular $\langle \omega(x), [a \otimes \mathbf{1}]_x \rangle_{\mathcal{H}_x} = 0$ for all $a \in \mathcal{A}_0$ for m-a.e. x. But finite linear combinations $\sum_i \lambda_i [a_i \otimes \mathbf{1}]_x$ with functions $a_i \in \mathcal{A}_0$ and rational coefficients λ_i are dense in the Hilbert space $\mathcal{H}_x$, therefore $\omega(x) = 0$ for m-a.e. x and consequently $\omega = 0$ in $\int_X^{\oplus} \mathcal{H}_x m(dx)$. This implies that the closure of the range $Im\, \chi$ of χ must be the entire direct integral. $\square$

Let us agree upon the notation

(22) $$\Gamma_{\mathcal{H},x}(\omega,\eta) := \langle\omega,\eta\rangle_{\mathcal{H}_x} \quad \text{for all } \omega,\eta\in\mathcal{H} \text{ and } x\in X.$$

Analogs of Lemma 2.1 and Corollary 2.1 now read as follows.

Corollary 2.2.

(i) *The measure $\Gamma_{\mathcal{H}}(\omega,\eta)$ from Lemma 2.1 is absolutely continuous with respect to m, and $\Gamma_{\mathcal{H},\cdot}(\omega,\eta)$ is a version of the Radon-Nikodym density $\frac{d\Gamma_{\mathcal{H}}(\omega,\eta)}{dm}$.*

(ii) *Definition (22) provides a well defined and uniquely determined bilinear mapping $\Gamma_{\mathcal{H}}:\mathcal{H}\times\mathcal{H}\to L_1(X,m)$ such that for any $\omega\in\mathcal{H}$, $\Gamma_{\mathcal{H},\cdot}(\omega)\geq 0$ m-a.e.*

Proof. (i) is obvious and (ii) is a simple consequence of Lemmas 2.3 and 2.4. □

Corollary 2.3.

(i) *If $\omega\in\mathcal{H}$ is such that $\|\omega\|_{\mathcal{H}}=0$, then $\Gamma_{\mathcal{H},\cdot}(\omega)=0$ in $L_1(X,m)$.*

(ii) *For $\omega,\eta\in\mathcal{H}$ with disjoint supports we have $\Gamma_{\mathcal{H}}(\omega,\eta)=0$ in $L_1(X,m)$.*

As in [11, 12] a *differential* $\partial:\mathcal{C}\to\mathcal{H}$ is defined by

$$\partial(a) = a\otimes\mathbf{1} \quad , \quad a\in\mathcal{C}.$$

The following properties are simple consequences of (4) and (5).

Corollary 2.4.

(i) *The operator ∂ is a derivation, i.e. it is linear and*

$$\partial(fg) = (\partial f)g + f\partial g \quad , \quad f,g\in\mathcal{C}.$$

(ii) *The operator ∂ is bounded, more precisely,*

$$\|\partial f\|_{\mathcal{H}} = \mathcal{E}(f)^{1/2} \quad , \quad f\in\mathcal{C}.$$

On the other hand, Eberle [15] calls a linear map d from an algebra C into a direct integral $\int_X^{\oplus} H_x m(dx)$ of Hilbert spaces an *L_2-differential* if

(i) the span of $\{fdg : f,g\in C\}$ is dense in $\int_X^{\oplus} H_x m(dx)$ and

(ii) $\partial(fg) = fdg + gdf$, $f,g\in C$.

Recall (16) and (19). The following result is immediate.

Corollary 2.5. *The operator ∂ is an L_2-differential on $\mathcal{C}$. Given $f,g\in\mathcal{A}$, we have $[\partial f]_x = \Psi_x(d_x f)$ and*

$$\langle\partial f,\partial g\rangle_{\mathcal{H}} = \int_X^{\oplus}(d_x f, d_x g)_{\mathcal{B}_x} m(dx).$$

Remark 2.6.

(i) Similar assumptions as in [15] would allow to extend formula (16) to the entire algebra $\mathcal{C}$, such that each element $f\in\mathcal{C}$ can be assigned classes $d_x f\in\mathcal{B}_x$, $x\in X$. Then, if *df* denotes the measurable vector field $x\mapsto d_x f$, $f\in\mathcal{C}$, the resulting mapping

$$d:\mathcal{C}\to\int_X^{\oplus}\mathcal{B}_x m(dx)$$

defines an L_2-differential. In this case also (19) extends to all of $\mathcal{C}$ and yields an isometry $\Psi = \int_X^{\oplus} \Psi_x m(dx)$ taking $\int_X \mathcal{B}_x m(dx)$ onto $\mathcal{H}$ such that $\partial = \Psi \circ d$. Note that this is closely related to the representation

$$\mathcal{H} = L_2(X, m, (\mathcal{H}_x)_{x \in X})$$

discussed in detail in Sections 3 and 6 below (see also Theorem 2.1).

(ii) If $(\mathcal{E}, \mathcal{F})$ is local, then we have

$$\Gamma_x(fg, h) = f(x)\Gamma_x(g, h) + g(x)\Gamma_x(f, h)$$

for all $f, g, h \in \mathcal{C}$ by the Leibniz rule for energy measures [19, Lemma 3.2.5]. This implies the *localized Leibniz rule*

$$d_x(fg) = f(x) d_x g + g(x) d_x f.$$

See for instance [15, p. 151] or [11, p. 112].

(iii) For the measurable field $(\mathcal{H}_x)_{x \in X}$ the function $x \mapsto d(x) = \dim \mathcal{H}_x$ from Remark 2.4 coincides with the *pointwise index* of $(\mathcal{E}, \mathcal{F})$ as introduced by Hino in [25] (also related to the *martingale dimension of fractals*, see [24]). There a detailed analysis of pointwise and global indices is provided and applied to first order derivatives of energy finite functions on a class of fractals.

Remark 2.7. The above construction has utilized the energy measures (2) to generate a related algebraic structure. We would like to remind the reader of the well known fact that they also generate metric structures: Given a symmetric local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$, consider

$$d(x, y) := \sup \left\{ f(x) - f(y) : f \in \widetilde{\mathcal{C}}, \Gamma(f) \le \mu \right\}, \quad x, y \in X, \tag{23}$$

where $\widetilde{\mathcal{C}}$ is a core of $(\mathcal{E}, \mathcal{F})$ and $\Gamma(f) \le \mu$ stands for the requirement that $\Gamma(f)$ is absolutely continuous with respect to μ having density $\frac{\Gamma(f)}{d\mu} \le 1$ μ-a.e. provides a pseudo-metric d on X, usually referred to as *Carnot-Caratheodory distance*. If $\widetilde{\mathcal{C}}$ separates the points of X, d is a metric in the wide sense (i.e. satisfies the axioms of a metric but may attain the value $+\infty$). To our knowledge, (23) has first been considered in the context of Dirichlet forms in [7, 8, 13] and [55, 56]. Under the assumptions that (X, d) is complete and the topology induced by d on X coincides with the original one, it had been shown in [55] (together with [56]) that (X, d) is a geodesic space. In [52] the completeness assumption had been dropped. Having in mind the constructions of the present paper, it would be interesting to know whether (or for which cores $\widetilde{\mathcal{C}}$) (X, d) is a geodesic space without any further topological assumptions.

3. Vector fields, gradient and divergence

As a Hilbert space $\mathcal{H}$ is self-dual. We therefore regard 1-forms also as *vector fields*, exact 1-forms ∂f also *gradients* and ∂ as the *gradient operator*. As $\mathcal{C}$ is dense in $\mathcal{F}$ which in turn is dense in $L_2(X, \mu)$, ∂ may be viewed as densely defined unbounded operator

$$\partial : L_2(X, \mu) \to \mathcal{H}$$

a priori equipped with the domain *dom* $\partial = \mathcal{C}$. As $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form, ∂ is closable and extends uniquely to a closed linear operator ∂ with domain $\mathcal{F}$.

In the sequel we inquire about the adjoint ∂^* of ∂. Let $\mathcal{C}^*$ denote the dual space of $\mathcal{C}$, normed by

$$\|w\|_{\mathcal{C}^*} = \sup\{|w(f)| : f \in \mathcal{C}, \|f\|_{\mathcal{C}} \leq 1\}$$

and automatically a Banach space. Given $f, g \in \mathcal{C}$, consider the mapping

$$u \mapsto -\langle g\partial f, \partial u\rangle_{\mathcal{H}} = -\int_X g\, d\Gamma(f,u) \tag{24}$$

on $\mathcal{C}$. By Cauchy-Schwarz in $\mathcal{H}$ and Corollary 2.4 (ii) we have

$$|\langle g\partial f, \partial u\rangle_{\mathcal{H}}| \leq \|g\partial f\|_{\mathcal{H}}\, \mathcal{E}(u)^{1/2}$$

which says that (24) defines an element $\partial^*(g\partial f)$ of $\mathcal{C}^*$ with norm bound

$$\|\partial^*(g\partial f)\|_{\mathcal{C}^*} \leq \|g\partial f\|_{\mathcal{H}} \ .$$

To

$$\partial^*(g\partial f) = -\int_X g\, d\Gamma(f,\cdot)$$

we refer as the *divergence of the vector field $g\partial f$.*

Lemma 3.1. *∂^* extends continuously to a bounded linear operator*

$$\partial^* : \mathcal{H} \to \mathcal{C}^*$$

with $\|\partial^* v\|_{\mathcal{C}^*} \leq \|v\|_{\mathcal{H}}$, $v \in \mathcal{H}$. Moreover,

$$\partial^* v(u) = -\langle v, \partial u\rangle_{\mathcal{H}}$$

for any $u \in \mathcal{C}$ and any $v \in \mathcal{H}$.

The operator ∂^* will be called the *divergence operator*. Note that this is a (global, non-local) definition in a *distributional sense*.

Proof. Let $a_i, b_j \in \mathcal{C}$, $i, j = 1, ..., N$. Then

$$\left\|\partial^*\left(\sum_i\sum_k a_i \otimes b_k\right)\right\|_{\mathcal{C}^*} \leq \left\|(\sum_i a_i) \otimes (\sum_k b_k)\right\|_{\mathcal{H}} \ .$$

Given a finite linear combination $\sum_k g_k\partial f_k$ of simple vector fields, consider the case $a_i = f_i$ and $b_k = g_i$ if $k = i$ and $b_k = 0$ otherwise to get

$$\left\|\partial^*\left(\sum_k g_k\partial f_k\right)\right\|_{\mathcal{C}^*} \leq \left\|\sum_k g_k\partial f_k\right\|_{\mathcal{H}} \ .$$

Such finite linear combinationes being dense in $\mathcal{H}$, we may extend ∂^* to the whole of $\mathcal{H}$ with the norm bound preserved. The last assertion is an immediate consequence. $\square$

In $X = \mathbb{R}^n$ we have the pointwise identity

$$div\,(g\; grad\; f) = g\Delta f + \nabla f \nabla g$$

for $f \in C^2(\mathbb{R}^n)$ and $g \in C^1(\mathbb{R}^n)$. Let $(L, dom\; L)$ denote the infinitesimal $L_2(X,\mu)$-generator of $(\mathcal{E}, \mathcal{F})$. For $f \in dom\; L$ and $g, u \in \mathcal{C}$ we have

$$(gLf)(u) = -\mathcal{E}(gu, f), \tag{25}$$

and if $f \in \mathcal{C}$, we may use (25) as a definition of gLf: Since

$$|(gLf)(u)| \leq \mathcal{E}(gu)^{1/2}\mathcal{E}(f)^{1/2} \leq \|u\|_{\mathcal{C}}\, \|g\|_{\mathcal{C}}\, \mathcal{E}(f)^{1/2}$$

for any $u \in \mathcal{C}$ by Cauchy-Schwarz and (1), gLf is a well defined member of $\mathcal{C}^*$. Similarly also the energy measure $\Gamma(f,g)$, seen as a linear functional

$$\Gamma(f,g)(u) := \int_X u \, d\Gamma(f,g)$$

on $\mathcal{C}$, is a member of $\mathcal{C}^*$, because $\|\Gamma(f)\|_{\mathcal{C}^*} \leq \mathcal{E}(f)$ and by polarization

$$\|\Gamma(f,g\|_{\mathcal{C}^*} \leq \frac{1}{2}(\mathcal{E}(f) + \mathcal{E}(g)).$$

Lemma 3.2. *For any simple vector field $g\partial f$, $f, g \in \mathcal{C}$, we have*

$$\partial^*(g\partial f) = gLf + \Gamma(f,g) \ , \tag{26}$$

seen as an equality in $\mathcal{C}^*$. In particular, $Lf = \partial^*\partial f$ for $f \in \mathcal{C}$.

Proof. This is now a simple consequence of the identity

$$-(gLf)(u) = \mathcal{E}(gu, f) = \int_X g d\Gamma(u,f) + \int_X u \, d\Gamma(f,g) \ ,$$

$u \in \mathcal{C}$, which itself may quickly be verified using (2). □

The preceding distributional definition can be complemented by a Hilbert space point of view. Generally the inclusions $\mathcal{C} \subset L_2(X,\mu) \subset \mathcal{C}^*$ are proper and seen as an operator

$$\partial^* : \mathcal{H} \to L_2(X,\mu),$$

the divergence ∂^* is unbounded. As usual $v \in \mathcal{H}$ is said to be a member of $dom\, \partial^*$ if there exists some (then automatically unique) $v^* \in L_2(X,\mu)$ such that $\langle u, v^* \rangle_{L_2(X,\mu)} = - \langle \partial u, v \rangle_{\mathcal{H}}$ for all $u \in \mathcal{C}$. In this case $\partial^* v := v^*$ and

$$\langle u, \partial^* v \rangle_{L_2(X,\mu)} = - \langle \partial u, v \rangle_{\mathcal{H}} \ , u \in \mathcal{C}, \tag{27}$$

i.e. $-\partial^*$ is the adjoint operator of ∂. It is immediate that $\{\partial f : f \in dom\, L\} \subset dom\, \partial^*$. As $-\partial^*$ is the adjoint of the densely defined and closable operator ∂ it is densely defined, see [48].

Probabilistic interpretations of ∂ and ∂^* are discussed in Section 10.

4. Applications to quasilinear PDE

The discussed setup will now be used to solve PDE by fixed point and monotonicity arguments. We focus on equations involving terms $u \mapsto div\, a(grad\, u)$ and $u \mapsto b(\nabla u)$, where a and b are possibly nonlinear transformations. In our context these expressions rewrite $u \mapsto \partial^*(a(\partial u))$ and $u \mapsto b(\partial u)$, respectively.

Throughout this section we assume that μ is an admissible reference measure on X, $(\mathcal{E}, \mathcal{F})$ is a symmetric local regular Dirichlet form on $L_2(X,\mu)$ and m is a measure satisfying Assumption 2.1.

Quasilinear elliptic PDE in divergence form. Consider the quasilinear PDE

$$\partial^* a(\partial u) = f. \tag{28}$$

We study (28) on the Hilbert space $L_2(X,\mu)$. The function f is assumed to be an element of $L_2(X,\mu)$ and the gradient ∂ and divergence ∂^* are interpreted as in Section 3. Let $Im\, \partial$ denote the image of $\mathcal{F}$ under ∂.

Assume that $a : \mathcal{H} \to \mathcal{H}$ satisfies the following monotonicity, growth and coercivity conditions:

$$\langle a(v) - a(w), v - w \rangle_{\mathcal{H}} \geq 0 \quad \text{for all } v, w \in Im\, \partial, \tag{29}$$

$$\|a(v)\|_{\mathcal{H}} \leq c_0(1 + \|v\|_{\mathcal{H}}) \quad \text{for all } v \in Im\, \partial \tag{30}$$

with some constant $c_0 > 0$,

$$\langle a(v), v \rangle_{\mathcal{H}} \geq c_1 \|v\|^2_{\mathcal{H}} - c_2 \quad \text{for all } v \in Im\, \partial \tag{31}$$

with constants $c_1 > 0$, $c_2 \geq 0$. Finally, suppose the validity of a *Poincaré inequality*,

$$\|f\|^2_{L_2(X,\mu)} \leq c_P\, \mathcal{E}(f) \tag{32}$$

with some constant $c_P > 0$ for all $f \in L_2(X, \mu)$ with $\int_X f d\mu = 0$. A function $u \in \mathcal{F}$ is called a *weak solution to (28)* if

$$\langle a(\partial u), \partial v \rangle_{\mathcal{H}} = - \langle f, v \rangle_{L_2(X,\mu)} \quad \text{for all } v \in \mathcal{F}.$$

The classical Brouwer-Minty monotonicity arguments based on Schauder's fixed point theorem, cf. [16, Section 9.1], now yield the following:

Theorem 4.1. *Assume a satisfies (29), (30) and (31) and suppose (32) holds. Then (28) has a weak solution. Moreover, if a is strictly monotone, i.e.*

$$\langle a(v) - a(w), v - w \rangle_{\mathcal{H}} \geq c_3 \|v - w\|^2_{\mathcal{H}} \quad \text{for all } v, w \in Im\, \partial \tag{33}$$

with some constant $c_3 > 0$, then (28) has a unique weak solution.

Remark 4.1. If a is a decomposable (non-linear) operator, that is if $a = (a_x)_{x \in X}$ with $a_x : \mathcal{H}_x \to \mathcal{H}_x$, $x \in X$ and $m - \operatorname{ess\,sup}_{x \in X} \|a_x\|_{\mathcal{H}_x \to \mathcal{H}_x} < \infty$, then to have (29) it is sufficient to have

$$\langle a_x(v(x)) - a_x(w(x)) \rangle_{\mathcal{H}_x} \geq 0$$

for all $v, w \in Im\, \partial$ and m-a.e. $x \in X$. Similarly for conditions (30), (31) and (33).

Quasilinear elliptic PDE in non-divergence form. Consider the PDE

$$- Lu + b(\partial u) + \varrho u = 0, \tag{34}$$

where $\varrho > 0$ and b is a generally non-linear function-valued mapping on $\mathcal{H}$. We assume that $b : \mathcal{H} \to L_2(X, \mu)$ is such that

$$\|b(v)\|_{L_2(X,\mu)} \leq c_4(1 + \|v\|_{\mathcal{H}}), \; v \in Im\, \partial, \tag{35}$$

with some $c_5 > 0$. A function $u \in \mathcal{F}$ is called a weak solution to (34) if

$$\mathcal{E}(u, v) + \langle b(\partial u), \partial v \rangle_{\mathcal{H}} + \varrho \langle u, v \rangle_{L_2(X,\mu)} = 0 \quad \text{for all } v \in \mathcal{F}.$$

Along the lines of [16, Section 9.2.2, Example 2], we obtain the following.

Theorem 4.2. *Assume that the embedding $\mathcal{F} \subset L_2(X, \mu)$ is compact and that (35) holds. Then for any sufficiently large $\varrho > 0$ there exists a weak solution to (34).*

For convenience we briefly comment on the proof.

Proof. Given $u \in \mathcal{F}$, note that $-b(\partial u) \in L_2(X,\mu)$ and denote by w the unique weak solution to the linear problem $-Lw + \varrho w = -b(\partial u)$, i.e. the unique function $w \in \mathcal{F}$ such that

$$\mathcal{E}(w,v) + \varrho \langle w, v\rangle_{L_2(X,\mu)} = -\langle b(\partial u), v\rangle_{L_2(X,\mu)} \tag{36}$$

for all $v \in \mathcal{F}$. From (35) we obtain $\|Lw\|_{L_2(X,\mu)} \leq c(1+\mathcal{E}_1(u)^{1/2})$. By the compact embedding, the mapping $u \mapsto \Phi(u) := w$ is easily seen to be continuous and compact from $\mathcal{F}$ into itself. See [16, Section 9.2.2, Theorem 5]. The set

$$\{u \in \mathcal{F} : u = \lambda\Phi(u) \ \text{ for some } 0 < \lambda \leq 1\}$$

is bounded in $\mathcal{F}$: For a member of this set, (36) implies

$$\begin{aligned}\mathcal{E}(u) + \varrho \|u\|^2_{L_2(X,\mu)} &= -\lambda \langle b(\partial u), u\rangle_{L_2(X,\mu)} \\ &\leq \|b(\partial u)\|_{L_2(X,\mu)} \|u\|_{L_2(X,\mu)} \\ &\leq c_4\varepsilon(1 + \|\partial u\|_{\mathcal{H}})\varepsilon^{-1} \|u\|_{L_2(X,\mu)} \\ &\leq c_4(\varepsilon + \varepsilon\mathcal{E}(u)^{1/2} + \varepsilon^{-1} \|u\|_{L_2(X,\mu)})^2 \\ &\leq c(1 + \varepsilon^2\mathcal{E}(u) + \varepsilon^{-2} \|u\|^2_{L_2(X,\mu)})\end{aligned}$$

for any $\varepsilon > 0$ and with a constant $c > 0$ independent of ε, λ and u. Now choose $\varepsilon > 0$ sufficiently small and $\varrho > 0$ sufficiently large to obtain

$$\mathcal{E}_1(u)^{1/2} \leq 2c.$$

Altogether this allows the application of Schaefer's fixed point theorem, cf. [16, Section 9.2.2, Theorem 4], to obtain the existence of a fixed point $u = \Phi(u)$ in $\mathcal{F}$. □

5. Change of proper speed measure and closability

As before let $(\mathcal{E},\mathcal{F})$ be a symmetric regular Dirichlet form on $L_2(X,\mu)$, where μ is an admissible reference measure on X. Recall that $\mathcal{C} = \{f \in C_0(X) : \mathcal{E}(f) < \infty\}$ and assume that m is a measure satisfying Assumption 2.1. We will now address the closability of $(\mathcal{E},\mathcal{C})$ on $L_2(X,m)$, first in the case of $(\mathcal{E},\mathcal{F})$ being local and transient and then in the case that $(\mathcal{E},\mathcal{F})$ is induced by a local regular resistance form.

A Dirichlet form $(\mathcal{E},\mathcal{F})$ is called *transient* relative to $L_2(X,\mu)$ if there is a bounded μ-integrable and μ-a.e. positive function γ on X such that

$$\int_X |u|\gamma d\mu \leq \mathcal{E}(u)^{1/2} \qquad \text{for all } u \in \mathcal{F}.$$

Now assume that the Dirichlet space $(\mathcal{F},\mathcal{E})$ is transient and let Cap_0 denote the corresponding 0-*capacity*, given by

$$\mathrm{Cap}_0(A) = \inf\{\mathcal{E}(u) : u \in \mathcal{F} : u \geq 1 \ m\text{-a.e on } A\}$$

for $A \subset X$ open and by $\mathrm{Cap}_0(B) = \inf\{\mathrm{Cap}_0(A) : \ A \subset X \text{ open}, B \subset A\}$ for general sets $B \subset X$. Any set of zero capacity is a null set for μ. A statement is said to hold q.e. (*quasi everywhere*) on a subset $A \subset X$ if there exists some set $N \subset A$ with $\mathrm{Cap}_0(N) = 0$ and the statement is valid for all $x \in A \setminus N$. A Borel function f is said to be *quasi-continuous* if for any $\varepsilon > 0$ there exists an open set $G \subset X$ such that $\mathrm{Cap}_0(G) < \varepsilon$ and f is continuous on $X \setminus G$. Any function u from the extended Dirichlet space $\mathcal{F}_e$ (i.e. any μ-equivalence class u of Borel functions from $\mathcal{F}_e$) possesses a Borel version (a representant of its class) $\widetilde{u}$ which is quasi-continuous. $\widetilde{\mathcal{F}_e}$ will denote the collection of all these versions $\widetilde{u}$.

Using the method in [50, Section 5] we can verify the following result.

Theorem 5.1. *Assume that $(\mathcal{E}, \mathcal{F})$ is local and transient relative to $L_2(X, \mu)$ and that m does not charge sets of zero 0-capacity. Then $(\mathcal{E}, \mathcal{C})$ is closable on $L_2(X, m)$, and its closure $(\mathcal{E}, \mathcal{F}^{(m)})$ is a symmetric local regular Dirichlet form.*

Note that in particular we may consider the measure $\widetilde{m}$ in Lemma 2.2, given by (14). The proof of Theorem 5.1 relies on the following observation.

Lemma 5.1. *Assume that $(\mathcal{E}, \mathcal{F})$ is local and transient relative to $L_2(X, \mu)$ and m does not charge sets of zero 0-capacity. Then for any $u \in \mathcal{F}_e$ we have*

$$\widetilde{u} = 0 \ m\text{-a.e. if and only if } \widetilde{u} = 0 \ \text{q.e.}$$

Proof. If $\widetilde{u} = 0$ q.e. then by hypothesis also $\widetilde{u} = 0$ m-a.e. Assume $m(\{\widetilde{u} \neq 0\}) = 0$. Then by Assumption 2.1 and (3) also $\Gamma(u)(\{\widetilde{u} \neq 0\}) = 0$. If

$$\Gamma(u)(\{\widetilde{u} = 0\}) = 0, \tag{37}$$

then $\mathcal{E}(u) = \Gamma(u)(\{\widetilde{u} \neq 0\}) + \Gamma(u)(\{\widetilde{u} = 0\}) = 0$ and accordingly $\mathrm{Cap}_0(\{\widetilde{u} \neq 0\}) = 0$ because

$$\mathrm{Cap}_0(\{|\widetilde{u}| \geq \varepsilon\}) \leq \frac{1}{\varepsilon^2} \mathcal{E}(u) = 0$$

for any $\varepsilon > 0$ by [19, Theorem 2.1.3 and Lemma 2.1.6]. We verify (37). For $f \in \mathcal{F}_e$ let σ^f denote the image measure of $\Gamma(f)$ under $f : X \to \mathbb{R}$, i.e.

$$\int_X \varphi d\sigma^f = \int_X \varphi(f) d\Gamma(f), \quad \varphi \in C_0(\mathbb{R}).$$

For $f \in \mathcal{F}$ [6, Theorem 5.2.3] shows that σ^f is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$, what implies (37) for $f := u \in \mathcal{F}$. This reasoning had already been used in [3, Lemma 2.7]. Transience together with the proof of [6, Theorem 5.2.3] will now produce the same absolute continuity of σ^f for general $f \in \mathcal{F}_e$: By (3) and bounded convergence the chain rule [19, Theorem 3.2.2] is seen to hold for compositions of functions from $\mathcal{F}_e$ with $C_b^1(\mathbb{R})$-functions. In particular,

$$\mathcal{E}(F(f)) = \int_{\mathbb{R}} \varphi^2 d\sigma^f$$

if $f \in \mathcal{F}_e$ and $F(y) = \int_0^y \varphi(t)dt$ with $\varphi \in C_0(\mathbb{R})$ supported in $(0, +\infty)$. If $K \subset (0, +\infty)$ is compact and of zero Lebesgue measure, we may approximate $\mathbf{1}_K$ pointwise by a sequence of functions $(\varphi_n)_n \subset C_0(\mathbb{R})$, $0 \leq \varphi_n \leq 1$ and $\operatorname{supp} \varphi_n \subset (0, +\infty)$. Since

$$\mathcal{E}(F_n(f) - F_k(f)) = \int_{\mathbb{R}} (\varphi_n - \varphi_k)^2 d\sigma^f,$$

where F_n is defined as F with φ_n in place of φ, $(F_n(f))_n$ is an $\mathcal{E}$-Cauchy sequence. By transience it has a unique $\mathcal{E}$-limit F_∞ in $\mathcal{F}_e$. Also by transience there exists a bounded μ-integrable and μ-a.e. strictly positive function γ on X such that $\widetilde{\mathcal{F}} \subset L_1(X, \gamma \cdot \mu)$ and for any $v \in \mathcal{F}_e$ we have

$$\int_X |\widetilde{v}| \gamma d\mu \leq \mathcal{E}(v)^{1/2}.$$

Hence we can find a subsequence $(\widetilde{F_{n_k}(f)})_k$ of $(\widetilde{F_n(f)})_n$ converging to $\widetilde{F_\infty}$ $\gamma \cdot \mu$-a.e. On the other hand $(F_{n_k}(f))_k$ converges to zero in $L_1(X, \gamma \cdot \mu)$. Necessarily $\widetilde{F_\infty} = 0$ $\gamma \cdot \mu$-a.e. and by the μ-a.e. strict positivity of γ also μ-a.e. $\widetilde{F}$ being quasi-continuous, this implies $\widetilde{F}=0$ q.e. by [19, Lemma 2.1.4], and therefore $\sigma^f(K)=\lim_n \mathcal{E}(F_n(f))=0$. □

Now the proof of Theorem 5.1 follows by the arguments in the proof of [50, Theorem 5.9].

Another case we are interested in arises if the regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on X is induced by a local regular resistance form $(\overline{\mathcal{E}},\overline{\mathcal{F}})$ on the set X, which is then equipped with the topology determined by the associated resistance metric, see [33, 34] and in particular [36, Definitions 3.1 and 9.5]. Regular resistance forms may for instance be obtained from regular harmonic structures on p.c.f. self-similar sets, [33], on finitely ramified fractals (not necessarily self-similar) [58] and on some infinitely ramified sets such as Sierpinski carpets [2]. A resistance form itself does not require the specification of a measure, and the conditions a measure must satisfy in order to obtain an induced Dirichlet form are rather weak. We quote the following result, which basically is a reformulation of [36, Lemma 9.2 and Theorem 9.4].

Theorem 5.2. *Assume that $(\overline{\mathcal{E}},\overline{\mathcal{F}})$ is a local regular resistance form on X and that X, equipped with the associated resistance metric, is locally compact, separable and complete. Assume further that $(\mathcal{E},\mathcal{F})$ is induced by $(\overline{\mathcal{E}},\overline{\mathcal{F}})$. Then for any admissible $\nu\in\mathcal{M}_+(X)$ we have $\mathcal{C}=\overline{\mathcal{F}}\cap C_0(X)$, the form $(\overline{\mathcal{E}},\mathcal{C})$ is closable on $L_2(X,\nu)$, and its closure $(\mathcal{E},\mathcal{F}^{(\nu)})$ is a symmetric local regular Dirichlet form.*

6. Sobolev spaces of functions and vector fields

We will now introduce L_p-spaces of vector fields and related Sobolev spaces of functions. Throughout this section it is assumed that $(\mathcal{E},\mathcal{F})$ is a symmetric regular Dirichlet form, m is a measure satisfying Assumption 2.1, and $(\mathcal{E},\mathcal{C})$ is closable on $L_2(X,m)$.

For a measurable section $v=(v(x))_{x\in X}$ let

$$\|v\|_{L_p(X,m,(\mathcal{H}_x)_{x\in X})}:=\left(\int_X \|v_x\|^p_{\mathcal{H}_x}\,m(dx)\right)^{1/p}$$

for $1\le p<\infty$ and

$$\|v\|_{L_\infty(X,m,(\mathcal{H}_x)_{x\in X})}:=\operatorname*{ess\,sup}_{x\in X}\|v_x\|_{\mathcal{H}_x}$$

and define the spaces $L_p(X,m,(\mathcal{H}_x)_{x\in X})$, $1\le p\le\infty$ as the collections of the respective equivalence classes of m-a.e. equal sections having finite norm. By a variant of the classical pointwise Riesz-Fischer argument they form Banach spaces, separable for $1\le p<\infty$. Note that $\mathcal{H}=L_2(X,m,(\mathcal{H}_x)_{x\in X})$.

For $1<p<\infty$ and $1/p+1/q=1$ the Hölder inequality

$$\left|\int_X \langle v_x,w_x\rangle_{\mathcal{H}_x}\,m(dx)\right|\leqslant\left(\int_X\|v_x\|^p_{\mathcal{H}_x}\,m(dx)\right)^{1/p}\left(\int_X\|w_x\|^q_{\mathcal{H}_x}\,m(dx)\right)^{1/q}\tag{38}$$

for $v\in L_p(X,m,(\mathcal{H}_x))$, $w\in L_q(X,m,(\mathcal{H}_x))$ follows from Cauchy-Schwarz in $\mathcal{H}$. We will write $\langle w,v\rangle$ for the the integral on the left hand side.

If $f\in\mathcal{B}_b(X)$ and $v=(v(x))_{x\in X}\in L_p(X,m,(\mathcal{H}_x))$ then the product fv is defined as the measurable section $x\mapsto f(x)v_x$, i.e. pointwise. Since

$$\|fv\|_{L_p(X,m,(\mathcal{H}_x)_x)}=\left(\int_X\|f(x)v_x\|^p_{\mathcal{H}_x}\,m(dx)\right)^{1/p}\le\|f\|_{L_\infty(X,m)}\|v\|_{L_p(X,m,(\mathcal{H}_x)_x)}$$

the operation $v \mapsto fv$ is linear and bounded in $L_p(X,m,(\mathcal{H}_x))$ and is continuous with respect to the pointwise convergence of uniformly bounded sequences, i.e. if $\sup_n \|f_n\|_{L_\infty(X,m)} < \infty$ and $\lim_n f_n = f$ pointwise m-a.e. on X, then $\lim_n f_n v = fv$ in $L_p(X,m,(\mathcal{H}_x))$ for all $v \in L_p(X,m,(\mathcal{H}_x))$. For $p = 2$ this multiplication coincides with (6).

We will make the following additional assumption.

Assumption 6.1. For any $1 < p < \infty$ there is a space $\mathcal{C}_p \subset \mathcal{C} \cap L_p(X,m)$ such that

(COREI) $\mathcal{C}_p$ is dense in $L_p(X,m)$,

(COREII) $\mathcal{C}_p \otimes \mathcal{C}_p$ is dense in $L_p(X,m,(\mathcal{H}_x)_x)$ and

(COREIII) for all $f \in \mathcal{C}_p$, the energy measure $\Gamma(f)$ is absolutely continuous with respect to m with density
$$\Gamma(f) = \frac{d\Gamma(f)}{dm} \in L_{p/2}(X,m).$$

Let ∂_p denote the restriction of ∂ to $\mathcal{C}_p$. By (COREIII) the operator ∂_p maps $\mathcal{C}_p$ into $L_p(X,m,(\mathcal{H}_x)_x)$. Recall that in this section we assume the closability of $(\mathcal{E},\mathcal{C})$ on $L_2(X,m)$. As an immediate consequence $(\partial_2,\mathcal{C}_2)$ is seen to be a closable operator from $L_2(X,m)$ to $L_2(X,m,(\mathcal{H}_x))$, because also $(\mathcal{E},\mathcal{C}_2)$ is closable on $L_2(X,m)$. For $2 < p < \infty$ we have the following result.

Theorem 6.1. *Let the conditions of Assumption 6.1 be valid and assume $m(X) < \infty$. Then $(\partial_p,\mathcal{C}_p)$ is a closable operator from $L_p(X,m)$ to $L_p(X,m,(\mathcal{H}_x))$ for any $2 < p < \infty$.*

Proof. Let $(u_n) \subset \mathcal{C}_p$ be a sequence of functions converging to zero in $L_p(X,m)$ and such that $(\partial_p u_n)$ is Cauchy in $L_p(X,m,(\mathcal{H}_x))$. As the latter space is complete, a unique limit $v := \lim_n \partial_p u_n \in L_p(X,m,(\mathcal{H}_x))$ exists. The measure m being finite, $(u_n)_n$ is seen to be $\mathcal{E}$-Cauchy and convergent to zero in $L_2(X,\mu)$ what implies that $\mathcal{E}(u_n)$ goes to zero. For an arbitrary member $f \otimes g$ of $\mathcal{C}_q \otimes \mathcal{C}_q$ with $1/p + 1/q = 1$ we have
$$\langle f \otimes g, v\rangle = \lim_n \langle f \otimes g, \partial_p u_n\rangle = \lim_n \langle f \otimes g, \partial u_n\rangle_{\mathcal{H}} = -\lim_n \partial^*(g\partial f)(u_n) = 0$$
because
$$|\partial^*(g\partial f)(u_n)| \le \int_X |g| d|\Gamma(u_n,f)| \le \sup_{x\in X} |g(x)| \mathcal{E}(u_n)^{1/2}\mathcal{E}(f)^{1/2}.$$
By (COREII) therefore $\lim_n \partial_p u_n = 0$ in $L_p(X,m,(\mathcal{H}_x))$. □

For the rest of this section we take Assumption 6.1 for granted and suppose that $2 \le p < \infty$ and $(\partial_p,\mathcal{C}_p)$ is closable. Its smallest closed extension is denoted by $(\partial_p, dom\,\partial_p)$, which then is a densely defined closed linear operator from $L_p(X,m)$ into $L_p(X,m,(\mathcal{H}_x))$. Note that for any simple vector field $g\partial f$ with $f,g \in \mathcal{C}_p$ we then have

$$\|g\partial f\|_{L_p(X,m,(\mathcal{H}_x)_x)} = \left(\int_X |g(x)|^p \Gamma_x(f)^{p/2} m(dx)\right)^{1/p}. \tag{39}$$

We write $H_0^{1,p}(X,m)$ for $dom\,\partial_p$, equipped with the norm
$$\|u\|_{1,p} := \left(\int_X \left(|u(x)|^p + \|\partial_x u\|^p_{\mathcal{H}_x}\right) m(dx)\right)^{1/p}, \quad u \in H_0^{1,p}(X,m).$$

As $\|\cdot\|_{1,p}$ is equivalent to the graph norm of ∂_p, $H_0^{1,p}(X,m)$ is a closed subspace of $L_p(X,m)$, clearly Banach, and continuously embedded in $L_p(X,m)$. For $p = 2$ we observe $H_0^{1,2}(X,m) = \mathcal{F}^{(m)}$.

Now the divergence operator ∂^* may be seen as an unbounded operator

$$\partial_q^* : L_q(X, m, (\mathcal{H}_x)) \to L_q(X, m),$$

where $1/p + 1/q = 1$, and similarly as in (27) we obtain an integration by parts formula by saying that an element $v \in L_q(X, m, (\mathcal{H}_x)_{x\in X})$ is in *dom* ∂_q^* if there is some $v^* \in L_q(X, m)$ such that $\langle u, v^* \rangle = -\langle \partial u, v \rangle$ for all $u \in \mathcal{C}_p$. We write $\partial_q^* v := v^*$ and

$$\left\langle u, \partial_q^* v \right\rangle = -\langle \partial u, v \rangle \ \ , u \in \mathcal{C}_p.$$

By duality *dom* ∂_q^* is then weakly dense in $L_q(X, m)$, cf. [48].

Examples 6.1. If $X \subset \mathbb{R}^n$ is a sufficiently nice domain, $m(dx) = \mu(dx) = dx$ is the n-dimensional Lebesgue measure on X and

$$\mathcal{E}(f, g) = \int_X \nabla f \nabla g dx$$

is the Dirichlet form of the standard Laplacian, then all fibers $\mathcal{H}_x = \mathbb{R}^n$ are constant, a canonical choice for $\mathcal{C}_p$ is $C_0^\infty(X)$ and $\Gamma(f) = |\nabla f|^2$. Similarly for more general second order elliptic differential operators.

7. Existence of continuous coordinates

One possible way to verify Assumption 6.1 in a non-classical contexts is to use abstract continuous coordinates. Let $(\mathcal{E}, \mathcal{F})$ be a symmetric local regular Dirichlet form and m is a measure according to Assumption 2.1. For the measure $\widetilde{m}$, as constructed in Lemma 2.2, we will actually prove the existence of coordinates.

Let Cap denote the 1-*capacity* corresponding to the Dirichlet space $(\mathcal{F}, \mathcal{E})$, given by

$$\mathrm{Cap}(A) = \inf \left\{ \mathcal{E}_1(u) : u \in \mathcal{F} : u \geq 1 \ m\text{-a.e on } A \right\}$$

for $A \subset X$ open and by $\mathrm{Cap}(B) = \inf \{\mathrm{Cap}(A) :\ A \subset X \text{ open}, B \subset A\}$ for general sets $B \subset X$. Quasi-notions can be defined and behave in a similar manner as written for the transient case after Theorem 5.1. See [19, Chapter 2]. Recall that previously m was required to be an energy dominated measure. We will now work under the following additional assumption:

Assumption 7.1. In addition to Assumption 2.1 we assume that the measure m is finite and does not charge sets of zero capacity.

Let $\{\varphi_i\}_{i\in I} \subset \mathcal{C}$ be a set of functions indexed by some set $I \neq \emptyset$. We say that $\{\varphi_i\}_{i\in I}$ is a set of *continuous coordinates* for $\mathcal{E}$ with respect to m if the following conditions are satisfied:

(COI) for all $i, j \in I$, $\Gamma(\varphi_i, \varphi_j) \in L_1(X, m) \cap L_\infty(X, m)$,

(COII) The space $\mathcal{F}C_b^1(X, \{\varphi_i\})$ of all cylinder functions of the form

$$f = F(\varphi_{i_1}, ..., \varphi_{i_m}), \quad i_1, ..., i_k \in I$$

with suitable $k \in \mathbb{N}$ and $F \in C_b^1(\mathbb{R}^k)$, $F(0) = 0$, is dense in $\mathcal{C}$ with respect to the norm in $\mathcal{F}$, i.e $\mathcal{E}_1$-dense.

For cylinder functions $f = F(\varphi_{i_1}, ...\varphi_{i_m})$ and $g = G(\varphi_{j_1}, ..., \varphi_{j_n})$ with $F \in C_b^1(\mathbb{R}^m)$ and $G \in C_b^1(\mathbb{R}^n)$ satisfying $F(0) = G(0) = 0$, we then have

$$\Gamma(f, g) = \sum_{k=1}^{m} \sum_{l=1}^{n} \frac{\partial F}{\partial x_k}(\varphi_{i_1}, ..., \varphi_{i_m}) \frac{\partial G}{\partial x_l}(\varphi_{j_1}, ..., \varphi_{j_n}) \Gamma(\varphi_k, \varphi_l)$$

by the chain rule, cf. [19, Theorem 3.2.2]. In particular, $\Gamma(f,g)$ is a member of $L_\infty(X,m)$ and has compact support.

From (COII) we can obtain further approximation and denseness results.

Lemma 7.1. *Suppose that m satisfies Assumption 7.1 and that $\{\varphi_i\}_{i\in I}$ is a set of continuous coordinates for $\mathcal{E}$ with respect to m. Then any function $g \in \mathcal{C}$ can be approximated pointwise m-a.e. by a uniformly bounded sequence of functions from $\mathcal{F}C_b^1(X,\{\varphi_i\})$.*

Proof. Let $g \in \mathcal{C}$. By (COII) there is a sequence $(g_n)_n \subset \mathcal{F}C_b^1(X,\{\varphi_i\})$ converging to g in $\mathcal{F}$ with respect to the $\mathcal{E}_1$-norm. Switching to a subsequence if necessary we may assume $(g_n)_n$ also converges to g q.e. by [19, Theorem 2.1.4]. As m does not charge sets of zero capacity, g is also the m-a.e. pointwise limit of $(g_n)_n$. Now set

$$s := \sup_{x\in X} |g(x)|$$

and let $\phi \in C_b^1(\mathbb{R})$ be a monotone function that satisfies

$$\phi(y) = \begin{cases} -2s & \text{if } y < -2s \\ y & \text{if } -s \le y \le s \\ 2s & \text{if } y > 2s. \end{cases}$$

Note that $\phi(g_n) \in \mathcal{F}C_b^1(X,\{\varphi_i\})$ for any n and $\sup_n \sup_X |\phi(g_n)| \le 2s$. Also the functions $\phi(g_n)$ converge to g m-a.e. pointwise. □

Corollary 7.1. *Suppose that m satisfies Assumption 7.1 and that $\{\varphi_i\}_{i\in I}$ is a set of continuous coordinates for $\mathcal{E}$ with respect to m. Then for any $1 < p < \infty$ conditions (COREI) and (COREIII) in Assumption 6.1 are satisfied with $\mathcal{C}_p = \mathcal{F}C_b^1(X,\{\varphi_i\})$.*

Proof. (COREIII) follows directly from (COI). Hence it suffices to prove that $\mathcal{F}C_b^1(X,\{\varphi_i\})$ is dense in $L_p(X,m)$, $1 < p < \infty$. By the denseness of $C_0(X)$ in $L_p(X,m)$, the finiteness of m and the regularity of $(\mathcal{E},\mathcal{F})$ it is enough to show any function $g \in \mathcal{C}$ can be approximated in $L_p(X,m)$-norm by a sequence of functions from $\mathcal{F}C_b^1(X,\{\varphi_i\})$. As m is finite, this follows from Lemma 7.1. □

Another consequence concerns the spaces $L_p(X,m,(\mathcal{H}_x)_x)$.

Lemma 7.2. *Suppose that m satisfies Assumption 7.1 and that $\{\varphi_i\}_{i\in I}$ is a set of continuous coordinates for $\mathcal{E}$ with respect to m. Then for any $1 < p < \infty$,*

$$\mathcal{S} := \operatorname{span}\left\{g\partial f : f,g \in \mathcal{F}C_b^1(X,\{\varphi_i\})\right\}$$

is a dense subspace of $L_p(X,m,(\mathcal{H}_x)_x)$. In particular, condition (CORE II) in Assumption 6.1 holds with $\mathcal{C}_p = \mathcal{F}C_b^1(X,\{\varphi_i\})$.

Proof. By (39) it is easily seen that for any $1 < p < \infty$, $\mathcal{S}$ is a subspace of $L_p(X,m,(\mathcal{H}_x)_x)$. Next observe that

$$\mathcal{S}_0 := \operatorname{span}\left\{g\partial f : f \in \mathcal{F}C_b^1(X,\{\varphi_i\}), g \in \mathcal{B}_b(X)\right\}$$

is dense in $\mathcal{H} = L_2(X,m,(\mathcal{H}_x)_x)$ because, by the definition of $\mathcal{H}$, it suffices to approximate finite linear combinations $\sum_i a_i \otimes b_i \in \mathcal{C} \otimes \mathcal{B}_b(X)$. For fixed i, let $(a_i^{(m)})_m \subset \mathcal{F}C_b^1(X,\{\varphi_i\})$ be a sequence approximating a_i in $\mathcal{E}$. Then

$$\left\|\sum_i a_i \otimes b_i - \sum_i a_i^{(m)} \otimes b_i\right\|_{\mathcal{H}}^2 = \sum_i \sum_j \int_X b_i b_j d\Gamma(a_i - a_i^{(m)}),$$

which is bounded by $2\max_i \sup_X |b_i|^2 \sum_i \mathcal{E}(a_i - a_i^{(m)})$ and therefore converges to zero as m goes to infinity. The space $\mathcal{S}_0$ is dense in $L_p(X, m, (\mathcal{H}_x)_x)$: Assume it were not, then by Hahn-Banach we could find some $\eta \in L_q(X, m, (\mathcal{H}_x)_x)$, $1/p+1/q = 1$, such that $\|\eta\|_{L_q(X,m,(\mathcal{H}_x)_x)} = 1$ and

$$\langle \omega, \eta \rangle = 0 \quad \text{for all } \omega \in \mathcal{S}_0. \tag{40}$$

For any $N \in \mathbb{N}$ let $K_N \subset X$ be compact such that $m(X \setminus K_N) < 1/N$ and set

$$S_N := \left\{ x \in X : \|\eta_x\|_{\mathcal{H}_x} < N \right\} \cap K_N.$$

Then

$$\lim_N \|\eta \mathbf{1}_{S_N} - \eta\|^q_{L_q(X,m,(\mathcal{H}_x)_x)} = \lim_N \int_X |\mathbf{1}_{S_N}(x) - \mathbf{1}(x)|^q \, \|\eta_x\|^q_{\mathcal{H}_x} \, m(dx) = 0$$

by dominated convergence and accordingly for fixed $\varepsilon > 0$ there exists some $N_\varepsilon > 0$ such that for any $N \geq N_\varepsilon$, $\|\eta \mathbf{1}_{S_N} - \eta\|_{L_q(X,m,(\mathcal{H}_x)_x)} < \varepsilon$. Note also that $\eta \mathbf{1}_{S_N} \in \mathcal{H}$ for all $N \in \mathbb{N}$ since

$$\int_X \|\eta_x \mathbf{1}_{S_N}(x)\|^2_{\mathcal{H}_x} \, m(dx) < N^2 m(K_N) < \infty.$$

As $\|\eta \mathbf{1}_{S_N}\|_{L_q(X,m,(\mathcal{H}_x)_x)} > 1 - \varepsilon$ necessarily also

$$\delta_N := \|\eta \mathbf{1}_{S_N}\|_{\mathcal{H}} > 0.$$

Now let $(\omega_n)_n \subset \mathcal{S}_0$, $\omega_n = \sum_i a_i^{(n)} \otimes b_i^{(n)}$ be a sequence that approximates $\eta \mathbf{1}_{S_N}$ in $\mathcal{H}$. Let $0 < \gamma < \delta_N$ and $n \in \mathbb{N}$ be so large that

$$\|\eta \mathbf{1}_{S_N} - \omega_n\|_{\mathcal{H}} \leq \gamma.$$

Since $|\langle \mathbf{1}_{S_N} \eta, \mathbf{1}_{S_N} \eta - \omega_n \rangle| \leq \gamma \delta_N$ we obtain

$$|\langle \mathbf{1}_{S_N} \omega, \eta \rangle| = |\langle \omega, \mathbf{1}_{S_N} \eta \rangle| > \delta_N(\delta_N - \gamma) > 0. \tag{41}$$

On the other hand

$$\mathbf{1}_{S_N} \omega_n = \sum_i \mathbf{1}_{S_N} (a_i^{(n)} \otimes b_i^{(n)}) = \sum_i a_i^{(n)} \otimes (\mathbf{1}_{S_N} b_i^{(n)})$$

itself is an element of $\mathcal{S}_0$. Therefore (41) contradicts (40). Note that $\mathcal{S}$ is $L_p(X, m, (\mathcal{H}_x)_x)$-dense in $\mathcal{S}_0$: Let $\sum_i a_i \otimes b_i \in \mathcal{F}C_b^1(X, \{\varphi_i\}) \otimes \mathcal{B}_b(X)$. Any b_i can be approximated uniformly by a sequence from $C_0(X)$, and by the regularity of $(\mathcal{E}, \mathcal{F})$ together with Lemma 7.1, any b_i can be approximated pointwise by a uniformly bounded sequence $(b_i^{(m)})_m$ of functions from $\mathcal{F}C_b^1(X, \{\varphi_i\})$. By (39),

$$\left\| \sum_i a_i \otimes b_i - \sum_i a_i \otimes b_i^{(m)} \right\|_{L_p(X,m,(\mathcal{H}_x)_x)} \leq \sum_i \left(\int_X |b_i(x) - b_i^{(m)}(x)|^p \Gamma_x(a_i)^{p/2} m(dx) \right)^{1/p},$$

which converges to zero since m is finite and $\Gamma_\cdot(a_i) \in L_\infty(X, m)$ for any i. □

In the following sense the existence of a countable set of continuous coordinates is always guaranteed. Recall that $\widetilde{m}$ denotes the measure (14) constructed in Lemma 2.2 as a sum of the energy measures of certain functions $f_n \in \mathcal{C}$, $n \in \mathbb{N}$, considered in (13).

Theorem 7.1. *Let $(\mathcal{E}, \mathcal{F})$ be a symmetric local regular Dirichlet form and $\widetilde{m}$ the measure given by (14). Then the set $\{f_n\}_{n \in \mathbb{N}}$ of functions f_n according to (13) is a set of continuous coordinates for $\mathcal{E}$ with respect to $\widetilde{m}$.*

Proof. Note first that given two measures $m_1, m_2 \in \mathcal{M}_+(X)$, m_1 is obviously absolutely continuous with respect to $m_1 + m_2$ and for the corresponding Radon-Nikodym density we observe

$$\frac{dm_1}{d(m_1+m_2)} \leq 1 \quad (m_1+m_2)\text{-a.e.} \tag{42}$$

An application of (42) with $m_1 := 2^{-n}\Gamma(f_n)$ and $m_2 := \widetilde{m} - 2^{-n}\Gamma(f_n)$ shows that

$$\frac{d\Gamma(f_n)}{d\widetilde{m}} \leq 2^n \quad \widetilde{m}\text{-a.e}$$

for any n, and by Cauchy-Schwarz (COI) follows. By construction span $(\{f_n\}_n)$ is $\mathcal{E}_1$-dense in $\mathcal{C}$, what implies (COII). Finally, recall that the finite measure $\widetilde{m}$ does not charge sets of zero capacity, [19, Lemma 3.2.4]. □

Remark 7.1. An alternative argument to prove at least (COI) directly may be obtained using the associated generator. Here we assume that $(\mathcal{E}, \mathcal{C})$ is closable on $L_2(X,m)$. Let $L^{(m)}$ be the infinitesimal generator of the Dirichlet form $(\mathcal{E}, \mathcal{F}^{(m)})$ in $L_2(X,m)$ and *dom* $L^{(m)}$ its domain. By $L^{(m),1}$ we denote the closure in $L_1(X,m)$ of the restriction of $L^{(m)}$ to

$$\left\{ f \in dom\, L^{(m)} \cap L_1(X,m) : L^{(m)} f \in L_1(X,m) \right\},$$

cf. [6, Section I.2.4]. As all energy measures $\Gamma(f)$, $f \in \mathcal{C}$ are absolutely continuous with respect to m, [6, Theorem I.4.2.2] tells that *dom* $L^{(m)}$ is an algebra and

$$\Gamma(\varphi, \psi) = L^{(m),1}(\varphi\psi) - \varphi L^{(m)}\psi - \psi L^{(m)}\varphi$$

for all $\varphi, \psi \in dom\, L^{(m)}$. Therefore, if $\{\varphi_i\}_{i\in I} \subset dom\, L^{(m)}$ and for all $i, j \in I$, the functions $L^{(m)}\varphi_i$ and $L^{(m),1}(\varphi_i\varphi_j)$ are members of $L_\infty(X,m)$, condition (COI) is obviously satisfied.

We conclude this section with a prototype example.

Examples 7.1. Let $(\overline{\mathcal{E}}, \overline{\mathcal{F}})$ be a local resistance form, [33, 34, 36], on a finitely ramified fractal X, which is defined as a specific type of cell-structured compact topological space, see [58] for the precise definition. Let $\varphi_1, ..., \varphi_k$ be a complete, up to constants, energy orthonormal set of harmonic functions and define a finite reference measure by

$$m := \sum_{j=1}^{k} \Gamma(\varphi_j) \tag{43}$$

(*Kusuoka measure*), where $\Gamma(\varphi_j)$ are the energy measures of the functions φ_j. Consider the map $\psi : X \to \mathbb{R}^m$, $\psi(x) = (\varphi_1(x), ..., \varphi_k(x))$, cf. [58]. We assume that $\psi : X \to \psi(X)$ is a homeomorphism. This implies that all φ_j are continuous on X, cf. [58, Proposition 5.3]. By [58, Theorem 3] or [58, Theorem 7] together with Theorem 5.2 we observe that $(\overline{\mathcal{E}}, \overline{\mathcal{F}})$ induces a Dirichlet form $(\mathcal{E}, \mathcal{F}^{(m)})$ on $L_2(X,m)$. The associated infinitesimal generator will be denoted by $\Delta^{(m)}$ and its domain by *dom* $\Delta^{(m)}$. Using Theorem 7.1 we may now conclude that $\{\varphi_i\}_i$ is a set of continuous coordinates for $\mathcal{E}$ with respect to m. Alternatively, we may use Remark 7.1: Clearly $\varphi_1, ..., \varphi_k \in dom\Delta^{(m)} \cap L_\infty(X,m)$ and $\Delta^{(m)}\varphi_j = 0$ for all j. By [58, Theorems 7 and 8 and Corollary 6.1, respectively], cylinder functions are dense in $\mathcal{F}$, $\varphi_i\varphi_j \in dom\Delta^{(m)}$ and $\Delta^{(m)}(\varphi_i\varphi_j) \in L_\infty(X,m)$. Consequently $\mathcal{E}$ admits continuous coordinates with respect to m.

8. Applications to SPDE

The results of Section 6 may for instance be used to study deterministic or stochastic evolution equations in the variational framework. To discuss a class of examples we assume throughout the entire section that $2 \le p < \infty$, m is finite and $(\partial_p, \mathcal{C}_p)$ is closable on $L_p(X,m)$.

SPDE in variational form. SPDE in variational form have been studied first in [37] and [45], a brief exposition is given in [47]. For simplicity consider Itô SPDE with additive Brownian noise of type

$$du(t) = \partial^* a(\partial u(t))dt + \sqrt{Q}dW(t) \tag{44}$$

on $(0,T) \times X$ with some initial condition $u(0) = u_0$. Here $(W(t))_{t \ge 0}$ is a cylindrical Wiener process on $L_2(X,m)$ of the form

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k$$

where $(\beta_k)_k$ is a sequence of mutually independent one-dimensional standard Brownian motions on a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$, and $\{e_k\}_k$ is an orthonormal basis in $L_2(X,m)$. Then $L_p(X,m) \subset L_2(X,m)$, and we have

$$H_0^{1,p}(X,m) \subset L_2(X,m) \subset (H_0^{1,p}(X,m))^* \ ,$$

where as before $(H_0^{1,p}(X,m))^*$ denotes the dual space of $H_0^{1,p}(X,m)$, and the embeddings are continuous. We write

$$\langle u, v\rangle := v(u) \quad \text{for } u \in (H_0^{1,p}(X,m))^* \text{ and } v \in H_0^{1,p}(X,m).$$

Generalizing the growth condition (30) we may require a to be a bounded operator

$$a : L_p(X,m,(\mathcal{H}_x)_x) \to L_q(X,m,(\mathcal{H}_x)_x)$$

with $1/p + 1/q = 1$ and such that

$$\|a(v)\|_{L_q(X,m,(\mathcal{H}_x)_x)} \le c_0(1 + \|v\|^{p-1}_{L_p(X,m,(\mathcal{H}_x)_x)}) \tag{45}$$

for all $v \in L_p(X,m,(\mathcal{H}_x)_x)$.

Remark 8.1. (45) is obviously valid with $p = 2$ if $a = (a_x)_x$ with bounded operators $a_x : \mathcal{H}_x \to \mathcal{H}_x$ such that $\text{m-ess}\sup_{x\in X} \|a_x\|_{\mathcal{H}_x \to \mathcal{H}_x} < \infty$.

The following is a simple consequence of the Hölder inequality (38):

Lemma 8.1. *If a satisfies (45) then $\partial^* a(\partial \cdot)$ defines a bounded operator from $H_0^{1,p}(X,m)$ into $(H_0^{1,p}(X,m))^*$ with*

$$\|\partial^* a(\partial u)\|_{(H_0^{1,p}(X,m))^*} \le c_6(1 + \|u\|^{p-1}_{H_0^{1,p}(X,m)}) \ ,$$

$u \in H_0^{1,p}(X,m)$, with a constant $c_6 > 0$.

Similarly as in the case of (28) we may invoke a general solution theory [37, 45], provided some regularity conditions are satisfied. In addition to (45) we will require the versions

$$\langle a(\partial f), \partial f\rangle \ge c_1 \|f\|^p_{1,p} - c_2 \|f\|^2_{L_2(X,m)} \quad \text{for all } f \in H_0^{1,p}(X,m) \tag{46}$$

with constants $c_1 > 0$, $c_2 \geq 0$ and

$$\langle a(\partial f) - a(\partial g), \partial f - \partial g\rangle \geq c_3 \|f-g\|^2_{L_2(X,m)} \quad \text{for all } f,g \in H_0^{1,p}(X,m), \tag{47}$$

with $c_3 > 0$ of (31) and (33) with the left hand sides interpreted in the sense of duality. Finally, assume that for all u, v, w from the image $Im\,\partial_p$ of $H_0^{1,p}(X,m)$ under ∂_p,

$$\text{the function} \quad \lambda \mapsto \langle a(u+\lambda v), w\rangle \quad \text{is continuous at zero.} \tag{48}$$

Remark 8.2. Note that if (45) is valid and $a = (a_x)_x$ is decomposable as before, the relation

$$\lim_{\lambda\to 0} \langle a_x(u(x)+\lambda v(x)), z(x)\rangle_{\mathcal{H}_x} = \langle a_x(v(x)), z(x)\rangle_{\mathcal{H}_x}$$

for m-a.e. $x \in X$ implies (48), because

$$|\langle a(\partial f + \lambda \partial g), \partial h\rangle| \leq c(1 + \|f\|_{1,p} + \|g\|_{1,p}) \|h\|_{1,p}, \quad f,g,h \in H_0^{1,p}(X,m),$$

as one can easily verify.

A continuous $(\mathcal{F}_t)$-adapted process $u = (u(t))_{t\in[0,T]}$ is called *a solution to (44) with initial condition* u_0 if

$$\mathbb{E}\int_0^T (\|u(t)\|^p_{1,p} + \|u(t)\|^2_{L_2(X,m)})dt < \infty$$

and

$$u(t) = u_0 + \int_0^t \partial^* a(\partial \widetilde{u}(s))ds + \int_0^t \sqrt{Q}dW(s)\ , \quad t \in [0,T],$$

seen as an identity of $(H_0^{1,p}(X,m))^*$-valued functions, where $\widetilde{u}$ is any $H_0^{1,p}(X,m)$-valued progressively measurable $dt \otimes d\mathbb{P}$-version of u.

The following is a special case of the classical results in [37, 47].

Theorem 8.1. *Let $2 \leq p < \infty$, $m(X) < \infty$ and assume that a satisfies (45), (46), (47) and (48). Let*

$$\mathbb{E}\int_X u_0^2(x)m(dx) < \infty.$$

Then (44) has a unique solution u with initial condition u_0.

Examples 8.1. A specific example is given by the following *stochastic p-Laplace equation*: Let $a = (a_x)_{x\in X}$ with

$$a_x(v(x)) := \|v(x)\|^{p-2}_{\mathcal{H}_x} v(x), \quad v \in Im\,\partial_p.$$

We have $\|a(v)\|_{L_q(X,m,(\mathcal{H}_x)_x)} = \|v\|^{p-1}_{L_p(X,m,(\mathcal{H}_x)_x)}$ by Hölder's inequality, hence (45) holds. Condition (48) rewrites

$$\lim_{\lambda\to 0}\int_X \left(\|\partial(u+\lambda v)(x)\|^{p-2}_{\mathcal{H}_x} \langle \partial(u+\lambda v)(x), \partial w(x)\rangle_{\mathcal{H}_x} - \|\partial u(x)\|^{p-2}_{\mathcal{H}_x}\langle \partial u(x), \partial w(x)\rangle_{\mathcal{H}_x}\right) m(dx) = 0.$$

But this follows by dominated convergence, the pointwise limit being obvious from the continuity of $\|\cdot\|_{\mathcal{H}_x}$ for fixed x and a dominating integrable function being provided by

$$c\left(\|\partial v(x)\|^{p-1}_{\mathcal{H}_x} + \|\partial v(x)\|^{p-1}_{\mathcal{H}_x}\right)\|\partial w(x)\|^{p-1}_{\mathcal{H}_x}\ .$$

(47) holds with $c_4 = 0$ because

$$\int_X \left(\|\partial f(x)\|_{\mathcal{H}_x}^p + \|\partial g(x)\|_{\mathcal{H}_x}^p - \|\partial f(x)\|_{\mathcal{H}_x}^{p-2} \langle \partial f(x), \partial g(x)\rangle_{\mathcal{H}_x} \right.$$
$$\left. - \|\partial g(x)\|_{\mathcal{H}_x}^{p-2} \langle \partial f(x), \partial g(x)\rangle_{\mathcal{H}_x}\right) m(dx)$$
$$\geq \int_X \left(\|\partial f(x)\|_{\mathcal{H}_x}^{p-1} - \|\partial g(x)\|_{\mathcal{H}_x}^{p-1}\right)\left(\|\partial f(x)\|_{\mathcal{H}_x} - \|\partial g(x)\|_{\mathcal{H}_x}\right) \geq 0.$$

Condition (46) follows immediately if a *p-Poincaré inequality* is satisfied,

$$\|f\|_{L_p(X,m)}^p \leq c_P \int_X \|\partial f(x)\|_{\mathcal{H}_x}^p \, m(dx) \tag{49}$$

with some $c_P > 0$ for all $f \in L_p(X, m)$ with $\int_X f dm = 0$. For smooth bounded Euclidean domains (49) follows by classical arguments. It also holds if $(\mathcal{E}, \mathcal{F})$ is induced by a regular resistance form [33, 34, 36] and is subject to Dirichlet boundary conditions in the sense that there is a point $p \in X$ such that $f(p) = 0$ for all $f \in \mathcal{F}$. As elements of $\mathcal{F}$ are continuous in the resistance metric, pointwise evaluation makes sense. Then (49) results from

$$\sup_x |u(x)| \leq c\, \mathcal{E}(u)^{1/2}, \quad x \in X, \tag{50}$$

with some $c > 0$ and for all $u \in \mathcal{F}$. (50) is verified for instance in [33, Lemma 5.2.8 and its proof]. Another situation where cases of (49) are easily verified arises if a *Sobolev inequality*

$$\|f\|_{2d_s/(d_s-2)} \leq c\, \mathcal{E}(f)^{1/2}$$

holds for all $f \in \mathcal{F}$ with $\int_X f dm = 0$, where $d_s > 2$ is the *spectral dimension*. Together with $m(X) < \infty$ this implies (49) for $2 \leq p \leq 2d_s/(d_s - 2)$.

9. p-Laplacian and p-energies

We provide a brief remark about related p-energies for $2 \leq p < \infty$. The mapping

$$f \mapsto \mathcal{E}_p(f) := \int_X \Gamma(f)^{p/2} dm, \; f \in H_0^{1,p}(X, m),$$

is usually referred to as the *p-energy functional*. One may define a functional of two arguments by

$$\mathcal{E}_p(f, g) := \int_X \Gamma(f)^{p/2-1}\Gamma(f, g) dm, \quad f, g \in H_0^{1,p}(X, m).$$

Note that $\mathcal{E}_p(f, f) = \mathcal{E}_p(f)$ and that by Hölder's inequality, $|\mathcal{E}_p(f, g)| \leq \mathcal{E}_p(f)^{(p-1)/p}\mathcal{E}_p(g)^{1/p}$. For functions $\varphi, \psi \in dom\, L^{(m)}$ we observe

$$\mathcal{E}_p(\varphi, \psi) := \frac{1}{p}\frac{d}{dt}\mathcal{E}_p(\varphi + t\psi)|_{t=0}.$$

A generalized p-Laplacian may be defined in the weak sense by associating to $f \in H_0^{1,p}(X, m)$ the element $\Delta_p f$ of the dual space $(H_0^{1,p}(X, m))^*$ given by

$$(\Delta_p f)(g) := -\mathcal{E}_p(f, g) = -\int_X \|\partial_x f\|_{\mathcal{H}_x}^{p-2} \langle \partial_x f, \partial_x g\rangle_{\mathcal{H}_x} m(dx) = -\left\langle \|\partial_\cdot f\|_{\mathcal{H}_\cdot}^{p-2} \partial f, \partial g\right\rangle_{\mathcal{H}},$$

$g \in H_0^{1,p}(X, m)$. Integrating by parts we obtain

$$\Delta_p f = \partial_p^*\left(\|\partial_\cdot f\|_{\mathcal{H}_\cdot}^{p-2}\partial f\right).$$

If $L = \Delta$ is the classical Laplacian on $\mathbb{R}^n$ and $m(dx) = dx$ the n-dimensional Lebesgue measure, then Δ_p is the usual p-Laplacian.

Remark 9.1. Another definition for a p-energy on Sierpinski gasket type fractals had been proposed in [21], a related p-Laplacian had been investigated in [54]. To attempt a comparison, we first rewrite our previous definition. For simplicity, let $X = K$ be the Sierpinski gasket $K = \bigcup_{i=1}^{3} F_i K$, where $\{F_1, F_2, F_3\}$ is the corresponding iterated function system of contractive similarities F_i with common contraction ratio $\frac{1}{2}$ and fixed points q_1, q_2, q_3 which are the vertices of an equilateral triangle. Let $V_0 = \{q_1, q_2, q_3\}$ and $V_m = \bigcup_{i=1}^{3} F_i V_{m-1}$, $m \geq 1$. Write F_w for $F_{w_1} \cdots F_{w_n}$ and a word $w = (w_1, ..., w_n)$ of length $|w| = n$ over the alphabet $\{1, 2, 3\}$. By Kusuoka's construction, [35, 38, 58], we have

$$\mathcal{E}(f) = \int_K \Gamma(f) dm = \int_K \langle \nabla F(\varphi_1(x), \varphi_2(x)), Z_x \nabla F(\varphi_1(x), \varphi_2(x)) \rangle_{\mathbb{R}^2} \, dm$$

for functions $f = F(\varphi_1, \varphi_2)$, where $\{\varphi_1, \varphi_2\}$ is an energy orthonormal basis (up to constants) of harmonic functions and $F \in C_b^1(\mathbb{R}^2)$. The measure $m = \Gamma(\varphi_1) + \Gamma(\varphi_2)$ is the Kusuoka measure as in (43). ∇F is the usual gradient of F in $\mathbb{R}^2$ and $Z = (Z_x)_{x \in K}$ is a measurable (2×2)-matrix valued function on K such that $rank\ Z = 1$ and $Tr\ Z = 1$ m-a.e. It arises as the m-a.e. limit of a bounded (2×2)-matrix valued m-martingale $(Z_n)_{n \in \mathbb{N}}$, cf. [38, 58]. Accordingly for fixed f as above, $(\langle \nabla F(\varphi_1, \varphi_2), Z_n \nabla F(\varphi_1, \varphi_2) \rangle_{\mathbb{R}^2})_{n \in \mathbb{N}}$ is a bounded m-martingale with m-a.s. limit $\langle \nabla F(\varphi_1, \varphi_2), Z \nabla F(\varphi_1, \varphi_2) \rangle_{\mathbb{R}^2}$. We have

$$\mathcal{E}(f) = \lim_{n \to \infty} \int_K \langle \nabla F(\varphi_1, \varphi_2), Z_n \nabla F(\varphi_1, \varphi_2) \rangle_{\mathbb{R}^2} \, dm$$

and

$$\langle \nabla F(\varphi_1, \varphi_2), Z_n \nabla F(\varphi_1, \varphi_2) \rangle_{\mathbb{R}^2} = \sum_{|w|=n} \mathbf{1}_{F_w K} \frac{r^{-n} \sum_{i=1}^{3} (f(F_w q_i) - f(F_w q_{i+1}))^2}{m(F_w K)}$$

(where $q_4 := q_1$) for $f = F(\varphi_1, \varphi_2)$, $F \in C^1(\mathbb{R}^2)$. Here $r = \frac{3}{5}$. By bounded convergence also

$$\begin{aligned}
\mathcal{E}_p(f) &= \int_K \Gamma(f)^{p/2} dm \\
&= \lim_{n \to \infty} \int_K \langle \nabla F(\varphi_1, \varphi_2), Z_n \nabla F(\varphi_1, \varphi_2) \rangle_{\mathbb{R}^2}^{p/2} \, dm \\
&= \lim_{n \to \infty} \sum_{|w|=n} \left(\frac{r^{-n} \sum_{i=1}^{3} (f(F_w q_i) - f(F_w q_{i+1})^2}{m(F_w K)} \right)^{p/2} m(F_w K),
\end{aligned}$$

and each member of this sequence is comparable to

$$r^{-np/2} \sum_{|w|=n} \frac{\sum_{i=1}^{3} |f(F_w q_i) - f(F_w q_{i+1})|^p}{m(F_w K)^{p/2}} m(F_w K).$$

The p-energy in [21] had been constructed by solving an abstract renormalization problem whose solution allows to define the p-energy as the limit of an increasing sequence of approximative p-energies on the sets V_m. Each member of this sequence is comparable to

$$r_p^{-n} \sum_{|w|=n} \frac{\sum_{i=1}^{3} |f(F_w q_i) - f(F_w q_{i+1})|^p}{m(F_w K)} m(F_w K),$$

where r_p is a scaling factor that is part of the solution of the renormalization problem. The construction is not very explicit and therefore it seems difficult to read off specific properties. However, since it is known that along different infinite words $w = w_1w_2w_3...$ the quantity $m(F_{w|_n}K)$, where $w|_n = w_1...w_n$, has different growth behaviour, cf. [5], one cannot expect the ratios

$$\frac{r^{-np/2}m(F_{w|_n}K)^{1-p/2}}{r_p^{-n}}$$

to have nontrivial and finite limits simultaneously for all infinite words w as n goes to infinity. Accordingly the domains of the two p-energies will most likely be disjoint.

10. Stochastic calculus

In this final section we comment on the natural connection between the approach to 1-forms by Cipriani and Sauvageot [11, 12] and the theory of stochastic integrals for continuous additive functionals as introduced by Nakao [44] and further investigated in [9, 10, 18, 41, 42]. Although this connection is known to experts (see for instance [42, p. 506] or [19, Example 5.6.1]), we would like to bring it to the attention of a broader audience. The setup studied in Sections 2 and 3 can be translated into probability and in particular, the notions of gradient and divergence have probabilistic counterparts. We finally point out that under mild conditions known perturbation results for symmetric Markov processes go well together with our fiberwise perspective on $\mathcal{H}$ and lead to some analogs of classical non-divergence form operators.

We assume that μ is an admissible reference measure and $(\mathcal{E},\mathcal{F})$ is a symmetric regular Dirichlet form on $L_2(X,\mu)$. Let $Y=(Y_t)_{t\geq 0}$ denote the *μ-symmetric Hunt process* on X uniquely associated with $(\mathcal{E},\mathcal{F})$. For background, notation and some important subtle details we refer to [19]. We consider *additive functionals (AF's)* of Y, see for instance [19, Section 5]. Given an AF $A=(A_t)_{t\geq 0}$ of Y, its *energy* is defined by

$$\mathbf{e}(A) := \lim_{t\to 0}\frac{1}{2t}\mathbb{E}_\mu(A_t^2).$$

Let

$$\mathring{\mathcal{M}} := \{M : M \text{ is a finite cadlag AF of } Y \text{ with } \mathbf{e}(M)<\infty \text{ such that} \\ \text{for any } t>0 \text{ we have } \mathbb{E}_x(M_t^2)<\infty \text{ and } \mathbb{E}_x(M_t)=0 \text{ for q.e. } x\in X\},$$

usually referred to as the *space of martingale additive functionals of finite energy.* By polarization the energy $\mathbf{e}$ provides a bilinear form on $\mathring{\mathcal{M}}$ such that $(\mathring{\mathcal{M}},\mathbf{e})$ is Hilbert. Given $M\in\mathring{\mathcal{M}}$ let $\mu_{<M>}$ denote its *energy measure* (the *Revuz measure* of its *sharp bracket* $<M>$) and for $M,N\in\mathring{\mathcal{M}}$ write $\mu_{<M,N>}$ for its bilinear version. For any $M\in\mathring{\mathcal{M}}$ and $f\in L_2(X,\mu_{<M>})$ the *stochastic integral* $f\bullet M\in\mathring{\mathcal{M}}$ is defined by the identity

$$\mathbf{e}(f\bullet M,N)=\frac{1}{2}\int_X f(x)\mu_{<M,N>}(dx),\quad N\in\mathring{\mathcal{M}}, \tag{51}$$

The map $f\mapsto f\bullet M$ from $L_2(X,\mu_{<M>})$ into $\mathring{\mathcal{M}}$ is linear and continuous. See [19, Theorem 5.6.1]. Note that by the nesting property of smooth measures, [19, Section 2.2], any $f\in C_0(X)$ is in $L_2(X,\mu_{<M>})$ for any $M\in\mathring{\mathcal{M}}$. To an energy finite function $u\in\mathcal{F}$ we can associate an additive functional $A^{[u]}$ by

$$A_t^{[u]} := \widetilde{u}(Y_t) - \widetilde{u}(Y_0), \quad t > 0,$$

where as before $\widetilde{u}$ denotes the quasi-continuous Borel version of u. According to Fukushima's theorem, [19, Theorem 5.2.2], the functional $A^{[u]}$ decomposes uniquely as

$$A^{[u]} = M^{[u]} + N^{[u]},$$

where $M^{[u]} \in \mathring{\mathcal{M}}$ and $N^{[u]}$ is a member of the space

$$\mathcal{N}_c := \{N : \ N \text{ is a finite continuous AF with } \mathbf{e}(N) = 0 \text{ and such that } \mathbb{E}_x(|N_t|) < \infty \text{ for q.e. } x \in X \text{ for each } t > 0.\}$$

of *continuous additive functionals of zero energy.* For $u, w \in \mathcal{F}$ we have

$$\mu_{<M^{[u]},M^{[w]}>} = \Gamma(u, w).$$

From (4) and (51) it is easily seen that

$$\Theta(f\partial u) := 2f \bullet M^{[u]} \tag{52}$$

defines a linear isometry from the subspace $\mathcal{C} \otimes \mathcal{C}$ of $\mathcal{H}$ into $\mathring{\mathcal{M}}$ satisfying

$$\mathbf{e}(\Theta(f\partial u)) = \|f\partial u\|_{\mathcal{H}}^2 .$$

(The factor 2 in (52) could easily be avoided by modifying related definitions, but to keep things simple we stick to those used in the literature.) Now recall that by Remark 2.1 the space $\mathcal{C} \otimes \mathcal{C}$ is dense in $\mathcal{H}$. Therefore Θ extends to a linear isometry from $\mathcal{H}$ onto its image. However, [19, Lemma 5.6.3] implies that the family

$$\left\{f \bullet M^{[u]} : f, u \in \mathcal{C}\right\}$$

is dense in $\mathring{\mathcal{M}}$. By a totality argument the range of Θ must therefore be all of $\mathring{\mathcal{M}}$, and we have reproved the following theorem, which is due to Nakao, [44, Theorem 5.1].

Theorem 10.1. *The spaces $\mathcal{H}$ and $\mathring{\mathcal{M}}$ are isometrically isomorphic under the map Θ determined by (52).*

By arguments similar to those in the proof of Lemma 2.1 we obtain the following result on energy measures.

Corollary 10.1. *For the energy measure of $M \in \mathring{\mathcal{M}}$ we have*

$$\mu_{<M>} = \Gamma_{\mathcal{H}}(\omega), \quad \textit{where} \quad \omega = \Theta^{-1}(M).$$

We can also reinterpret the gradient ∂u of an energy finite function $u \in \mathcal{F}$ as the element uniquely corresponding to the martingale part of $A^{[u]}$.

Corollary 10.2. *The image of the space $Im\,\partial$ under Θ is the subspace $\left\{M^{[u]} : u \in \mathcal{F}\right\}$ of $\mathring{\mathcal{M}}$, and for any $u \in \mathcal{F}$ we have $M^{[u]} = \frac{1}{2}\Theta(\partial u)$.*

Also the divergence $\partial^* v$ of a vector field $v \in \mathcal{H}$ has probabilistic meaning. We briefly recall a construction from [44, Section 3]. To simplify notation let us assume that $(\mathcal{E}, \mathcal{F})$ is conservative. Set

$$\lambda(h; M) := \frac{1}{2}\mu_{<M^h,M>}(X) \quad \text{for } h \in \mathcal{F} \text{ and } M \in \mathring{\mathcal{M}}.$$

For fixed $M \in \mathring{\mathcal{M}}$ define a continuous additive functional $\Gamma(M)$ by

$$\Gamma(M)_t = N_t^{[w]} - \int_0^t w(Y_s)ds,$$

where $w \in \mathcal{F}$ is the unique function such that

$$\lambda(h; M) = \mathcal{E}_1(w, h).$$

The functional $\Gamma(M)$ is characterized by the validity of

$$\lim_{t \to 0} \frac{1}{t} \mathbb{E}_{h \cdot \mu}[\Gamma(M)_t] = -\lambda(h; M) \tag{53}$$

for any $h \in \mathcal{C}$. In [44] the functional $\Gamma(M)$ had been used to define Itô and Stratonovich type integrals with respect to continuous additive functionals, see also [9, 10]. The following first observation is immediate.

Lemma 10.1. *Let $M \in \mathring{\mathcal{M}}$ and $\omega = \Theta^{-1}(M)$. Then we have*

$$\partial^* \omega(h) = -\lambda(h; M)$$

for all $h \in \mathcal{C}$.

Recall that to each continuous additive functional $A = (A_t)_{t \geq 0}$ there exists a unique signed smooth Borel measure μ_A such that

$$\lim_{t \to 0} \frac{1}{t} \mathbb{E}_{h \cdot \mu}((f \cdot A)_t) = \int_X \widetilde{h} f d\mu_A \tag{54}$$

for any $f \in \mathcal{B}_b(X)$ and any nonnegative $h \in \mathcal{F}$, as follows from [19, Theorem 5.1.4]. In this case μ_A is called the *signed Revuz measure of A*. Given $M \in \mathring{\mathcal{M}}$, let $\mu_{\Gamma(M)}$ denote the signed Revuz measure of $\Gamma(M)$. From (53) together with the previous lemma we obtain the following.

Corollary 10.3. *Let $M \in \mathring{\mathcal{M}}$ and $\omega = \Theta^{-1}(M)$. Then the element $\partial^* \omega$ of $\mathcal{C}^*$ can be represented by integration with respect to $\mu_{\Gamma(M)}$,*

$$\partial^* \omega(h) = \int_X h d\mu_{\Gamma(M)}, \quad h \in \mathcal{C}.$$

Moreover, if $\omega \in dom\ \partial^*$, then $\mu_{\Gamma(M)}$ is absoulutely continuous with respect to μ, and its density is $\partial^* \omega \in L_2(X, \mu)$.

Proof. For nonnegative $h \in \mathcal{C}$ identity (53) together with (54) yields $\int_X h d\mu_{\Gamma(M)} = -\lambda(h; M)$, and the first statement follows by linearity and the previous lemma. The second statement is an immediate consequence. □

Examples 10.1. For $u \in \mathcal{C}$ and $M = 2M^{[u]} = \Theta(\partial u)$ we have

$$\partial^* \partial u(h) = -\mathcal{E}(u, h) = \lim_{t \to 0} \frac{1}{t} \mathbb{E}_{h \cdot \mu}[N_t^{[u]}],$$

and for $-Lu$, viewed as a member of $\mathcal{C}^*$ as in (25), we obtain the measure representation

$$(-Lu)(h) = \int_X f d\mu_{\Gamma(M)}.$$

If in addition $u \in dom\ L$, then $\partial u \in dom\ \partial^*$ and we have $d\mu_{\Gamma(M)} = -Lu\, d\mu$, seen as an equality of signed measures.

We finally consider a simple version of a related perturbation result. To do so, we assume that m is a measure satisfying Assumption 2.1 and that $(\mathcal{E},\mathcal{C})$ is closable on $L_2(X,m)$ with closure $(\mathcal{E},\mathcal{F}^{(m)})$. Recall that this is particularly true for $m=\mu$ if μ itself satisfies Assumption 2.1. Let

$$\mathcal{H}_\infty := \{v\in\mathcal{H} : \Gamma_{\mathcal{H},\cdot}(v)\in L_\infty(X,m)\}$$

denote the space of *vector fields of bounded length*. Given $b\in\mathcal{H}_\infty$ and $u\in\mathcal{F}^{(m)}$, Cauchy-Schwarz yields

$$\int_X \widetilde{u}^2\Gamma_{\mathcal{H},\cdot}(b)dm \leq \|\Gamma_{\mathcal{H},\cdot}(b)\|_{L_\infty(X,m)}\|u\|_{L_2(X,m)}.$$

(Here and in the following the quasi-continuous versions are taken with respect to $(\mathcal{E},\mathcal{F}^{(m)})$.) In particular, the measure $\Gamma_{\mathcal{H},\cdot}dm$ is of the *Hardy class*, cf. [18, p. 141].

If $b,\hat{b}\in\mathcal{H}_\infty$ and $c\in L_\infty(X,m)$ then the bilinear form

$$(55)\quad \mathcal{Q}(f,g) := \mathcal{E}(f,g) - \int_X \widetilde{g}(x)\left\langle b_x,\partial_x f\right\rangle_{\mathcal{H}_x} m(dx) - \int_X \widetilde{f}(x)\left\langle \hat{b}_x,\partial_x g\right\rangle_{\mathcal{H}_x} m(dx) - \int_X f(x)g(x)c(x)m(dx),$$

$f,g\in\mathcal{F}^{(m)}$, is closed on $L_2(X,m)$, as can be seen by standard perturbation arguments. Moreover, the strongly continuous L_2-semigroup associated with

$$\mathcal{Q}_\alpha(f,g) := \mathcal{Q}(f,g) + \alpha\left\langle f,g\right\rangle_{L_2(X,m)}$$

is *positivity preserving* if $\alpha>0$ is large enough. See for instance [18] (in particular p. 142) and the references therein. In general this semigroup may possibly fail to be Markovian, cf. [41].

If $\hat{b}\in dom\,\partial^*$, then we have

$$\begin{aligned}\partial^*(u\hat{b})(h) &= -\left\langle \hat{b},u\partial h\right\rangle_{\mathcal{H}}\\ &= -\left\langle \hat{b},\partial(uh)\right\rangle_{\mathcal{H}} + \left\langle \hat{b},h\partial u\right\rangle_{\mathcal{H}}\\ &= \int_X h(x)(\partial^*\hat{b})(x)u(x)m(dx) + \int_X h(x)\left\langle \hat{b}_x,\partial_x u\right\rangle_{\mathcal{H}_x} m(dx)\end{aligned}$$

for any $h,u\in\mathcal{C}$. That is, we can reinterpret $\partial^*(u\hat{b})$ as a measure that is absolutely continuous with respect to m and has a density given by

$$\partial^*(u\hat{b})(x) = (\partial^*\hat{b})(x)u(x) + \left\langle b_x\partial_x u\right\rangle_{\mathcal{H}_x}$$

for m-a.a. $x\in X$. For the generator $L^{\mathcal{Q}}$ of $\mathcal{Q}$ and any function $u\in\mathcal{C}\cap dom\,L^{(m)}$ we therefore have

$$L^{\mathcal{Q}}u(x) = L^{(m)}u(x) + \left\langle b_x,\partial_x u\right\rangle_{\mathcal{H}_x} - \partial^*(u\hat{b})(x) + c(x)u(x)$$

for m-a.a. $x\in X$. This is a generalization of formula (1.12) in [18, Example 1.1]. The operator $L^{\mathcal{Q}}$ may be seen as the analog of a non-divergence form operator in our context.

References

[1] M. T. Barlow, *Diffusions on fractals.* Lectures on Probability Theory and Statistics (Saint-Flour, 1995), 1–121, Lecture Notes in Math., **1690**, Springer, Berlin, 1998.

[2] M.T. Barlow, R.F. Bass, *Brownian motion and harmonic analysis on Sierpinski carpets*, Canad. J. Math. 51 (1999), 673-744.

[3] M. T. Barlow, R. F. Bass, T. Kumagai, and A. Teplyaev, *Uniqueness of Brownian motion on Sierpinski carpets.* J. Eur. Math. Soc. **12** (2010), 655-701.

[4] V. Barbu, G. DaPrato, M. Röckner, *Stochastic porous media equation and self-organized criticality*, Comm. Math. Phys. 285 (2009) no.3, 901-923.

[5] O. Ben-Bassat, R.S. Strichartz, A. Teplyaev, *What is not in the domain of Sierpinski gasket type fractals*, J. Funct. Anal. 166 (1999), 197-217.

[6] N. Bouleau, F. Hirsch, *Dirichlet Forms and Analysis on Wiener Space*, deGruyter Studies in Math. 14, deGruyter, Berlin, 1991.

[7] M. Biroli, U. Mosco, *Formes de Dirichlet et estimation structurelles dans les milieux discontinus*, C.R. Acad. Sci. Paris 313 (1991), 593-598.

[8] M. Biroli, U. Mosco, *A Saint-Venant principle for Dirichlet forms on discontinuous media*, Ann. Mat. Pura Appl. 169 (4) (1995), 125-181.

[9] Z.-Q. Chen, P.J. Fitzsimmons, K. Kuwae, T.-S. Zhang, *Stochastic calculus for symmetric Markov processes*, Ann. Probab. **36** (3) (2008), 931-970.

[10] Z.-Q. Chen, P.J. Fitzsimmons, K. Kuwae, T.-S. Zhang, *Perturbations of symmetric Markov processes*, Probab. Th. Relat. Fields **140** (2008), 239-275.

[11] F. Cipriani, J.-L. Sauvageot, *Derivations as square roots of Dirichlet forms*, J. Funct. Anal. 201 (2003), 78-120.

[12] F. Cipriani, J.-L. Sauvageot, *Fredholm modules on p.c.f. self-similar fractals and their conformal geometry*, Comm. Math. Phys. 286 (2009), 541-558.

[13] E.B. Davies, *Heat kernels and spectral theory*, Cambridge Univ. Press, Cambridge, 1989.

[14] J. Dixmier, *Von Neumann Algebras*, North-Holland Math. Lib. 27, North-Holland, Amsterdam, 1981.

[15] A. Eberle, *Uniqueness and non-uniqueness of semigroups generated by singular diffusion operators*, Springer LNM 1718, Springer, New York, 1999.

[16] L.C. Evans, *Partial Differential Equations*, Grad. Stud. Math. vol 19, AMS, Providence, RI, 1998.

[17] K.J. Falconer, J. Hu, *Nonlinear Diffusion Equations on unbounded fractal domains*, J. Math. Anal. Appl. 256 (2001), 606-624.

[18] P.J. Fitzsimmons, K. Kuwae, *Non-symmetric perturbations of symmetric Dirichlet forms*, J. Funct. Anal. **208** (2004), 140-162.

[19] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet forms and symmetric Markov processes*, deGruyter, Berlin, New York, 1994.

[20] B. M. Hambly, V. Metz and A. Teplyaev, *Self-similar energies on post-critically finite self-similar fractals.* J. London Math. Soc. **74** (2006), 93–112.

[21] P.E. Herman, R. Peirone, R.S. Strichartz, *p-energy and p-harmonic functions on Sierpinski gasket type fractals*, Pot. Anal. 20 (2004), 125-148.

[22] M. Hino, *On singularity of energy measures on self-similar sets*, Probab. Theory Relat. Fields 132 (2005), 265-290.

[23] M. Hino, K. Nakahara, *On singularity of energy measures on self-similar sets II*, Bull. London Math. Soc. **38** (2006), 1019-1032.

[24] M. Hino, *Martingale dimensions for fractals.*, Ann. Probab. **36** (2008), 971–991.

[25] M. Hino, *Energy measures and indices of Dirichlet forms, with applications to derivatives on some fractals*, Proc. London Math. Soc. **100** (2010), 269-302.

[26] M. Hinz, 1-*forms and polar decomposition on harmonic spaces*, Potential Anal. DOI 10.1007/s11118-012-9272-2 (2012).

[27] M. Hinz, A. Teplyaev, *Local Dirichlet forms, Hodge theory, and the Navier-Stokes equations on topologically one-dimensional fractals*, preprint (2012) arXiv:1206.6644

[28] M. Hinz, A. Teplyaev, *Dirac and magnetic Schrödinger operators on fractals*, preprint (2012) arXiv:1207.3077

[29] M. Hinz, A. Teplyaev, *Vector analysis on fractals and applications*, preprint (2012) arXiv:1207.6375

[30] M. Ionescu, L. Rogers, A. Teplyaev, *Derivations, Dirichlet forms and spectral analysis.* Journal of Functional Analysis **263** (2012), 2141–2169

[31] D. Kelleher and A. Teplyaev, *Tangent bundles, Banach algebras and sub-Riemanian geometry of fractals*, preprint.

[32] J. Kigami, *Harmonic metric and Dirichlet form on the Sierpiński gasket.* Asymptotic problems in probability theory: stochastic models and diffusions on fractals (Sanda/Kyoto, 1990), 201–218, Pitman Res. Notes Math. Ser., **283**, Longman Sci. Tech., Harlow, 1993.

[33] J. Kigami, *Analysis on Fractals*, Cambridge Univ. Press, Cambridge, 2001.

[34] J. Kigami, *Harmonic analysis for resistance forms*, J. Funct. Anal. 204 (2003), 525–544.

[35] J. Kigami, *Measurable Riemannian geometry on the Sierpinski gasket: the Kusuoka measure and the Gaussian heat kernel estimate*, Math. Ann. 340 (2008), 781-804.

[36] J. Kigami, *Resistance forms, quasisymmetric maps and heat kernel estimates*, Mem. Amer. Math. Soc. **216** (2012).

[37] N.V. Krylov, B.L. Rozovskij, *Stochastic evolution equations*, Journal Soviet Math. 16(4) (1981), 1233-1277.

[38] S. Kusuoka, *Dirichlet forms on fractals and products of random matrices*, Publ. Res. Inst. Math. Sci. 25 (1989), 659-680.

[39] Y. LeJan, *Mesures associées à une forme de Dirichlet. Applications.*, Bull. Soc. Math. France 106 (1978), 61-112.

[40] T. Lindstrøm, *Brownian motion on nested fractals.* Mem. Amer. Math. Soc. **420**, 1989.

[41] J. Lunt, T.J. Lyons, T.-S. Zhang, *Integrability of functionals of Dirichlet processes, probabilistic representations of semigroups, and estimates of heat kernels*, J. Funct. Anal. **153** (1998), 320-342.

[42] T.J. Lyons, T.-S. Zhang, *Decomposition of Dirichlet processes*, Ann. Probab **22** (1) (1994), 494-524.

[43] F.-Y. Maeda, *Dirichlet Integrals on Harmonic Spaces*, Lecture Notes in Math. 803, Springer, New York, 1980.

[44] Sh. Nakao, *Stochastic calculus for continuous additive functionals*, Z. Wahrsch. verw. Geb. **68** (1985), 557-578.

[45] E. Pardoux, *Équations aux dérivées partielles stochastiques de type monotone*, Séminaire sur les Équations aux Dérivées Partielles (1974-1975), III, Exp. no. 2, Collège de France, Paris, 1975, p.10.

[46] R. Peirone, *Existence of eigenforms on nicely separated fractals.*, Analysis on graphs and its applications, 231–241, Proc. Sympos. Pure Math., **77**, Amer. Math. Soc., Providence, RI, 2008. *Existence of Self-similar Energies on Finitely Ramified Fractals*, preprint, 2011.

[47] C. Prévôt, M. Röckner, *A Concise Course on Stochastic Partial Differential Equations*, Springer LNM 1905, Springer, Berlin, 2007.

[48] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, vol. 1*, Acad. Press, San Diego 1980.

[49] J. Ren, M. Röckner, F.-Y. Wang, *Stochastic generalized porous media and fast diffusion equations*, J. Diff. Equations 238 (2007) no.1, 118-152.

[50] M. Röckner and N. Wielens, *Dirichlet forms—closability and change of speed measure*, Infinite-dimensional analysis and stochastic processes (Bielefeld, 1983), Res. Notes in Math., **124**, 119–144, Pitman, Boston, MA, 1985.

[51] B. Steinhurst, *Diffusions and Laplacians on Laakso, Barlow-Evans and other Fractals*, Thesis (Ph.D.) University of Connecticut, 2010.

[52] P. Stollmann, *A dual characterization of length spaces with application to Dirichlet forms*, Studia Math. 198 (3) (2010), 221-233.

[53] R.S. Strichartz, *Differential Equations on Fractals: A Tutorial*, Princeton Univ. Press, Princeton 2006.

[54] R.S. Strichartz, C. Wong, *The p-Laplacian on the Sierpinski gasket*, Nonlinearity 17 (2004), 595-616.

[55] K.-Th. Sturm, *Analysis on local Dirichlet spaces - I. Recurrence, conservativeness and L^p-Liouville properies*, J. reine angew. Math. 456 (1994), 173-196.

[56] K.-Th. Sturm, *On the geometry defind by Dirichlet forms*, In: Progress in Probability, Vol. 36, Birkhäuser, Basel 1995, p. 231-242.

[57] M. Takesaki, *Theory of Operator Algebras I*, Encycl. Math. Sci. 124, Springer, New York, 2002.

[58] A. Teplyaev, *Harmonic coordinates on fractals with finitely ramified cell structure*, Canad. J. Math. 60 (2008), 457–480.

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