GIBBS MEASURES OF CONTINUOUS SYSTEMS: AN ANALYTIC APPROACH

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Abstract. We present a new method to prove existence and uniform *à-priori* estimates for Gibbs measures associated with classical particle systems in continuum. The method is based on the choice of appropriate Lyapunov functionals and on the corresponding exponential bounds for the local Gibbs specification.

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Contents

1. INTRODUCTION

In this paper we study Gibbs states of classical particle systems in continuum. Our main goal is to establish a *new* analytical method for proving existence and *à-priori* bounds for the corresponding Gibbs measures $\mu \in \mathcal{G}$. This method uses only elementary constructions from probability and infinite dimensional analysis. To illustrate the main ideas, we restrict ourselves to the case of a pair interaction $V(x, y)$ having finite range. Nevertheless, the method straightforwardly extends to the case of multi-particle interactions of possibly infinite range as well as to the marked Gibbs measures or to the configuration spaces over manifolds, which will be done in subsequent papers.

Key words and phrases. Configurations spaces; Continuous Systems; Gibbs measures; DLR equation.

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An initial step in the study of Gibbs measures is the existence problem. This is, however, far from evident in our setting, namely for measures supported by the space Γ of locally finite sets (i.e., configurations) γ over $\mathbb{R}^d$. Let us recall that for classical continuous systems there are two basic and commonly used approaches employing respectively *Ruelle's superstability estimates* [37, 38] and *Dobrushin's existence criterion* [7, 8, 3, 22] (see the comments in Subsection 2.4). As it is usual for non-compact spin spaces, we have to restrict ourselves to a set $\mathcal{G}^{\mathrm{t}}$ of tempered Gibbs measures, but defined in a more general way than in the existing literature (cf. (2.54), (2.55)). The key idea of our method is to prove the exponentially integrability of a certain *Lyapunov functional*, which is given by the energy $H(\gamma_{\Lambda_k})$ of a configuration γ restricted to a small cube $\Lambda_k \subset \mathbb{R}^d$, see Lemma 3.1. An important issue is the *weak* dependence of such bound on the boundary values $\xi \in \Gamma$ fixed outside Λ_k. Using the consistency property of the local Gibbs specification, in Lemma 3.3 we then extend the above estimate to large volumes $\Lambda \subset \mathbb{R}^d$ constructed by means of the partition $\mathbb{R}^d = \coprod_k \Lambda_k$. For the kernels of the Gibbs specifications $\pi_\Lambda(\gamma|\xi)$, this implies the necessary tightness to prove that $\mathcal{G}^{\mathrm{t}} \neq \varnothing$ (cf. Theorem 2.6). Since the bounds obtained are asymptotically uniform, as $\Lambda \nearrow \mathbb{R}^d$, for μ-almost *all* $\xi \in \Gamma$, from the *DLR* equation we immediately get similar exponential bounds to be hold *uniformly* for all $\mu \in \mathcal{G}^{\mathrm{t}}$ (cf. Theorem 2.7). At this point we have a principal difference from Dobrushin's approach, which just ensures the existence of some tempered Gibbs states μ without any information about the whole set $\mathcal{G}^{\mathrm{t}}$. Furthermore, instead of inductive and combinatorical techniques which were typical in the above mentioned approaches, we directly control weak dependence, contractivity and compactness properties in suitable weighted seminorms over Γ. Actually, the class of measures $\mathcal{G}^{\mathrm{t}}$ we constructed includes all μ having Ruelle's support, cf. Remark 3.9. For a more comprehensive reference to Ruelle's and Dobrushin's techniques see Remarks 2.11 and 3.2.

The paper is organized as follows. In Section 2 we fix the standard framework for the Gibbs measures on configuration spaces and present our main results. In Subsection 2.1 we shortly recall some facts about the Poisson measure $\pi_{z\sigma}$ on Γ which we use later on. The local Gibbs specification $\Pi = \{\pi_\Lambda(\mathrm{d}\gamma|\xi)\}$ and the corresponding Gibbs measures $\mu \in \mathcal{G}$ as solutions to the *DLR* equation are defined in Subsection 2.2. In Subsection 2.3 we make precise the conditions on the interaction and relate them with Ruelle's superstability. Next, in Subsection 2.4 we introduce the set of tempered configurations $\gamma \in \Gamma^{\mathrm{t}}$ and, respectively, the set of tempered Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ obeying $\mu(\Gamma^{\mathrm{t}}) = 1$. Then, we formulate our main Theorems 2.6 and 2.7 on the existence and *à-priori* estimates for $\mu \in \mathcal{G}^{\mathrm{t}}$. Comments, which in particular relate these results with the previous ones obtained by other methods, conclude this section.

In Section 3 are proven main Theorems 2.6 and 2.7 and discussed their possible improvements and applications. In Subsection 3.1 we prepare the key technical Lemmas 3.1 and 3.3 about the integrability properties of the specification kernels $\pi_\Lambda(\mathrm{d}\gamma|\xi)$. Thereafter, in Subsection 3.2 are given complete elementary proofs of Theorems 2.6 and 2.7. In Subsection 3.3 we establish the so-called *Feller* regularity property of the specification, which is crucial to ensure that all its limit points are Gibbs measures. A natural question how the singularity of the potential $V(x,y)$ may improve the regularity properties of the corresponding $\mu \in \mathcal{G}^{\mathrm{t}}$ is clarified in Subsection 3.4. Possible refinements of Theorems 2.6 and 2.7 in the case of *strong* *superstable* interactions are outlined in Subsection 3.5. Afterwards, in Subsection 3.6 we obtain precise information on support properties of all tempered Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$.

Finally, we emphasize that our method seems to be quite universal and applicable to different classes of models: classical or quantum, on a lattice, on an infinite graph, or in the continuum. We mention here some recent papers where a proper modification of this technique was firstly implemented to quantum anharmonic crystals [23, 30, 31] and to classical spin systems on graphs [21]. Therefore, we hope that this work contributes to make the theory of Gibbs measures more accessible and friendly to a wider audience, in particular, for specialists interesting in applications of the infinite dimensional analysis in problems of mathematical physics.

2. DESCRIPTION OF THE MODEL AND MAIN RESULTS

In this section we recall the standard setting of classical statistical mechanics and present the main theorems obtained in the paper.

2.1. Preliminaries on configuration spaces.

We consider a system of identical particles in $\mathbb{R}^d$, which interact via the pair potential $V(x,y)$ obeying certain stability properties to be specified below. The position of each particle is described by a vector $x = (x^{(i)})_{i=1}^d$ in the d-dimensional Euclidean space $(\mathbb{R}^d, |\cdot|)$, $d \in \mathbb{N}$. By $\mathcal{B}(\mathbb{R}^d)$ we denote its Borel σ-algebra and by $\mathcal{B}_c(\mathbb{R}^d)$ respectively the family of all *bounded Borel* sets (having compact closure). As usual, $|\Lambda|$ stands for the cardinality, Λ^c – for the complement, $\bar{\Lambda}$ – for the topological closure, and $\partial\Lambda$ – for the topological boundary of a set $\Lambda \subseteq \mathbb{R}^d$. Subsets constituted by points $k = (k^{(i)})_{i=1}^d$of the integer lattice $\mathbb{Z}^d \subseteq \mathbb{R}^d$ will be denoted by $\mathcal{K}$; for shorthand we write $\mathcal{K} \Subset \mathbb{Z}^d$ if $0 < |\mathcal{K}| < \infty$. A sequence of $\Lambda_N \in \mathcal{B}_c(\mathbb{R}^d)$ (or $\mathcal{K}_N \Subset \mathbb{Z}^d$), $N \in \mathbb{N}$, is called *order generating* if it is ordered by inclusion and exhausts the whole $\mathbb{R}^d$ (respectively, $\mathbb{Z}^d$). Furthermore, $\Lambda \nearrow \mathbb{R}^d$ (or $\mathcal{K} \nearrow \mathbb{Z}^d$) means the limit taken along any unspecified sequence of this type. Finally, we denote $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ and $\overline{\mathbb{Z}_+} := \mathbb{Z}_+ \cup \{\infty\}$, where $\mathbb{N} := \{1, 2, \ldots\}$.

The *configuration space* $\Gamma := \Gamma_{\mathbb{R}^d}$ over $\mathbb{R}^d$ consists of all locally finite subsets, i.e.,

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d \,\middle|\, |\gamma_\Lambda| < \infty, \quad \forall \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \right\}, \tag{2.1}$$

where $|\gamma_\Lambda|$ is the total number of points in the restriction $\gamma_\Lambda := \gamma \cap \Lambda$. The elements $\gamma \in \Gamma$ are called (*simple*) *configurations*. Each γ can be identified with the $\overline{\mathbb{Z}_+}$-valued Radon measure $\sum_{x\in\gamma} \delta_x$, where δ_x is the Dirac distribution with mass at point x. In this sense we have a natural embedding of the configuration space Γ into the space $\mathcal{M}(\mathbb{R}^d)$ of all Radon measures on $\mathbb{R}^d$. Note that Γ becomes a *Polish* (i.e., separable metrizable) space in the *vague topology* $\mathcal{O}(\Gamma)$ inherited from $\mathcal{M}(\mathbb{R}^d)$. This is the weakest topology in which the following mappings are continuous

$$\Gamma \ni \gamma \mapsto \langle f, \gamma \rangle := \int_{\mathbb{R}^d} f(x)\gamma(\mathrm{d}x) = \sum\nolimits_{x\in\gamma} f(x), \quad f \in C_0(\mathbb{R}^d), \tag{2.2}$$

where $C_0(\mathbb{R}^d)$ stands for the set of all continuous functions with compact support. As usual, all sums over the empty index set are agreed to be zero, so that $\sum_{x\in\gamma} f(x) = 0$ if $\gamma = \varnothing$. The sequential convergence in $(\Gamma, \mathcal{O}(\Gamma))$ can be characterized as follows:

$$\gamma^{(N)} \to \gamma, \text{ as } N \to \infty, \text{ iff } |\gamma_\Lambda^{(N)}| \to |\gamma_\Lambda|, \text{ for all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \text{ with } |\gamma_{\partial\Lambda}| = 0.$$

By $\mathcal{B}(\Gamma)$ we denote the corresponding *Borel* σ-algebra, which also coincides with the smallest σ-algebra generated by the cylinder sets

$$\{\gamma \in \Gamma \mid |\gamma_\Lambda| = n\}, \quad \Lambda \in \mathcal{B}_c(\mathbb{R}^d), \quad n \in \mathbb{Z}_+.$$

The family of all probability measures μ on $(\Gamma, \mathcal{B}(\Gamma))$ (also called *simple point processes* [15, 16]) will be denoted by $\mathcal{P}(\Gamma)$.

The space of configurations located in $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ is defined as the disjoint union

$$\Gamma_\Lambda := \left\{\gamma \in \Gamma \,\middle|\, \gamma_{\mathbb{R}^d \setminus \Lambda} = \varnothing\right\} = \bigsqcup\nolimits_{n \in \mathbb{Z}_+} \Gamma_\Lambda^{(n)} \tag{2.3}$$

of the n-particle subsets

$$\Gamma_\Lambda^{(n)} := \{\gamma \in \Gamma_\Lambda \mid |\gamma| = n\}, \quad \Gamma_\Lambda^{(0)} := \{\varnothing\}. \tag{2.4}$$

Each Γ_Λ is equipped with the topology $\mathcal{O}(\Gamma_\Lambda)$ induced from $\mathcal{O}(\Gamma)$ under the natural projection

$$\mathbb{P}_\Lambda : \Gamma \to \Gamma_\Lambda, \quad \mathbb{P}_\Lambda \gamma := \gamma_\Lambda, \tag{2.5}$$

and with the corresponding Borel σ-algebra $\mathcal{B}(\Gamma_\Lambda) := \mathcal{B}(\Gamma) \cap \Gamma_\Lambda$. This yields us the *"localized"* description of $(\Gamma, \mathcal{B}(\Gamma))$ as the projective limit of the measurable spaces $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ as $\Lambda \nearrow \mathbb{R}^d$. We also shall need the subset of configurations *finite* in the whole $\mathbb{R}^d$

$$\Gamma_0 := \bigcup\nolimits_{\Lambda \in \mathcal{B}_c(\mathbb{R}^d)} \Gamma_\Lambda = \bigsqcup\nolimits_{n \in \mathbb{Z}_+} \Gamma_0^{(n)}, \tag{2.6}$$

where, similarly to (2.4),

$$\Gamma_0^{(n)} := \{\gamma \in \Gamma_0 \mid |\gamma| = n\} = \bigcup\nolimits_{\Lambda \in \mathcal{B}_c(\mathbb{R}^d)} \Gamma_\Lambda^{(n)}, \quad \Gamma_0^{(0)} := \{\varnothing\}.$$

For more details on topological structure of the above configuration spaces see e.g. [1, 18, 20, 24, 27, 28].

In statistical physics, the state of an ideal gas is described by a *Poisson* random point field $\pi_{z\sigma}$ on Γ. We next recall the well-known explicit construction of $\pi_{z\sigma}$ (see e.g. Section 2.1 in [1] or Section 2.4 in [9]). Let us fix a chemical *activity* parameter $z > 0$ and a *non-atomic*, locally finite measure σ on the underlying phase space $\left(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\right)$ such that $\sigma(\mathbb{R}^d) = \infty$. For each $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, the corresponding *Lebesgue-Poisson* measure $\lambda_{z\sigma} := \lambda_{z\sigma}^\Lambda$ on $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ is defined by the identity

$$\int_{\Gamma_\Lambda} F(\gamma_\Lambda) \mathrm{d}\lambda_{z\sigma}(\gamma_\Lambda) := F(\{\varnothing\}) + \sum_{n \in \mathbb{N}} \frac{z^n}{n!} \int_{\Lambda^n} F(\{x_1, \ldots, x_n\}) \mathrm{d}\sigma(x_1) \ldots \mathrm{d}\sigma(x_n), \tag{2.7}$$

which holds for all bounded measurable functions $F \in L^\infty(\Gamma_\Lambda)$. Taking into account that $\lambda_{z\sigma}^\Lambda(\Gamma_\Lambda) = e^{z\sigma(\Lambda)}$, we then introduce a probability measure on $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$

$$\pi_{z\sigma}^\Lambda := e^{-z\sigma(\Lambda)} \lambda_{z\sigma}^\Lambda. \tag{2.8}$$

The *Poisson* measure $\pi_{z\sigma} \in \mathcal{P}(\Gamma)$ is given as the projective limit of the consistent family $\left(\pi_{z\sigma}^\Lambda\right)_{\Lambda \in \mathcal{B}_c(\mathbb{R}^d)}$, that is to say

$$\pi_{z\sigma} := \mathbb{P}_\Lambda^{-1} \circ \pi_{z\sigma}^\Lambda, \quad \Lambda \in \mathcal{B}_c(\mathbb{R}^d). \tag{2.9}$$

An equivalent way of defining $\pi_{z\sigma}$ is to claim that, for any collection of disjoint domains $(\Lambda_j)_{j=1}^N \subset \mathcal{B}_c(\mathbb{R}^d)$, the variables $|\gamma_{\Lambda_j}|$ should be mutually independent and distributed by the Poissonian law with parameters $z\sigma(\Lambda_j)$, i.e.,

$$\pi_{z\sigma}\left(\left\{\gamma \in \Gamma \,\middle|\, |\gamma_{\Lambda_j}| = n\right\}\right) = \frac{z^n \sigma^n(\Lambda_j)}{n!} e^{-z\sigma(\Lambda_j)}, \quad n \in \mathbb{Z}_+. \tag{2.10}$$

Another well known analytic characterization of $\pi_{z\sigma}$ is through its Laplace transform, see e.g. [10],

$$\int_{\Gamma_\Lambda} \exp\langle f,\gamma\rangle \mathrm{d}\pi_{z\sigma}(\gamma) := \exp\left\{\int_{\mathbb{R}^d}\left(e^{f(x)}-1\right)\mathrm{d}\sigma(x)\right\}, \quad f\in C_0(\mathbb{R}^d).$$

2.2. **Gibbsian formalism.** Now we define Gibbs reconstructions of the *"free"* measure $\pi_{z\sigma}$. A *pair potential* (without hard core) is a symmetric measurable function

$$V:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}\cup\{+\infty\}, \quad V(x,y)=V(y,x) \quad \text{for all } x,y\in\mathbb{R}^d, \tag{2.11}$$

which is *continuous* everywhere on $\mathbb{R}^d\times\mathbb{R}^d$ *except* possibly at the diagonal $\mathcal{D}:=\left\{x,y\in\mathbb{R}^d \,|\, x=y\right\}$. For simplicity, we impose the following technical restriction on the potential:

(**FR**)**:** ***Finite range*****:** *There exists* $0<R<\infty$ *such that*

$$V(x,y)=0 \quad \text{if } \ |x-y|\geq R. \tag{2.12}$$

The case of $R=\infty$ needs more accurate analysis, which will be done in the forthcoming paper [21].

The *Hamiltonian* (or *energy functional*) $H:\Gamma_0\to\mathbb{R}$ is given on finite configurations by

$$H(\gamma):=\sum_{\{x,y\}\subset\gamma} V(x,y)\in\mathbb{R}, \tag{2.13}$$

where the sum runs over all (unordered) pairs of distinct points x,y from the support of $\gamma\in\Gamma_0$. By convention, this functional is vanishing at the empty and one-point configurations, i.e.,

$$H(\{\varnothing\})=0, \quad H(\{x\})=0, \quad x\in\mathbb{R}^d. \tag{2.14}$$

Respectively, for each $\Lambda\in\mathcal{B}_c(\mathbb{R}^d)$ and $\xi\in\Gamma$,

$$W(\gamma_\Lambda|\xi):=\sum_{x\in\gamma_\Lambda,\, y\in\xi_{\Lambda^c}} V(x,y) \tag{2.15}$$

can be viewed as the *interaction energy* between $\gamma_\Lambda\in\Gamma_\Lambda$ and $\xi_{\Lambda^c}:=\xi\cap\Lambda^c$. Combining (2.13) and (2.15), we introduce the *local conditional Hamiltonians* $H_\Lambda(\cdot|\xi):\Gamma_\Lambda\to\mathbb{R}$ by

$$H_\Lambda(\gamma_\Lambda|\xi):=H(\gamma_\Lambda)+W(\gamma_\Lambda|\xi), \tag{2.16}$$

$$H_\Lambda(\gamma_\Lambda|\varnothing):=H(\gamma_\Lambda). \tag{2.17}$$

Fixed the inverse temperature $\beta:=1/T>0$, the *local Gibbs state* with boundary condition ξ is defined by

$$\mu_\Lambda(\mathrm{d}\gamma_\Lambda|\xi):=\left[Z_\Lambda(\xi)\right]^{-1}\exp\left\{-\beta H_\Lambda(\gamma_\Lambda|\xi)\right\}\lambda_{z\sigma}(\mathrm{d}\gamma_\Lambda), \tag{2.18}$$

provided the *partition function* (cf. (2.7), (2.14))

$$\begin{aligned} Z_\Lambda(\xi) \quad &:= \int_{\Gamma_\Lambda}\exp\left\{-\beta H_\Lambda(\gamma_\Lambda|\xi)\right\}\mathrm{d}\lambda_{z\sigma}(\gamma_\Lambda) \qquad (2.19)\\ &= 1+z+\sum_{n\geq 2}\frac{z^n}{n!}\int_{\Lambda^n}\exp\left\{-\beta H_\Lambda(\{x_1,\dots,x_n\}|\xi)\right\}\mathrm{d}\sigma(x_1)\dots\mathrm{d}\sigma(x_n) \end{aligned}$$

is finite. In the case of $Z_\Lambda(\xi) = +\infty$ we set respectively $\mu_\Lambda(\mathrm{d}\gamma_\Lambda|\xi) = 0$. From (2.19) we see that it always should hold $Z_\Lambda(\xi) > 1$, which will be repeatedly used in the subsequent estimates.

The *local specification* $\Pi = \{\pi_\Lambda\}_{\Lambda\in\mathcal{B}_c(\mathbb{R}^d)}$ is a family of stochastic kernels

$$\mathcal{B}(\Gamma)\times\Gamma \ni (\Delta, y) \mapsto \pi_\Lambda(\Delta|\xi) \in [0,1] \tag{2.20}$$

given by

$$\pi_\Lambda(\Delta|\xi) := \mu_\Lambda(\Delta'|\xi), \quad \Delta' := \{\gamma_\Lambda \in \Gamma_\Lambda \,|\gamma' := \gamma_\Lambda \cup \xi_{\Lambda^c} \in \Delta\} \in \mathcal{B}(\Gamma_\Lambda). \tag{2.21}$$

By construction (cf. Proposition 2.6 in [35]), the family (2.21) obeys the *consistency property* which means that for all $\Delta \in \mathcal{B}(\Gamma)$ and $\xi \in \Gamma$

$$\int_\Gamma \pi_\Lambda(\Delta|\gamma)\pi_{\Lambda'}(\mathrm{d}\gamma|\xi) = \pi_{\Lambda'}(\Delta|\xi), \quad \Lambda \subseteq \Lambda'. \tag{2.22}$$

In Subsection 3 we shall impose some natural conditions on V which guarantee that $Z_\Lambda(\xi) < \infty$ (cf. (2.49)), so that each specification kernel $\pi_\Lambda(\mathrm{d}\gamma|\xi)$ would be certainly a *probability* distribution on $(\Gamma, \mathcal{B}(\Gamma))$.

Given $F \in L^\infty(\Gamma)$ and $\mu \in \mathcal{P}(\Gamma)$, let us define *"convolutions"* $\pi_\Lambda F \in L^\infty(\Gamma)$ and $\pi_\Lambda\mu \in \mathcal{P}(\Gamma)$ respectively by

$$(\pi_\Lambda F)(\xi) \quad : \quad = \int_\Gamma F(\gamma)\pi_\Lambda(\mathrm{d}\gamma|\xi), \quad \xi \in \Gamma, \tag{2.23}$$

$$(\pi_\Lambda \mu)(\Delta) \quad : \quad = \int_\Gamma \pi_\Lambda(\Delta|\gamma)\mu(\mathrm{d}\gamma), \quad \Delta \in \mathcal{B}(\Omega). \tag{2.24}$$

They are related by the duality $\langle \pi_\Lambda F, \mu\rangle = \langle F, \pi_\Lambda\mu\rangle$, where we use a shorthand for expectation

$$\langle F, \mu\rangle := \mu\,(F) = \int_\Omega F \mathrm{d}\mu. \tag{2.25}$$

Definition 2.1. *A probability measure $\mu \in \mathcal{P}(\Gamma)$ is called a grand canonical* ***Gibbs measure*** *(or* ***state****) with the pair potential V and activity $z > 0$ if it satisfies the Dobrushin-Lanford-Ruelle (****DLR****) equilibrium equation*

$$(\pi_\Lambda \mu)(\Delta) := \int_\Gamma \pi_\Lambda(\Delta|\gamma)\mu(\mathrm{d}\gamma) = \mu(\Delta) \tag{2.26}$$

valid for all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $\Delta \in \mathcal{B}(\Omega)$. Fixed an inverse temperature β, the associated set of all Gibbs states will be denoted by $\mathcal{G}$.

In practice, it is more convenient to deal with the equivalent to (2.26) equation

$$\langle \pi_\Lambda F, \mu\rangle := \int_\Gamma\int_\Gamma F(\gamma)\pi_\Lambda(\mathrm{d}\gamma|\xi)\mu(\mathrm{d}\xi) = \int_\Omega F(\gamma)\mathrm{d}\mu(\gamma) = \langle F, \mu\rangle, \tag{2.27}$$

which has to be valid for all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $F \in L^\infty(\Gamma)$. Furthermore, it would suffice to check (2.27) on all bounded continuous functions $F \in C_b(\Gamma)$, which constitute a measure determining class for every $\mu \in \mathcal{P}(\Gamma)$. In what follows, we consider $C_b(\Gamma)$ as a Banach space with the norm $||F||_{C_b(\Gamma)} := \sup_{\gamma\in\Gamma} |F(\gamma)|$.

We recall that the standard sources on the *DLR* approach in statistical mechanics are the monographs [11, 34].

2.3. **Assumptions on the interaction and superstability.** Let us make precise the further conditions on the interaction to be used throughout the paper.

(**LB**)**:** ***Lower boundedness***: *There exist* $M \geq 0$ *and* $r_1, r_2 \in [0, R]$, $r_1 \leq r_2$, *such that*

$$\inf_{x,y\in\mathbb{R}^d} V(x,y) \geq -M, \tag{2.28}$$

$$\text{and} \quad V(x,y) \geq 0 \quad \text{if} \;\; |x-y| < r_1 \;\; \text{or} \;\; |x-y| > r_2. \tag{2.29}$$

(**RC**)**:** ***Repulsion condition***: *There exist* $A \geq 0$ *and* $\delta > 0$ *such that*

$$\inf_{x,y:\,|x-y|\leq\delta} V(x,y) \geq A \geq 3Mm, \tag{2.30}$$

where

$$m := m(r_1, r_2; \delta) := \left\{ 2\sqrt{d} \left[(r_2 - r_1)/\delta + 2 \right] \right\}^d. \tag{2.31}$$

Remark 2.2. The role of the parameter m will be clear from (2.37). A classical example which fulfills (2.30), with an arbitrary large $A > 0$ and a corresponding $\delta := \delta(A)$, is potentials V of the so-called *Dobrushin-Fisher-Ruelle* (*DFR*) type obeying asymptotical behavior with some $C, \varkappa > 0$

$$V(x,y) \geq C|x-y|^{-(d+\varkappa)}, \quad \text{as } |x-y| \to 0. \tag{2.32}$$

The trivial situation when $V(x,y) \geq 0$ for all $x, y \in \mathbb{R}^d$ is described by the choice of $r_1 = r_2$ and $A = M = 0$. Merely speaking, Condition (**RC**) means that the *repulsive* part $V^+ := \max\{V, 0\}$ of the pair interaction dominates the *attractive* one $V^- := \min\{V, 0\}$. In the case of translation invariant potentials, a similar assumption was employed in [33]. For a better control of V in (2.29), one can use a finite system of intervals $(r_1^{(n)}, r_2^{(n)})$, such that $V(x,y)$ is nonnegative for $|x-y|$ outside each of them. Furthermore, proceeding in the spirit of [8], it is possible to refine the global bounds like (2.28) and (2.30) by certain integral conditions on V^+ and V^-.

To analyze the consequences of the above assumptions let us consider a *partition* of the phase space $\mathbb{R}^d = \bigsqcup_{k\in\mathbb{Z}^d} Q_{gk}$ by the *cubes* of edge length $g > 0$

$$Q_{gk} := \left\{ x = (x^{(i)})_{i=1}^d \in \mathbb{R}^d \,\middle|\, g\left(k^{(i)} - 1/2\right) \leq x^{(i)} < g\left(k^{(i)} + 1/2\right) \right\}, \tag{2.33}$$

which are centered at the points gk, $k \in \mathbb{Z}^d$. For each configuration $\gamma \in \Gamma$ and $k \in \mathbb{Z}^d$, we define

$$\gamma_k := \gamma \cap Q_{gk} \;\in\; \Gamma_{Q_{gk}} =: \Gamma_k. \tag{2.34}$$

In what follows, we pick the parameter $g := \delta/\sqrt{d}$ so that

$$\operatorname{diam}(Q_{gk}) := \sup_{x,y\in Q_{gk}} |x-y| = \delta, \tag{2.35}$$

and hence $V(x,y) \geq A$ for all $x, y \in \gamma_k$. Technically we need to control those pairs $\{x, y\} \subset \gamma$ for which $V(x,y) < 0$. It is clear that $V(x,y)$ might be negative for some $x \in \gamma_k$ and $y \in \gamma_j$ if only

$$j \in \partial_g k := \left\{ j \neq k \mid (r_1/\delta - 1)\sqrt{d} < |k-j| < (r_2/\delta + 1)\sqrt{d} \right\}. \tag{2.36}$$

The total number of such *"neighbor"* cubes Q_{gj} can be roughly estimated by the constant m in (2.30), i.e.,

$$|\partial_g k| \leq m(r_1, r_2; \delta). \tag{2.37}$$

To each *"index"* set $\mathcal{K} \Subset \mathbb{Z}^d$ there corresponds the cubic domain

$$\Lambda_{\mathcal{K}} := \bigsqcup_{k \in \mathcal{K}} Q_{gk} \in \mathcal{B}_c(\mathbb{R}^d). \tag{2.38}$$

For any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ we denote

$$\mathcal{U}(\Lambda) := \{ y \in \Lambda^c \mid r_1 \leq \mathrm{dist}(y, \Lambda) \leq r_2 \} \in \mathcal{B}_c(\mathbb{R}^d), \tag{2.39}$$

so that for all $x \in \Lambda$ and $y \notin \mathcal{U}(\Lambda)$ it surely holds $V(x,y) \geq 0$. Finally, letting

$$\mathcal{K}_\Lambda := \left\{ k \in \mathbb{Z}^d \middle| \ \Lambda \cap Q_{gk} \neq \varnothing \right\}, \tag{2.40}$$

we construct the *"minimal"* covering of the volume Λ

$$\Lambda_g := \bigsqcup_{k \in \mathcal{K}_\Lambda} Q_{gk} \ \supseteq \ \Lambda, \tag{2.41}$$

whereby $|\mathcal{K}_\Lambda|$ is the number of cubes Q_{gk} having nonvoid intersection with Λ.

The first claim of Lemma 2.3 says that under the above assumptions the interaction is *superstable* in the usual sense of *Ruelle* [38] (this gives a positive answer on the question posed on page 146 of [33]). The second claim (playing a crucial role in our approach) establishes a *lower bound* on the local Hamiltonians $H_\Lambda(\gamma|\xi)$ in terms of boundary condition ξ_{Λ^c}, which is valid in *small volumes* Λ.

Lemma 2.3. *(a) For any partition of* $\mathbb{R}^d$ *by the cubes (2.33) with edge length* $g > 0$ *there exist proper* D_g, $E_g \geq 0$, *such that for all* $\gamma \in \Gamma_0$ *the following holds:*

$$(\mathbf{RSS})\textit{: Ruelle's Superstability:} \quad H(\gamma) \geq D_g \sum_{k \in \mathbb{Z}^d} |\gamma_k|^2 - E_g |\gamma|. \tag{2.42}$$

Furthermore, this D_g *can be chosen positive if* $A > 0$.

(b) Let a volume $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ *be such that* $\mathrm{diam}(\Lambda) \leq \delta$, *then for any* $\gamma, \xi \in \Gamma$

$$H_\Lambda(\gamma_\Lambda|\xi) \geq \frac{A}{2} \left(|\gamma_\Lambda|^2 - |\gamma_\Lambda| \right) - M |\gamma_\Lambda| \cdot |\xi_{\mathcal{U}(\Lambda)}|. \tag{2.43}$$

Proof. Because of the elementary inequality

$$\sum_{k=1}^{K} |\gamma_k|^2 \leq \left(\sum_{k=1}^{K} |\gamma_k| \right)^2 \leq K \sum_{k=1}^{K} |\gamma_k|^2, \quad K \in \mathbb{N}, \tag{2.44}$$

it suffices to check (2.42) for the fixed $g := \delta/\sqrt{d}$. By (2.28), (2.30), (2.37), and (2.40) we have that for each $\gamma \in \Gamma_\Lambda$ and $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$

$$\begin{aligned} H(\gamma_\Lambda) &= \sum_{k \in \mathcal{K}_\Lambda} \sum_{\{x,y\} \subset \gamma_k} V(x,y) + \sum_{\substack{k,j \in \mathcal{K}_\Lambda \\ k \neq j}} \sum_{\substack{x \in \gamma_k \\ y \in \gamma_j}} V(x,y) \\ &\geq A \sum_{k \in \mathcal{K}_\Lambda^+} \binom{|\gamma_k|}{2} - M \sum_{\substack{k,j \in \mathcal{K}_\Lambda \\ j \in \partial_g k}} |\gamma_k| \cdot |\gamma_j| \\ &\geq \left(\frac{A}{2} - Mm \right) \sum_{k \in \mathcal{K}_\Lambda} |\gamma_k|^2 - \frac{A}{2} |\gamma|. \end{aligned} \tag{2.45}$$

For the above value of g, this yields the claim (2.42) with $D_g := A/2 - Mm$ and $E_g := -A/2$.

The proof of (b) is similar, see also Lemma 1 in [33]. □

Remark 2.4. (i) For a domain $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ let us consider its cubic covering Λ_g having volume $\mathrm{vol}(\Lambda_g) := g^d|\mathcal{K}_\Lambda|$, cf. (2.41). Using (2.44) with $K := |\mathcal{K}_\Lambda^+|$, one can continue (2.42) as

$$\begin{aligned} H(\gamma_\Lambda) &\geq D_g \frac{g^d}{\mathrm{vol}(\Lambda_\delta)}|\gamma_\Lambda|^2 - E_g|\gamma_\Lambda| \qquad (2.46) \\ &\geq D_g \left(\frac{g}{2g+\mathrm{diam}\Lambda}\right)^d |\gamma_\Lambda|^2 - E_g|\gamma_\Lambda|. \end{aligned}$$

Setting $B_g := D_g\,(g/3)^d > 0$, we get that for every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ with $\mathrm{diam}\Lambda \geq g$ and for all $\gamma \in \Gamma_\Lambda$

$$(\mathbf{GSS}):\quad H(\gamma_\Lambda) \geq B_g \frac{1}{(\mathrm{diam}\Lambda)^d}|\gamma_\Lambda|^2 - E_g|\gamma_\Lambda|, \qquad (2.47)$$

which means the superstability in the sense of *Ginibre*, see Condition B on page 29 of [13].

(ii) The superstability (**RSS**) immediately implies the usual *stability* property

$$(\mathbf{S}):\quad H(\gamma) \geq -E_g|\gamma|, \quad \text{for all } \gamma \in \Gamma_0, \qquad (2.48)$$

which is necessary for correct thermodynamic description of infinite particle systems.

It is well known (see e.g. Section 3.2 of [37]) that the stability of the interaction (2.48) provides that the partition function (2.19) is finite for all $\xi \in \Gamma$. Indeed, we have the following bound in terms of the interaction parameters

$$\begin{aligned} Z_\Lambda(\xi) &\leq \sum_{n=0}^{\infty} \frac{z^n}{n!}\sigma(\Lambda)^n \exp\left[n\beta\left(\frac{A}{2} + M|\xi_{\mathcal{U}(\Lambda)}|\right)\right] \qquad (2.49) \\ &= \exp\left\{z\sigma(\Lambda)\exp\left[\beta\left(\frac{A}{2} + M|\xi_{\mathcal{U}(\Lambda)}|\right)\right]\right\}. \end{aligned}$$

The latter means that each $\mu_\Lambda(\mathrm{d}\gamma_\Lambda|\xi)$ (and hence, $\pi_\Lambda(\mathrm{d}\gamma|\xi)$) is actually a probability measure on Γ_Λ (respectively, on Γ); see the discussion in Subsection 2.2. Furthermore, the following *exponential integrability* property of $\mu_\Lambda(\mathrm{d}\gamma_\Lambda|\xi)$ will be strongly used below. For $\kappa, \lambda \geq 0$, let us define

$$\Gamma_0 \ni \gamma \mapsto \Phi(\gamma) := \kappa H(\gamma) + \lambda|\gamma|^2, \qquad (2.50)$$

which will play a role of *Lyapunov functional* in studying stability properties of our model. According to the hypothesis (2.30) and (2.35),

$$\Phi(\gamma) \geq 0, \quad \text{for any } \gamma \in \Gamma_k,\ k \in \mathbb{Z}^d.$$

Lemma 2.5. *Let the parameters $\kappa \in [0,\beta]$ and $\lambda \geq 0$ obey the relation*

$$\frac{1}{2}(\beta - \kappa)A \;\geq\; \lambda + \frac{3}{2}\beta M m. \qquad (2.51)$$

In particular, one may choose here either $\kappa = 0$ and $\lambda = \beta(A - 3Mm)/2$ or $\lambda = 0$ and $\kappa = \beta(1 - 3Mm/A)$. Then, for any $k \in \mathbb{Z}^d$, $\xi \in \Gamma$ and all sets $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ containing Q_{gk},

$$\int_{\Gamma_\Lambda} \exp\{\Phi(\gamma_k)\}\,\mu_\Lambda(\mathrm{d}\gamma|\xi) < \infty. \qquad (2.52)$$

Proof. Proceeding similarly to (2.45) and (2.49), we immediately obtain that

$$\begin{aligned}
&\int_{\Gamma_\Lambda} \exp\{\Phi(\gamma_k)\}\,\mu_\Lambda(\mathrm{d}\gamma_\Lambda|\xi) \le \int_{\Gamma_\Lambda} \exp\{\Phi(\gamma_k) - \beta H_\Lambda(\gamma_\Lambda|\xi)\}\,\mathrm{d}\lambda_{z\sigma}(\gamma_\Lambda) \\
\le\ & \int_{\Gamma_\Lambda} \exp\left\{\left[\lambda - \frac{A}{2}(\beta-\kappa) + \beta M m\right]|\gamma_k|^2\right. \\
& \left. +\beta\left(-\frac{A}{2} + Mm\right)\sum_{j\in\mathcal{K}_\Lambda\cap\partial_g k}|\gamma_j|^2 + \beta\left(\frac{A}{2} + M|\xi_{\mathcal{U}(\Lambda)}|\right)|\gamma_\Lambda|\right\}\mathrm{d}\lambda_{z\sigma}(\gamma_\Lambda) \\
\le\ & \exp\left\{z\sigma(\Lambda)\exp\left[\beta\left(\frac{A}{2} + M|\xi_{\mathcal{U}(\Lambda)}|\right)\right]\right\}. \qquad (2.53)
\end{aligned}$$

□

2.4. **Main theorems and comments.** The paper contains two principal results, Theorems 2.6 and 2.7, describing the set $\mathcal{G}^{\mathrm{t}}$ of *tempered* (or *physically relevant*) Gibbs measures. Possible improvements of these theorems will be discussed in Subsections 3.4 and 3.5 below.

A starting point in any theory of unbounded spin systems is the proper notion of *temperedness*. Let us introduce the subsets of *tempered configurations*

$$\Gamma^{\mathrm{t}} := \bigcap_{\alpha>0}\Gamma_\alpha, \qquad (2.54)$$

$$\Gamma_\alpha := \left\{\gamma\in\Gamma\ \middle|\ ||\gamma||_\alpha := \left[\sum_{k\in\mathbb{Z}^d}|\gamma_k|^2\exp\{-\alpha|k|\}\right]^{1/2} < \infty\right\}, \quad \alpha>0.$$

Respectively, the subset $\mathcal{G}^{\mathrm{t}}$of *tempered Gibbs states* consists of all $\mu\in\mathcal{G}$ which are supported by Γ^{t}, i.e.,

$$\mathcal{G}^{\mathrm{t}} := \mathcal{G}\cap\mathcal{P}^{\mathrm{t}}(\Gamma), \quad \text{where} \quad \mathcal{P}^{\mathrm{t}}(\Gamma) := \left\{\mu\in\mathcal{P}(\Gamma)\ \middle|\ \mu(\Gamma^{\mathrm{t}}) = 1\right\}. \qquad (2.55)$$

Note that $||\gamma||_\alpha$ extends to a seminorm on the linear space $\mathcal{M}(\mathbb{R}^d)$ ($\supset\Gamma$) of all signed Radon measures on $\mathbb{R}^d$. It is clear that the sets Γ_α and Γ^{t} do not depend on a value of parameter g, which is the edge length of the partition cubes Q_{gk}. Each of Γ_α and Γ^{t} is closed in $(\Gamma,\mathcal{O}(\Gamma))$ (this follows e.g. from (3.31), (3.32) describing convergence in $\mathcal{O}(\Gamma_\Lambda)$ for a compact $\Lambda=\bar\Lambda\in\mathcal{B}_c(\mathbb{R}^d)$). Equipped with the induced vague topology, Γ^{t} becomes a Polish space as well.

We also emphasize that our notion of temperedness is more general than those used in the earlier papers. In particular, $\mathcal{G}^{\mathrm{t}}$ contains the class $\mathcal{G}^{\mathrm{st}}$ of the so-called *Ruelle* type *"superstable"* Gibbs states μ, which are characterized by the support condition (see Equation (5.10) in [38])

$$\sup_{K\in\mathbb{N}}\left\{K^{-d}\sum_{|k|\le K}|\gamma_k|^2\right\} =: C(\gamma) < \infty, \quad \forall\gamma\in\Gamma \ \ (\mu-\text{a.e.}). \qquad (2.56)$$

We begin by verifying the *existence* of tempered Gibbs measures.

Theorem 2.6. *Under hypotheses* (**FR**), (**LB**) *and* (**RC**), *the set of tempered Gibbs measures is not empty, i.e.,* $\mathcal{G}^{\mathrm{t}}\neq\varnothing$.

Next we establish an exponential moment estimate similar to (2.52), which holds *for all* tempered Gibbs measures.

Theorem 2.7. *Let the parameters $\kappa \in [0, \beta]$ and $\lambda \geq 0$ be related by (2.51). Then there exists a positive constant $\mathcal{C} := \mathcal{C}(\kappa, \lambda)$ such that uniformly for all $\mu \in \mathcal{G}^{\mathrm{t}}$*

$$\sup_{k \in \mathbb{Z}^d} \int_{\Omega} \exp \left\{ \kappa H(\gamma_k) + \lambda |\gamma_k|^2 \right\} \mu(\mathrm{d}x) \leq \mathcal{C}. \tag{2.57}$$

Corollary 2.8. *$\mathcal{G}^{\mathrm{t}}$ is the compact set in the vague topology $\mathcal{O}(\Gamma)$.*

The proof of the above assertions essentially relies on certain regularity properties of the specification $\Pi = \{\pi_\Lambda\}$. Because of independent interest, we separately state them as Propositions 2.9 and 2.10 below.

Proposition 2.9. *The specification $\Pi = \{\pi_\Lambda\}$ is Feller in the following sense: for every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, the operator $F \mapsto \pi_\Lambda F$ defined by (2.23) is a contraction in the Banach space $C_b(\Gamma)$.*

Proposition 2.10. *The specification $\Pi = \{\pi_\Lambda\}$ is compact in the following sense: for each $\xi \in \Gamma^{\mathrm{t}}$, the family $\{\pi_\Lambda(\mathrm{d}\gamma_{\Lambda_\mathcal{K}}|\xi)\}_{\mathcal{K} \Subset \mathbb{Z}^d}$ defined by (2.26), (2.40) is relatively compact, as $\mathcal{K} \nearrow \mathbb{Z}^d$, in the topology $\mathcal{O}(\Gamma)$. Furthermore, all its limit points belong to $\mathcal{G}^{\mathrm{t}}$.*

Remark 2.11. (i) The existence problem for $\mu \in \mathcal{G}^{\mathrm{t}}$ goes back to the pioneering works of *R. Dobrushin* [7, 8] and *D. Ruelle* [37, 38]. It is well known that Stability Condition (**S**) enables to construct the Gibbs measures at *small* values of the inverse temperature β and activity z. This can be done either by the method of *cluster expansions* or by the contraction method for the *Kirkwood-Salsburg equation* (see respectively Sections 4.4 and 4.2 of [37]). In order to solve the existence problem for $\mu \in \mathcal{G}^{\mathrm{st}}$ at *all* values of β and z, one typically has to impose Ruelle's Superstability Condition (**RSS**), see [13] and Theorem 5.5 of [38]. The famous *Ruelle's approach* then applies, which is based on certain *à-priori* bounds on correlation functions of the Gibbs measures. In turn, these fundamental bounds are derived from the superstability properties of the interaction. An alternative way is given by *Dobrushin's approach*, which relies on the reduction to the associated lattice system with the further use of the general Dobrushin existence criterion for lattice Gibbs fields. Since the original papers [7, 8], the latter approach has been further developed in [3, 22, 33] (see Section 3 for more details). Of course, the existence result of Theorem 2.6 can be obtained within the both of Ruelle's and Dobrushin's approaches. Our aim here is however not to improve the known existence results (which in particular will be done in the forthcoming paper [21]), but to present a *new* elementary technique of getting the existence as well as *à-priori* bounds for $\mu \in \mathcal{G}^{\mathrm{t}}$. Its conceptual difference from the previous schemes is the systematic use of (infinite dimensional) stochastic analysis substituting long combinatorical arguments. Since we are working directly with the initial continuous system in $\mathbb{R}^d$, no reduction to its lattice counterpart in $\mathbb{Z}^d$ is more needed. As an immediate outcome of our approach, in Theorem 2.7 we get the description of the set of all tempered Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$, which seems to be entirely new in the literature.

(ii) Up to our knowledge, the statements of Propositions 2.9, 2.10 about the *Feller* property and compactness of the specification $\Pi = \{\pi_\Lambda\}$ were not yet available for continuous systems. This opens a natural way of constructing $\mu \in \mathcal{G}^{\mathrm{t}}$ as limit points of the specification kernels $\pi_\Lambda(\mathrm{d}\gamma|\xi)$ as $\Lambda \nearrow \mathbb{R}^d$, and thus allows us to avoid the highly nontrivial analysis of their correlations functions. We call (2.57) the *à-priori* bound, since it is proven simultaneously with the fact of existence of $\mu \in \mathcal{G}^{\mathrm{t}}$. The constant in the right-hand side in (2.57), which is uniform for all $\mu \in \mathcal{G}^{\mathrm{t}}$, depends on the inverse temperature β and parameters of the model only. About possible generalizations of these results see Subsection 3.5.

3. EXISTENCE AND A-PRIORI BOUNDS FOR $\mu \in \mathcal{G}^{\mathrm{t}}$

3.1. **Moment estimates for local Gibbs measures.** This subsection plays a key role in carrying out our strategy. Here we establish the integrability properties of the kernels $\pi_\Lambda(\mathrm{d}\gamma|\xi)$ needed later for proving Theorems 2.6 and 2.7.

Recall that the Lyapunov function $\Phi(\gamma)$ was defined in (2.50). Our aim is to show that the left-hand side in (2.53) can be estimated *uniformly* in volume. To this end we shall perform an inductive scheme based on the consistency property (2.22). A starting point is the following exponential bound for the *"one-point"* kernels $\mu_k(\mathrm{d}\gamma_k|\xi) := \mu_{Q_{gk}}(\mathrm{d}\gamma_{Q_{gk}}|\xi)$ subject to the fixed boundary condition $\xi \in \Gamma$.

Lemma 3.1. *Let $\kappa \in [0, \beta]$ and $\lambda \geq 0$ be the same as in Lemma 2.5. Then, there exists a corresponding $\Upsilon := \Upsilon(\kappa, \lambda) > 0$ such that for all $k \in \mathbb{Z}^d$ and $\xi \in \Gamma$*

$$\int_{\Gamma_k} \exp\{\Phi(\gamma_k)\}\, \mu_k(\mathrm{d}\gamma_k|\xi) \leq \exp\left\{ \Upsilon + \frac{1}{2}\beta M \sum_{j\in\partial_g k} |\xi_j|^2 \right\}. \tag{3.1}$$

Proof. Repeating the estimate (2.53) for $\Lambda := Q_{gk}$, we get by means of the Cauchy inequality that

$$\begin{aligned}
&\int_{\Gamma_k} \exp\{\Phi(\gamma_k)\}\, \mu_k(\mathrm{d}\gamma_k|\xi) \leq \int_{\Gamma_k} \exp\{\Phi(\gamma_k) - \beta H_k(\gamma_k|\xi)\}\, \mathrm{d}\lambda_{z\sigma}(\gamma_k) \\
&\leq \int_{\Gamma_k} \exp\left\{ \left[\lambda - \frac{A}{2}(\beta-\kappa)\right] |\gamma_k|^2 + \left[\frac{A}{2}(\beta-\kappa) + \beta M \sum_{j\in\partial_g k} |\xi_j|\right] |\gamma_k| \right\} \mathrm{d}\lambda_{z\sigma}(\gamma_k) \\
&\leq \int_{\Gamma_k} \exp\left\{ \left[\lambda - \frac{A}{2}(\beta-\kappa) + \frac{1}{2}\beta M m\right] |\gamma_k|^2 + \frac{A}{2}(\beta-\kappa)\, |\gamma_k| \right\} \mathrm{d}\lambda_{z\sigma}(\gamma_k) \\
&\quad \times \quad \exp\left\{ \frac{1}{2}\beta M \sum_{j\in\partial_g k} |\xi_j|^2 \right\}.
\end{aligned} \tag{3.2}$$

In view of (2.51) the claim holds with

$$\Upsilon := \log \int_{\Gamma_k} \exp\left\{ \frac{A}{2}(\beta-\kappa)\, |\gamma_k| \right\} \mathrm{d}\lambda_{z\sigma}(\gamma_k) = z g^d \exp\left\{ \frac{A}{2}(\beta-\kappa) \right\}. \tag{3.3}$$

□

Remark 3.2. (i) A subsequent application of Jensen's inequality to the both sides in (3.1) gives us the following *"Dobrushin-type"* estimate (cf. Equation (4.9) in [8])

$$\int_{\Gamma_k} \Phi(\gamma_k)\mu_k(\mathrm{d}\gamma_k|\xi) \le \Upsilon + \frac{\beta M}{2}\sum_{j\in\partial_g k}|\xi_j|^2 \le \Upsilon + \frac{\beta M}{2\lambda}\sum_{j\in\partial_g k}\Phi(\gamma_j). \tag{3.4}$$

By considering a lattice counterpart of our continuous model (as it was done in [3, 8, 22, 33]), this opens a possibility to apply here the general *Dobrushin's existence criterion*, Theorem 1 in [7]). However, the later scheme is rather cumbersome and leads to the theory of Gibbs measures on a larger space of *multiple* configurations $\ddot{\Gamma} \supset \Gamma$, see [3, 22, 25, 33]. Such technical extension to $\ddot{\Gamma}$ contradicts a physical intuition and leaves open the initial question about the existence of $\mu \in \mathcal{G}^{\mathrm{t}}$.

(ii) Although the bound (3.1) is stronger than (3.4), actually it is derived much easily in view of the additive structure of the Hamiltonian $H(\gamma)$ and the multiplication rule for exponents (see the proof of Lemma 3.1). So, the novelty of our approach will be to use the *exponential* bound (3.1) and to work *directly* with the specification $\pi_\Lambda(\mathrm{d}\gamma|\xi)$ in continuum.

(iii) Other principle difference from the previous schemes is that the function $\Phi(\gamma_k)$ in (2.57) is constructed as a linear combination of the local *energy* $H(\gamma_k)$ and *stabilizing* factor $|\gamma_k|^2$. This not only greatly simplifies all calculations, but also provides optimal estimates on $\mu \in \mathcal{G}^{\mathrm{t}}$ as well. For translation invariant ferromagnets on a lattice, the exponential bound (3.1) with a standard choice of $\Phi(x_k) := \lambda|x_k|$, $\lambda > 0$, was first appeared in [4], whereas the validity of Dobrushin's criterion was checked in [39]. For continuous particle systems in $\mathbb{R}^d$, Dobrushin's bounds like (3.4) with $\Phi(\gamma_k) := \exp\{\lambda|\gamma_k|^\alpha\}$ or $\Phi(\gamma_k) := |\gamma_k|^\alpha$, $\alpha \ge 1$, have been established in [3, 8, 12, 22, 33]. First attempts to modify the former scheme by considering compact functions like $\Phi(\gamma_k) := \exp\{\lambda H(\gamma_k)\}$ can be found in Section 5.2 of [25]. Finally let us note that there is a certain analogy with the lattice case, whereby instead of the single spins $x_k \in \mathbb{R}^d$ one has to control *"collective"* variables $|\gamma_k|$ which are the number of particles in the partition cubes Q_{gk}, $k \in \mathbb{Z}^d$.

Consider the cubic domains $\Lambda_{\mathcal{K}} := \bigsqcup_{k\in\mathcal{K}} Q_k \in \mathcal{B}_c(\mathbb{R}^d)$ indexed by $\mathcal{K} \Subset \mathbb{Z}^d$. Note that $\Lambda_{\mathcal{K}} \nearrow \mathbb{R}^d$ as $\mathcal{K} \nearrow \mathbb{Z}^d$. Using the *"one-point"* estimate (3.1), our next step will be to get similar moment estimates for all specification kernels $\pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi) := \pi_{\Lambda_{\mathcal{K}}}(\mathrm{d}\gamma|\xi)$. Let us define for $k \in \mathbb{Z}^d$

$$n_k(\mathcal{K}|\xi) := \log\left\{\int_{\Gamma} \exp\left\{\Phi(\gamma_k)\right\}\pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi)\right\}, \tag{3.5}$$

which are nonnegative and finite by Lemma 3.1. A natural idea is to establish certain *weighted* l^1-bounds on the sequence $(n_k(\mathcal{K}|\xi))_{k\in\mathbb{Z}^d}$, which then imply the required bounds on every its component in (3.5).

Lemma 3.3. *Under the assumptions of Lemma 3.1, there exists a finite* $\Upsilon_\alpha := \Upsilon_\alpha(\kappa, \lambda) > 0$ *such that uniformly for all* $k_0 \in \mathbb{Z}^d$ *and* $\xi \in \Gamma^{\mathrm{t}}$

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d}\left[\sum_{k\in\mathcal{K}} n_k(\mathcal{K}|\xi)\exp\{-\alpha|k-k_0|\}\right] \le \Upsilon_\alpha. \tag{3.6}$$

Herefrom, in particular,

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d}\int_{\Gamma}\exp\left\{\kappa H(\gamma_k) + \lambda|\gamma_k|^2\right\}\pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi) \le \exp\Upsilon_\alpha, \quad \textit{for any } k. \tag{3.7}$$

Proof. Without loss of generality, we may assume (due to (2.30) and (2.51)) that

$$\frac{1}{2}\beta M m e^{\alpha} < \lambda \le \frac{1}{2}(\beta-\kappa) A - \beta M m, \tag{3.8}$$

which is achieved by a small enough $\alpha > 0$. Integrating both sides of (3.1) with respect to $\pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi)$ with an arbitrary $\xi \in \Gamma^{\mathrm{t}}$ and taking into account the consistency property (2.22), we arrive at the following estimate

$$\begin{aligned} n_k(\mathcal{K}|\xi) \quad &\le \quad \Upsilon + \frac{\beta M}{2} \sum_{j \in \mathcal{K}^c \cap \partial_g k} |\xi_j|^2 \\ &\quad + \log\left\{ \int_{\Gamma} \exp\left(\frac{\beta M}{2\lambda} \sum_{j\in\mathcal{K}\cap\partial_g k} \lambda |\gamma_j|^2 \right) \pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi) \right\} \\ &\le \quad \Upsilon + \frac{\beta M}{2} \sum_{j \in \mathcal{K}^c \cap \partial_g k} |\xi_j|^2 + \frac{\beta M}{2\lambda} \sum_{j\in\mathcal{K}\cap\partial_g k} n_j(\mathcal{K}|\xi), \end{aligned} \tag{3.9}$$

where the constant $\Upsilon := \Upsilon(\kappa,\lambda)$ is the same as in (3.3). Here we have used the multiple Hölder inequality

$$\mu\left(\prod\nolimits_{j=1}^{K} f_j^{s_j}\right) \le \prod\nolimits_{j=1}^{K} \mu^{s_j}(f_j), \qquad \mu(f_j) := \int f_j \mathrm{d}\mu, \tag{3.10}$$

valid for any probability measure μ, nonnegative functions f_j, and numbers $s_j \ge 0$ such that $\sum_{j=1}^{K} s_j \le 1$. In our context, $f_j := \exp\left(\lambda|\gamma_j|^2\right)$ and $s_j := \beta M/2\lambda < 1/m$ for all $j \in \partial_g k$. After summing in (3.9) over $k \in \mathcal{K}$ with the weights $\exp\{-\alpha|k - k_0|\}$, we get that

$$\begin{aligned} n_{k_0}(\mathcal{K}|\xi) &\le \sum_{k\in\mathcal{K}} n_k(\mathcal{K}|\xi)\exp\{-\alpha|k_0-k|\} \\ &\le \left(1 - \frac{1}{2\lambda}\beta M m e^{\alpha}\right)^{-1} \left[\Upsilon \sum_{k\in\mathcal{K}} \exp\{-\alpha|k|\} + \beta M m e^{\alpha} \|\xi_{\mathcal{K}^c}\|_{\alpha}^2 \right]. \end{aligned} \tag{3.11}$$

Since $\|\xi_{\mathcal{K}^c}\|_{\alpha}$ tends to zero as $\mathcal{K} \nearrow \mathbb{Z}^d$, we obtain that for each $k_0 \in \mathbb{Z}^d$

$$\begin{aligned} \limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} n_{k_0}(\mathcal{K}|\xi) \quad &\le \quad \limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} \left[\sum_{k\in\mathcal{K}} n_k(\mathcal{K}|\xi)\exp\{-\alpha|k_0-k|\} \right] \\ &\le \quad \Upsilon \left(1 - \frac{1}{2\lambda}\beta M m e^{\alpha}\right)^{-1} \sum_{k\in\mathbb{Z}^d} \exp\{-\alpha|k|\} =: \Upsilon_{\alpha}, \end{aligned} \tag{3.12}$$

which completes the proof of (3.6) and (3.7). □

3.2. **Proof of Theorems 2.6 and 2.7.** Here we prove our main Theorems 2.6 and 2.7 describing the set $\mathcal{G}^{\mathrm{t}}$. Instead of reducing to the lattice case and then using the general Dobrushin criterion, we prefer to follow a direct way. To some extent this strategy is motivated by the paper of J. Bellissard and R. Høegh-Krohn [4], dealing however with a much different and simpler model of lattice ferromagnets. The main idea is to show that the uniform bounds (3.7) for $\pi_{\Lambda}(\mathrm{d}\gamma|\xi)$, combined with the compactness argument in the topology $\mathcal{O}(\Gamma)$, readily imply the *existence* of $\mu \in \mathcal{G}^{\mathrm{t}}$. Furthermore, on this way we also get *à-priori* moment bounds like (2.52) to be valid uniformly *for all* measures $\mu \in \mathcal{G}^{\rm t}$. Note that the latter information cannot be straightforwardly extracted from Dobrushin's criterion, which just provides existence of *some* $\mu \in \mathcal{G}^{\rm t}$ being limit points of $\pi_\Lambda(\mathrm{d}\gamma|\xi)$ as $\Lambda \nearrow \mathbb{R}^d$.

Proof of Theorem 2.6 (including **Proposition 2.10**)**.** For $\kappa = 0$ and a fixed positive $\lambda \le \frac{1}{2}\beta\,(A - 3Mm)$, by (3.6) and Jensen's inequality we conclude that

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} \int_\Gamma ||\gamma||_\alpha^2 \pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi) \le\, \Upsilon_\alpha/\lambda, \quad \text{for all } \alpha > 0, \tag{3.13}$$

where $||\gamma||_\alpha$is defined by (2.54). Hence, for each $\xi \in \Gamma^{\rm t}$ one finds a finite $\mathcal{C}_\alpha(\xi) > 0$ such that

$$\sup_{|\mathcal{K}|<\infty} \int_\Gamma ||\gamma||_\alpha^2 \pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi) \le \mathcal{C}_\alpha(\xi). \tag{3.14}$$

Now we would like to apply Prokhorov's criterion on weak convergence of measures on Polish spaces, see e.g. Theorem 6.1 in [5]. A technical problem is that the level sets

$$\{\gamma \in \Gamma \mid ||\gamma||_\alpha \le C < \infty\}$$

are not relatively compact in the vague topology on Γ. Nevertheless, one easily observes (by Proposition 4.1.3 of [16]) that they are relatively compact in a larger space of *multiple* configurations

$$\ddot{\Gamma} := \left\{ \ddot{\gamma} = (\gamma, n)\,,\ \gamma \subset \mathbb{R}^d,\ n : \gamma \to \mathbb{N} \;\middle|\; \sum_{x\in\gamma_\Lambda} n(x) < \infty, \quad \forall \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \right\}. \tag{3.15}$$

Recall (see Proposition 1.9.1 of [16]) that $\ddot{\Gamma}$ is a Polish space with respect to the vague topology $\mathcal{O}(\ddot{\Gamma})$ induced from $\mathcal{M}(\mathbb{R}^d)$, whereby $\Gamma \subset \ddot{\Gamma}$ is a measurable set in the Borel σ-algebra $\mathcal{B}(\ddot{\Gamma})$ (the latter is also clear from (3.25)). Thus, by Prokhorov's criterion there exists a limit probability measure $\mu_* \in \mathcal{P}(\ddot{\Gamma})$ such that

$$\int_\Gamma F(\gamma)\pi_{\mathcal{K}_N}(\mathrm{d}\gamma|\xi) \to \int_{\ddot{\Gamma}} F(\ddot{\gamma})\mu_*(\mathrm{d}\ddot{\gamma}), \quad N \to \infty, \tag{3.16}$$

along some order generating sequence $\mathcal{K}_N \nearrow \mathbb{Z}^d$. The convergence (3.16) holds for all bounded continuous functions $F \in C_b(\ddot{\Gamma})$, which constitute a measure defining class for every $\mu \in \mathcal{P}(\ddot{\Gamma})$. Note that $C_b(\ddot{\Gamma})$ is a Banach space with the sup-norm $||F||_{C_b(\ddot{\Gamma})}$ and there is an inclusion $C_b(\ddot{\Gamma}) \subset C_b(\Gamma)$. Below we show that the above μ_* is indeed supported by $\Gamma^{\rm t}$ and satisfies the *DLR* equation (2.26).

Consider the following extension of the specification $\Pi = \{\pi_\Lambda\}$ to the multiple configurations (cf. (2.18), (2.21)). For each $\Delta \in \mathcal{B}(\ddot{\Gamma})$ and $\ddot{\eta} \in \ddot{\Gamma}$ we set

$$\pi_\Lambda(\Delta|\ddot{\eta}) := [Z_\Lambda(\ddot{\eta})]^{-1} \int_{\Gamma_\Lambda} \mathbf{1}_\Delta(\gamma_\Lambda \cup \ddot{\eta}_{\Lambda^c}) \exp\{-\beta H_\Lambda(\gamma_\Lambda|\ddot{\eta})\}\, \lambda_{z\sigma}(\mathrm{d}\gamma_\Lambda), \tag{3.17}$$

$$Z_\Lambda(\ddot{\eta}) := \int_{\Gamma_\Lambda} \exp\{-\beta H_\Lambda(\gamma_\Lambda|\ddot{\eta})\}\, \lambda_{z\sigma}(\mathrm{d}\gamma_\Lambda), \tag{3.18}$$

where (analogously to (2.15)–(2.17))

$$H_\Lambda(\gamma_\Lambda|\ddot{\eta}) := H(\gamma_\Lambda) + W(\gamma_\Lambda|\ddot{\eta}), \tag{3.19}$$

$$W(\gamma_\Lambda|\ddot{\eta}) := \sum_{x\in\gamma_\Lambda,\ y\in\ddot{\eta}_{\Lambda^c}} n(y)V(x,y).$$

The partition function $Z_\Lambda(\ddot\eta)$ is finite in accordance with (2.49), so that each $\pi_\Lambda(\mathrm{d}\ddot\gamma|\ddot\eta)$ will be a probability measure on $\ddot\Gamma$ actually supported by configurations of the form $\ddot\gamma := \gamma_\Lambda \cup \ddot\eta_{\Lambda^c}$ with $\gamma_\Lambda \in \Gamma_\Lambda$. In particular,

$$\pi_\Lambda(\Delta_\Lambda|\ddot\eta) = 1, \quad \Delta_\Lambda := \left\{\ddot\gamma := \ddot\gamma_\Lambda \cup \ddot\gamma_{\Lambda^c} \in \ddot\Gamma \mid \ddot\gamma_\Lambda \in \Gamma_\Lambda\right\} \in \mathcal{B}(\ddot\Gamma). \tag{3.20}$$

One can directly check that the family of probability kernels (3.17) obeys the consistency property like (2.22). The latter means that for any $F \in C_b(\ddot\Gamma)$

$$\int_{\ddot\Gamma} (\pi_\Lambda F)(\ddot\gamma)\pi_{\Lambda'}(\mathrm{d}\ddot\gamma|\ddot\eta) = (\pi_{\Lambda'}F)(\ddot\eta), \quad \Lambda \subseteq \Lambda', \tag{3.21}$$

where we denote

$$(\pi_\Lambda F)(\ddot\eta) := \int_{\ddot\Gamma} F(\ddot\gamma)\pi_{\Lambda'}(\mathrm{d}\ddot\gamma|\ddot\eta) \tag{3.22}$$

$$= [Z_\Lambda(\ddot\eta)]^{-1} \int_{\Gamma_\Lambda} F(\gamma_\Lambda \cup \ddot\eta_{\Lambda^c}) \exp\left\{-\beta\left[H(\gamma_\Lambda) + \beta W(\gamma_\Lambda|\ddot\eta_{\Lambda^c})\right]\right\} \lambda_{z\sigma}(\mathrm{d}\gamma_\Lambda).$$

For every function $F \in C_b(\ddot\Gamma)$ we claim that $\pi_\Lambda F \in C_b(\ddot\Gamma)$ as well; for the proof of this fact see Lemma 3.4 below (which, in turn, generalizes Proposition 2.9). Thus, having regard of (3.16) and (3.21), one can perform the limit procedure

$$\begin{aligned}\int_{\ddot\Gamma} (\pi_\Lambda F)(\ddot\gamma)\mu_*(\mathrm{d}\ddot\gamma) &= \lim_{N\to\infty} \int_\Gamma (\pi_\Lambda F)(\gamma)\pi_{\mathcal{K}_N}(\mathrm{d}\gamma|\xi) \\ &= \lim_{N\to\infty} \int_\Gamma F(\gamma)\pi_{\mathcal{K}_N}(\mathrm{d}\gamma|\xi) = \int_{\ddot\Gamma} F(\ddot\gamma)\mu_*(\mathrm{d}\ddot\gamma),\end{aligned} \tag{3.23}$$

for all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $F \in C_b(\ddot\Gamma)$. Thus, μ_* satisfies the *DLR* equation with respect to the extended specification (3.17)

$$\mu_*(\Delta) = \int_{\ddot\Gamma} \pi_\Lambda(\Delta|\ddot\gamma)\mu_*(\mathrm{d}\ddot\gamma), \quad \Lambda \Subset \mathbb{L}, \ \Delta \in \mathcal{B}(\ddot\Gamma). \tag{3.24}$$

It remains to check that $\mu_*(\Gamma^{\mathrm{t}}) = 1$ for any $\mu_* \in \mathcal{P}(\ddot\Gamma)$ which solves (3.24). Indeed, let us take a sequence of compacts $\bar\Lambda_N := \bigcup_{k\in\mathcal{K}_N} \bar Q_k$ indexed by $\mathcal{K}_N \nearrow \mathbb{Z}^d$. Then, the subset of simple configurations can be represented as

$$\Gamma = \bigcap_{N\in\mathbb{N}} \Gamma_N, \quad \Gamma_N := \left\{\ddot\gamma \in \ddot\Gamma \,\middle|\, \ddot\gamma_{\bar\Lambda_N} \in \Gamma_{\bar\Lambda_N}\right\} \in \mathcal{B}(\ddot\Gamma), \tag{3.25}$$

which together with (3.20) and (3.24) implies that

$$\mu_*(\Gamma) = \lim_{N\to\infty} \mu_*(\Gamma_N) = \lim_{N\to\infty} \int_{\ddot\Gamma} \pi_{\Lambda_N}(\Gamma_N|\ddot\gamma)\mu_*(\mathrm{d}\ddot\gamma) = 1. \tag{3.26}$$

Finally, by means of (3.14), (3.31), and Fatou's lemma we conclude that

$$\begin{aligned}\int_\Gamma ||\gamma||_\alpha^2 \mu_*(\mathrm{d}\gamma) &= \lim_{K\to\infty} \sum_{|k|\le K} \exp\{-\alpha|k|\} \int_\Gamma |\gamma \cap \bar Q_{gk}|^2 \mu_*(\mathrm{d}\gamma) \\ &= \lim_{K\to\infty} \lim_{N\to\infty} \sum_{|k|\le K} \exp\{-\alpha|k|\} \int_\Gamma |\gamma \cap \bar Q_{gk}|^2 \pi_{\mathcal{K}^{(N)}}(\mathrm{d}\gamma|\xi) \le \mathcal{C}_\alpha(\gamma),\end{aligned} \tag{3.27}$$

which in particular yields that $\mu_*(\Gamma_\alpha) = 1$ for each $\alpha > 0$. $\square$

Proof of Theorem 2.7. For each $\mu \in \mathcal{G}^{\mathrm{t}}$, with help of (2.26), (3.14), and Fatou's lemma we have the following estimate with arbitrary $N > 0$

$$\begin{aligned} &\int_{\Gamma_\alpha} \exp\left\{\Phi(\gamma_k) \wedge N\right\} \mu(\mathrm{d}\gamma) \qquad (3.28)\\ = &\lim_{\mathcal{K} \nearrow \mathbb{Z}^d} \int_{\Gamma_\alpha} \int_{\Gamma_\alpha} \left\{\exp \Phi(\gamma_k) \wedge N\right\} \pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi) \mu(\mathrm{d}\xi)\\ \leq &\int_{\Gamma_\alpha} \left[\limsup_{\mathcal{K} \nearrow \mathbb{Z}^d} \int_{\Gamma_\alpha} \exp\left\{\Phi(\gamma_k)\right\} \pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi)\right] \mu(\mathrm{d}\xi) \leq \exp \Upsilon_\alpha, \end{aligned}$$

where Υ_α was introduced in (3.12). Applying Fatou's lemma once more, we conclude from (3.28) that

$$\begin{aligned} &\int_{\Gamma_\alpha} \exp\left\{\Phi(\gamma_k)\right\} \mu(\mathrm{d}\gamma) \qquad (3.29)\\ \leq &\limsup_{N \to \infty} \int_{\Gamma_\alpha} \exp\left\{\Phi(\gamma_k) \wedge N\right\} \mu(\mathrm{d}x) \leq \exp \Upsilon_\alpha. \end{aligned}$$

Hence, (3.29) yields us the desired estimate (2.57) with the constant

$$\mathcal{C} := \inf_{\alpha > 0} \left\{\exp \Upsilon_\alpha\right\}, \qquad (3.30)$$

which is the same for all $\mu \in \mathcal{G}^{\mathrm{t}}$. □

Proof of Corollary 2.8. The line of reasoning is similar to that proved Theorem 2.6. From Prokhorov's criterion and the uniform estimate

$$\sup_{\mu \in \mathcal{G}^{\mathrm{t}}} \int_{\Gamma} ||\gamma||_\alpha^2 \mu(\mathrm{d}\gamma) < \infty$$

we get the relative compactness of the set $\mathcal{G}^{\mathrm{t}}$ in the topology $\mathcal{O}(\ddot{\Gamma})$. Let $\mu_* \in \mathcal{P}(\ddot{\Gamma})$ be one of its limit points. Due to the Feller property of the specification (3.19) established by Lemma 3.4, this μ_* ought to satisfy the *DLR* equation (3.24) on $\ddot{\Gamma}$. Repeating the arguments (3.25)–(3.27), we conclude that $\mu_*(\Gamma^{\mathrm{t}}) = 1$ and hence $\mu_* \in \mathcal{G}^{\mathrm{t}}$. □

3.3. **Feller property of the specification.** In the proof of Theorem 2.6 we have crucially used the so-called *Feller regularity* of the specification $\Pi = \{\pi_\Lambda\}$, which makes the contest of Lemma 3.4 below and extends Proposition 2.9. Somewhat surprisingly, it turns out that such regularity of the kernels $\pi_\Lambda(\mathrm{d}\gamma|\xi)$ holds true, even though the potential $V(x, y)$ itself may be *singular* at the diagonal.

Lemma 3.4. *(generalizing **Proposition 2.9**) For every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, the mapping $F \mapsto \pi_\Lambda F$ defined by (3.22) (respectively, by (2.23)) is a contraction in $C_b(\ddot{\Gamma})$ (respectively, in $C_b(\Gamma)$).*

Proof. A non-trivial issue is to check that π_Λ preserves the class $C_b(\ddot{\Gamma})$. We start by the observation that, for every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $\gamma_\Lambda \in \Gamma_\Lambda$, the interaction energy (cf. (2.15))

$$\ddot{\Gamma} \ni \ddot{\eta} \mapsto W(\gamma_\Lambda|\ddot{\eta}) := \sum_{x \in \gamma_\Lambda,\ y \in \ddot{\eta}_{\Lambda^c}} n(y) V(x, y)$$

is a cylinder continuous function depending on the restriction $\ddot{\eta}_{\mathcal{V}(\Lambda)} := \ddot{\eta} \cap \mathcal{V}(\Lambda)$ to the domain $\mathcal{V}(\Lambda) := \{y \in \Lambda^c \mid \mathrm{dist}(y, \Lambda) \leq R\}$. Indeed, let a sequence $\left\{\ddot{\eta}^{(N)},\ N \in \mathbb{N}\right\} \subset$

$\ddot{\Gamma}$ converge to some $\ddot{\eta} \in \ddot{\Gamma}$ in the vague topology $\mathcal{O}(\ddot{\Gamma})$. For each compact set $\bar{\Delta} \in \mathcal{B}_c(\mathbb{R}^d)$, we thus have the convergence of $\ddot{\eta}^{(N)}_{\bar{\Delta}}$ to $\ddot{\eta}_{\bar{\Delta}} := \{x_j\}_{j=1}^n$ in the induced topology $\mathcal{O}(\ddot{\Gamma}_{\bar{\Lambda}'})$. The latter is *equivalent* to the following (cf. Proposition 1.9.10 of [16]):

(a) There exists $N_0 \in \mathbb{N}$ such that the particle numbers $n := |\ddot{\eta}_{\bar{\Delta}}|$, $n_N := |\ddot{\eta}^{(N)}_{\bar{\Delta}}|$ obey

$$n = n_N, \quad \text{for all } N \geq N_0. \tag{3.31}$$

(b) In each configuration $\ddot{\eta}^{(N)}_{\bar{\Delta}}$, $N \geq N_0$, one can choose a numeration of its particles $\{x_j^{(N)}\}_{j=1}^n$ (with possible repetitions) in a way that

$$\left| x_j^{(N)} - x_j \right| \to 0, \quad \text{for all } 1 \leq j \leq n. \tag{3.32}$$

By picking some compact $\bar{\Delta} \supseteq \mathcal{V}(\Lambda)$, we observe from (3.32) that

$$\lim_{N\to\infty} \operatorname{dist}(\gamma_\Lambda, \ddot{\eta}^{(N)}_{\mathcal{V}(\Lambda)}) = \operatorname{dist}(\gamma_\Lambda, \ddot{\eta}_{\mathcal{V}(\Lambda)}) > 0$$

for every fixed $\gamma_\Lambda \in \Gamma_\Lambda$. Due to the continuity of the potential $V(x,y)$ for $x \neq y$, this yields the desired convergence

$$\lim_{N\to\infty} W(\gamma_\Lambda|\ddot{\eta}^{(N)}) = W(\gamma_\Lambda|\ddot{\eta}).$$

Let us check that $\lim_{N\to\infty}(\pi_\Lambda F)(\ddot{\eta}^{(N)}) = (\pi_\Lambda F)(\ddot{\eta})$. By (2.28), (2.45), and (3.19) there is a uniform bound

$$\begin{aligned} &|F(\gamma_\Lambda \cup \ddot{\eta}^{(N)}_{\Lambda^c})| \exp\left\{-\beta H(\gamma_\Lambda|\ddot{\eta}^{(N)})\right\} \\ \leq \quad & ||F||_{C_b(\ddot{\Gamma})} \exp\left\{\left(\frac{A}{2} + M|\ddot{\eta}_\Lambda|\right)|\gamma_\Lambda^{(N)}|\right\}, \quad \gamma \in \Gamma_\Lambda,\ \ddot{\eta} \in \ddot{\Gamma}, \end{aligned}$$

which enables us to apply Lebesgue's dominated convergence theorem to the integral with respect to $\lambda_{z\sigma}(\mathrm{d}\gamma_\Lambda)$ in the last line in (3.22). The partition functions $Z_\Lambda(\ddot{\eta}^{(N)}) > 1$ are considered in the same way. Thus we obtain the required continuity of $\ddot{\Gamma} \ni \ddot{\eta} \mapsto (\pi_\Lambda F)(\ddot{\eta})$. Obviously,

$$||\pi_\Lambda F||_{C_b(\ddot{\Gamma})} \leq ||F||_{C_b(\ddot{\Gamma})} := \sup_{\ddot{\eta}\in\ddot{\Gamma}} |F(\ddot{\eta})|,$$

which completes the proof. In a similar way one can check the continuity of π_Λ in $C_b(\Gamma)$. □

3.4. **Regularity of $\mu \in \mathcal{G}^{\mathrm{t}}$ due to singular potentials.** Of special interest in physical applications is the case when

$$V(x,y) \geq v(|x-y|), \quad x, y \in \mathbb{R}^d, \tag{3.33}$$

with a majorating function $v \in C(\mathbb{R}_+ \to \mathbb{R})$ such that $\lim_{t\to+0} v(t) = +\infty$. Theorem 2.7 then says that all Gibbs measures $\mu \in \mathcal{G}$ which are initially supported by the set Γ^{t} (cf. (2.54)) *à-posteori* has to obey the much stronger integrability property

$$\sup_{k\in\mathbb{Z}^d} \int_\Gamma \exp\left\{\beta \sum_{\{x,y\}\subset\gamma_k} v(|x-y|)\right\} \mu(\mathrm{d}\gamma) \leq \mathcal{C}. \tag{3.34}$$

¿From a technical point of view, the possibility of such regularization relies on the fact that the right-hand side in our basic estimates (3.1) and (3.11) does *not* depend at all on the values of $v(|x-y|)$ for $x, y \in \xi$. For example, let us suppose (similarly to (2.32)) that for some $C, \varkappa > 0$

$$v(x) \geq C|x|^{-(d+\varkappa)}, \quad \text{as} \quad |x| \to 0. \tag{3.35}$$

Then, according to (3.34) one finds a certain $\kappa > 0$ such that

$$\sup_{\mu \in \mathcal{G}^{\mathrm{t}}} \sup_{k \in \mathbb{Z}^d} \int_{\Gamma} \exp\left\{ \kappa \sum_{\{x,y\} \subset \gamma_k} |x-y|^{-(d+\varkappa)} \right\} \mu(\mathrm{d}\gamma) < \infty. \tag{3.36}$$

Note that the proof of Theorem 2.6 in this situation can be essentially simplified, as it is possible to employ the compactness argument directly in the vague topology $\mathcal{O}(\Gamma)$. To this end, let us consider a finite covering of the cube $[-g, g]^d$ by the translates $Q_0(a_l) := a_l + Q_0$ with some $a_l \in \mathbb{R}^d$ and $1 \leq l \leq L < \infty$. Recall that Q_0 is given by (2.33) with $k = 0$. Without loss of generality, we may assume that each $x \in [-g, g]^d$ is an inner point at least in one of $Q(a_l)$. Setting

$$Q_{gk}(a_l) := gk + Q_0(a_l), \quad \gamma_{k,l} := \gamma \cap Q_{gk}(a_l), \quad k \in \mathbb{Z}^d,\ 1 \leq l \leq L,$$

let us define for $\gamma \in \Gamma_0$

$$\Psi(\gamma) := \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq l \leq L} \Phi(\gamma_{k,l}) \exp\{-\alpha|k|\}, \tag{3.37}$$

$$\Phi(\gamma) := \kappa \sum_{\{x,y\} \subset \gamma} v(|x-y|) + \lambda|\gamma|^2.$$

An important observation based on (3.33) is that $\Psi : \Gamma_{\bar{\Lambda}} \to \mathbb{R}_+ \cup \{+\infty\}$ is a *compact function* in the vague topology $\mathcal{O}(\Gamma_{\bar{\Lambda}})$ for each closed $\bar{\Lambda} \in \mathcal{B}_c(\mathbb{R}^d)$ (about the compactness property in configuration spaces, see e.g. [14, 22, 25]). Therefore, we can immediately apply Prokhorov's criterion in the space $\Gamma := \mathrm{pr\,lim}_{\bar{\Lambda} \in \mathcal{B}_c(\mathbb{R}^d)} \Gamma_{\bar{\Lambda}}$, so that no more intermediate steps with extension to the multiple configurations $\ddot{\gamma} \in \ddot{\Gamma}$ are needed at all.

3.5. **Strong superstable interactions.** Here we outline possible improvements of Theorems 2.6 and 2.7. Let us substitute Repulsion Assumption (**RC**) from Subsection 2.3 by the following one:

(**SSS**)**:** ***Strong Superstability***: *There exist constants $D, E > 0$ and $P > 2$ such that for some partition $\mathbb{R}^d = \bigsqcup_{k \in \mathbb{Z}^d} Q_{gk}$ with $g > 0$, cf. (2.33),*

$$H(\gamma) \;\geq\; D \sum_{k \in \mathbb{Z}^d} |\gamma_k|^P - E|\gamma|, \quad \text{for all } \gamma \in \Gamma_0. \tag{3.38}$$

Since $|\gamma_k|^P \geq |\gamma_k|^2$ for any $|\gamma_k| \in \mathbb{N} \cup \{0\}$, it is obvious that (**SSS**) is stronger than Ruelle's superstability (**RSS**) resulting from (**RC**), see Lemma 2.3 (a). According to (3.38), the function V should be bounded from below, i.e.,

$$V(x,y) = H(\{x,y\}) \geq -M, \quad \text{with} \quad M := 2E, \tag{3.39}$$

which agrees with the other initial Assumption (**LB**). In the present form the notion of strong superstability was firstly used in [29], although some arguments leading to better analysis of stability can be already found in the earlier contributions [6, 8, 32, 37]. For a historical survey and a list of sufficient conditions for (**SSS**) in terms of the potentials V we refer to [36]. In particular, Theorem 2.3 there says that the asymptotical behavior of V like as in (2.32) and (3.35) ensures the fulfillment of (3.38) with $P = 2 + \varkappa/d$.

Under Assumptions (**FR**), (**LB**), and (**SSS**) we can get a substantial refinement of the previous results. For fixed

$$0 \le \kappa < \beta \quad \text{and} \quad 0 \le \lambda < (\beta - \kappa)\, D, \tag{3.40}$$

let us define the Lyapunov function

$$\mathit{\Gamma}_k \ni \gamma_k \mapsto \Phi(\gamma_k) := \kappa H(\gamma_k) + \lambda |\gamma_k|^P \ge 0. \tag{3.41}$$

A starting point is the following modification of the exponential bounds (3.1), (3.2)

$$\begin{aligned} &\int_{\mathit{\Gamma}_k} \exp\{\Phi(\gamma_k)\}\, \mu_k(\mathrm{d}\gamma_k|\xi) \\ \le\;& \int_{\mathit{\Gamma}_k} \exp\left\{[\lambda - (\beta-\kappa)\, D]\, |\gamma_k|^P + \left[\frac{1}{2}(\beta-\kappa) + \beta|\xi_{\partial_g k}|\right] M|\gamma_k|\right\} \mathrm{d}\lambda_{z\sigma}(\gamma_k) \\ \le\;& \exp\left\{ \mathit{\Upsilon}_\epsilon + \epsilon\beta M \sum_{j\in\partial_g k} |\xi_j|^P \right\}, \end{aligned} \tag{3.42}$$

with a small enough

$$0 < \epsilon < (\beta M m)^{-1} \min\{\lambda;\ \ [(\beta-\kappa)\, D - \lambda]\}$$

and a corresponding

$$\begin{aligned} \mathit{\Upsilon}_\epsilon :=\; & \mathit{\Upsilon}_\epsilon(\kappa,\lambda) := \beta M \epsilon^{\frac{2}{2-P}} \qquad (3.43) \\ + \log & \int_{\mathit{\Gamma}_k} \exp\left\{[\lambda - (\beta-\kappa)\, D + \epsilon\beta M m]\cdot|\gamma_k|^P + \frac{1}{2}(\beta-\kappa)\, M|\gamma_k|\right\} \mathrm{d}\lambda_{z\sigma}(\gamma_k). \end{aligned}$$

In deriving (3.42) we have used Young's inequality in the form

$$st \le \epsilon\left(s^P + t^P\right) + \epsilon^{\frac{2}{2-P}}, \quad \text{for any } \epsilon, s, t > 0. \tag{3.44}$$

Thereafter, we can follow step by step the arguments used in proving Theorems 2.6 and 2.7. The statement of Theorem 2.7 now reads as

$$\sup_{k\in\mathbb{Z}^d} \int_{\mathit{\Omega}} \exp\left\{\kappa H(\gamma_k) + \lambda|\gamma_k|^P\right\} \mu(\mathrm{d}x) \le \mathcal{C}(\kappa,\lambda), \tag{3.45}$$

which holds uniformly for all $\mu \in \mathcal{G}^{\mathrm{t}}$ supported by

$$\left\{ \gamma \in \mathit{\Gamma} \;\middle|\; \sum_{k\in\mathbb{Z}^d} |\gamma_k|^P \exp\{-\alpha|k|\} < \infty, \quad \forall \alpha > 0 \right\}. \tag{3.46}$$

It is clear that both (2.54) and (3.46) define the same set $\mathit{\Gamma}^{\mathrm{t}}$. Finally we note that the value of $D > 0$ in Assumption (**SSS**) is not relevant for the existence result of Theorem 2.6.

Remark 3.5. (i) In the proof of (3.41) we used only the lower bound (2.28) and the *"local"* version of (3.38) just for all $\gamma \in \mathit{\Gamma}_k$, $k \in \mathbb{Z}^d$.

(ii) The *most general* setup which completely includes the previous Subsections 2.3–3.5 can be given as follows. Suppose that there exist a function $\phi : \mathit{\Gamma}_0 \to \mathbb{R}_+$

and constants $M, A, B \geq 0$ such that for all $\gamma_k \in \Gamma_k$ and $\xi_j \in \Gamma_j$ with $k \neq j$,

$$H(\gamma_k) \geq \frac{A}{2}\phi(\gamma_k) - B|\gamma_k|, \tag{3.47}$$

$$W(\gamma_k, \xi_j) := \sum_{x\in\gamma_k,\ y\in\xi_j} W(x,y) \geq -\frac{M}{2}\left[\phi(\gamma_k) + \phi(\xi_j)\right]. \tag{3.48}$$

In addition, let $A \geq 3Mm$ with the parameter m defined in (2.31). Note that the standard *superstability* (or *strong superstability*) properties analyzed before are reduced to the *particular* choice of $\phi(\gamma_k) = |\gamma_k|^2$ (or $\phi(\gamma_k) = |\gamma_k|^P$ with $P > 2$). Respectively, our main statements will concern the Lyapunov function

$$\Gamma_0 \ni \gamma \mapsto \Phi(\gamma) := \kappa H(\gamma) + \lambda\phi(\gamma),$$

where $\kappa \in [0, \beta]$ and $\lambda \geq 0$ are related by (2.51). Going through the proof of Lemma 3.1 and taking use of (3.47), (3.48), we obtain the key bound

$$\int_{\Gamma_k} \exp\left\{\Phi(\gamma_k)\right\}\mu_k(\mathrm{d}\gamma_k|\xi) \leq \exp\left\{\Upsilon + \frac{1}{2}\beta M \sum_{j\in\partial_g k}\phi(\xi_j)\right\}. \tag{3.49}$$

In turn, (3.49) implies the existence of the Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ (cf. Theorem 2.6) and the following uniform estimate for them (cf. Theorem 2.7)

$$\sup_{k\in\mathbb{Z}^d}\int_{\Omega}\exp\left\{\kappa H(\gamma_k) + \lambda\phi(\gamma_k)\right\}\mu(\mathrm{d}x) \leq \mathcal{C}(\kappa, \lambda). \tag{3.50}$$

3.6. **Support properties of $\mu \in \mathcal{G}^{\mathrm{t}}$.** There are few important sequels from the *à-priori* bounds (2.57), (3.45). Let us recall that the set of tempered Gibbs measures was introduced by means of the rather moderate restrictions (2.54), (2.55). We now show that all $\mu \in \mathcal{G}^{\mathrm{t}}$ indeed are carried by a much smaller *universal subset*, which is known for $P = 2$ as the *Lanford–Lebowitz–Presutti support* (see Definition 3.2 of [26] in the lattice case and respectively Definition 5.2.1 of [25] for configuration spaces). Keeping fixed the edge length $g := \delta/\sqrt{d}$ of the partition cubes Q_{gk}, let us define for $b > 0$

$$\Gamma(b) := \left\{\gamma \in \Gamma |\ \exists K_\gamma \in \mathbb{Z}_+ :\ |\gamma_k|^P \leq b\log(1 + |k|) \ \text{ if } |k| \geq K_\gamma\right\}. \tag{3.51}$$

Proposition 3.6. *For given $P \geq 2$ and $\lambda > 0$, let us consider $\mu \in \mathcal{P}(\Gamma)$ which fulfill*

$$\sup_{k\in\mathbb{Z}^d}\int_{\Gamma}\exp\left\{\lambda|\gamma_k|^P\right\}\mu(\mathrm{d}\gamma) =: \mathcal{C}(\mu, \lambda) < \infty. \tag{3.52}$$

Then, simultaneously for all such measures (and hence by Theorem 2.7, for all $\mu \in \mathcal{G}^{\mathrm{t}}$), one has $\mu[\Gamma(b)] = 1$ as soon as $b > d/\lambda$.

Proof. We proceed similarly to the proof of Lemma 3.1 in [26]. The complement set to (3.51) can be written as

$$[\Gamma(b)]^c = \bigcap\nolimits_{K\in\mathbb{N}}\bigcup\nolimits_{|k|\geq K}[\Gamma_k(b)]^c, \tag{3.53}$$

where

$$\Gamma_k(b) := \left\{\gamma \in \Gamma \,\middle|\, |\gamma_k|^P \leq b\log(1 + |k|)\right\}.$$

By Chebyshev's inequality and the estimate (3.52)

$$\mu\left([\Gamma_k(b)]^c\right) \leq \mathcal{C}(\mu, \lambda)\cdot(1 + |k|)^{-b\lambda}, \tag{3.54}$$

and therefore by (3.53) and (3.54)

$$\mu\left([\Gamma(b)]^c\right) \le \mathcal{C}(\mu,\lambda) \lim_{K\in\mathbb{N}} \sum\nolimits_{|k|\ge K} (1+|k|)^{-b\lambda}. \tag{3.55}$$

By setting $b > d/\lambda$ one makes the series in (3.55) convergent, which yields the result $\mu\left([\Gamma(b)]^c\right) = 0$. □

Next, we claim that all finite volume projections

$$\mu_\Lambda := \mu \circ \mathbb{P}_\Lambda^{-1}, \quad \mu_k := \mu \circ \mathbb{P}_{Q_k}^{-1}, \quad k \in \Lambda \in \mathcal{B}_c(\mathbb{R}^d),$$

of the Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ satisfy a certain *Ruelle-type bound* (cf. Proposition 5.2 in [38]).

Proposition 3.7. *Under Assumptions* (**FR**), (**LB**), *and* (**SSS**), *each* $\mu \in \mathcal{G}^{\mathrm{t}}$ *is locally absolutely continuous with respect to the Lebesgue-Poisson measure* $\lambda_{z\sigma}$. *The corresponding Radon–Nikodym derivatives are bounded by*

$$\begin{aligned} \frac{\mathrm{d}\mu_\Lambda(\gamma_\Lambda)}{\mathrm{d}\lambda_{z\sigma}(\gamma_\Lambda)} &=: \rho_{\mu,\Lambda}(\gamma_\Lambda) \\ &\le \exp\left\{-E\sum_{k\in\mathcal{K}_\Lambda}|\gamma_k|^P + G_\Lambda|\gamma_\Lambda|\right\} \le (C_\Lambda)^{|\gamma_\Lambda|}, \quad \gamma_\Lambda \in \Gamma_\Lambda, \end{aligned} \tag{3.56}$$

with any positive $E < \beta D$ *and certain* $G_\Lambda, C_\Lambda > 0$ *being the same for all such* μ.

Proof. Since (3.56) certainly holds for $\gamma_\Lambda = \emptyset$, below we restrict ourselves to the case of $|\gamma_\Lambda| \ge 1$. From the *DLR* equation (2.26) it is easily to see that the Radon–Nikodym derivatives, if such exist, should have the form

$$\begin{aligned} \rho_{\mu,\Lambda}(\gamma_\Lambda) &= \exp\{-\beta H(\gamma_\Lambda)\}\int_\Gamma [1/Z_\Lambda(\xi)]\exp\{-\beta W(\gamma_\Lambda|\xi)\}\,\mu(\mathrm{d}\xi) \\ &\le \exp\{-\beta H(\gamma_\Lambda)\}\int_\Gamma \exp\left\{-\beta\sum_{x\in\gamma_\Lambda,\ y\in\xi_{\Lambda^c}} V(x,y)\right\}\mu(\mathrm{d}\xi), \quad \gamma\in\Gamma_\Lambda. \end{aligned} \tag{3.57}$$

So, the only thing needed to check is the validity of the upper bound (3.56), which in turn implies $\rho_{\mu,\Lambda} \in L^1(\lambda_{z\sigma})$ and hence $\mu_\Lambda(\mathrm{d}\gamma_\Lambda) \ll \lambda_{z\sigma}(\mathrm{d}\gamma_\Lambda)$. The integral in the last line in (3.57) can be estimated by means of the Hölder inequality (3.10) and the exponential bound (3.45)

$$\begin{aligned} &\int_\Gamma \exp\left\{-\beta\sum_{x\in\gamma_\Lambda,\ y\in\xi_{\Lambda^c}} V(x,y)\right\}\mu(\mathrm{d}\xi) \le \int_\Gamma \exp\left\{\beta M\sum_{\substack{k\in\mathcal{K}_\Lambda,\\ j\in\partial_g k}} |\gamma_k|\cdot|\xi_j|\right\}\mu(\mathrm{d}\xi) \\ &\le \exp\left\{\frac{1}{4\lambda}(\beta M m)^2|\mathcal{K}_\Lambda|\sum_{k\in\mathcal{K}_\Lambda}|\gamma_k|^2\right\}\int_\Gamma \exp\left\{\frac{\lambda}{m|\mathcal{K}_\Lambda|}\sum_{\substack{k\in\mathcal{K}_\Lambda,\\ j\in\partial_g k}}|\xi_j|^2\right\}\mu(\mathrm{d}\xi) \\ &\le \mathcal{C}(\lambda)\exp\left\{\frac{1}{4\lambda}(\beta M m)^2|\mathcal{K}_\Lambda|\sum_{k\in\mathcal{K}_\Lambda}|\gamma_k|^2\right\}, \end{aligned} \tag{3.58}$$

where we fixed some positive $\lambda \in (0, \beta D)$ the same as in (3.40) and used the notation (2.36)–(2.40). Thus

$$\rho_{\mu,\Lambda}(\gamma_\Lambda) \le \exp\left\{-\beta H(\gamma_\Lambda) + \frac{1}{4\lambda}\left(\beta M m\right)^2 |\mathcal{K}_\Lambda| \sum_{k\in\mathcal{K}_\Lambda} |\gamma_k|^2 + \log \mathcal{C}(\lambda)\cdot|\gamma_\Lambda|\right\}, \tag{3.59}$$

which together with the strong superstability (3.38) and Young's inequality (3.44) yields us the required bound on $\rho_{\mu,\Lambda}$. □

Remark 3.8. (i) In general, the constants G_Λ, C_Λ in (3.56) may *depend* on geometry of Λ. If $\Lambda := \Lambda_N := [-N, N]^d$, the best control we can get here is that G_{Λ_N} behave like $\mathcal{O}\left(N^{dp/(p-2)}\right)$ as $N \to \infty$.

(ii) Assume that (**RC**) holds with $A > 4Mm$, and let us pick $\lambda := \beta\,(A - 3Mm)\,/2$ which is the largest possible value in (2.51). Then, (3.59) combined with (2.45) and (2.51), yields the following *"one-point"* estimate with some $E, G > 0$

$$\begin{aligned}\rho_{\mu,k}(\gamma_k) \quad &\le \quad \exp\left\{\left(\frac{1}{4\lambda}\left(\beta M m\right)^2 - \frac{1}{2}\beta A\right)|\gamma_k|^2 + \left(\frac{1}{2}\beta A + \log\mathcal{C}(\lambda)\right)|\gamma_k|\right\}\\ &\le \quad \exp\left\{-E|\gamma_k|^2 + G|\gamma_k|)\right\}, \quad \text{for all } \gamma_k \in \Gamma_k,\ k \in \mathbb{Z}^d.\end{aligned} \tag{3.60}$$

In line with the n-particle decomposition $\Gamma_\Lambda = \bigsqcup_{n\in\mathbb{Z}_+} \Gamma_\Lambda^{(n)}$, we have the induced representation $\mu_\Lambda = \sum_{n\in\mathbb{Z}_+} \mu_\Lambda^{(n)}$, where

$$\mathrm{d}\mu_\Lambda^{(n)}(\{x_1, ..., x_n\}) = \frac{z^n}{n!}\rho_{\mu,\Lambda}^{(n)}(\{x_1, ..., x_n\})\mathrm{d}\sigma_{\mathrm{sym}}^{\otimes n}(x_1, ..., x_n). \tag{3.61}$$

According to (3.56), the *system of density distribution* $\rho_{\mu,\Lambda}^{(n)} : \Gamma_\Lambda^{(n)} \to \mathbb{R}_+$, $n \in \mathbb{Z}_+$, obeys the local bound

$$\rho_{\mu,\Lambda}^{(n)}(\{x_1, ..., x_n\}) \le C_\Lambda^n, \quad \{x_1, ..., x_n\} \in \Gamma_\Lambda^{(n)}. \tag{3.62}$$

In a similar way one can derive estimates on the *correlation functional* $k_\mu : \Gamma_0 \to \mathbb{R}_+$ of the measures $\mu \in \mathcal{G}^{\mathrm{t}}$ (see [15, 38]). It can be written in the form

$$k_\mu(\gamma) = \int_\Gamma \exp\left\{-\beta H(\gamma) - \beta W(\gamma,\xi)\right\}\mu(\mathrm{d}\xi), \quad \gamma \in \Gamma_0, \tag{3.63}$$

where, cf. (2.15),

$$W(\gamma,\xi) := \sum_{x\in\gamma,\ y\in\xi} V(x,y), \quad \gamma \in \Gamma_0,\ \xi \in \Gamma, \tag{3.64}$$

stands for the interaction energy between configurations γ and ξ. By analogy with (3.58), we get that for all $\gamma \in \Gamma_\Lambda$ and $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$

$$\begin{aligned}k_\mu(\gamma) \quad &\le \quad \exp\left\{-\beta H_\Lambda(\gamma) + \frac{1}{4\lambda}\left(\beta M m\right)^2 |\mathcal{K}_\Lambda| \sum_{k\in\mathcal{K}_\Lambda} |\gamma_k|^2 + \log\mathcal{C}(\lambda)\cdot|\gamma|\right\}\\ &\le \quad \exp\left\{-E\sum_{k\in\mathbb{Z}^d} |\gamma_k|^P + G_\Lambda|\gamma|)\right\} \le (C_\Lambda)^{|\gamma|}.\end{aligned} \tag{3.65}$$

In particular,

$$k_\mu^{(n)}(\{x_1, ..., x_n\}) \le C_\Lambda^n, \quad \{x_1, ..., x_n\} \subset \Lambda \in \mathcal{B}_c(\mathbb{R}^d), \tag{3.66}$$

where the family $k_\mu^{(n)} := k_\mu \restriction_{\Gamma_0^{(n)}}$, $n \in \mathbb{Z}_+$, are the well-known *correlation functions* of statistical physics (see Section 4.1 of [37]). Suppose we are able to show that $C := \sup_{\Lambda \in \mathcal{B}_c(\mathbb{R}^d)} C_\Lambda < \infty$, then this would mean the *global* Ruelle's bound, cf. Proposition 2.6 in [38]. For a detailed discussion of different *à-priori* bounds and their consequences for measures on configuration spaces see [19].

Remark 3.9. It is still an *open question* whether $\mathcal{G}^{\mathrm{t}} = \mathcal{G}^{\mathrm{st}}$. Nevertheless, it is clear that any *translation invariant* measure $\mu \in \mathcal{P}(\Gamma)$ obeying the exponential bound (3.52), ought to fulfill

$$\sup_{K \in \mathbb{N}} \left\{ K^{-d} \sum_{|k| \le K} \exp \left[\kappa \sum_{\{x,y\} \subset \gamma_k} V(x,y) + \lambda |\gamma_k|^2) \right] \right\} \le C(\gamma), \quad \forall \gamma \in \Gamma \ (\mu - \text{a.e.}), \tag{3.67}$$

which is much stronger than the original Ruelle's support condition (2.56). The latter assertion immediately follows from the multidimensional ergodic theorem (cf. e.g. Theorem 14.A8 in [11]) applied to the stationary process $\Phi_k(\gamma) := \kappa H(\gamma_k) + \lambda |\gamma_k|^2$, $k \in \mathbb{Z}^d$, defined on the probability space $(\Gamma, \mathcal{B}(\Gamma), \mu)$.

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