Existence and Uniqueness of Invariant Measures for Stochastic Evolution Equations with Weakly Dissipative Drifts

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Abstract

In this paper, a new decay estimate for a class of stochastic evolution equations with weakly dissipative drifts is established, which directly implies the uniqueness of invariant measures for the corresponding transition semigroups. Moreover, the existence of invariant measures and the convergence rate of corresponding transition semigroup to the invariant measure are also investigated. As applications, the main results are applied to singular stochastic p-Laplace equations and stochastic fast diffusion equations, which solves an open problem raised by Barbu and Da Prato in [Stoc. Proc. Appl. 120(2010), 1247-1266].

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1 Introduction

In recent years, the variational approach has been used intensively by many authors to analyze semilinear and quasilinear stochastic partial differential equations. This approach was first investigated by Pardoux [19] to study SPDE, carried on by Krylov and Rozovoskii [13] who further developed it and applied it to nonlinear filtering problems. We refer to [17, 20, 23] for a more detailed exposition and references. Within this framework, various types of both analytic and probabilistic properties such as the large deviation principle, discretized approximation of solutions, ergodic properties and existence of random attractors have already been established for different types of nonlinear SPDE (cf. [12, 16, 17] and references therein). In particular, the existence and uniqueness of invariant measures and

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the asymptotic behavior of the corresponding transition semigroups have been studied for stochastic porous medium equations and stochastic p-Laplace equations, see e.g. [1, 5, 9, 14, 20, 22].

The principal aim of this work is to show the uniqueness of invariant measures for a class of stochastic evolution equations with weakly dissipative drifts such as stochastic fast diffusion equations and singular stochastic p-Laplace equations (where *singular* means $1 < p < 2$ here). The existence of an invariant measure has been established by Wang and the first named author in [18, 15], and recently by Barbu and Da Prato in [2] for stochastic fast diffusion equations under some weaker assumptions. It is more difficult, however, to derive the uniqueness of invariant measures for this type of stochastic equations due to the lack of strong dissipativity for the drift. Under some non-degeneracy assumption on the noise, Wang and the first named author have established the Harnack inequality for the associated transition semigroup in [18, 15], which implies the uniqueness of invariant measures and some heat kernel estimates. In this work we prove the uniqueness of invariant measures in a more general setting and do not assume any non-degeneracy of the noise. Inspired by the recent work of Es-Sarhir, von Renesse and Stannat [11], we establish a decay estimate for stochastic evolution equations with weakly dissipative drifts (see $(A2)$ below), which directly implies the uniqueness of invariant measures. The result is applied to stochastic p-Laplace equations and stochastic fast diffusion equations, which also solves an open problem raised by Barbu and Da Prato (see [2, Remark 3.3]). Further applications of this new decay estimate to asymptotic behavior of corresponding transition semigroups and the construction of the corresponding Kolmogorov operators will be investigated in a separate paper.

Let us describe our framework in detail. Let H be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and dual H^*. Let V be a reflexive Banach space such that the embedding $V \subseteq H$ is continuous and dense. Then for its dual space V^* it follows that the embedding $H^* \subseteq V^*$ is also continuous and dense. Identifying H and H^* via the Riesz isomorphism we know that

$$V \subseteq H \equiv H^* \subseteq V^*$$

forms a so-called Gelfand triple. If the dualization between V^* and V is denoted by ${}_{V^*}\langle\cdot,\cdot\rangle_V$ we have

$${}_{V^*}\langle u, v\rangle_V = \langle u, v\rangle_H \ \text{ for all } u \in H, v \in V.$$

Suppose $\{W_t\}$ is a cylindrical Wiener process on a separable Hilbert space U w.r.t a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. We consider the following stochastic evolution equation

$$dX_t = A(X_t)\,dt + B\,dW_t,\ X_0 = x \in H, \tag{1.1}$$

where B is a Hilbert-Schmidt operator from U to H and $A : V \to V^*$ is measurable.

Suppose that for a fixed $\alpha > 1$ there exist constants $\delta > 0$, $\beta \in (0, \alpha], \gamma \geq 0$ and $K \in \mathbb{R}$ such that the following conditions hold for all $v, v_1, v_2 \in V$.

$(A1)$ *Hemicontinuity* of A: The map $\lambda \mapsto {}_{V^*}\langle A(v_1 + \lambda v_2), v\rangle_V$ is continuous on $\mathbb{R}$.

$(A2)$ *(Weak) dissipativity* of A:

$$2_{V^*}\langle A(v_1) - A(v_2), v_1 - v_2\rangle_V \le -\delta \frac{\|v_1 - v_2\|_H^2}{\|v_1\|_V^\beta + \|v_2\|_V^\beta}.$$

$(A3)$ *Coercivity* of A:

$$2_{V^*}\langle A(v), v\rangle_V \le -\delta\|v\|_V^\alpha + K.$$

$(A4)$ *Boundedness* of A:

$$\|A(v)\|_{V^*} \le K\left(1 + \|v\|_V^{\alpha-1}\right).$$

Remark 1.1. (1) It is easy to check that $(A1)$–$(A4)$ hold for some concrete examples such as the stochastic fast diffusion equation and the singular stochastic p-Laplace equation (i.e. $1 < p \le 2$). We refer to Section 3 for more details.

(2) $(A2)$ resembles the following stronger dissipativity condition ($\alpha > 2$):

$$2_{V^*}\langle A(v_1) - A(v_2), v_1 - v_2\rangle_V \le -\delta\|v_1 - v_2\|_V^\alpha,\ v_1, v_2 \in V. \tag{1.2}$$

This type of dissipativity condition holds for the stochastic porous medium equation, the stochastic p-Laplace equation ($p \ge 2$) and some other equations with similar degenerate drifts (cf. [12, 14, 16]). The condition (1.2) has been used for the investigation of the ergodicity in [14], large deviation principle in [16] and the existence of random attractors in [12] for a large class of stochastic evolution equations.

Note that $(A2)$ is stronger than the classical (weak) monotonicity condition (cf.[13, 20]), hence for any $T > 0$ and any $x \in H$, (1.1) has a unique solution $\{X_t(x)\}_{t\in[0,T]}$ which is an adapted continuous process on H such that $\mathbb{E}\int_0^T \|X_t\|_V^\alpha\, dt < \infty$ and

$$\langle X_t, v\rangle_H = \langle x, v\rangle_H + \int_0^t {}_{V^*}\langle A(X_s), v\rangle_V\, ds + \langle BW_t, v\rangle_H$$

holds for all $v \in V$ and $(t, \omega) \in [0, T] \times \Omega$.

Let us define the corresponding transition semigroup

$$P_tF(x) := \mathbb{E}F(X_t(x)),\ t \ge 0,\ x \in H,$$

where F is a bounded measurable function on H.

Theorem 1.1. *Suppose* $(A1)$–$(A4)$ *hold for* (1.1).

(i) There exists a constant $C > 0$ *such that*

$$\mathbb{E}\left[\|X_t(x) - X_t(y)\|_H^{\frac{2\alpha}{\beta}}\right] \le C\left(\frac{\|x-y\|_H^2}{t}\right)^{\frac{\alpha}{\beta}}\left(1 + \frac{\|x\|_H^2}{t} + \frac{\|y\|_H^2}{t}\right),\ x, y \in H, t > 0, \tag{1.3}$$

where $X_t(y)$ *denotes the solution of* (1.1) *with starting point* $y \in H$.

(ii) $\{P_t\}$ *is a Feller semigroup. Moreover, there exists* $C > 0$ *such that for any Lipschitz function* $F : H \to \mathbb{R}$ *we have*

$$(1.4)\quad |P_tF(x) - P_tF(y)| \le \frac{C\mathcal{L}(F)\|x-y\|_H}{\sqrt{t}}\left(1 + \frac{\|x\|_H}{\sqrt{t}} + \frac{\|y\|_H}{\sqrt{t}}\right)^{\frac{\beta}{\alpha}},\ x, y \in H, t > 0,$$

where $\mathcal{L}(F)$ *is the Lipschitz constant of* F.

(iii) If $\beta \in (0, \alpha)$*, then* $\{P_t\}$ *has at most one invariant measure.*

Remark 1.2. (1) This type of decay estimate (1.3) is new for stochastic fast diffusion equations and singular stochastic p-Laplace equations. For stochastic porous media equations, the following type of estimate is established in [9, Theorem 1.3] ($\alpha > 2$):

$$\|X_t(x) - X_t(y)\|_H^2 \le \|x-y\|_H^2 \wedge \left\{Ct^{-\frac{2}{\alpha-2}}\right\},\ t > 0,\ x, y \in H.$$

Moreover, it has been proved in [14] that the above estimate holds for a large class of stochastic evolution equations with strong dissipativity condition (1.2).

(2) For the plane stochastic curve shortening flow (cf.[10]), a similar type of polynomial decay estimate is established by Es-Sarhir, von Renesse and Stannat for the ergodic measure in [11].

Theorem 1.2. *Suppose that (A1)–(A4) hold with* $\beta \in (0, \alpha)$ *and the embedding* $V \subseteq H$ *is compact. Then the Markov semigroup* $\{P_t\}$ *has an unique invariant probability measure* μ*, which satisfies*

$$\int_H \|x\|_V^\alpha\, \mu(dx) < \infty.$$

Moreover,

(i) if $\alpha \ge \sqrt{2}$*, there exists* $C > 0$ *such that the following estimate holds:*

$$(1.5)\qquad |P_tF(x) - \mu(F)| \le \frac{C\mathcal{L}(F)\,(1+\|x\|_H)}{\sqrt{t}}\left[1 + \left(\frac{1+\|x\|_H}{\sqrt{t}}\right)^{\frac{\beta}{\alpha}}\right],\ x \in H, t > 0;$$

(ii) if $1 < \alpha \le \sqrt{2}$*, then for any* $\gamma \in \left(0, \frac{\alpha^2}{\alpha+\beta}\right]$*, there exists* $C > 0$ *such that*

$$(1.6)\qquad |P_tF(x) - \mu(F)| \le \frac{C|F|_\gamma\,(1+\|x\|_H^\gamma)}{t^{\frac{\gamma}{2}}}\left(1 + \frac{1+\|x\|_H^{\frac{\beta\gamma}{\alpha}}}{t^{\frac{\beta\gamma}{2\alpha}}}\right),\ x \in H, t > 0,$$

where F *is any* γ*-Hölder continuous function and*

$$|F|_\gamma := \sup_{x \ne y \in H} \frac{|F(x) - F(y)|}{\|x-y\|_H^\gamma}.$$

The rest of the paper is organized as follows: the proofs of the main theorems are given in the next section. In Section 3, we apply the main results to some concrete examples of SPDE.

2 Proof of the main results

2.1 Proof of Theorem 1.1

(i) Let $X_t(x), X_t(y)$ denote the solution of (1.1) starting from x, y respectively. Then by $(A2)$ and the chain rule, we have

$$\|X_t(x) - X_t(y)\|_H^2 \;\leq\; \|x-y\|_H^2 - \delta \int_0^t \frac{\|X_s(x) - X_s(y)\|_H^2}{\|X_s(x)\|_V^\beta + \|X_s(y)\|_V^\beta}\, ds.$$

Note that by

$$\frac{d}{dt}\|X_t(x) - X_t(y)\|_H^2 = 2_{V^*}\langle A(X_t(x)) - A(X_t(y)), X_t(x) - X_t(y)\rangle_V \leq 0,$$

the map $t \mapsto \|X_t(x) - X_t(y)\|_H^2$ is decreasing. Hence we have that

$$\|X_t(x) - X_t(y)\|_H^2 \;\leq\; \|x-y\|_H^2 - \delta \|X_t(x) - X_t(y)\|_H^2 \int_0^t \frac{1}{\|X_s(x)\|_V^\beta + \|X_s(y)\|_V^\beta}\, ds.$$

Furthermore,

$$\|X_t(x) - X_t(y)\|_H^2 \leq \|x-y\|_H^2 \left(1 + \int_0^t \frac{\delta}{\|X_s(x)\|_V^\beta + \|X_s(y)\|_V^\beta}\, ds\right)^{-1}.$$

By Jensen's inequality, we get that

$$\int_0^t \frac{\delta}{\|X_s(x)\|_V^\beta + \|X_s(y)\|_V^\beta}\, ds \geq \frac{\delta t^2}{\int_0^t \left(\|X_s(x)\|_V^\beta + \|X_s(y)\|_V^\beta\right) ds}.$$

Which leads to

$$\begin{aligned}
\|X_t(x) - X_t(y)\|_H^2 \;&\leq\; \|x-y\|_H^2 \left(1 + \frac{\delta t^2}{\int_0^t \left(\|X_s(x)\|_V^\beta + \|X_s(y)\|_V^\beta\right) ds}\right)^{-1} \\
&=\; \|x-y\|_H^2 \frac{\int_0^t \left(\|X_s(x)\|_V^\beta + \|X_s(y)\|_V^\beta\right) ds}{\delta t^2 + \int_0^t \left(\|X_s(x)\|_V^\beta + \|X_s(y)\|_V^\beta\right) ds} \\
&\leq\; \frac{\|x-y\|_H^2}{\delta t} \left(\frac{1}{t}\int_0^t \|X_s(x)\|_V^\beta\, ds + \frac{1}{t}\int_0^t \|X_s(y)\|_V^\beta\, ds\right).
\end{aligned}$$

Using Jensen's inequality again, we get that
(2.1)

$$\|X_t(x) - X_t(y)\|_H^{\frac{2\alpha}{\beta}} \leq \left(\frac{2\|x-y\|_H^2}{\delta t}\right)^{\frac{\alpha}{\beta}} \left(\frac{1}{t}\int_0^t \|X_s(x)\|_V^\alpha\, ds + \frac{1}{t}\int_0^t \|X_s(y)\|_V^\alpha\, ds\right).$$

By applying Itô's formula to $\|\cdot\|_H^2$ and using $(A3)$, one can easily get the estimates

$$\mathbb{E}\left(\|X_t(x)\|_H^2 + \delta\int_0^t \|X_s(x)\|_V^\alpha\, ds\right) \le \|x\|_H^2 + t\left(K + \|B\|_{HS}^2\right);$$

$$\mathbb{E}\left(\|X_t(y)\|_H^2 + \delta\int_0^t \|X_s(y)\|_V^\alpha\, ds\right) \le \|y\|_H^2 + t\left(K + \|B\|_{HS}^2\right),$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm from U to H.

Hence there exists a constant $C > 0$ such that

$$\mathbb{E}\left[\|X_t(x) - X_t(y)\|_H^{\frac{2\alpha}{\beta}}\right] \le C\left(\frac{\|x-y\|_H^2}{t}\right)^{\frac{\alpha}{\beta}}\left(1 + \frac{\|x\|_H^2}{t} + \frac{\|y\|_H^2}{t}\right).$$

(ii) It is obvious that (1.3) implies that $\{P_t\}$ is a Feller semigroup. Moreover, for any Lipschitz function $F: H \to \mathbb{R}$ we have

$$\begin{aligned} |P_tF(x) - P_tF(y)| &\le \mathcal{L}(F)\,\mathbb{E}\,\|X_t(x) - X_t(y)\|_H \\ &\le \frac{C\mathcal{L}(F)\|x-y\|_H}{\sqrt{t}}\left(1 + \frac{\|x\|_H}{\sqrt{t}} + \frac{\|y\|_H}{\sqrt{t}}\right)^{\frac{\beta}{\alpha}}, \end{aligned}$$

where $\mathcal{L}(F)$ is the Lipschitz constant of F and $C > 0$ is a constant (independent of x, y, t and F).

(iii) Let us prove that (1.4) is sufficient for the uniqueness of invariant measures. It is well known that one only need to show the uniqueness of ergodic invariant measures (cf. [7]).

In fact, if there exist two ergodic invariant measures μ and ν, then for any bounded Lipschitz function F we get in the limit $T \to \infty$,

$$\frac{1}{T}\int_0^T P_tF(x)\, dt \to \int_H F\, d\mu \text{ for } \mu\text{-a.e. } x;$$

$$\frac{1}{T}\int_0^T P_tF(y)\, dt \to \int_H F\, d\nu \text{ for } \nu\text{-a.e. } y.$$

Since $\beta < \alpha$, by (1.4) we have that

$$\begin{aligned} &\left|\frac{1}{T}\int_0^T P_tF(x)\, dt - \frac{1}{T}\int_0^T P_tF(y)\, dt\right| \\ \le\ & \frac{1}{T}\int_0^T |P_tF(x) - P_tF(y)|\, dt \\ \le\ & \frac{C\mathcal{L}(F)\|x-y\|_H}{T}\int_0^T \frac{1}{\sqrt{t}}\left(1 + \frac{\|x\|_H}{\sqrt{t}} + \frac{\|y\|_H}{\sqrt{t}}\right)^{\frac{\beta}{\alpha}} dt \\ \longrightarrow\ & 0 \text{ as } T \to \infty. \end{aligned}$$

Therefore, for any bounded Lipschitz function F on H we have that

$$\int_H F\,d\mu = \int_H F\,d\nu,$$

i.e. $\mu = \nu$. Therefore, $\{P_t\}$ has at most one invariant measure.

2.2 Proof of Theorem 1.2

Note that $\{P_t\}$ is a Markov semigroup (cf.[13, 20]) and Feller by Theorem 1.1. Therefore, the existence of an invariant measure μ can be proved by the standard Krylov-Bogoliubov procedure (cf. [20, 14]). Let

$$\mu_n := \frac{1}{n}\int_0^n \delta_0\, P_t\, dt,\ \ n \geq 1,$$

where δ_0 is the Dirac measure at 0.

Hence for the existence of an invariant measure, one only needs to verify the tightness of $\{\mu_n : n \geq 1\}$.

By using Itô's formula and $(A3)$, we have the following estimate:

$$\|X_t\|_H^2 \leq \|x\|_H^2 + \int_0^t (K + \|B\|_{HS}^2 - \delta\|X_s\|_V^\alpha)\, ds + 2\int_0^t \langle X_s, B\, dW_s\rangle_H. \tag{2.2}$$

Note that $M_t := \int_0^t \langle X_s, B\, dW_s\rangle_H$ is a martingale, then (2.2) implies that

$$\mu_n(\|\cdot\|_V^\alpha) = \frac{1}{n}\int_0^n \mathbb{E}\|X_t(0)\|_V^\alpha\, dt \leq \frac{(K + \|B\|_{HS}^2)}{\delta},\ \ n \geq 1. \tag{2.3}$$

Note that the embedding $V \subseteq H$ is compact, then for any constant C the set

$$\{x \in H :\ \|x\|_V \leq C\}$$

is relatively compact in H. Therefore, (2.3) implies that $\{\mu_n\}$ is tight, hence the limit of a convergent subsequence provides an invariant measure μ of $\{P_t\}$.

The uniqueness of μ follows from Theorem 1.1. And the concentration property for μ follows from (2.3), since μ is the weak limit of μ_n.

(i) If $\alpha \geq \sqrt{2}$, then it is also easy to show (1.5) by using (1.4) and $\mu(\|\cdot\|_V^\alpha) < \infty$.

(ii) For the case $1 < \alpha \leq \sqrt{2}$, one can consider a smaller class of test function in the estimate (1.5). More precisely, for any γ-Hölder continuous function $F : H \to \mathbb{R}$, by Hölder's inequality and (1.3) we have

$$\begin{aligned}
|P_tF(x) - P_tF(y)| &\leq |F|_\gamma\, \mathbb{E}\left(\|X_t(x) - X_t(y)\|_H^\gamma\right) \\
&\leq |F|_\gamma \left[\mathbb{E}\left(\|X_t(x) - X_t(y)\|_H^{\frac{2\alpha}{\beta}}\right)\right]^{\frac{\beta\gamma}{2\alpha}} \\
&\leq \frac{C|F|_\gamma \|x-y\|_H^\gamma}{t^{\frac{\gamma}{2}}}\left(1 + \frac{\|x\|_H}{\sqrt{t}} + \frac{\|y\|_H}{\sqrt{t}}\right)^{\frac{\beta\gamma}{\alpha}},
\end{aligned}$$

where $|F|_\gamma$ is the γ-Hölder norm of F and $C > 0$ is a constant (independent of x, y, t, F).

Hence for any $0 < \gamma \leq \frac{\alpha^2}{\alpha+\beta}$, we have that

$$|P_t F(x) - \mu(F)| \leq \frac{C|F|_\gamma \left(1 + \|x\|_H^\gamma\right)}{t^{\frac{\gamma}{2}}} \left(1 + \frac{1 + \|x\|_H^{\frac{\beta\gamma}{\alpha}}}{t^{\frac{\beta\gamma}{2\alpha}}}\right), \ x \in H, t > 0,$$

where F is any γ-Hölder continuous function.

3 Applications

In order to verify $(A2)$ for concrete examples of stochastic evolution equations, we first recall the following inequality in Hilbert space proved in [15].

Lemma 3.1. *Let $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a Hilbert space, then for any $0 < r \leq 1$ we have*

$$\langle \|a\|^{r-1}a - \|b\|^{r-1}b, a - b\rangle \geq r\|a - b\|^2 \left(\|a\| \vee \|b\|\right)^{r-1}, \ a, b \in H. \tag{3.1}$$

The first example is the stochastic p-Laplace equation, which arises from geometry, plasma physics and fluid dynamics etc (cf. [8, 15]). In particular, Ladyzenskaja suggests the p-Laplace equation as a model for the motion of non-Newtonian fluids.

Let Λ be an open bounded domain in $\mathbb{R}^d$ with a sufficiently smooth boundary. We consider the following Gelfand triple

$$V := W_0^{1,p}(\Lambda) \subseteq H := L^2(\Lambda) \subseteq (W_0^{1,p}(\Lambda))^*$$

and the stochastic p-Laplace equation

$$dX_t = \left[\mathbf{div}(|\nabla X_t|^{p-2}\nabla X_t)\right] dt + B\, dW_t, \ X_0 = x \in L^2(\Lambda), \tag{3.2}$$

where $p \in (1 \vee \frac{2d}{2+d}, 2)$, B is a Hilbert-Schmidt operator on $L^2(\Lambda)$ and $\{W_t\}$ is a cylindrical Wiener process on $L^2(\Lambda)$ w.r.t. a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$.

Proposition 3.2. *The Markov semigroup $\{P_t\}$ associated with* (3.2) *has a unique invariant probability measure.*

Proof. According to Theorem 1.2, we need to show $(A1)$-$(A4)$ hold for (3.2). It is well known that (3.2) satisfies $(A1), (A3)$ and $(A4)$ with $\alpha = p$ (cf. [15, 20]). Let us verify $(A2)$ with $\beta = 2 - p$.

By Lemma 3.1 and Hölder's inequality we have

$$
\begin{aligned}
&{}_{V^*}\langle \mathbf{div}(|\nabla v_1|^{p-2}\nabla v_1) - \mathbf{div}(|\nabla v_2|^{p-2}\nabla v_2), v_1 - v_2\rangle_V \\
=& -\int_\Lambda \left(|\nabla v_1|^{p-2} v_1 - |\nabla v_2|^{p-2} v_2\right)(\nabla v_1 - \nabla v_2)\, d\xi \\
\le& -(p-1)\int_\Lambda |\nabla v_1 - \nabla v_2|^2 \left(|\nabla v_1| + |\nabla v_2|\right)^{p-2} d\xi \\
\le& -(p-1)\frac{\|v_1 - v_2\|_V^2}{\left(\int_\Lambda (|\nabla v_1| + |\nabla v_2|)^p\, d\xi\right)^{\frac{2-p}{p}}} \\
\le& -C\frac{\|v_1 - v_2\|_H^2}{\|v_1\|_V^{2-p} + \|v_2\|_V^{2-p}},\ v_1, v_2 \in V,
\end{aligned}
$$

where $C > 0$ is some constant derived from the Poincaré inequality used in last step.

Note that the embedding $W_0^{1,p}(\Lambda) \subseteq L^2(\Lambda)$ is compact, hence the conclusion follows from Theorem 1.2. Now the proof is complete. □

Remark 3.1. In [6], Ciotir and the second named author show the convergence of solutions, corresponding semigroups and invariant measures for stochastic p-Laplace equations as $p \to p_0$, where $p_0 \in [1, 2]$. In their result (see [6, Theorem 1.5]) they assume that the transition semigroup of (3.2) has a unique invariant measure for p_0. From the above result we know that this assumption always holds for $p_0 \in (1, 2]$ (since $d = 1, 2$ is assumed in [6]). However, the uniqueness of invariant measures in the limit case (i.e. $p_0 = 1$) is still open.

The second example is the stochastic fast diffusion equation, which models diffusion in plasma physics, curvature flows and self-organized criticality in sandpile models, e.g see [3, 4, 21] and the references therein. We consider the following stochastic fast diffusion equation in an open bounded domain Λ of $\mathbb{R}^d$ with sufficiently smooth boundary (cf. [2, 18]):

$$
\begin{cases}
dX_t(\xi) = \Delta\left(|X_t(\xi)|^{r-1} X_t(\xi)\right) dt + B\, dW_t,\ \xi \in \Lambda, \\
X_t(\xi) = 0,\ \forall \xi \in \partial\Lambda, \\
X_0(\xi) = x(\xi),\ \forall \xi \in \Lambda,
\end{cases}
\tag{3.3}
$$

where $r \in (0, 1)$, B is a Hilbert-Schmidt operator from $L^2(\Lambda)$ to $W^{-1,2}(\Lambda)$ and $\{W_t\}$ is a standard cylindrical Wiener process on $L^2(\Lambda)$ w.r.t. a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$.

According to the classical Sobolev embedding theorem, if $r > \max\{0, \frac{d-2}{d+2}\}$, then the embedding $L^{r+1}(\Lambda) \subseteq W^{-1,2}(\Lambda)$ is compact. By using the following Gelfand triple

$$
V := L^{r+1}(\Lambda) \subseteq H := W^{-1,2}(\Lambda) \subseteq \left(L^{r+1}(\Lambda)\right)^*,
$$

we can rewrite the stochastic fast diffusion equation (3.3) into the following form:

$$
dX_t = \Delta\left(|X_t|^{r-1} X_t\right) dt + B\, dW_t,\ X_0 = x \in H. \tag{3.4}
$$

Proposition 3.3. *If* $r \in (0 \vee \frac{d-2}{d+2}, 1)$*, then the Markov semigroup* $\{P_t\}$ *associated with* (3.4) *has a unique invariant probability measure.*

Proof. According to Theorem 1.2, we only need to show $(A1)$–$(A4)$ hold for (3.4). It is easy to see that (3.4) satisfies $(A1), (A3)$ and $(A4)$ with $\alpha = r+1$ (cf. [18, 2, 20]). We will show that $(A2)$ holds with $\beta = 1-r$.

Combining Lemma 3.1 with Hölder's inequality, we get that

$$\begin{aligned}
&{}_{V^*}\langle \Delta\left(|v_1|^{r-1}v_1\right) - \Delta\left(|v_2|^{r-1}v_2\right), v_1 - v_2\rangle_V \\
&= -\int_\Lambda \left(|v_1|^{r-1}v_1 - |v_2|^{r-1}v_2\right)(v_1 - v_2)\, d\xi \\
&\leq -r \int_\Lambda |v_1 - v_2|^2 \left(|v_1| + |v_2|\right)^{r-1} d\xi \\
&\leq -r \frac{\|v_1 - v_2\|_{L^{r+1}}^2}{\left(\int_\Lambda (|v_1| + |v_2|)^{r+1}\, d\xi\right)^{\frac{1-r}{1+r}}} \\
&\leq -C \frac{\|v_1 - v_2\|_H^2}{\|v_1\|_V^{1-r} + \|v_2\|_V^{1-r}}, \ v_1, v_2 \in L^{r+1}(\Lambda),
\end{aligned}$$

where $C > 0$ is some constant derived from the Sobolev inequality used in last step.

Therefore, P_t has a unique invariant measure. □

Remark 3.2. For more general existence results of invariant measures for stochastic fast diffusion equations we refer to [2, 5, 18]. If the noise in (3.4) is non-degenerate, then the uniqueness of invariant measures, some concentration property and heat kernel estimates has been established in [18] by using Harnack inequality. In this paper, the uniqueness of invariant measures is established without any non-degeneracy assumption on the noise, which answers the problem raised by Barbu and Da Prato in [2] (see Remark 3.3 therein).

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References

[1] V. Barbu and G. Da Prato, *Ergodicity for Nonlinear Stochastic Equations in Variational Formulation,* Appl. Math. Optim. **53** (2006), 121–139.

[2] V. Barbu and G. Da Prato, *Invariant measures and the Kolmogorov equation for the stochastic fast diffusion equation,* Stoc. Proc. Appl. **120** (2010), 1247–1266.

[3] V. Barbu, G. Da Prato and M. Röckner, *Stochastic porous media equation and self-organized criticality,* Comm. Math. Phys. **285** (2009), 901–923.

[4] J. Berryman and C. Holland, *Stability of the separabel solutions fro fast diffusion,* Arch. Ration. Mech. Anal. **74** (1980), 379–388.

[5] V.I. Bogachev, G. Da Prato and M. Röckner, *Invariant measures of generalized stochastic porous medium equations,* Dokl. Math. **69** (2004), 321–325.

[6] I. Ciotir and J. M. Tölle, *Convergence of solutions to the stochastic p-Laplace equations as p goes to 1,* BiBoS-Preprint 11-01-371.

[7] G. Da Prato and J. Zabczyk, *Ergodicity for infinite-dimensional systems,* London Mathematical Society Lecture Note Series, 229. Cambridge University Press, Cambridge, 1996.

[8] E. DiBenedetto, *Degenerate Parabolic Equations,* Springer-Verlag, New York, 1993.

[9] G. Da Prato , M. Röckner, B.L. Rozovskii and F.-Y. Wang, *Strong solutions to stochastic generalized porous media equations: existence, uniqueness and ergodicity,* Comm. Part. Diff. Equat. **31** (2006), 277–291.

[10] A. Es-Sarhir and M. von Renesse, *Ergodicity of Stochastic Curve Shortening Flow in the Plane,* SIAM J. Math. Analysis, In press.

[11] A. Es-Sarhir, M. von Renesse and W. Stannat, *Estimates for the Ergodic Measure and Polynomial Stability of Plane Stochastic Curve Shortening Flow,* arXiv:1008.1961.

[12] B. Gess, W. Liu and M. Röckner, *Random attractors for a class of stochastic partial differential equations driven by general additive noise*, J. Differential Equations **251** (2011), 1225–1253.

[13] N.V. Krylov and B.L. Rozovskii, *Stochastic evolution equations,* Translated from Itogi Naukii Tekhniki, Seriya Sovremennye Problemy Matematiki **14**(1979), 71–146, Plenum Publishing Corp.

[14] W. Liu, *Harnack inequality and applications for stochastic evolution equations with monotone drifts,* J. Evol. Equ. **9** (2009), 747–770.

[15] W. Liu, *On the stochastic p-Laplace equation,* J. Math. Anal. Appl. **360** (2009), 737–751.

[16] W. Liu, *Large deviations for stochastic evolution equations with small multiplicative noise*, Appl. Math. Optim. **61** (2010), no. 1, 27–56.

[17] W. Liu and M. Röckner, *SPDE in Hilbert space with locally monotone coefficients*, J. Funct. Anal. **259** (2010), 2902–2922.

[18] W. Liu and F.-Y. Wang, *Harnack inequality and Strong Feller property for stochastic fast diffusion equations,* J. Math. Anal. Appl. **342** (2008), 651–662.

[19] E. Pardoux, *Equations aux dérivées partielles stochastiques non linéaires monotones,* Thesis, Université Paris XI, 1975.

[20] C. Prévôt and M. Röckner, *A Concise Course on Stochastic Partial Differential Equations,* Lecture Notes in Mathematics 1905, Springer, 2007.

[21] Ph. Rosenan, *Fast and super fast diffusion processes,* Phys. Rev. Lett. **74** (1995), 7–14.

[22] F.-Y. Wang, *Harnack Inequality and Applications for Stochastic Generalized Porous Media equations,* Ann. Probab. **35** (2007), 1333–1350.

[23] X. Zhang, *On stochastic evolution equations with non-Lipschitz coefficients,* Stoch. Dyn. **9** (2009), 549–595.