Vladimir I. Bogachev[1], Michael Röckner,
and Stanislav V. Shaposhnikov

On uniqueness of solutions to the Cauchy problem for degenerate Fokker–Planck–Kolmogorov equations

Abstract. We prove a new uniqueness result for highly degenerate second order parabolic equations on the whole space. A novelty is our class of solutions in which uniqueness holds, including also a form of the initial condition.

Keywords: degenerate parabolic equation, Cauchy problem, uniqueness.

AMS Subject Classification: 35K10, 35K15, 35K65, 35K67

1. Introduction

Let us consider the second order operator

$$Lu = \varrho \operatorname{div}\bigl(A\nabla u\bigr) + \sqrt{\varrho}(b, \nabla u) + \sqrt{\varrho} cu$$

acting on functions u on $\mathbb{R}^d \times (0,T)$, where $\varrho(x,t) \ge 0$, $A(x,t)$ is a nonnegative $d \times d$-matrix, $b(x,t) \in \mathbb{R}^d$ and $c(x,t) \le 0$; div and ∇ are taken with respect to x. We study uniqueness of solutions to the Cauchy problem

$$\partial_t z = L^* z, \quad z|_{t=0} = 0. \tag{1}$$

We shall say that a function z belonging to the class $L^2(\mathbb{R}^d \times [\kappa, T])$ for every number $\kappa \in (0,T)$ satisfies the Fokker–Planck–Kolmogorov equation $\partial_t z = L^* z$ if

$$\int_0^T \int_{\mathbb{R}^d} [\partial_t u(x,t) + Lu(x,t)] z(x,t)\, dx\, dt = 0 \quad \forall\, u \in C_0^\infty(\mathbb{R}^d \times (0,T)).$$

A function z satisfies the initial condition $z|_{t=0} = 0$ if $(I-\Delta)^{-1} z(x,t) \in L^1(\mathbb{R}^d)$ for almost all $t \in (0,T)$ and

$$\lim_{n\to\infty} \operatorname{ess\,sup}_{t\in(0,1/n)} \|(I-\Delta)^{-1} z(\,\cdot\,,t)\|_{L^1} = 0, \tag{2}$$

$$\lim_{n\to\infty} \operatorname{ess\,sup}_{t\in(0,1/n)} \|(I-\Delta)^{-1/2} z(\,\cdot\,,t)\|_{L^2} = 0. \tag{3}$$

The last equality means convergence to zero in the negative Sobolev space $H^{2,-1}(\mathbb{R}^d)$ dual to the Sobolev space $H^{2,1}(\mathbb{R}^d)$, so that $z(\,\cdot\,,t)$ belongs to this space once it is in L^2; equality (2) corresponds to the norm in the space $H^{1,-2}(\mathbb{R}^d)$ also belonging to the scale of spaces

$$H^{p,r}(\mathbb{R}^d) := (I-\Delta)^{-r/2}(L^p(\mathbb{R}^d)), \quad \|(I-\Delta)^{-r/2} f\|_{p,r} = \|f\|_p,$$

where $\|f\|_p$ is the norm in $L^p(\mathbb{R}^d)$. In terms of the function $g = (I-\Delta)^{-1} z$ condition (3) means that $\lim\limits_{t\to 0} \|g(\,\cdot\,,t)\|_{2,1} = 0$ provided we consider t in some full measure set in $(0,T)$, while (2) is simply $\lim\limits_{t\to 0} \|g(\,\cdot\,,t)\|_1 = 0$ with the same interpretation.

Let $c = 0$. Similarly, one can consider the Cauchy problem

$$\partial_t z = L^* z, \quad z|_{t=0} = \nu \tag{4}$$

[1]Vladimir I. Bogachev: Department of Mechanics and Mathematics, Moscow State University, 119991 Moscow, Russia, vibogach@mail.ru; the author for correspondence
Michael Röckner: Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany, roeckner@math.uni-bielefeld.de
Stanislav V. Shaposhnikov: Department of Mechanics and Mathematics, Moscow State University, 119991 Moscow, Russia, starticle@mail.ru

in the class of functions z such that $z(\,\cdot\,,t)$ is a probability density for a.e. t (such solutions will be called probability solutions) with the initial condition ν that is a probability measure on $\mathbb{R}^d$. In that case the definition of the initial condition in (4) is this: there is a full measure set $S_0 \subset (0,T)$ such that, as $t \to 0$ in S_0, the measures $z(\,\cdot\,,t)dx$ converge to the measure ν weakly, that is,

$$\lim_{t\to 0, t\in S_0} \int \varphi(x) z(x,t)\, dx = \int \varphi(x)\, \nu(dx), \quad \varphi \in C_b(\mathbb{R}^d). \tag{5}$$

We recall that in the case of probability measures it is enough to have this equality just for all $\varphi \in C_0^\infty(\mathbb{R}^d)$ (which is convergence in the sense of distributions), so that it follows from (2) or (3) (the latter, of course, requires also the membership in L^2 for the solution).

Note that (4) holds for transition densities of diffusions processes, but we do not discuss such processes here and do not assume their existence.

In this paper we give a sufficient condition that the Cauchy problems (1) and (4) have unique solutions.

Let us list our assumptions about the coefficients of the operator L:

(H1) ϱ is a nonnegative measurable function on $\mathbb{R}^d \times [0,T]$ (neither boundedness nor strict positivity or smoothness is assumed);

(H2) $A(x,t) = (a^{ij}(x,t))_{i,j\le d}$ is a symmetric positive definite matrix with bounded and Lipschitzian in the variables (x,t) components a^{ij} such that for some number $\lambda > 0$ one has $A(x,t) \ge \lambda I$;

(H3) $b = (b^i)_{i\le d}$ is a vector mapping with components $b^i \in L^\infty(\mathbb{R}^d \times [0,T])$, $c \in L^2(\mathbb{R}^d \times [0,T]) \bigcap L^\infty(\mathbb{R}^d \times [0,T])$ and $c \le 0$.

Our main result is the following theorem.

Theorem 1. *Let Conditions* (H1), (H2), *and* (H3) *be fulfilled. Then the Cauchy problem* (1) (*with initial condition in the sense of* (2), (3)) *has the unique solution* $z = 0$ *in the class of solutions such that, for every* $\kappa \in (0,T)$ *and every ball* $U \subset \mathbb{R}^d$, *one has* $z \in L^2(\mathbb{R}^d \times [\kappa,T])$, $\varrho z \in L^2(U \times [\kappa,T])$ *and*

$$\lim_{N\to\infty} \int_\kappa^T \int_{N\le|x|\le 2N} \frac{\sqrt{\varrho(x,t)}|z(x,t)|}{1+|x|} + \frac{\varrho(x,t)|z(x,t)|}{1+|x|} + \frac{\varrho(x,t)^2|z(x,t)|^2}{1+|x|^2}\, dx\, dt = 0. \tag{6}$$

If (6) *holds for* $\kappa = 0$, *then the same is true if in place of* (2), (3) *we have convergence* $z(\,\cdot\,,t) \to 0$ *as* $t \to 0$, $t \in S_0$ *in the sense of distributions.*

Let $c = 0$. *If* z_1 *and* z_2 *are probability solutions to the Cauchy problem* (4) *such that, letting* $z = z_1 - z_2$, *we have* $z \in L^2(\mathbb{R}^d \times [\kappa,T])$ *for every number* $\kappa \in (0,T)$ *and also* (3) *and* (6) *hold, then* $z_1 = z_2$. *If* (6) *holds with* $\kappa = 0$, *then, for any probability measure* ν *on* $\mathbb{R}^d$, *the Cauchy problem* (4) *with* (5) *in place of* (2), (3) *has at most one probability solution belonging to* $L^2(\mathbb{R}^d \times [\kappa,T])$ *for each* $\kappa \in (0,T)$.

In the case of coefficients of the class C^2 the uniqueness of solutions to the Cauchy problem for degenerate Fokker–Planck–Kolmogorov equations was proved, e.g., in [19]. The case of integrable and Sobolev coefficients was studied in [14], [9], [16], [3], and [2]. In all these papers the existence of densities of solutions was assumed. Without this assumption in the case of a degenerate diffusion matrix there is no uniqueness of solutions; see Girsanov's example [10], where $d = 1$, $b = 0$, $c = 0$,

$A = 1$, $\varrho(x) = |x|^\gamma$, $\gamma \in (0, 1/2)$, Dirac's measure at 0 is a solution and there is another solution (an example with bounded ϱ is given in Remark 3.11 in [3]).

In [3] in the one-dimensional case and in [2] in the multi-dimensional case, uniqueness was studied for the equation $\partial_t z = \Delta(\varrho z)$, where no smoothness of ϱ was assumed, but only its measurability. Thus, we extend the results from [3], [2] to the case of more general operators L. We split the problem in the two parts: degeneracy of ϱ and a connection with the initial data; conditions (2)–(3) and (6) are our principal novelties. We remark that we follow a completely different approach to the proof of uniqueness, which is a modification of the classical Holmgren principle. The case of a nondegenerate diffusion matrix was investigated in many works, among which we note [1], [11], [15], [4], [17], [18], and [7]. A survey of recent progress in the study of equation (4) for measures (its definition is similar) is given in [6]. It is worth noting that a probability measure satisfying (4) is given by a density if $\varrho > 0$ (see [5] and [6]), otherwise singular solutions may exist.

2. Proof and additional remarks

The proof of Theorem 1 is based on a number of auxiliary results.

Lemma 1. *Suppose that z is a solution to the equation $\partial_t z = L^* z$ and $\alpha \in (0, T)$. Then, for every function $u \in C_0^\infty(\mathbb{R}^d \times (-1, \alpha])$, there exists a set J_u of full measure in $[0, \alpha]$ such that for every pair $s, t \in J_u$ with $s < t$*

$$\int_{\mathbb{R}^d} u(x,t) z(x,t)\, dx = \int_{\mathbb{R}^d} u(x,s) z(x,s)\, dx + \int_s^t \int_{\mathbb{R}^d} [\partial_t u(x,t) + Lu(x,t)] z(x,t)\, dx\, d\tau.$$

Proof. Let $\eta \in C_0^\infty((0, \alpha))$. By definition

$$\int_0^\alpha \int_{\mathbb{R}^d} [\partial_t(u\eta) + L(u\eta)] z\, dx\, dt = 0.$$

Hence we obtain

$$-\int_0^\alpha \eta'(t) \int_{\mathbb{R}^d} u(x,t) z(x,t)\, dx\, dt = \int_0^\alpha \eta(t) \int_{\mathbb{R}^d} [\partial_t u + Lu] z\, dx\, dt.$$

Therefore, the function

$$t \mapsto \int_{\mathbb{R}^d} u(x,t) z(x,t)\, dx$$

has an absolutely continuous version and

$$\frac{d}{dt} \int_{\mathbb{R}^d} u(x,t) z(x,t)\, dx = \int_{\mathbb{R}^d} [\partial_t u + Lu] z\, dx.$$

Then for some number $C \in \mathbb{R}$

$$\int_{\mathbb{R}^d} u(x,t) z(x,t)\, dx = C + \int_0^t \int_{\mathbb{R}^d} [\partial_s u + Lu] z\, dx\, ds$$

for almost all $t \in [0, \alpha]$. Let J_u denote the set of all those t for which the last equality is fulfilled. Then, subtracting from this equality for t the equality for s, for all $t, s \in J_u$ we obtain the desired identity. □

Remark 1. (i) If $u(x,t) = \psi(x)$ with $\psi \in C_0^\infty(\mathbb{R}^d)$, then there exists a full measure set J_ψ in $[0, \alpha]$ such that for all $t, s \in J_\psi$

$$\int_{\mathbb{R}^d} \psi(x) z(x,t)\, dx = \int_{\mathbb{R}^d} \psi(x) z(x,s)\, dx + \int_s^t \int_{\mathbb{R}^d} L\psi(x) z(x,\tau)\, dx\, d\tau.$$

(ii) Let $u \in C_0^\infty(\mathbb{R}^d \times (-1,\alpha])$ and $\alpha \in J_{u(\,\cdot\,,\alpha)}$. Then for all $s \in J_{u(\,\cdot\,,\alpha)}$

$$\int_{\mathbb{R}^d} u(x,\alpha)z(x,\alpha)\,dx = \int_{\mathbb{R}^d} u(x,s)z(x,s)\,dx + \int_s^\alpha \int_{\mathbb{R}^d} [\partial_t u + Lu]z\,dx\,d\tau.$$

We need an estimate on solutions to the backward Cauchy problem for the adjoint operator.

Lemma 2. *Let $q \in C_0^\infty(\mathbb{R}^d \times (-1,\alpha])$, $q \ge 0$, r^{ij}, h^i, $g \in C^\infty(\mathbb{R}^d \times (-1,\alpha])$, $g \le 0$, and $g \in L^2(\mathbb{R}^d \times (-1,\alpha])$. Suppose also that $R = (r^{ij})$ is a symmetric matrix such that for some number $\lambda > 0$ one has $R \ge \lambda I$. Then the Cauchy problem*

$$\partial_t f + q\,\mathrm{div}(R\nabla f) + \sqrt{q}(h,\nabla f) + \sqrt{q} g f = 0, \quad f|_{t=\alpha} = \psi, \tag{7}$$

where $\psi \in C_0^\infty(\mathbb{R}^d)$, has a solution of the class $C^\infty(\mathbb{R}^d \times (-1,\alpha])$ such that

$$\int_{\mathbb{R}^d} |\nabla f(x,t)|^2\,dx + \lambda^{-1}\int_t^\alpha \int_{\mathbb{R}^d} q\bigl|\mathrm{div}(R\nabla f)\bigr|^2\,dx\,ds \le$$
$$\le \lambda^{-1}M \int_{\mathbb{R}^d} |\nabla\psi(x)|^2\,dx + \lambda^{-1}M \max_x |\psi(x)|^2 \int_{-1}^\alpha \int_{\mathbb{R}^d} |g(x,t)|^2\,dx\,dt, \tag{8}$$

where $M = 2\sup_{x,t}|R(x,t)|\exp\Bigl(2\alpha\lambda^{-1}\sup_{x,t}|\partial_t R(x,t)| + 2\alpha\lambda^{-1}\sup_{x,t}|h(x,t)|^2\Bigr)$ and $\partial_t R$ denotes the matrix $(\partial_t r^{ij})_{i,j\le d}$.

Proof. 1. Let us prove the existence of a solution. According to [13, Chapter IV, § 5, Theorem 5.1] for every natural number $n \ge 1$ there exists a bounded solution $f_n \in C^\infty(\mathbb{R}^d \times (-1,\alpha])$ of the backward Cauchy problem

$$\partial_t f_n + (q + n^{-1})\,\mathrm{div}(R\nabla f_n) + \sqrt{q}(h,\nabla f_n) + \sqrt{q} g f_n = 0, \quad f_n|_{t=\alpha} = \psi.$$

Recall that $g \le 0$. By the maximum principle (see [13, Chapter I, § 2, Theorem 2.5]) we have $\max_{x,t}|f_n(x,t)| \le \max_x |\psi(x)|$. Applying [19, Section 3.2, Theorem 3.2.4] we obtain the following estimate for every fixed pair of natural numbers m and l:

$$\max_{x,t} |\partial_t^m \partial_x^l f_n(x,t)| \le C_{m,l},$$

where $C_{m,l}$ does not depend on n. Hence there exists a subsequence $\{f_{n_k}\}$ which converges together with all derivatives to a function f. It is clear that f belongs to $C^\infty(\mathbb{R}^d \times (-1,\alpha])$ and is a solution of the Cauchy problem (7).

2. Let us prove our main estimate (8). Let U be a ball in $\mathbb{R}^d$ such that $q = 0$ and $\psi = 0$ outside of U. Then, whenever $x \notin U$, we have $\partial_t f(x,t) = 0$ and $f(x,\alpha) = \psi(x) = 0$. Therefore, $f(x,t) = 0$ for all $t \in (-1,\alpha]$ whenever x does not belong to U. Multiplying the equation by $\mathrm{div}(R\nabla f)$ and integrating we obtain

$$\int_{\mathbb{R}^d} \partial_t f\,\mathrm{div}(R\nabla f)\,dx + \int_{\mathbb{R}^d} q\bigl|\mathrm{div}(R\nabla f)\bigr|^2\,dx =$$
$$= -\int_{\mathbb{R}^d} \sqrt{q}(h,\nabla f)\mathrm{div}(R\nabla f)\,dx - \int_{\mathbb{R}^d} \sqrt{q} g f \mathrm{div}(R\nabla f)\,dx. \tag{9}$$

To simply long formulas we omit indication of variables of functions like $f(x,t)$, $R(x,t)$, etc. Integrating by parts (f has compact support) we have

$$2\int_{\mathbb{R}^d} \partial_t f\,\mathrm{div}(R\nabla f)\,dx = -\frac{d}{dt}\int_{\mathbb{R}^d} |\sqrt{R}\nabla f|^2\,dx + \int_{\mathbb{R}^d} (\partial_t R\nabla f, \nabla f)\,dx.$$

Applying the inequality $2xy \le 2^{-1}x^2 + 2y^2$, we find that

$$2\left|\int_{\mathbb{R}^d} \sqrt{q}(h, \nabla f)\mathrm{div}(R\nabla f)\,dx\right| \le$$
$$\le 2^{-1}\int_{\mathbb{R}^d} q\big|\mathrm{div}(R\nabla f)\big|^2\,dx + 2\sup_{x,t}|R(x,t)^{-1/2}h(x,t)|^2\int_{\mathbb{R}^d}|\sqrt{R}\nabla f|^2\,dx.$$

By the maximum principle $\max_{x,t}|f(x,t)| \le \max_x|\psi(x)|$. Hence

$$2\left|\int_{\mathbb{R}^d}\sqrt{q}gf\mathrm{div}(R\nabla f)\,dx\right| \le 2^{-1}\int_{\mathbb{R}^d} q\big|\mathrm{div}(R\nabla f)\big|^2\,dx + 2\max_x|\psi(x)|^2\int_{\mathbb{R}^d}|g(x,t)|^2\,dx.$$

Using these relations in (9) and integrating over $[t,\alpha]$, we arrive at the inequality

$$\int_{\mathbb{R}^d}|\sqrt{R}(x,t)\nabla f(x,t)|^2\,dx + \int_t^\alpha\int_{\mathbb{R}^d} q\big|\mathrm{div}(R\nabla f)\big|^2\,dx\,ds \le$$
$$\le \int_{\mathbb{R}^d}|\sqrt{R}(x,\alpha)\nabla\psi(x)|^2\,dx + 2\max_x|\psi(x)|^2\int_t^\alpha\int_{\mathbb{R}^d}|g|^2\,dx\,ds +$$
$$+ M_1\int_t^\alpha\int_{\mathbb{R}^d}|\sqrt{R}\nabla f|^2\,dx\,ds,$$

where $M_1 = 2\lambda^{-1}\sup_{x,t}|\partial_t R(x,t)| + 2\lambda^{-1}\sup_{x,t}|h(x,t)|^2$. In particular,

$$\int_{\mathbb{R}^d}|\sqrt{R}(x,t)\nabla f(x,t)|^2\,dx \le \int_{\mathbb{R}^d}|\sqrt{R}(x,\alpha)\nabla\psi(x)|^2\,dx +$$
$$+ 2\max_x|\psi(x)|^2\int_t^\alpha\int_{\mathbb{R}^d}|g|^2\,dx\,ds + M_1\int_t^\alpha\int_{\mathbb{R}^d}|\sqrt{R}\nabla f|^2\,dx\,ds.$$

Applying the Gronwall inequality and the estimate $R \ge \lambda I$, we obtain the required estimate. □

Remark 2. We observe that subtracting the function ψ from f we may assume that f solves the Cauchy problem for a non-homogeneous equation, but with zero initial condition. Applying inner estimates for strong solutions to parabolic equations (see [12, Section 4 in Chapter 2]), we can assert the following. Let $q > 0$ on some ball U. Then for every ball U' with $\overline{U'} \subset U$ one has

$$\int_0^\alpha\int_{U'}\Big[|\partial_{x_i} f(x,t)|^2 + |\partial_{x_i}\partial_{x_j} f(x,t)|^2\Big]\,dx\,dt \le C_1,$$

where C_1 depends on U, U', q, ψ, and $\sup_{x,t}(|R| + |\nabla R| + |\partial_t R| + |h| + |g|)$.

Now let us prove Theorem 1. Informally speaking, the idea is to multiply the solution by the solution to the backward Cauchy problem with the adjoint operator and integrate, but due to rather weak assumptions we end up with inequalities.

Proof. Let $\psi \in C_0^\infty(\mathbb{R}^d)$, $|\psi(x)| \le 1$ and

$$\int_{\mathbb{R}^d}|\nabla\psi(x)|^2\,dx \le 1.$$

Let also $\varphi_N(x) = \varphi(x/N)$, where $\varphi \in C_0^\infty(\mathbb{R}^d)$ is such that $\varphi \ge 0$, $\varphi(x) = 1$ if $|x| < 1$ and $\varphi(x) = 0$ if $|x| > 2$. Fix a natural number N. Let $U_{2N} = \{x\colon\ |x| < 2N\}$. Let a sequence of functions $q_n \in C_0^\infty(\mathbb{R}^d \times (-1,\alpha])$ be such that $q_n \ge 0$, $q_n(x) > 0$ if $|x| < 2(N+1)$ and

$$\lim_{n\to\infty} \int_\kappa^T \int_{U_{2N}} \frac{|\varrho - q_n|^2}{q_n} z^2\,dx\,dt = 0 \quad \forall\,\kappa \in (0,T).$$

Note that existence of such a sequence follows from the condition that ϱz belongs to $L^2(U_{2N} \times [\kappa, T])$ for every $\kappa \in (0,T)$. Indeed, for every $\kappa = 1/n$ there is a smooth nonnegative function $\widetilde{q}_n$ with compact support such that

$$\int_{1/n}^T \int_{U_{2N}} |\varrho - \widetilde{q}_n|^2 z^2\,dx\,dt < 1/n^2.$$

Set $q_n = \widetilde{q}_n + n^{-1}\omega$, where $\omega \in C_0^\infty(\mathbb{R}^d)$, $\omega \ge 0$, $\omega(x) = 1$ if $|x| < 2(N+1)$ and $\omega(x) = 0$ if $|x| > 2(N+2)$. It is clear that $\{q_n\}$ is the required sequence.

We recall that $b \in L^\infty(\mathbb{R}^d \times [0,T])$. Therefore, there exists a sequence of functions $h_m \in C^\infty(\mathbb{R}^d \times (-1,T))$ such that $|h_m(x,t)| \le \|b\|_{L^\infty}$ and $h_m(x,t) \to b(x,t)$ for almost all (x,t) in $\mathbb{R}^d \times [0,T]$.

We also recall that $c \in L^2(\mathbb{R}^d \times [0,T]) \bigcap L^\infty(\mathbb{R}^d \times [0,T])$ and $c \le 0$. Therefore, there exists a sequence of functions $g_m \in C^\infty(\mathbb{R}^d \times (-1,T))$ such that $g_m \le 0$, $|g_m(x,t)| \le \|c\|_{L^\infty}$, $\|g_m\|_{L^2(\mathbb{R}^d\times[-1,T])} \le 2\|c\|_{L^2(\mathbb{R}^d\times[0,T])}$ and $g_m(x,t) \to c(x,t)$ for almost all (x,t) in $\mathbb{R}^d \times [0,T]$.

By assumption the matrix A has Lipschitzian (in (x,t)) entries a^{ij}. Let Λ be a common Lipschitz constant for the functions a^{ij}. There is a sequence of functions $R_m^{ij} \in C^\infty(\mathbb{R}^d \times (-1,T))$ such that the operators $R_m := (R_m^{ij})_{i,j\le d}$ are strictly positive, the functions R_m^{ij} converge to a^{ij} uniformly on $U \times [0,T]$ for every ball U, there holds the estimate $|\partial_t R_m^{ij}| + |\nabla R_m^{ij}| \le 2\Lambda$ and $\nabla R_m^{ij}(x,t) \to \nabla a^{ij}(x,t)$ for almost all $(x,t) \in \mathbb{R}^d \times [0,T]$.

Let $\alpha \in \bigcap_N J_{\psi\varphi_N}$ and $0 < \alpha < T$. We observe that $\bigcap_N J_{\psi\varphi_N}$ is a full measure set in the interval $(0,T)$. Let $f_{n,m}$ be the solution to the following backward Cauchy problem in $\mathbb{R}^d \times (-1,\alpha]$:

$$\partial_t f_{n,m} + q_n \mathrm{div}\big(R_m \nabla f_{n,m}\big) + \sqrt{q_n}\big(h_m, \nabla f_{n,m}\big) + \sqrt{q_n} g_m f_{n,m} = 0, \quad f_{n,m}(x,\alpha) = \psi(x).$$

Set

$$\widehat{L}u = q_n \mathrm{div}\big(R_m \nabla u\big) + \sqrt{q_n}\big(h_m, \nabla u\big) + \sqrt{q_n} g_m u,\ \ L_0 u = \varrho\,\mathrm{div}\big(A\nabla u\big) + \sqrt{\varrho}(b, \nabla u).$$

Applying Lemma 1 to the function $u = f_{n,m}\varphi_N$, we obtain the equality

$$\int_{\mathbb{R}^d} \psi(x)\varphi_N(x) z(x,\alpha)\,dx = \int_{\mathbb{R}^d} f_{n,m}(x,s)\varphi_N(x) z(x,s)\,dx +$$
$$+ \int_s^\alpha \int_{\mathbb{R}^d} [(Lf_{n,m} - \widehat{L}f_{n,m})\varphi_N + f_{n,m} L_0\varphi_N + 2\varrho(A\nabla f_{n,m}, \nabla\varphi_N)] z\,dx\,dt \quad (10)$$

for almost all $s \in (0,\alpha)$. We observe that by the maximum principle

$$|f_{n,m}(x,t)| \le \sup_x |\psi(x)| \le 1.$$

Hence

$$\int_s^\alpha \int_{\mathbb{R}^d} f_{n,m} L_0\varphi_N z\,dx\,dt \le \int_s^T \int_{\mathbb{R}^d} |L_0\varphi_N||z|\,dx\,dt.$$

By Lemma 2 there exists a number C, which does not depend on n, m, N, and t, such that

$$\int_{\mathbb{R}^d} |\nabla f_{n,m}(x,t)|^2\,dx + \int_0^\alpha \int_{\mathbb{R}^d} q_n \bigl|\mathrm{div}(R_m\nabla f_{n,m})\bigr|^2\,dx\,dt \le C. \tag{11}$$

Then, taking only the first term in this estimate and using the Cauchy inequality, we have

$$\int_s^\alpha \int_{\mathbb{R}^d} \varrho(A\nabla f_{n,m}, \nabla\varphi_N) z\,dx\,dt \le \sqrt{CT}\biggl(\int_s^T \int_{\mathbb{R}^d} \varrho^2 z^2 |A\nabla\varphi_N|^2\,dx\,dt\biggr)^{1/2}.$$

Let us prove that for all fixed s and N one has

$$\lim_{n\to\infty}\biggl(\lim_{m\to\infty}\int_s^\alpha \int_{\mathbb{R}^d} (Lf_{n,m} - \widehat{L}f_{n,m})\varphi_N z\,dx\,dt\biggr) = 0.$$

To this end we observe that

$$\begin{aligned} Lf_{n,m} - \widehat{L}f_{n,m} &= (\varrho - q_n)\mathrm{div}(R_m\nabla f_{n,m}) + \varrho\,\mathrm{div}\bigl((A - R_m)\nabla f_{n,m}\bigr) + \\ &\quad + (\sqrt{\varrho} - \sqrt{q_n})(h_m, \nabla f_{n,m}) + \sqrt{\varrho}(b - h_m, \nabla f_{n,m}) + \\ &\quad + (\sqrt{\varrho} - \sqrt{q_n})g_m f_{n,m} + \sqrt{\varrho}(c - g_m)f_{n,m}. \end{aligned} \tag{12}$$

Let us consider the summands separately. We recall that $0 \le \varphi_N \le 1$. Applying estimate (11), where we take only the second term, we obtain

$$\begin{aligned} &\int_s^\alpha \int_{\mathbb{R}^d} (\varrho - q_n) z\varphi_N \mathrm{div}(R_m\nabla f_{n,m})\,dx\,dt \le \\ &\le \biggl(\int_s^T \int_{U_{2N}} \frac{|\varrho - q_n|^2}{q_n} z^2\,dx\,dt\biggr)^{1/2} \biggl(\int_s^\alpha \int_{\mathbb{R}^d} q_n |\mathrm{div}(R_m\nabla f_{n,m})|^2\,dx\,dt\biggr)^{1/2} \le \\ &\qquad \le C^{1/2}\biggl(\int_s^T \int_{U_{2N}} \frac{|\varrho - q_n|^2}{q_n} z^2\,dx\,dt\biggr)^{1/2}. \end{aligned}$$

Noting that $|\sqrt{\varrho} - \sqrt{q_n}|^2 \le |\varrho - q_n|^2 q_n^{-1}$ and $|h_m| \le \|b\|_{L^\infty}$, we obtain the estimate

$$\begin{aligned} &\int_s^\alpha \int_{\mathbb{R}^d} (\sqrt{\varrho} - \sqrt{q_n})(h_m, \nabla f_{n,m}) z\varphi_N\,dx\,dt \le \\ &\qquad \le C^{1/2}\|b\|_{L^\infty}\biggl(\int_s^T \int_{U_{2N}} \frac{|\varrho - q_n|^2}{q_n} z^2\,dx\,dt\biggr)^{1/2}. \end{aligned}$$

We now estimate the term in (12) with $\sqrt{\varrho}(b - h_m, \nabla f_{n,m})$. We have

$$\int_s^T \int_{\mathbb{R}^d} \sqrt{\varrho}(b - h_m, \nabla f_{n,m}) z\varphi_N\,dx\,dt \le C^{1/2}\biggl(\int_s^T \int_{U_{2N}} \varrho z^2 |b - h_m|^2\,dx\,dt\biggr)^{1/2}.$$

In the same way we estimate the terms with $(\sqrt{\varrho} - \sqrt{q_n})g_m f_{n,m}$ and $\sqrt{\varrho}(c - g_m)f_{n,m}$. We obtain

$$\int_s^\alpha \int_{\mathbb{R}^d} (\sqrt{\varrho} - \sqrt{q_n})g_m f_{n,m}\varphi_N z\,dx\,dt \le 2\|c\|_{L^2}\biggl(\int_s^T \int_{U_{2N}} \frac{|\varrho - q_n|^2}{q_n} z^2\,dx\,dt\biggr)^{1/2}$$

and

$$\int_s^\alpha \int_{\mathbb{R}^d} \sqrt{\varrho}(c - g_m)f_{n,m}\varphi_N z\,dx\,dt \le \|c - g_m\|_{L^2(U_{2N}\times[0,T])}\biggl(\int_s^T \int_{U_{2N}} \varrho z^2\,dx\,dt\biggr)^{1/2}.$$

According to Remark 2,

$$\int_0^\alpha \int_{U_{2N}} |\partial_{x_i}\partial_{x_j} f_{n,m}(x,t)|^2\,dx\,dt \le C_1(n),$$

where $C_1(n)$ does not depend on m. Then

$$\begin{aligned}\int_s^\alpha \int_{\mathbb{R}^d} \varrho z \varphi_N \operatorname{div}\bigl((A-R_m)\nabla f_{n,m}\bigr)\,dx\,dt \le \\ \le C_1(n)^{1/2}\Bigl(\int_s^T \int_{U_{2N}} \varrho^2 z^2 |A-R_m|^2\,dx\,dt\Bigr)^{1/2} + \\ + C^{1/2}\Bigl(\int_s^T \int_{U_{2N}} \varrho^2 z^2 |\nabla(A-R_m)|^2\,dx\,dt\Bigr)^{1/2}.\end{aligned}$$

Therefore, we arrive at the estimate

$$\begin{aligned}\Bigl|\int_s^\alpha \int_{\mathbb{R}^d} (Lf_{n,m} - \widehat{L}f_{n,m})\varphi_N z\,dx\,dt\Bigr| \le \\ \le C^{1/2}(1+\|b\|_{L^\infty} + 2\|c\|_{L^2}) \int_s^T \int_{U_{2N}} \frac{|\varrho - q_n|^2}{q_n} z^2\,dx\,dt + \\ + C^{1/2}\Bigl(\int_s^T \int_{U_{2N}} \varrho z^2 |b-h_m|^2\,dx\,dt\Bigr)^{1/2} + \\ + \|c-g_m\|_{L^2(U_{2N}\times[0,T])}\Bigl(\int_s^T \int_{U_{2N}} \varrho z^2\,dx\,dt\Bigr)^{1/2} + \\ + C_1(n)^{1/2}\Bigl(\int_s^T \int_{U_{2N}} \varrho^2 z^2 |A-R_m|^2\,dx\,dt\Bigr)^{1/2} + \\ + C^{1/2}\Bigl(\int_s^T \int_{U_{2N}} \varrho^2 z^2 |\nabla(A-R_m)|^2\,dx\,dt\Bigr)^{1/2}.\end{aligned}$$

It remains to observe that letting first $m\to\infty$ and then $n\to\infty$, we see that the right-hand side of the latter inequality vanishes in the limit taken in the indicated order. Finally, let us consider in (10) the term

$$\int_{\mathbb{R}^d} f_{n,m}(x,s)\varphi_N(x) z(x,s)\,dx.$$

Let $g(x,s) = (I-\Delta)^{-1} z(x,s)$. Then

$$g(\,\cdot\,,s) - \Delta g(\,\cdot\,,s) = z(\,\cdot\,,s), \quad g(\,\cdot\,,s) \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d).$$

By the definition of the initial condition there is a full measure set $S_0 \subset (0,T)$ such that

$$\lim_{s\to 0, s\in S_0} \|g(\,\cdot\,,s)\|_1 = 0, \qquad \lim_{s\to 0, s\in S_0} \|g(\,\cdot\,,s)\|_{2,1} = 0.$$

Substituting $g-\Delta g$ in place of z and integrating by parts (which is possible, since $f_{n,m}$ and g are Sobolev in x and $\varphi_N\in C_0^\infty$), we arrive at the equality

$$\int_{\mathbb{R}^d} f_{n,m}(x,s)\varphi_N(x)z(x,s)\,dx =$$
$$= \int_{\mathbb{R}^d} f_{n,m}(x,s)\varphi_N(x)g(x,s)\,dx + \int_{\mathbb{R}^d} (\nabla f_{n,m}(x,s), \nabla g(x,s))\varphi_N(x)\,dx -$$
$$- \int_{\mathbb{R}^d} (\nabla f_{n,m}(x,s), \nabla \varphi_N(x))g(x,s)\,dx - \int_{\mathbb{R}^d} f_{n,m}(x,s)\Delta\varphi_N(x)g(x,s)\,dx.$$

Since the sequences $\{\varphi_N\}, \{\nabla\varphi_N\}, \{\Delta\varphi_N\}$, and $\{f_{n,m}\}$ are uniformly bounded and $\|\nabla f_{n,m}(\,\cdot\,,t)\|^2_{L^2(\mathbb{R}^d)}\le C$, there exists a number $C_2>0$, which does not depend on N, n, and m, such that

$$\int_{\mathbb{R}^d} f_{n,m}(x,s)\varphi_N(x)z(x,s)\,dx \le C_2\bigl(\|g(\,\cdot\,,s)\|_1 + \|g(\,\cdot\,,s)\|_{2,1}\bigr).$$

Summing the obtained inequalities and letting first $m\to\infty$ and then $n\to\infty$, on the basis of (10) we arrive at the estimate

$$\int_{\mathbb{R}^d} \psi(x)\varphi_N(x)z(x,\alpha)\,dx \le C_2\bigl(\|g(\,\cdot\,,s)\|_1 + \|g(\,\cdot\,,s)\|_{2,1}\bigr)+$$
$$+\int_s^T\int_{\mathbb{R}^d} |L_0\varphi_N||z|\,dx\,dt + 2\sqrt{CT}\biggl(\int_s^T\int_{\mathbb{R}^d} \varrho^2 z^2|A\nabla\varphi_N|^2\,dx\,dt\biggr)^{1/2}.$$

Note that for some number $C_3>0$

$$\int_s^T\int_{\mathbb{R}^d} |L_0\varphi_N||z|\,dx\,dt \le C_3\int_s^T\int_{N\le|x|\le 2N} \frac{\sqrt{\varrho(x,t)}|z(x,t)|}{1+|x|} + \frac{\varrho(x,t)|z(x,t)|}{1+|x|}\,dx\,dt,$$

since $|\nabla\varphi_N|\le N^{-1}\max_x|\nabla\varphi(x)|$, $|\partial_{x_i}\partial_{x_j}\varphi_N|\le N^{-2}\max_x|\partial_{x_i}\partial_{x_j}\varphi(x)|$. Letting now $N\to\infty$ and then $s\to 0$, $s\in S_0$, we obtain

$$\int_{\mathbb{R}^d} \psi(x)z(x,\alpha)\,dx \le 0$$

for almost all $\alpha\in(0,T)$. Since ψ was arbitrary, we obtain that $z=0$. Hence the first assertion of the theorem is proven.

Let $c=0$. Let z_1 and z_2 be two probability solutions ($z_i\ge 0$ and $\|z_i(\,\cdot\,,t)\|_1=1$, $i=1,2$) to the Cauchy problem $\partial_t z = L^*z$, $z|_{t=0}=\nu$, where the initial condition is understood in the sense of (5). Then the difference $z=z_1-z_2$ satisfies the Cauchy problem with zero initial condition. We recall that $z\in L^2(\mathbb{R}^d\times[\kappa,T])$ for every $\kappa\in(0,T)$. Let us fix a full measure set $S_0\subset(0,T)$ such that for all $t\in S_0$ one has $z(\,\cdot\,,t)\in L^2(\mathbb{R}^d)$ and $\lim\limits_{t\to 0, t\in S_0}\|(I-\Delta)^{-1/2}z(\,\cdot\,,t)\|_2=0$. Let us show that (2) holds automatically:

$$\lim_{t\to 0, t\in S_0}\|(I-\Delta)^{-1}z(\,\cdot\,,t)\|_1=0.$$

Indeed, let $g_1(x,t)=(I-\Delta)^{-1}z_1(x,t)$. Then $g_1(\,\cdot\,,t)$ belongs to $H^{2,2}(\mathbb{R}^d)$ and satisfies the equation $g_1-\Delta g_1=z_1$ for each fixed $t\in S_0$. By the maximum principle $g_1\ge 0$. Let us consider the Fourier transform:

$$\widehat{g_1}(y,t)=(1+|y|^2)^{-1}\widehat{z_1}(y,t).$$

We observe that $\widehat{g_1}(0,t)=1$, hence each $g_1(x,t)\,dx$, $t\in S_0$, is a probability measure. Similarly, $g_2(\,\cdot\,,t)\,dx$ is a probability measure, $g_2(x,t)=(I-\Delta)^{-1}z_2(x,t)$ if $t\in S_0$.

In addition, for every y the function $\widehat{z_1}(y,t)$ tends to $\widehat{\nu}(y)$ as $t \to 0$, $t \in S_0$, which implies convergence of $\widehat{g_1}(y,t)$ to $\widehat{g_1}(y,0)$. Hence the measures $g_1(x,t)\,dx$ converge weakly to $g_1(x,0)\,dx$ as $t \to 0$, $t \in S_0$. Since $L^1(\mathbb{R}^d) \subset H^{q,-1}(\mathbb{R}^d)$ with $q = p/(p-1)$, $p > d$ and $(I-\Delta)^{-1}(H^{q,-1}(\mathbb{R}^d)) = H^{q,1}(\mathbb{R}^d)$, we have $\|g_1(\,\cdot\,,t)\|_{q,1} \le C$ for some number C that does not depend on $t \in S_0$. Therefore, one has convergence of $g_1(x,t)$ to $g(x)$ in $L^1(U)$ on every ball $U \subset \mathbb{R}^d$ as $t \to 0$, $t \in S_0$. This implies convergence of $g_1(x,t)$ to $g_1(x,0)$ in $L^1(\mathbb{R}^d)$, since $g_1(x,t)$ are probability densities in x. The same reasoning applies to g_2. Therefore, the difference $g_1(\,\cdot\,,t) - g_2(\,\cdot\,,t)$ converges to zero in $L^1(\mathbb{R}^d)$ as $t \to 0$, $t \in S_0$. Thus, in the considered situation of probability solutions it is enough to keep only condition (3).

Finally, we observe that if in (6) we allow $\kappa = 0$, then (2), (3) can be replaced by convergence $z(\,\cdot\,,t) \to 0$ as $t \to 0$, $t \in S_0$ in the sense of distributions (in the case of probability solutions by (5)). To this end, it suffices to apply the same reasoning as above, but with $s = 0$, when the first integral on the right in (10) vanishes at $s = 0$, so that (2) and (3) are not used. □

It is worth noting that in the first assertion in Theorem 1 a solution need not be integrable on the whole space. Certainly, this assertion can be applied to obtain uniqueness in the class of signed bounded measures possessing square integrable densities satisfying (6). In particular, it gives uniqueness in the class of bounded integrable functions satisfying (6) (and the latter holds automatically in this class if ϱ is uniformly bounded).

Remark 3. Let $V \in C^2(\mathbb{R}^d)$, $V \ge 0$ and $\lim\limits_{|x|\to\infty} V(x) = +\infty$. Repeating the proof of the theorem with the function $\varphi_N(x) = \varphi(V(x)/N)$, we see that Theorem 1 remains valid if condition (6) is replaced by the following one:

$$\lim_{N\to\infty} N^{-1} \int_\kappa^T \int_{N \le V(x) \le 2N} \Big(\big(|L_0 V| + \varrho|\sqrt{A}\nabla V|^2\big)|z| + \varrho^2 |z|^2 |A\nabla V|^2 \Big)\, dx\, dt = 0$$

for every $\kappa \in (0,T)$, where $L_0 V = \varrho \operatorname{div}\big(A\nabla V\big) + \sqrt{\varrho}(b, \nabla V)$.

Note also that if $\varrho(x,t) \le C(1+|x|)$ and $z \in L^1(\mathbb{R}^d \times [\kappa,T]) \cap L^2(\mathbb{R}^d \times [\kappa,T])$, then (6) holds automatically. If ϱ is bounded and $d \le 2$, then (6) holds automatically too.

Remark 4. We observe that for the functions $f_{n,m}$ constructed in the proof (solutions to the backward equation) the Sobolev inequality yields

$$\int_{\mathbb{R}^d} |f_{n,m}|^{2d/(d-2)}\, dx \le C(d) \biggl(\int_{\mathbb{R}^d} |\nabla f_{n,m}|^2\, dx \biggr)^{(d-2)/d}.$$

Therefore, for some number $C > 0$

$$\int_s^\alpha \int_{\mathbb{R}^d} f_{n,m} L_0 \varphi_N z\, dx\, dt \le$$

$$\le C N^{-1} \biggl(\int_s^\alpha \int_{N \le |x| \le 2N} |\varrho|^{2d/(d+2)} |z|^{2d/(d+2)}\, dx\, dt \biggr)^{(d+2)/2d} +$$

$$+ C N^{-1} \biggl(\int_s^\alpha \int_{N \le |x| \le 2N} |\varrho|^{d/(d+2)} |z|^{2d/(d+2)}\, dx\, dt \biggr)^{(d+2)/2d},$$

whose right-hand side is estimated by

$$C_1\Bigl(\int_s^\alpha\int_{N\le|x|\le 2N}|\varrho|^2|z|^2\,dx\,dt\Bigr)^{1/2}+$$
$$+\,C_1\Bigl(\int_s^\alpha\int_{N\le|x|\le 2N}|\varrho|^2|z|^2\,dx\,dt\Bigr)^{1/4}\Bigl(\int_s^\alpha\int_{N\le|x|\le 2N}|z|^2\,dx\,dt\Bigr)^{1/4}.$$

Therefore, condition (6) can be also replaced by

$$\lim_{N\to\infty}\int_\kappa^T\int_{N\le|x|\le 2N}\varrho^2|z|^2\,dx\,dt=0\quad\forall\,\kappa\in(0,T).$$

Remark 5. Let us also note that (3) is equivalent to

$$\lim_{n\to\infty}\operatorname{ess\,sup}_{t\in(0,1/n)}\|(1+|y|^2)^{-1/2}\widehat{z}(y,t)\|_{L^2}=0, \tag{13}$$

where $\widehat{z}(y,t)$ denotes the Fourier transform in the first variable. In particular, due to the dominated convergence theorem (13) holds in the case of probability solutions on $\mathbb{R}^1$; for probability solutions in case $d=2$ it is ensured by an estimate

$$|\widehat{z}(y,t)|\le C/\log(|y|+1),$$

for probability solutions in any dimension by an estimate

$$|\widehat{z}(y,t)|^2\le(1+|y|)^{3-d}\Phi(|y|)$$

with an integrable function Φ on $[0,+\infty)$. For general solutions the latter estimate and (2) also yield (13).

If $d=1$, then, as noted above, condition (3) can be omitted (since it holds automatically), however, the requirement that admissible solutions are in L^2 is essential for uniqueness, as shows the example from [3] with bounded ϱ mentioned above.

This work was supported by the RFBR projects 10-01-00518, 11-01-90421-Ukr-f-a, 11-01-12104-ofi-m, the Russian President Grant MK-3674.2011.1, and the DFG through the programme SFB 701 at Bielefeld University.

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