Addendum to : " Stochastic nonlinear diffusion equations with singular diffusivity "

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Abstract In this addendum, we improve the results of the article [V. Barbu, G. Da Prato, M. Rockner, SIAM J. Math. Anal. 41(2009), pp.1106-1120) on existence and uniqueness of solutions to stochastic nonlinear diffusion equations and complete them with a new result on finite time extinction of the solution. Also, some technical points are clarified and a misleading conclusion is corrected.

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Consider the stochastic singular diffusion equation in $H = L^2(\mathcal{O})$

$$
(1) \qquad \begin{aligned} &dX = \mathrm{div}\ \mathrm{sgn}(\nabla(X))dt + XdW \quad \text{in} \ \ (0,\infty)\times\mathcal{O}, \\ &X = 0 \quad \text{on} \ \ (0,\infty)\times\partial\mathcal{O}, X(0) = x \quad \text{in} \ \ \mathcal{O}, \end{aligned}
$$

where $\mathcal{O}$ is a bounded and open domain of $\mathbb{R}^d$ and $W(t)$ is a Wiener process of the form $W(t) = \sum_{k=1}^{\infty} \mu_k e_k \beta_k(t), \{\beta_k\}$ is a sequence of independent real-valued Brownian motions on a filtered probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t>0}, \mathbb{P}\}$ and $\{e_k\}$ is an orthonormal basis in $H = L^2(\mathcal{O})$. The multi-valued function $u \to \mathrm{sgn}\, u$ from $\mathbb{R}^d$ to $\mathbb{R}^d$ is defined by

$$
\mathrm{sgn}\, u = \frac{u}{|u|_d} \ \text{ for } u \neq 0; \qquad \mathrm{sgn}\, 0 = \{v \in \mathbb{R}^d;\ |v|_d \leq 1\},
$$

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where $|\cdot|_d$ is the Euclidean norm of $\mathbb{R}^d$.

Equation (1) is not well posed in the sense of the classical Ito integral but only in a generalized (variational) sense to be recalled below ([3]). Let $BV(\mathcal{O})$ be the space of functions u with bounded variations on $\mathcal{O}$, that is (see [1]))

$$\|Du\| = \sup\left\{\int_{\mathcal{O}} u \operatorname{div} \varphi \, d\xi;\ \varphi \in C_0^\infty(\mathcal{O};\mathbb{R}^d), |\varphi|_\infty \leq 1\right\} < \infty.$$

Consider the function $\phi : L^2(\mathcal{O}) \to \overline{\mathbb{R}} =]-\infty, +\infty]$ defined by

$$\phi(u) = \begin{cases} \|Du\| + \displaystyle\int_{\partial\mathcal{O}} |\gamma_0(u)| d\mathcal{H}^{d-1} & \text{if } u \in BV(\mathcal{O}) \cap L^2(\mathcal{O}), \\ +\infty & \text{otherwise,} \end{cases}$$

where γ_0 is the trace operator on the boundary of $\mathcal{O}$ .Equivalently,

$$\phi(u) = \|D\widetilde{u}\| \text{ if } \widetilde{u} \in BV(\mathbb{R}^d);\ +\infty \text{ otherwise,}$$

where $\widetilde{u}$ is the extension of u by zero outside $\mathcal{O}$.

The function ϕ is lower-semicontinuous on $L^2(\mathcal{O})$ and, as a matter of fact, it is the closure in $L^1(\mathcal{O})$ of the norm of the Sobolev space $W_0^{1,1}(\mathcal{O})$. For this reason, we may interpret $\phi(u) < \infty$ as a Dirichlet boundary condition.

Definition 1 Let $0 < T < \infty$ and let $x \in L^2(\mathcal{O})$. A stochastic process $X : [0,T] \to L^2(\mathcal{O})$ is said to be a variational solution (or strong solution) to (1), if the following conditions hold.

(i) X is $(\mathcal{F}_t)$-adapted and has $\mathbb{P}$-a.s. continuous sample paths in $L^2(\mathcal{O})$, $X(0) = x$.

(ii) $X \in C([0,T]; L^2(\Omega; L^2(\mathcal{O}))) \cap L^1((0,T)\times\Omega; BV(\mathcal{O})), \phi(X) \in L^1((0,T)\times\Omega)$.

(iii) For all $(\mathcal{F}_t)$ adapted processes $G \in L^2(0,T; L^2(\Omega; L^2(\mathcal{O})))$ and $Z \in C([0,T]; L^2(\Omega, L^2(\mathcal{O}))), \phi(Z) \in L^1(0,T;\Omega)$ solving the equation

$$dZ(t) + G(t)dt = Z(t)dW(t),\ t \in [0,T],\ Z(0) \in L^2(\Omega, \mathcal{F}_0, L^2(\mathcal{O})), \quad (2)$$

we have

$$
\begin{aligned}
&\frac{1}{2}\,\mathbb{E}|X(t)-Z(t)|_2^2+\mathbb{E}\int_0^t \phi(X(\tau))d\tau\\
&\quad\le \frac{1}{2}\,\mathbb{E}|x-Z(0)|_2^2+\mathbb{E}\int_0^t \phi(Z(\tau))d\tau\\
&\quad+\frac{1}{2}\,\mathbb{E}\sum_{k=1}^{\infty}\mu_k^2\int_0^t\int_{\mathcal{O}}(e_k(X(\tau)-Z(\tau)))^2 d\xi\, d\tau\\
&\quad+\mathbb{E}\int_0^t \langle X(\tau)-Z(\tau),G(\tau)\rangle\, d\tau, t\in[0,T].
\end{aligned}
\tag{3}
$$

Here, $\langle\cdot,\cdot\rangle$ is the pairing in duality with the pivot space $L^2(\mathcal{O})$ and $|\cdot|_2$ is its norm .

The definition of a variational (strong) solution to (1) with additive noise is completely similar except that the quadratic term from the right hand side of (3) is missing and the inequality is taken $\mathbb{P}$-a.s.

It should be said that this definition of a strong solution was given in [3] for $d=1,2$, but with a different function ϕ, namely, for

$$
\phi_0(u)=\begin{cases}\|Du\| & \text{if } u\in BV(\mathcal{O}),\ \gamma_0(u)=0,\\ +\infty & \text{otherwise.}\end{cases}
$$

Though ϕ_0 is not l.s.c in $L^2(\mathcal{O})$, its l.s.c. closure is just ϕ, and so the definitions are equivalent. It is true however that $\phi_0(u)<+\infty$ does not mean that $u\in BV_0(\mathcal{O})$ as was erroneously claimed in [3]). As regards existence for (1) we have:

Theorem 2 *Assume that* $d\ge 1$ *and*

$$
C^*=\frac{1}{2}\sum_{k=1}^{\infty}\mu_k^2|e_k|_\infty^2<+\infty \tag{4}
$$

and that $x\in L^2(\mathcal{O})$. *Then, there is a unique variational solution* $X\in L^2(\Omega;C([0,T];L^2(\mathcal{O})))$ *to* (1) *such that*

$$
\lim_{\lambda\to 0}\mathbb{E}\{\sup_{t\in[0,T]}|X(t)-X_\lambda(t)|_2\}=0,\ \forall T>0, \tag{5}
$$

where $X_\lambda \in L^2(\Omega; C([0,T]; L^2(\mathcal{O})))$ *is the solution to the equation*

$$(6)\qquad \begin{aligned} &dX_\lambda - (I+\lambda A)^{-1}(\operatorname{div}\psi_\lambda(\nabla(I+\lambda A)^{-1}X_\lambda))dt = X_\lambda dW_t \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,\infty)\times\mathcal{O}, \\ &X_\lambda(0) = x. \end{aligned}$$

Here, $A = -\Delta$, $D(A) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$ and ψ_λ is the Yosida approximation of the sgn-multivalued function.

The existence part of Theorem 2 was established in [3] for $d = 1, 2$, but the proof is exactly the same for a general d. (The choice of $d = 1, 2$ in [3] was dictated by the inclusion of $BV(\mathcal{O})$ into $L^2(\mathcal{O})$ for $p = 1, 2$ but this is not essential for the existence and uniqueness proof.) As regards the uniqueness of the solution X, it was established in [3] only for (1) with additive noise, but it was recently proved for the case of multiplicative noise as in [5]. It is claimed in [3] that $X(t) \in BV^0(\mathcal{O}) = \{u \in BV(\mathcal{O}); \gamma_0(u) = 0\}$ as a consequence of the fact that

$$\mathbb{E}\int_0^T \phi((1+\lambda A)^{-1}X_\lambda(t))dt \le C, \quad \forall \varepsilon > 0,$$

which implies by lower semicontinuity $\phi(X) \in L^1((0,T)\times\Omega)$. As mentioned earlier, this is false and Theorem 2 is the correct formulation while the proof is exactly the same as in [3].

Equation (1) with the Neumann boundary condition $\nabla X \cdot \vec{n} = 0$ can be similarly treated by taking in Definition 1 the functional $\phi = \phi_N : L^2(\mathcal{O}) \to \overline{\mathbb{R}}$,

$$\phi_N(u) = \|Du\| \text{ if } u \in L^2(\mathcal{O}) \cap BV(\mathcal{O}), \ +\infty \text{ otherwise.}$$

Also, the periodic boundary conditions $X(t, \xi + \pi) \equiv X(t, \xi)$ can be incorporated into Definition 1 by a suitably chosen function ϕ. (See, e.g., [6].)

A striking feature of solutions to singular nonlinear diffusion stochastic equations is the extinction in finite time with positive probability. (See [4])

Theorem 3 *Let* $d = 1, 2$ *and let* X *be the variational solution to* (1) *given by Theorem* 2. *Let* $\tau = \inf\{t;\ |X(t)|_2 = 0\}$. *Then, we have*

$$(7)\qquad \mathbb{P}[\tau \le t] \ge 1 - \rho^{-1}\left(\int_0^t e^{-C^* s}ds\right)^{-1} s|x|_2, \ \forall t \ge 0,$$

where $\rho = \sup\{|y|_2/|y|_{W_0^{1,1}(\mathcal{O})}; y \in W_0^{1,1}(\mathcal{O})\}$ *and* C^* *is as in* (4).

Proof. Let X_λ be the solution to (6) and $\widetilde{X}_\lambda = (I+\lambda A)^{-1}X_\lambda$. We note that

$$dX_\lambda - (I+\lambda A)^{-1}\text{div}\ \psi_\lambda(\nabla\widetilde{X}_\lambda)dt = X_\lambda dW. \tag{8}$$

We apply Itô's formula to$|X_\lambda|_2^2$ and subsequently for $\varepsilon > 0$ to the function $\varphi(r) = (r+\varepsilon)^{\frac{1}{2}}, r \in \mathbb{R}$, and obtain

$$\begin{aligned} &d\varphi_\varepsilon(|X_\lambda(t)|_2^2) + \left(\int_{\mathcal{O}} \psi_\lambda(\nabla\widetilde{X}_\lambda)\cdot\nabla\widetilde{X}_\lambda d\xi\right)(|X_\lambda|_2^2+\varepsilon)^{-\frac{1}{2}}dt \\ &\quad\le C^*|X_\lambda(t)|_2^2(|X_\lambda(t)|_2^2+\varepsilon)^{-\frac{1}{2}}dt \\ &\quad +2\left\langle X_\lambda(t)dW(t), \varphi'_\varepsilon(|X_\lambda(t)|_2^2)X_\lambda(t)\right\rangle_2. \end{aligned} \tag{9}$$

Recalling that, by the Sobolev embedding theorem for $d \ge 1$

$$|\nabla y|_{L^1(\mathcal{O})} \ge \rho|y|_{\frac{d}{d-1}},\quad \forall y \in W_0^{1,1}(\mathcal{O})$$

and that $\psi_\lambda(r)\cdot r \ge |r|_d^2, \forall r \in \mathbb{R}^d$, we get by (9) that

$$\begin{aligned} &d\varphi_\varepsilon(|X_\lambda(t))|_2^2 + \rho|\widetilde{X}_\lambda(t)|_2(|X_\lambda(t)|_2+\varepsilon)^{-\frac{1}{2}}dt \\ &\le C^*|X_\lambda(t))|_2dt + \left\langle X_\lambda(t)dW(t), X_\lambda(t)\right\rangle_2(|X_\lambda(t)|_2^2+\varepsilon)^{-\frac{1}{2}}. \end{aligned}$$

Integrating from s to t and letting first λ and then ε tend to zero, we obtain $\mathbb{P}$-a.s. for all $0 \le s \le t$

$$\begin{aligned} &e^{-C^*t}|X(t)|_2 + \rho\int_s^t \mathbb{1}_{[|X(\theta)|_2>0]}e^{-C^*\theta}d\theta \\ &\quad\le e^{-C^*s}|X(s)|_2 + \int_s^t \mathbb{1}_{[|X(\theta)|_2>0]}e^{-C^*\theta}|X(\theta)|_2^{-1}\left\langle X(\theta), dW(\theta)\right\rangle_2. \end{aligned} \tag{10}$$

In particular, this implies that the process $t \to e^{-\theta^*t}|X(t)|_2$ is an $\{\mathcal{F}_t\}$-supermartingale and, therefore,

$$|X(t)|_2 = 0 \text{ for } t \ge \tau = \inf\{t \ge 0;\ |X(t)|_2 = 0\}.$$

If we take expectation and set $s = 0$, we see that

$$e^{-C^*t}\mathbb{E}|X(t)|_2 + \rho\int_0^t e^{-C^*\theta}\mathbb{P}[\tau > \theta]d\theta \le |x|_2,\ \forall t > 0.$$

This yields

$$\mathbb{P}[\tau > t] \le \left(\rho\int_0^t e^{-C^*\theta}d\theta\right)^{-1}|x|_2,\ \forall\lambda > 0$$

as claimed.This completes the proof.

Remark 4 *In particular, taking in* (4) $\mu_k = 0$ *for all* k, *implying* $C^* = 0$, *we have* $\tau \leq |x|_d/\rho$ *and recover the deterministic case for* $d = 1, 2$*(see [2].) As in deterministic case that is for* $C^* = 0$ *there is an analogous extinction result for all dimensions* $d \geq 1$ *also in the stochastic case. The proof,however, is much more involved than the above and would go beyond the scope of this Addendum.It will be contained instead in a forthcomong paper which is in preparation.*

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