Finite time extinction of solutions to fast diffusion equations driven by linear multiplicative noise

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Abstract

In this paper one surveys and sharpens some recent results obtained by the authors on the finite time extinction property of solutions to stochastic diffusion equations of the form $dX - \rho\Delta X^m dt = XdW$ and $dX - \rho\Delta\ \mathrm{sgn} X dt = XdW$, where $0 < m < 1,\ \rho > 0$. These equations arise as models for nonlinear diffusion processes in porous media, plasma and self-organized criticality under stochastic Gaussian perturbation. These equations can be also viewed as control systems governed by fast diffusion porous media equations with a stochastic feedback controller XdW.

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1 Introduction

We here consider stochastic differential equations of the form

$$\begin{cases} dX(t) - \Delta\Psi(X(t))dt \ni X(t)\, dW(t), & \text{in } (0,\infty)\times\mathscr{O}, \\ \Psi(X(t)) \ni 0, & \text{on } (0,\infty)\times\partial\mathscr{O}, \\ X(0) = x, & \text{in } \mathscr{O}, \end{cases} \tag{1.1}$$

where $\mathscr{O}$ is a bounded open domain of $\mathbb{R}^d$ with smooth boundary $\partial\mathscr{O}$ and $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$, that is

$$(s_1 - s_2)(r_1 - r_2) \geq 0, \quad \forall\, r_1, r_2 \in \mathbb{R},\ s_1 \in \Psi(r_1),\ s_2 \in \Psi(r_2)$$

and the range $R(I + \Psi)$ of $r \mapsto r + \Psi(r)$ covers $\mathbb{R}$. $W(t)$ is an $\mathscr{F}_t$–Wiener process on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with filtration $(\mathscr{F}_t)_{t\geq 0}$, which will be specified later.

Equation (1.1) arises in the description of a large variety of physical phenomena and processes including the following: fluid flows in porous media, diffusion processes in kinetic gas theory, heat transfer in plasmas, population dynamics and heat phase transitions.

If Ψ is differentiable, we may rewrite equation (1.1) in the more common form

$$dX(t) - \operatorname{div}(\Psi'(X(t))\nabla X(t))dt = X(t)dW(t). \tag{1.2}$$

Let us denote by j the potential corresponding to Ψ, that is $j(r) = \int_0^r \Psi(r)dr$. In most applications $j(X(t))$ is the diffusion coefficient and $X(t)$ represents the mass concentration. In other physical models $j(X(t))$ represents the conductivity coefficient. (We refer to [24] for presentations of physical models described by deterministic nonlinear diffusion equations of this form.)

The case considered here, that is (1.1), represents the classical porous media equation perturbed by a Gaussian process XdW which is proportional to the state X of the system. By this the solution X to (1.1), is a stochastic flow on the stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. There is strong physical evidence justifying this stochastic model of nonlinear diffusions in which the stochastic perturbation is of the form $\sigma(X)dW$ where σ is a Lipschitz function such that $\sigma(r)r \geq 0$ for all $r \in \mathbb{R}$. One of the main reasons to consider this stochastic model is that in this case the flow $X(t,x)$ leaves invariant the set of all nonnegative states x which in applications is an essential feature of the model.

The standard case considered in the literature is

$$\Psi(r) = \rho|r|^{m-1}r, \quad \forall\, r \in \mathbb{R}, \tag{1.3}$$

where $1 < m < \infty$ or more generally,

$$r\Psi(r) \geq \rho |r|^{m+1}, \quad \forall\, r \in \mathbb{R}, \tag{1.4}$$

$$|\Psi(r)| \leq b|r|^q + b, \quad \forall\, r \in \mathbb{R}, \tag{1.5}$$

where $b, \rho > 0$, $q \geq m$.

The case $m > 1$, *low diffusion*, arises as a model of diffusion of a gas inside a porous medium equation, whereas $0 < m < 1$ describes *fast diffusion* processes. The case $-1 < m < 0$ corresponds to *superfast diffusion* (see [13], [23]). However, there are important cases described by other nonlinearities Ψ which are briefly presented below. For instance

$$\Psi(r) = \rho \,\mathrm{sign}\, r + \nu(r), \quad \forall\, r \in \mathbb{R}, \tag{1.6}$$

where $\rho > 0$, ν is maximal monotone and

$$\mathrm{sign}\, r = \begin{cases} \frac{r}{|r|} & \text{for } r \neq 0, \\ [-1, 1] & \text{for } r = 0. \end{cases} \tag{1.7}$$

In this case equation (1.1) is used to describe the self-organized criticality in the evolution of a large variety of systems (see e.g. [4], [5], [14]) and its main feature is that the diffusion coefficient $j(r) = \rho|r|$ is singular in the origin. Taking into account (1.2) we may formally represent equation (1.1) in this case as

$$dX - \rho\, \mathrm{div}(\delta(X)\nabla X)dt + \nu(X)dt = XdW(t),$$

where δ is the Dirac function, which places the equation in the class of superfast diffusions.

In the case where $\Psi : \mathbb{R} \to \mathbb{R}$ is given by

$$\Psi(r) = \begin{cases} \log r & \text{for } r > 0, \\ \varnothing & \text{for } r \leq 0, \end{cases} \tag{1.8}$$

equation (1.1) arises in plasma physics as well as in the approximation of Carleman's model of the Boltzmann equation (see [23] for the corresponding deterministic model).

Moreover, in the deterministic case this equation can be used to describe the evolution of a conformally flat metric by its curvature flow (see [24]). As a matter of fact this is the limit case $m = 1$ of the superfast diffusion equation

$$div\ (X^{-m}\nabla X) = \Delta X^{1-m}, \quad 1 < m < 2. \tag{1.9}$$

Also note that in the special case where

$$\Psi(r) = \begin{cases} br & \text{for } r \le 0, \\ 0 & \text{for } 0 < r < \rho, \\ a(r-\rho) & \text{for } r \ge \rho, \end{cases} \tag{1.10}$$

the following version of equation (1.1)

$$dX - \Delta\Psi(X)dt = \Psi(X)dW(t),$$

represents a model for the stochastic two phase transition Stefan problem

$$\begin{cases} d\theta - a\Delta\theta dt = \theta dW(t) \quad \text{for } \theta > 0, \\ \\ d\theta - b\Delta\theta dt = \theta dW(t) \quad \text{for } \theta < 0, \\ \\ (a\nabla\theta^+ - b\nabla\theta^-)\cdot\nabla\ell = -\rho \quad \text{in } \{(t,\xi):\ \theta(t,\xi)=0\}, \end{cases} \tag{1.11}$$

where

$$\{(t,\xi):\ \theta(t,\xi)=0\} = \{(t,\xi):\ t=\ell(\xi)\},$$

and $\theta = \Psi(X)$ is the temperature. (See [7]).

There is now a quite complete theory for equation (1.1) in the low and fast diffusion cases treated in [8], [10], [11], [12], [20], [22], which cover equation (1.1) as well as that with additive noise, that is

$$dX - \Delta\Psi(X)dt = dW(t). \tag{1.12}$$

Our concern here is the long time behavior and in particular the finite time extinction property of solutions when (1.1) is a *fast diffusion equation.* From the deterministic theory of nonlinear diffusion (porous media) equations one knows that in this case the process $X = X(t)$ terminates within a finite time and in [5], [9] a similar result was obtained for the stochastic fast diffusion equation. In Section 2 we treat this problem in detail, following [9] .

The case (1.6) which as mentioned earlier is relevant in self-organized criticality models, is another situation where *finite time extinction* happens with high probability and will be treated in Section 3.

1.1 Notation

We here use the standard notations for spaces $L^p(\mathscr{O})$, $1 \le p \le \infty$, of Lebesgue integrable functions and Sobolev spaces $H^k(\mathscr{O}) \subset L^2(\mathscr{O}),\ k=1,2,\ H^1_0(\mathscr{O}) = \{u \in H^1(\mathscr{O}):\ u=0 \text{ on } \partial\mathscr{O}\}$. The norm of $L^p(\mathscr{O})$ is denoted by $|\cdot|_p$ and the scalar product in $L^2(\mathscr{O})$ by $\langle\cdot,\cdot\rangle_2$. Let $H^{-1}(\mathscr{O})$ denote the dual space of $H^1_0(\mathscr{O})$ with norm

$$\|u\|_{-1}=\left(\int_{\mathscr{O}}(-\Delta)^{-1}u\,u\,d\xi\right)^{1/2}$$

and scalar product

$$\langle u,v\rangle_{-1}=\int_{\mathscr{O}}(-\Delta)^{-1}u\,v\,d\xi.$$

(Here $-\Delta$ is the Laplace operator in $L^2(\mathscr{O})$ with domain $H^2(\mathscr{O})\cap H^1_0(\mathscr{O})$.)

For $p,q\in[1,+\infty]$ and H a Hilbert space let $L^q_W(0,T;L^p(\Omega,H))$ denote the space of all q-integrable processes $u:[0,T]\to L^p(\Omega,H)$ which are adapted to the filtration $\{\mathscr{F}_t\}_{t\geq 0}$. By $C_W([0,T];L^p(\Omega,H))$ we denote the space of all H-valued adapted processes which are p mean square continuous. The space $L^2_W(0,T;L^2(\Omega,H))$ is sometimes simply denoted $L^2_W(0,T;H)$.

Finally, by $L^2_W(\Omega;C([0,T];H))$ we denote the space of all H-valued adapted processes which are H-continuous on $[0,T]$ and

$$\mathbb{E}|X|^2_{C([0,T];H)}<\infty.$$

(We refer to [15] and [19] for basic notations and results on infinite dimensional stochastic equations.)

As the Wiener process W is concerned, we shall assume that it is of the form

$$W(t)=\sum_{k=1}^{\infty}\mu_k e_k\beta_k(t),\quad t\geq 0, \tag{1.13}$$

where $\{e_k\}$ is an orthonormal basis in $L^2(\mathscr{O})$ (there are $\alpha_k\in\mathbb{R}$ such that $\{\alpha_k e_k\}$ is an orthonormal basis in $H^{-1}(\mathscr{O})$), $\{\mu_k\}$ a sequence of real numbers and $\{\beta_k\}$ a mutually independent sequence of real Brownian motions in a stochastic basis $(\Omega,\mathscr{F},\{\mathscr{F}_t\},\mathbb{P})$. We shall always assume that for some positive constants c_1,c_2 we have

$$\sum_{k=1}^{\infty}\mu_k^2|xe_k|_2^2\leq c_1|x|_2^2,\quad\forall\, x\in L^2(\mathscr{O}) \tag{1.14}$$

and

$$\sum_{k=1}^{\infty}\mu_k^2|xe_k|_{-1}^2\leq c_2|x|_{-1}^2,\quad\forall\, x\in H^{-1}(\mathscr{O}). \tag{1.15}$$

By (1.14) it follows that if $X\in L^2_W(\Omega;C([0,T];H))$ we have

$$\mathbb{E}\left|\int_0^t X(s)dW(s)\right|_2^2\leq c_1^2\mathbb{E}\int_0^t|X(s)|_2^2\,ds \tag{1.16}$$

Similarly, by (1.15) it follows that if $X \in L^2_W(C(\Omega;[0,T]);H))$ we have

$$\mathbb{E}\left|\int_0^t X(s)dW(s)\right|_{-1}^2 \le c_1^2\mathbb{E}\int_0^t |X(s)|_{-1}^2\,ds. \tag{1.17}$$

Example 1.1. Let us consider the Wiener process (1.13) where $\{e_k\}$ is the orthonormal basis of eigenfunctions of the Laplace operator in $L^2(\mathscr{O})$ with homogeneous boundary conditions, that is

$$-\Delta e_k = \lambda_k e_k \text{ in } \mathscr{O}, \quad e_k = 0 \text{ on } \partial\mathscr{O}, \tag{1.18}$$

where $\mathscr{O} \subset \mathbb{R}^d$, $d \ge 3$, $\mathscr{O}$ open, bounded. Clearly $\{\lambda_k^{1/2} e_k\}$ is an orthonormal system in $H^{-1}(\mathscr{O})$. We assume that $\partial\mathscr{O}$ is sufficiently regular (for instance of class C_2) in order to apply [16]. Then for each $x \in L^2(\mathscr{O})$ we have

$$|xe_k|_2 \le |x|_2\,|e_k|_\infty \le c\lambda_k^{\frac{d-1}{2}}\,|x|_2, \quad \forall\, k \in \mathbb{N},$$

because by [16] we have $|e_k|_\infty \le c\lambda_k^{\frac{d-1}{2}}$ for all $k \in \mathbb{N}$.

Therefore, (1.14) is fulfilled provided

$$\sum_{k=1}^\infty \mu_k^2 \lambda_k^{d-1} < \infty. \tag{1.19}$$

Let us now consider (1.15). Since $H^{-1}(\mathscr{O})$ is the dual of $H_0^1(\mathscr{O})$ we have

$$|xe_k|_{-1}^2 = \sup\left\{|\langle xe_k,\varphi\rangle|_2^2:\ \varphi \in H_0^1(\mathscr{O}),\ |\varphi|_{H_0^1(\mathscr{O})} \le 1\right\}. \tag{1.20}$$

But

$$|\langle xe_k,\varphi\rangle|_2^2 = |\langle x, e_k\varphi\rangle|_2^2 \le |x|_{-1}^2 |e_k\varphi|^2_{H_0^1(\mathscr{O})}$$

On the other hand, for all $k \in \mathbb{N}$

$$|e_k\varphi|^2_{H_0^1(\mathscr{O})} = |\nabla(e_k\varphi)|_2^2 = -\int_{\mathscr{O}} e_k\,\varphi\,\Delta(e_k\,\varphi)\,d\xi$$

$$= -\int_{\mathscr{O}} (e_k\,\varphi^2\,\Delta e_k + e_k^2\varphi\,\Delta\varphi + \tfrac{1}{2}\nabla(e_k^2)\cdot\nabla(\varphi^2))d\xi$$

$$= -\int_{\mathscr{O}} (e_k\,\varphi^2\,\Delta e_k + e_k^2\varphi\,\Delta\varphi - \tfrac{1}{2}\,e_k^2\,\Delta(\varphi^2))d\xi$$

Since

$$\Delta(\varphi^2) = 2\varphi\,\Delta\varphi + 2|\nabla\varphi|^2,$$

we have

$$|e_k\varphi|^2_{H^1_0} = \int_{\mathscr{O}} (\lambda_k\varphi^2 + |\nabla\varphi|^2))e_k^2 d\xi, \quad \forall\, k \in \mathbb{N}. \tag{1.21}$$

Therefore,

$$|e_k\varphi|^2_{H^1_0(\mathscr{O})} \le \lambda_k|\varphi e_k|_2^2 + |\varphi|^2_{H^1_0(\mathscr{O})}|e_k|^2_\infty, \quad \forall\, k \in \mathbb{N}. \tag{1.22}$$

Now by the Sobolev embedding theorem we have $H^1_0(\mathscr{O}) \subset L^{\frac{2d}{d-2}}(\mathscr{O})$ with continuous embedding. Then, using Hölder in the first term of (1.22) we see that there is a constant $c > 0$ such that

$$|e_k\varphi|^2_{H^1_0} \le (c\lambda_k|e_k|_d^2 + |e_k|^2_\infty)|\varphi|^2_{H^1_0(\mathscr{O})}. \tag{1.23}$$

Now as mentioned earlier we know that

$$|e_k|^2_\infty \le c_1\lambda_k^{d-1}, \quad \forall\, k \in \mathbb{N},$$

so that, by interpolation [(1)]

$$|e_k|^2_d \le c_2\lambda_k^{\frac{(d-1)(d-2)}{d}}, \quad \forall\, k \in \mathbb{N},$$

Finally, we find

$$|e_k\varphi|^2_{H^1_0} \le c(\lambda_k^{1+\frac{(d-1)(d-2)}{d}} + \lambda_k^{d-1})|\varphi|^2_{H^1_0} \le c_1\lambda_k^{d-1}|\varphi|^2_{H^1_0}, \quad \forall\, k \in \mathbb{N}, \tag{1.24}$$

and therefore by (1.20)

$$|xe_k|_{-1} \le C_1\lambda_k^{\frac{d-1}{2}}|x|_{-1}, \quad \forall\, k \in \mathbb{N}. \tag{1.25}$$

In conclusion (1.15) is fulfilled provided

$$\sum_{k=1}^\infty \mu_k^2\lambda_k^{d-1} < \infty, \quad \forall\, k \in \mathbb{N}. \tag{1.26}$$

Recalling finally that λ_k behaves as $k^{\frac{2}{d}}$ as $k \to \infty$ (see e.g. [1]) we conclude that assumptions (1.14) and (1.15) are fulfilled choosing

$$\mu_k = k^{-\alpha}$$

where $\alpha > \frac{3d-2}{2d}$. □

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(1) $|f|_p \le |f|_2^{\frac{2}{p}}\, |f|_\infty^{\frac{p-2}{p}}$.

The existence and uniqueness of a strong solution to (1.1) follows for instance under the following assumptions on $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$.

Hypothesis 1.2. *(i)* Ψ *is a maximal monotone multivalued function (graph) such that* $0 \in \Psi(0)$ *and*

$$\sup\{|\theta| : \theta \in \Psi(r)\} \leq C(1 + |r|^q), \quad \forall\, r \in \mathbb{R},$$

where $q \geq 1$ *and* $C > 0$.

(ii) (1.26) *is fulfilled.*

We note that assumption (ii) is fulfilled in all situations encountered above and it is more general than (1.4) which for instance is not satisfied by (1.6).

Definition 1.3. *A continuous,* $\mathscr{F}_t$*-adapted process* $X : [0,T] \to H^{-1}(\mathscr{O})$ *is said to be a solution of equation* (1.1) *if*

(i) $X \in L^2(0,T; L^2(\Omega; L^2(\mathscr{O})))$.

(ii) There is $\eta \in L^1_W(0,T; L^1(\Omega; H))$ *such that* $\eta \in \Psi(X)$*, a.e. in* $(0,T) \times \mathscr{O} \times \Omega$.

(iii) We have

$$X(t) = x + \Delta \int_0^t \eta(s)ds + \int_0^t X(s)dW(s), \quad t \in [0,T]. \tag{1.27}$$

(iv) $\int_0^t \eta(s)ds \in C([0,T]; H^1_0(\mathscr{O}))$, $\mathbb{P}$*-a.s.*

There is an equivalent formulation of (1.27) in terms of the orthonormal basis $\{e_k\}$ used in [8] and [10]. Namely, $\mathbb{P}$-a.s., one has

$$\begin{aligned}\langle X(t), e_j\rangle_2 &= \langle x, e_j\rangle_2 + \int_0^t \int_{\mathscr{O}} \eta(s,\xi)\Delta e_j(\xi)ds\, d\xi \\ &\quad + \sum_{k=1}^{\infty} \mu_k \int_0^t \langle X(s)e_k, e_j\rangle_2\, d\beta_k(s), \quad \forall\, j \in \mathbb{N},\ t \in (0,T).\end{aligned} \tag{1.28}$$

It should be emphasized that the space $H^{-1}(\mathscr{O})$ is the basic functional space for the treatment of equation (1.1) because in this space the operator $F : D(F) \subset H^{-1}(\mathscr{O}) \to H^{-1}(\mathscr{O})$ defined by

$$\begin{cases} Fy = -\Delta\Psi(y), \quad \forall\, y \in D(F), \\ D(F) = \{y \in H^{-1}(\mathscr{O}) \cap L^1(\mathscr{O}) :\ \exists\, \eta \in H^1_0(\mathscr{O}) \text{ such that } \eta \in \Psi(y), \text{ a. e. in } \mathscr{O}\}, \end{cases}$$

$$(1.29)$$

is maximal monotone. (If Ψ is multivalued one understands by $\Psi(y)$ one of its sections.)

The standard way to study existence for equation (1.1) is to approximate it by the equation

$$\begin{cases} dX_\lambda - \Delta(\Psi_\lambda(X_\lambda) + \lambda X_\lambda)dt = X_\lambda dW(t), & \text{in } (0,\infty)\times\mathscr{O}, \\ \Psi_\lambda(X_\lambda)) + \lambda X_\lambda = 0, & \text{on } (0,\infty)\times\partial\mathscr{O}, \\ X_\lambda(0) = x, & \text{in } \mathscr{O}, \end{cases} \tag{1.30}$$

where

$$\Psi_\lambda(r) = \frac{1}{\lambda}\,(r - (1+\lambda\Psi)^{-1}r), \quad \lambda > 0,\ r \in \mathbb{R},$$

is the Yosida approximation of Ψ (see e.g. [6].) Since Ψ_λ is Lipschitzian and non decreasing, equation (1.30) has a unique solution

$$X_\lambda \in L^2(\Omega\times[0,T]\times\mathscr{O}) \cap L^2_W(\Omega; C([0,T]; H^{-1}(\mathscr{O}))),$$

and $X_\lambda, \Psi_\lambda(X_\lambda) \in L^2_W(0,T; H^1_0(\mathscr{O}))$ for each $x \in H^{-1}(\mathscr{O})$. (It should be emphasized that the Sobolev space $H^{-1}(\mathscr{O}) = (H^1_0(\mathscr{O}))'$ is the basic space to study well posedness of problem (1.30) as well as (1.1) because in this space the operator $x \mapsto -\Delta\Psi(x)$ defined in (1.29) is m-accretive.)

The existence and uniqueness of a solution X to (1.1) under Hypothesis 1.2 was estabilshed in [10] (see also [8], [11] for other growth conditions on Ψ.)

Theorem 1.4. *Let $x \in L^p(\mathscr{O})$ where $p \geq \max\{4, 2q\}$ and $0 < T < \infty$. Then there is a unique solution X to* (1.1) *which additionally satisfies*

(i) $X \in L^\infty_W(0,T; L^p(\Omega, L^p(\mathscr{O}))) \cap L^2(\Omega; C([0,T]; H^{-1}(\mathscr{O})))$.

(ii) $\eta \in L^{p/q}(\Omega\times[0,T]\times\mathscr{O})$.

(iii) If $x \geq 0$ a.e. in $\mathscr{O}$ then $X(t,x) \geq 0$ in $\Omega\times[0,T]\times\mathscr{O}$.

(iv) $X_\lambda \to X$ strongly in $L^2(\Omega; C([0,T]; H^{-1}(\mathscr{O})))$, weakly in $L^p(\Omega\times[0,T]\times\mathscr{O})$ and weak star in $L^\infty(0,T; L^p(\Omega; L^p(\mathscr{O}))$.

(v) $\Psi_\lambda(X_\lambda) \to \eta$ weakly in $L^{p/q}(\Omega\times[0,T]\times\mathscr{O})$.

2 Finite time extinction for fast diffusion stochastic equation

We consider here equation (1.1) under the following conditions on the maximal monotone graph $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$,

$$\rho|r|^{m+1} \leq \Psi(r)r \leq b|r|^q + c|r|, \quad \forall\, r \in \mathbb{R}, \tag{2.1}$$

where $0 < m < 1$ and $q \geq m+1$, $\rho, b > 0$, $c \in \mathbb{R}$.

Typical examples are

$$\Psi(r) = \rho|r|^{m-1}r, \quad \forall\, r \in \mathbb{R}, \tag{2.2}$$

$$\Psi(r) = \rho|r|^{m-1}r\log(1+|r|) + ar, \quad \forall\, r \in \mathbb{R}, \tag{2.3}$$

$$\Psi(r) = \begin{cases} a_1 r^m & \text{for } r > 0, \\ [-L, 0] & \text{for } r = 0, \\ a_2|r|^{m-1}r - L & \text{for } r < 0, \end{cases} \tag{2.4}$$

where $a_1, a_2 > 0$, $a \geq 0$ and $0 < m < 1$, $L \geq 0$.

Then Theorem 1.4 is applicable in the present situation and so, equation (1.1) has a strong solution X. We shall prove here that the process $X = X(t)$ terminates within finite time with positive probability. Namely one has:

Theorem 2.1. *Assume that $x \in L^{\max\{4,2q\}}(\mathscr{O})$, that $d \leq 3$ and $0 < m < 1$ if $d = 1, 2$, $\frac{1}{5} \leq m < 1$ if $d = 3$. Let $\tau = \inf\{t \geq 0 : \, |X(t,x)|_{-1} = 0\}$. Then we have*

$$|X(t,x)|_{-1} = 0 \quad \forall\, t \geq \tau, \ \mathbb{P}\text{-a.s.}.$$

Moreover, for each $t > 0$ we have

$$\mathbb{P}(\tau \leq t) \geq 1 - (\rho\gamma^{m+1})^{-1}|x|_{-1}^{1-m}\left(\int_0^{(1-m)t} e^{-C^* s} ds\right)^{-1}. \tag{2.5}$$

In particular, if $|x|_{-1}^{1-m} < \frac{\rho\gamma^{m+1}}{C^*}$ then $\mathbb{P}(\tau < \infty) > 0$.

Here

$$C^* = \frac{1}{2}\sup\left\{\sum_{k=1}^{\infty} \mu_k^2 |x e_k|_{-1}^2 : \ |x|_{-1} \leq 1\right\}$$

and γ is the inverse of the norm of the Sobolev embedding $L^{m+1}(\mathscr{O}) \subset H^{-1}(\mathscr{O})$, that is

$$\gamma^{-1} = \sup\{|u|_{-1}\, |u|_{m+1}^{-1} : \ u \in L^{m+1}(\mathscr{O})\}. \tag{2.6}$$

Proof. We first establish the estimate

$$
\begin{aligned}
&|X(t)|_{-1}^{1-m} + \rho(1-m)\gamma^{m+1} \int_r^t 1\!\!1_{\{|X(s)|_{-1}>0\}}\, ds \\
&\le |X(r)|_{-1}^{1-m} + C^*(1-m) \int_r^t |X(s)|_{-1}^{1-m}\, ds \\
&+(1-m) \int_r^t \langle |X(s)|_{-1}^{-(m+1)} X(s), X(s)dW(s)\rangle_{-1}, \quad \mathbb{P}\text{-a.s., } r<t<\infty.
\end{aligned}
\tag{2.7}
$$

A formal argument to show (2.7) is to apply in equation (1.1) Itô's formula to the function $\varphi(x) = |x|_{-1}^{1-m}$. Then

$$
\begin{aligned}
&D\varphi(x) = (1-m)x|x|_{-1}^{-m-1}, \\
&D^2\varphi(x)h = (1-m)|x|_{-1}^{-m-1}h - (1-m^2)|x|_{-1}^{-m-2}\langle x,h\rangle_{-1}, \quad \forall\, h \in H^{-1}(\mathscr{O})
\end{aligned}
$$

and we get therefore

$$
\begin{aligned}
&d\varphi(X(t)) + (1-m)\int_{\mathscr{O}} \Psi(X(t))X(t)d\xi\; |X(t)|_{-1}^{-m-1} \\
&= \frac{1}{2}\sum_{k=1}^{\infty} \Big((1-m)|X(t)|_{-1}^{-m-1}|X(t)e_k|_{-1}^2 \\
&-(1-m^2)|X(t)|_{-1}^{-m-3}|X(t)e_k|_{-1}^2\, |X(t)|_{-1}^2 \Big)\mu_k^2 dt \\
&+(1-m)\langle X(t)dW(t), X(t)\rangle_{-1}|X(t)|_{-1}^{-m} \\
&\le C^*(1-m)|X(t)|_{-1}^{-m-1}dt + (1-m)\langle X(t), X(t)dW(t)\rangle_{-1}|X(t)|_{-1}^{-m},
\end{aligned}
$$

because in virtue of (1.25) $|X(t)e_k|_{-1} \le \lambda_k^{\frac{d-1}{2}} |X(t)|_{-1}$. By (2.1) and (2.6) we have

$$
\int_{\mathscr{O}} \Psi(X(t))X(t)\, d\xi \ge \rho\gamma^{1+m}|X(t)|_{-1}^{m+1}.
$$

This yields by integration from r to t the inequality (2.7). A rigorous argument to prove (2.7) uses the same approach but for the equation (1.30). Namely applying in (2.6) the Itô formula to the semi-martingale $|X_\lambda(t)|_{-1}^2$

and to the function $\varphi_\epsilon(r)=(r+\epsilon^2)^{\frac{1-m}{2}},\ r>-\epsilon^2$, yields

$$d\varphi_\epsilon(|X_\lambda(t)|_{-1}^2)+(1-m)(|X_\lambda(t)|_{-1}^2+\epsilon^2)^{-\frac{m+1}{2}}\langle(\Psi_\lambda+\lambda)(X_\lambda(t)),X_\lambda(t)\rangle_2\,dt$$

$$=\frac{1}{2}\sum_{k+1}^{\infty}\mu_k^2\left[(1-m)\,\frac{|X_\lambda(t)e_k|_{-1}^2}{(|X_\lambda(t)|_{-1}^2+\epsilon^2)^{\frac{1+m}{2}}}-(1-m^2)\,\frac{|X_\lambda(t)e_k|_{-1}^2\,|X_\lambda(t)|_{-1}^2}{(|X_\lambda(t)|_{-1}^2+\epsilon^2)^{\frac{3+m}{2}}}\right]dt$$

$$+2\langle\varphi_\epsilon'(|X_\lambda(t)|_{-1}^2)X_\lambda(t),X_\lambda(t)dW(t)\rangle_{-1}.$$

This yields as above

$$\begin{aligned}&\varphi_\epsilon(|X_\lambda(t)|_{-1}^2)\\&+(1-m)\int_r^t(|X_\lambda(s)|_{-1}^2+\epsilon^2)^{-\frac{m+1}{2}}\langle(\Psi_\lambda+\lambda)(X_\lambda(s)),X_\lambda(s)\rangle_2\,ds\\&\le\varphi_\epsilon(X_\lambda(r)|_{-1}^2+C^*(1-m)\int_r^t\frac{|X_\lambda(s)|_{-1}^2}{(|X_\lambda(s)|_{-1}^2+\epsilon^2)^{\frac{m+1}{2}}}\,ds\\&+2\int_r^t\langle\varphi_\epsilon'(|X_\lambda(s)|_{-1}^2)X_\lambda(s),X_\lambda(s)dW(s)\rangle_{-1},\quad t\ge r\end{aligned}\tag{2.8}$$

We are going to let $\lambda\to0$ in (2.8). Since, as seen earlier in Theorem 1.4(iv), $|X_\lambda(s)|_{-1}\to|X(s)|_{-1}$ uniformly on $[0,T]$, we also have

$$\lim_{\lambda\to0}\varphi_\epsilon(|X_\lambda(s)|_{-1}^2)=\varphi_\epsilon(|X(s)|_{-1}^2),\quad\text{uniformly on }[0,T].$$

Moreover

$$\begin{aligned}&\int_r^t\varphi_\epsilon'(|X_\lambda(s)|_{-1}^2)\langle\Psi_\lambda(X_\lambda(s))+\lambda X_\lambda(s),X_\lambda(s)\rangle_2\,ds\\&\ge\int_r^t\varphi_\epsilon'(|X_\lambda(s)|_{-1}^2)\langle\Psi(1+\lambda\Psi)^{-1}(X_\lambda(s)),(1+\lambda\Psi)^{-1}(X_\lambda(s))\rangle_2\,ds\\&\ge\rho\int_r^t\varphi_\epsilon'(|X_\lambda(s)|_{-1}^2)|(1+\lambda\Psi)^{-1}(X_\lambda(s))|_{m+1}^{m+1}\,ds.\end{aligned}$$

We claim that

$$\begin{aligned}&\liminf_{\lambda\to0}\int_r^t\varphi_\epsilon'(|X_\lambda(s)|_{-1}^2)\langle\Psi_\lambda(X_\lambda(s))+\lambda X_\lambda(s),X_\lambda(s)\rangle_2\,ds\\&\ge\rho\int_r^t\varphi_\epsilon'(|X(s)|_{-1}^2)|X(s)|_{m+1}^{m+1}\,ds,\quad\mathbb{P}\text{-a.s.}\end{aligned}\tag{2.9}$$

We note first that, since as seen in the proof of Theorem 1.4

$$\mathbb{E}\int_{\mathscr{O}}(X_\lambda(t,\xi))^{m+1}\,d\xi \le C, \quad \forall\,\lambda>0,$$

and $|X_\lambda-(1+\lambda\Psi)^{-1}X_\lambda| = \lambda|\Psi_\lambda(X_\lambda))|$ while $\{\Psi_\lambda(X_\lambda))\}$ is bounded in $L^2((0,T)\times\mathscr{O}\times\Omega)$, we have for $\lambda\to 0$

$$(1+\lambda\Psi)^{-1}X_\lambda \to X, \quad \text{weakly in } L^{m+1}((0,T)\times\mathscr{O}\times\Omega)$$

and therefore

$$\varphi'_\epsilon(|X_\lambda|^2_{-1})^{\frac{1}{m+1}}(1+\lambda\Psi)^{-1}(X_\lambda) \to (\varphi'_\epsilon(|X|^2_{-1})^{\frac{1}{m+1}}X,$$

weakly in $L^{m+1}((0,T)\times\mathscr{O}\times\Omega)$. This implies that for each $\chi\in L^\infty(\Omega)$ with $\chi\ge 0$, $\mathbb{P}$-a.s. in Ω, we have

$$\chi^{\frac{1}{m+1}}\varphi'_\epsilon(|X_\lambda|^2_{-1})^{\frac{1}{m+1}}(1+\lambda\Psi)^{-1}(X_\lambda) \to \chi^{\frac{1}{m+1}}(\varphi'_\epsilon(|X|^2_{-1})^{\frac{1}{m+1}}X,$$

weakly in $L^{m+1}((0,T)\times\mathscr{O}\times\Omega)$. By the weak lower semicontinuity of the norm $|\cdot|_{m+1}$ we have

$$\liminf_{\lambda\to 0}\int_\Omega\int_r^t\int_{\mathscr{O}}\chi\varphi'_\epsilon(|X_\lambda|^2_{-1})|(1+\lambda\Psi)^{-1}(X_\lambda)|^{m+1}\;d\xi\,dt\,d\mathbb{P}(\omega)$$

$$\ge \int_\Omega\int_r^t\int_{\mathscr{O}}\chi\varphi'_\epsilon(|X|^2_{-1})|X|^{m+1}\;d\xi\,dt\,d\mathbb{P}(\omega)$$

and, since χ is arbitrary, this implies the pointwise inequality (2.9) as claimed.

Now it remains to be shown that

$$\begin{aligned}&\lim_{\lambda\to 0}\int_r^t\langle\varphi'_\epsilon(|X_\lambda(s)|^2_{-1})X_\lambda(s),X_\lambda(s)dW(s)\rangle_{-1}\\ &=\int_r^t\langle\varphi'_\epsilon(|X(s)|^2_{-1})X(s),X(s)dW(s)\rangle_{-1},\quad \mathbb{P}\text{ -a.s.}\end{aligned} \tag{2.10}$$

To prove (2.10) we note that

$$\mathbb{E}\left|\int_r^t\langle\varphi'_\epsilon(|X_\lambda(s)|^2_{-1})X_\lambda(s),X_\lambda(s)dW(s)\rangle_{-1}-\int_r^t\langle\varphi'_\epsilon(|X(s)|^2_{-1})X(s),X(s)dW(s)\rangle_{-1}\right|^2$$

$$=\frac{(1-m)^2}{4}\mathbb{E}\int_r^t\sum_{k=1}^\infty\mu_k^2\Big|(|X_\lambda(s)|^2_{-1}+\epsilon^2)^{-\frac{m+1}{2}}\langle X_\lambda(s),X_\lambda(s)e_k\rangle_{-1}$$

$$-(|X(s)|^2_{-1}+\epsilon^2)^{-\frac{m+1}{2}}\langle X(s),X(s)e_k\rangle_{-1}\Big|^2\,ds\le\frac{(1-m)^2}{2}(J_1+J_2),$$

where

$$J_1 = \mathbb{E}\int_r^t \sum_{k=1}^{\infty} \mu_k^2 \left|(|X_\lambda(s)|_{-1}^2 + \epsilon^2)^{-\frac{m+1}{2}} - (|X(s)|_{-1}^2 + \epsilon^2)^{-\frac{m+1}{2}}\right|^2$$

$$\times |\langle X_\lambda(s), X_\lambda(s) e_k\rangle_{-1}|^2 ds$$

and

$$J_2 \le \mathbb{E}\int_r^t \sum_{k=1}^{\infty} \mu_k^2 (|X(s)|_{-1}^2 + \epsilon^2)^{-(m+1)} \, |\langle X(s), X(s)e_k\rangle_{-1} - \langle X_\lambda(s), X_\lambda(s) e_k\rangle_{-1}|^2.$$

λ

Taking into account that $X_\lambda \to X$ in $L^2(\Omega; C([0,T]; H))$, (2.10) follows.

Now we note that the integral inequality (2.7) by Itô's rule product implies

$$\begin{aligned} & e^{-C^*(1-m)t} |X(t)|_{-1}^{1-m} + \rho(1-m)\gamma^{m+1} \int_r^t e^{-C^*(1-m)s} 1\!\!1_{|X(s)|_{-1} > 0} \, ds \\ & \le e^{-C^*(1-m)r} |X(r)|_{-1}^{1-m} \\ & +(1-m) \int_r^t e^{-C^*(1-m)s} \langle |X(s)|_{-1}^{-(m+1)} X(s), X(s) dW(s)\rangle_{-1}, \\ & \mathbb{P}\text{-a.s.}, \ r < t < \infty. \end{aligned} \tag{2.11}$$

This clearly implies that the process

$$t \mapsto e^{-C^*(1-m)t} |X(t)|_{-1}^{1-m},$$

is an $\{\mathscr{F}_t\}$- supermartingale and therefore by a well known result (see e.g. [21] or [18, Chapter 4, Lemma 3.19]) $|X(t)|_{-1} = 0$ for $t \ge \tau$.

If in (2.11) we take expectation and set $r = 0$ we obtain that for all $t \ge 0$,

$$e^{-C^*(1-m)t} \mathbb{E}|X(t)|_{-1}^{1-m} + \rho(1-m)\gamma^{m+1} \int_0^t e^{-C^*(1-m)s} \mathbb{P}(\tau > s) \, ds \le |x|_{-1}^{1-m}.$$

This yields

$$\mathbb{P}(\tau > t) \le \left(\rho(1-m)\gamma^{m+1} \int_0^t e^{-C^*(1-m)s} \, ds\right)^{-1} |x|_{-1}^{1-m}, \quad \forall\, t > 0$$

and (2.5) follows. This completes the proof. □

Remark 2.2. It is easily seen that Theorem 2.1 extends to all bounded $\mathscr{O} \subset \mathbb{R}^d$ with $d \geq 4$ if $m \in (\frac{d-2}{d+2}, 1)$ but we confined to $1 \leq d \leq 3$ because this is really the interesting case in the applications. On the other hand, the condition $d > \frac{1}{5}$ in 3-D seems to be merely a technical one.

In the deterministic case the finite time extinction happens at time

$$T = \left(\rho(1-m)\gamma^{m+1}\right)^{-1} |x|_{-1}^{1-m},$$

while in the present situation it seems that the probability given by formula (2.5) is always strictly less than 1 (see the discussion below), though so far we have not been able to prove this.

The main conclusion of Theorem 2.1 is that fast diffusion processes perturbed by a Gaussian multiplicative noise terminates within a finite time with strictly positive probability which is close to 1 if the initial datum x has small norm in $H^{-1}(\mathscr{O})$.

The extinction in finite time of the process is due to the fact that the diffusion coefficient $|X|^{m-1}$ is large for small concentration X and so causes a faster speed of mass. As a matter of fact the finite time extinction in the fast diffusion equation (1.1) is due to a loss of mass during the diffusion process. In fact if we apply Itô's formula in (1.1) to the function $x \mapsto \varphi(x) \equiv 1$ we get (formally, but this can be proven rigorously by a regularization procedure)

$$\int_{\mathscr{O}} X(t,\xi)d\xi + \int_0^t \int_{\partial\mathscr{O}} \frac{\partial}{\partial\nu} X^m(s,\xi)d\sigma\, ds$$

$$= \int_{\mathscr{O}} x(\xi)d\xi + \sum_{k=1}^{\infty} \mu_k \int_0^t \int_{\mathscr{O}} X(s,\xi)e_k(\xi)d\beta_k(s), \quad \forall\, t \geq 0,\ \mathbb{P}\text{-a.s.}$$

At time t the loss of mass is just

$$R(t,\omega) = -\int_0^t \int_{\partial\mathscr{O}} \frac{\partial}{\partial\nu} X^m(s,\xi)d\sigma\, ds$$

$$+ \sum_{k=1}^{\infty} \mu_k \int_0^t \int_{\mathscr{O}} X(s,\xi)e_k(\xi)d\beta_k(s), \quad \forall\, t \geq 0,\ \mathbb{P}\text{-a.s.}$$

(Here ν is the exterior normal to $\mathscr{O}$ and the integral $\int_{\partial\mathscr{O}} \frac{\partial}{\partial\nu} X^m(s,\xi)d\sigma$ should be taken in the distributional sense.)

Of course, the process terminates in $t = \tau$ if $R(t) \geq |x|_1$ and so, Theorem 2.1 amounts to saying that the probability that this happens before time T is estimated by (2.5). The expression of R explains why, contrary to the deterministic case, one cannot expect this probability to be one for some finite time T.

Remark 2.3. In a slightly weaker form Theorem 2.1 was formulated and its proof outlined in [9].

3 Finite time extinction for the stochastic self-organized criticality equation

We consider here equation (1.1) where Ψ is given by (1.6), that is

$$\begin{cases} dX(t) - \rho\Delta\,\mathrm{sign}(X(t))dt - \Delta\nu(X(t))dt = X(t)\,dW(t), \quad \text{in } (0,\infty)\times\mathscr{O}, \\ \\ \rho\,\mathrm{sign}(X(t)) + \nu(X(t)) = 0, \quad \text{on } (0,\infty)\times\partial\mathscr{O}, \\ \\ X(0) = x, \quad \text{in } \mathscr{O}, \end{cases} \tag{3.1}$$

where $\rho > 0$, ν is a maximal monotone graph satisfying (2.1) such that $\nu(0) \in 0$ and sign is defined by (1.7). Since the mapping $x \mapsto \rho\,\mathrm{sign}(x)+\nu(x)$ is multivalued, equation (3.1) and the boundary condition are understood to be satisfied by a section $\eta \in \rho\,\mathrm{sign}(x) + \nu(x)$ as specified in Definition 1.3.

Equation (3.1) is referred to a *stochastic self-organized citicality equation* (SOC) and it is generally accepted as a mathematical model in a large variety of physical systems which develop a spontaneous mechanism to reach critical states. The standard SOC is the so-called sand-pile model due to Bak, Tang and Wiesenfeld [2], [3] and likewise the fast diffusion equation (3.1) is a nonlinear diffusion equation with singularity in the diffusion coefficient and this is the source of finite time extinction phenomenon of the process $X = X(t)$. As seen earlier in Section 1, (3.1) can be viewed as a super fast diffusion equation.

Here we briefly describe two results of this type already studied in [10], [12]. We assume everywhere in the following that ν is a maximal monotone graph possibly multivalued which satisfies Hypothesis 1.2.

Theorem 3.1. *Assume that $d = 1$. Let $\tau = \inf\{t \geq 0 : \ |X(t)|_{-1} = 0\}$. Then for each $t > 0$ we have*

$$\mathbb{P}(\tau \leq t) \geq 1 - |x|_{-1}\left(\rho\gamma\int_0^t e^{-C^* s}ds\right)^{-1} \tag{3.2}$$

where

$$\gamma = \inf\{|x|_1\,|x|_{-1}^{-1} :\ x \in L^1(\mathscr{O})\}$$

and

$$C^* = \frac{1}{2}\sum_{k=1}^{\infty}\mu_k^2\lambda_k^2$$

In particular, if $|x|_{-1} < \frac{\rho\gamma}{C^*}$ *then* $\mathbb{P}(\tau < \infty) > 0$.

Proof. We proceed as in the proof of Theorem 2.1, namely, we first prove the inequality

$$\begin{aligned}|X(t)|_{-1}\rho\gamma\int_r^t 1\!\!1_{\{|X(s)|_{-1}>0\}}\,ds &\le |X(r)|_{-1} + C^*\int_r^t |X(s)|_{-1}\,ds \\ &+ \int_r^t \langle |X(s)|_{-1}X(s), X(s)dW(s)\rangle_{-1}, \quad \mathbb{P}\text{-a.s.},\ r<t<\infty.\end{aligned} \tag{3.3}$$

To this end we apply Itô's formula in equation (1.30) to the function $\varphi_\epsilon(x) = (|x|_{-1}^2 + \epsilon^2)^{1/2}$ and obtain as in the previous case

$$\varphi_\epsilon(|X_\lambda(t)|_{-1}^2) + \int_r^t (|X_\lambda(s)|_{-1}^2 + \epsilon^2)^{-\frac{1}{2}}\langle(\Psi_\lambda + \lambda)(X_\lambda(s)), X_\lambda(s)\rangle_2\,ds$$

$$= \varphi_\epsilon(|X_\lambda(r)|_{-1}^2) + \frac{1}{2}\sum_{k=1}^{\infty}\mu_k^2\int_r^t \frac{|X_\lambda(s)e_k|_{-1}^2(|X_\lambda(s)|_{-1}^2+\epsilon^2) - \langle X_\lambda(s), X_\lambda(s)e_k\rangle_{-1}}{(|X_\lambda(s)|_{-1}^2+\epsilon^2)^{\frac{3}{2}}}\,ds$$

$$+ \int_r^t \frac{\langle X_\lambda(s), X_\lambda(s)dW(s)\rangle_{-1}}{(|X_\lambda(s)|_{-1}^2+\epsilon^2)^{\frac{1}{2}}}\,, \quad t \ge r,$$

and letting $\lambda \to 0$ we obtain by virtue of Theorem 1.4(iv), by the same arguments as in the proof of Theorem 2.1, that $\mathbb{P}$-a.s.

$$\varphi_\epsilon(|X(t)|_{-1}^2) + \rho\int_r^t (|X(s)|_{-1}^2 + \epsilon^2)^{-\frac{1}{2}}|X(s)|_{-1}^2, ds$$

$$\le \varphi_\epsilon(|X(t)|_{-1}^2) + \frac{1}{2}\sum_{k=1}^{\infty}\mu_k^2\int_r^t \frac{|X(s)e_k|_{-1}^2(|X(s)|_{-1}^2+\epsilon^2) - \langle X(s), X(s)e_k\rangle_{-1}}{(|X(s)|_{-1}^2+\epsilon^2)^{\frac{3}{2}}}\,ds$$

$$+ \int_r^t \frac{\langle X(s), X(s)dW(s)\rangle_{-1}}{(|X(s)|_{-1}^2+\epsilon^2)^{\frac{1}{2}}}\,, \quad t \ge r, \ \ \mathbb{P}\text{-a.s.}$$

Now taking into account that $\gamma|x|_{-1} \le |x|_1$ and letting $\epsilon \to 0$ we obtain (3.3) as claimed. By (3.3) and Itô's product rule we get

$$\begin{aligned}e^{-C^*t}|X(t)|_{-1}+\rho\gamma\int_r^t e^{-C^*s}\mathbb{1}_{\{|X(s)|_{-1}>0\}}\,ds &\le e^{-C^*r}|X(r)|_{-1}\\ +\int_r^t e^{-C^*s}\langle |X(s)|_{-1}^{-1}X(s),X(s)dW(s)\rangle_{-1}, &\quad \mathbb{P}\text{-a.s., } r<t<\infty.\end{aligned} \tag{3.4}$$

This implies that the process $t\mapsto e^{-C^*(1-m)t}|X(t)|_{-1}$, is an $\{\mathscr{F}_t\}$- supermartingale and so, $|X(t)|_{-1}=0$ for $t\ge\tau$. Then taking expectation in (3.4) and setting $r=0$ we find that

$$e^{-C^*t}\mathbb{E}|X(t)|_{-1}+\rho\gamma\int_0^t e^{-C^*s}\mathbb{P}(\tau>s)\,ds\le |x|_{-1},\quad \mathbb{P}\text{-a.s., } t>0.$$

The latter clearly implies (3.2) thereby completing the proof. □

For technical reasons (the Sobolev embedding Theorem), Theorem 3.1 is confined to 1-D case. We present below an asymptotical extinction result which works in all dimensions d.

Remark 3.2. By the previous proof it is easily seen that Theorem 3.1 remains true for more general equations of the form

$$dX-\Delta\Psi(X)dt=XdW(t),$$

where Ψ is a maximal monotone graph satisfying (2.1) and such that $[-\rho,\rho]\subset\Psi(0)$.

3.1 Asymptotic extinction to SOC

Theorem 3.3. *Let $x\in L^4(\mathscr{O})$, $d=1,2,3$, $x\ge 0$ and $\nu=0$. Then the solution X to equation (3.1) satisfies $X\ge 0$, a.e. $(0,\infty)\times\mathscr{O}\times\Omega$ and*

$$\lim_{t\to\infty}\int_{\mathscr{O}} X(t,\xi)d\xi=l<\infty,\quad \mathbb{P}\text{-}a.s., \tag{3.5}$$

$$\int_0^\infty m(\mathscr{O}\setminus\mathscr{O}_0^t)dt<\infty,\quad \mathbb{P}\text{-}a.s., \tag{3.6}$$

where m is the Lebesgue measure and

$$\mathscr{O}_0^t=\{\xi\in\mathscr{O}:\ X(t,\xi)=0\},\quad t\ge 0. \tag{3.7}$$

By (3.7) it follows that for "almost all" sequences $t_n \to \infty$ we have $m(\mathscr{O} \setminus \mathscr{O}_0^{t_n}) \to 0$. Roughly speaking this means that for t large enough $X(t,\xi) = 0$ on a set $\mathscr{O}_0^t$ which differs from $\mathscr{O}$ by a set of small Lebesgue measure. In other words, for t large enough the *non critical zone* $\mathscr{O} \setminus \mathscr{O}_0^t$ of $X(t)$ is "arbitrarily small". (3.5) means that the total mass associated with the process $X(t)$ is $\mathbb{P}$-a.s. convergent as $t \to \infty$. One might suspect that $l = 0$ (as it happens in deterministic case [5]) and we shall see that this is indeed the case for a special form of the Wiener process $W(t)$.

We now prove Theorem 3.3.

Proof. Since the complete proof of the theorem is given in work [12] here we confine ourselves to sketch it and we refer to [12] for details.

We come back to the approximating equation (1.30) where $\Psi = \rho\,\mathrm{sgn}$ and show first via a standard martingale integral inequality that for each $T > 0$

$$\mathbb{E} \sup_{t\in[0,T]} |X(t)|_2^2 \leq C_T |x|_2^2. \tag{3.8}$$

(The details are omitted.)

Next we consider a function $\varphi_\lambda \in C_b^3(\mathbb{R})$ such that $\varphi_\lambda(0) = 0$ and

$$\begin{cases} \varphi_\lambda'(r) = \frac{r}{\lambda} \text{ for } |r| \leq \lambda, \quad \varphi_\lambda'(r) = 1+\lambda \text{ for } |r| \geq 2\lambda \\ \\ \varphi_\lambda'(r) = -(1+\lambda) \text{ for } |r| \leq -2\lambda, \quad 0 \leq \varphi_\lambda''(r) \leq \frac{C}{\lambda}, \end{cases} \tag{3.9}$$

for all $r \in \mathbb{R}$ and some $C > 0$.

It is easily seen that φ_λ is a smooth approximation of the function $r \mapsto |r|$ and

$$|\varphi_\lambda'(r) - (\mathrm{sign})_\lambda(r)| \leq C\lambda, \quad \forall\, r \in \mathbb{R},\ \lambda > 0, \tag{3.10}$$

where $(\mathrm{sign})_\lambda$ is the Yosida approximation of the sign graph (1.7), i.e. $\Psi_\lambda = \rho(\mathrm{sign})_\lambda$.

Next we set $Y_\lambda^\epsilon := (1+\epsilon A)^{-1} X_\lambda$, where $A = -\Delta$, $D(A) := H^2(\mathscr{O}) \cap H_0^1(\mathscr{O})$, $\epsilon > 0$ and rewrite (1.30) in terms of Y_λ^ϵ. We obtain that

$$\begin{cases} dY_\lambda^\epsilon + A(1+\epsilon A)^{-1}(\Psi_\lambda(X_\lambda) + \lambda X_\lambda)dt = (1+\epsilon A)^{-1} X_\lambda dW(t), \quad \text{in } (0,\infty)\times\mathscr{O}, \\ \\ \Psi_\lambda(X_\lambda)) + \lambda X_\lambda = 0, \quad \text{on } (0,\infty)\times\partial\mathscr{O}, \\ \\ Y_\lambda^\epsilon(0) = (1+\epsilon A)^{-1} x, \quad \text{in } \mathscr{O}. \end{cases} \tag{3.11}$$

The process $t \mapsto Y^\epsilon_\lambda(t)$ is $H^1_0(\mathscr{O})$-valued and continuous on $[0,T]$ and so, applying Itô's formula in (3.11)and letting $\epsilon \to 0$, yields

$$\int_{\mathscr{O}} \varphi_\lambda(X_\lambda)d\xi + \int_0^t \int_{\mathscr{O}} \nabla(\Psi_\lambda(X_\lambda)+\lambda X_\lambda)\cdot \nabla\varphi'_\lambda(X_\lambda)ds\,d\xi$$

$$= \int_{\mathscr{O}} \varphi_\lambda(x)d\xi + \sum_{k=1}^\infty \mu_k^2 \int_0^t \int_{\mathscr{O}} \varphi''_\lambda(X_\lambda)|(X_\lambda e_k)|^2 d\xi\,ds \tag{3.12}$$

$$+ \int_0^t \langle \varphi'_\lambda(X_\lambda), X_\lambda dW(s)\rangle_2.$$

We also note that by (3.9) we have

$$\sum_{k=1}^\infty \mu_k^2 \int_0^t \int_{\mathscr{O}} \varphi''_\lambda(X_\lambda)|(X_\lambda e_k)|^2 d\xi\,ds$$

$$\le 4C\lambda \sum_{k=1}^\infty \mu_k^2 \int_0^t \int_{\mathscr{O}} \mathbb{1}_\lambda(s,\xi)d\xi\,ds, \tag{3.13}$$

where $\mathbb{1}_\lambda$ is the characteristic function of the set

$$\{(s,\xi,\omega)\in(0,\infty)\times\mathscr{O}\times\Omega:\ 0\le X_\lambda(s,\xi,\omega)\le 2\lambda\}.$$

It follows also that

$$\lim_{\lambda\to 0}\int_{\mathscr{O}} \varphi_\lambda(X_\lambda(t,\xi))d\xi = \int_{\mathscr{O}} X(t,\xi)d\xi,\ \text{weakly in } L^2(\Omega),\ \forall\, t\ge 0. \tag{3.14}$$

We set

$$I_\lambda(t) = \int_0^t \int_{\mathscr{O}} \nabla(\Psi_\lambda(X_\lambda)+\lambda X_\lambda)\cdot\nabla\varphi'_\lambda(X_\lambda)d\xi\,ds,$$

$$M_\lambda(t) \;=\; \int_0^t \langle \varphi'_\lambda(X_\lambda), X_\lambda)dW\rangle_2 = \sum_{k=1}^\infty \int_0^t \langle \varphi'_\lambda(X_\lambda), X_\lambda e_k)d\beta_k(s)\rangle_2$$

and so we rewrite (3.12) as

$$\int_{\mathscr{O}} \varphi_\lambda(X_\lambda(t))d\xi + I_\lambda(t)$$

$$= \int_{\mathscr{O}} \varphi_\lambda(x)d\xi + \sum_{k=1}^\infty \mu_k^2 \int_0^t \int_{\mathscr{O}} \varphi''_\lambda(X_\lambda(t))|(X_\lambda(t)e_k)|^2 d\xi\,ds + M_\lambda(t). \tag{3.15}$$

Taking into account that

$$X_\lambda \to X, \quad \varphi'_\lambda(X_\lambda) \to \eta \in \rho\,\text{sign}\,x, \text{ weakly in } L^2((0,\infty)\times\mathscr{O}\times\Omega),$$

it follows after some calculations (see [12]) that $\mathbb{P}$-a.s.

$$\lim_{\lambda\to 0} M_\lambda(t) = M(t) = \int_0^t \langle \eta, X(s)dW(s)\rangle_2, \quad \forall\, t \geq 0 \tag{3.16}$$

Then by (3.13)–(3.16) we see that

$$\int_{\mathscr{O}} \varphi(X(t,\xi))d\xi + \widetilde{I}(t) = \int_{\mathscr{O}} \varphi(x)d\xi + M(t), \quad \forall\, t \geq 0, \tag{3.17}$$

where

$$\widetilde{I}(t) = w - \lim_{\lambda\to 0} I_\lambda(t) \quad \text{in } L^2(\Omega). \tag{3.18}$$

We set

$$Z(t) = \int_{\mathscr{O}} \varphi(X(t,\xi))d\xi$$

and note that it is a nonnegative semimartingale with $\mathbb{E}Z(t) < \infty, \forall\, t \geq 0$. Since the function $t \mapsto X(t)$ is a weakly continuous $L^2(\mathscr{O})$-valued function it follows also that $t \mapsto Z(t)$ is continuous. Then we may define a continuous version $I(t)$ of $\widetilde{I}(t)$

$$I(t) = Z(0) - Z(t) + M(t), \quad \forall\, t \geq 0 \tag{3.19}$$

and it follows also that I is a nondecreasing process on $(0,\infty)$. Moreover $M(t)$ is a continuous semimartingale. Then we shall apply the following version of a martingale convergence result (see [17, page 139]).

Lemma 3.4. *Let Z be a nonnegative semimartingale with $\mathbb{E}Z(t) < \infty, \forall\, t \geq 0$ and let I be a nondecreasing continuous process such that*

$$Z(t) + I(t) = Z(0) + I_1(t) + M(t), \quad \forall\, t \geq 0, \tag{3.20}$$

where M is a local martingale. Then if $\lim_{t\to\infty} I_1(t) < \infty$, $\mathbb{P}$-a.s., we have

$$\lim_{t\to\infty} Z(t) + I(\infty) < \infty, \quad \mathbb{P}\text{-a.s.} \tag{3.21}$$

Applying Lemma 3.4 to (3.19) we infer that

$$\lim_{t\to\infty}\int_{\mathscr{O}}\varphi(X(t,\xi))d\xi = l < \infty,$$

exists $\mathbb{P}$-a.s.

Now coming back to I_λ we see that $\mathbb{P}$-a.s.

$$\begin{aligned} I_\lambda(t) &\geq \int_0^t\int_{\mathscr{O}}\nabla(\Psi_\lambda(X_\lambda)+\lambda X_\lambda)\cdot\nabla\varphi'_\lambda(X_\lambda)d\xi\,ds \\ &\geq \int_0^t\int_{\mathscr{O}}|\nabla\Psi_\lambda(X_\lambda)|^2 d\xi\,ds. \end{aligned}$$

Taking into account that $\Psi_\lambda(X_\lambda)\to\eta\in\rho\,\mathrm{sign}\,X$ weakly in $L^2((0,\infty)\times\mathscr{O}\times\Omega)$ as $\lambda\to 0$ we infer that

$$\int_0^t|\nabla\eta|_2^2\,dt \leq I(t),\quad t\geq 0,\ \mathbb{P}\text{-a.s.}$$

and therefore

$$\int_0^\infty|\nabla\eta|_2^2\,dt \leq I(\infty),\quad \mathbb{P}\text{-a.s.}$$

Next by the Sobolev embedding theorem we have

$$|\eta(t)|_{p^*}\leq C|\nabla\eta|_2,\quad \forall\, t\geq 0,$$

where $p^*=\frac{2d}{d-2}$ for $d>2$, p^* arbitrary in $[2,\infty)$ for $d=2$ and $p^*=\infty$ for $d=1$. Hence

$$\int_0^\infty|\eta|_{p^*}^2\,dt\leq\infty,\quad t\geq 0,\ \mathbb{P}\text{a.s.} \tag{3.22}$$

Taking into account that $\eta\in\rho\,\mathrm{sign}\,X$ a.e. in $(0,\infty)\times\mathscr{O}\times\Omega$, we have $\eta=\rho$ a.e. in $\{(t,\xi,\omega):\ X((t,\xi,\omega))>\}$ and so (3.22) yields

$$\int_0^\infty(m(\mathscr{O}\setminus\mathscr{O}_0^t)^{\frac{2}{p^*}}\,dt<\infty,$$

as claimed. □

We shall now assume that the noise is finite dimensional,

$$W(t,\xi)=\sum_{k=1}^N\mu_k e_k(\xi)\beta_k(t),\quad t\geq 0,\ \xi\in\mathscr{O}, \tag{3.23}$$

and set

$$\widetilde{\mu}(\xi)) = \sum_{k=1}^{N} \mu_k^2 e_k^2(\xi), \quad \xi \in \mathscr{O}. \tag{3.24}$$

In this case Theorem 3.3 is completed by the following asymptotic result.

Theorem 3.5. *Under the assumptions of Theorem 3.3 assume further that W is of the form (3.23). Then we have*

$$\lim_{t\to\infty} e^{-W(t)} X(t) = 0 \quad \text{in } L^1(\mathscr{O}),\ \mathbb{P}\text{-a.s.} \tag{3.25}$$

and if $\widetilde{\mu}(\xi) > 0$ for all $\xi \in \mathscr{O}$

$$\lim_{t\to\infty} X(t) = 0 \quad \text{in } L^1_{loc}(\mathscr{O}),\ \mathbb{P}\text{-a.s.} \tag{3.26}$$

Moreover, for each compact subset $K \subset \mathscr{O}$ we have

$$\int_K X(t,\xi)d\xi \le (m(K))^{\frac{1}{2}} |x|_2 \exp\left\{ \sup_K (\widetilde{\mu})^{\frac{1}{2}} \left(\sum_{k=1}^{N} |\beta_k(t)| \right)^{\frac{1}{2}} \right\} e^{-\frac{t}{2}\inf_{K'} \widetilde{\mu}}, \tag{3.27}$$

where K' is any compact neighborhood of K. In particular, one has

$$\int_K X(t,\xi,\omega)d\xi \le (m(K))^{\frac{1}{2}} |x|_2 e^{-\rho_K t}, \quad \forall\, t \ge t_0(\omega),\ \omega \in \Omega, \tag{3.28}$$

for some $\rho_K > 0$.

It should be noted that the condition $\widetilde{\mu} > 0$ on $\mathscr{O}$ automatically holds if $\mu_1 > 0$ because the first eigenfunction e_1 of the Laplace operator with homogeneous boundary conditions is positive on $\mathscr{O}$.

Proof. We proceed by rescaling equation (1.1) via the transformation

$$X(t) = e^{W(t)} Y(t)$$

and so to reduce it to the random differential equation

$$\begin{cases} \frac{\partial Y(t)}{\partial t} - e^{-W(t)} \Delta \Psi(e^{W(t)} Y(t)) + \frac{1}{2}\, \widetilde{\mu}\, Y(t) = 0 \quad \text{in } (0,\infty) \times \mathscr{O} \\ \psi(e^{W(t)} Y(t))) \in H_0^1(\mathscr{O}), \quad \forall\, t \ge 0,\ \mathbb{P}\text{-a.s.}, \\ Y(0) = x. \end{cases} \tag{3.29}$$

(here$\frac{\partial Y}{\partial t}$ is taken in $H^{-1}(\mathscr{O})$.)

We first note that via the regularized equation we have that

$$|Y(t)|_2 \leq |x|_2, \quad \forall \ \mathbb{P}\text{-a.s.}. \tag{3.30}$$

To prove this we consider the solution Y_λ to approximating equation

$$\begin{cases} \dfrac{\partial Y_\lambda}{\partial t} - e^{-W}\Delta(\Psi_\lambda(e^W Y_\lambda) + \lambda e^W Y_\lambda)) + \dfrac{1}{2}\,\widetilde{\mu}\, Y_\lambda = 0 \quad \text{in } (0,\infty)\times\mathscr{O} \\ \\ Y_\lambda(0) = x. \end{cases} \tag{3.31}$$

and get appropriate estimates.

Now let us prove (3.25). Assume that this is not true, that is, there exists $\delta > 0$ such that for some $\{t_n\} \to \infty$

$$|Y(t_n)|_1 \geq \delta > 0, \quad \forall\, n \in \mathbb{N}, \tag{3.32}$$

where $Y = Y(t,\omega)$ and $\omega \in \Omega$ is arbitrary but fixed. By estimate (3.30) it follows that there is $f \in L^2(\mathscr{O})$ such that $Y(t_n) \to f$ weakly in $L^2(\mathscr{O})$ (possibly on a subsequence of $\{t_n\}$.) Clearly by (3.32) we have

$$0 < \delta \leq \int_{\mathscr{O}} f(\xi) d\xi, \quad f \geq 0 \text{ a.e. in } \mathscr{O}$$

and so $f \neq 0$. On the other hand, for each $n \in \mathbb{N}$ there is $t_n > 0$ such that

$$\left| \int_{\mathscr{O}} Y(t,\xi) d\xi - \int_{\mathscr{O}} Y(t_n,\xi) d\xi \right| \leq \frac{1}{n}\,, \quad \forall\, t \in (t_n - \epsilon_n, t_n + \epsilon_n). \tag{3.33}$$

By (3.6) it follows that there is a subsequence $\{t_{n_k}\} \to \infty$ and $s_k \in (t_{n_k} - \epsilon_{n_k}, t_{n_k} + \epsilon_{n_k})$ such that

$$\int_{\mathscr{O}} 1\!\!1_{\{X(s_k)\neq 0\}}\, d\xi = m(\mathscr{O} \setminus \mathscr{O}_0^{s_k}) \to 0$$

as $k \to \infty$. Hence (selecting a further subsequence if necessary) we have

$$1\!\!1_{\{X(s_k)\neq 0\}} \to 0, \quad \text{a.s. as } k \to \infty.$$

Once again by (3.30) we have that $X(s_k) \to \widetilde{f}$ weakly in $L^2(\mathscr{O})$ and this clearly implies that $Y(s_k) = Y(s_k) 1\!\!1_{\{X(s_k)\neq 0\}} \to 0$ a.e. as $k \to \infty$ and so $\widetilde{f} = 0$ a.e. This yields (see (3.33))

$$\int_{\mathscr{O}} f(\xi)\, d\xi = \lim_{k\to\infty} \int_{\mathscr{O}} Y(t_{n_k},\xi)\, d\xi = \int_{\mathscr{O}} \widetilde{f}(\xi)\, d\xi = 0.$$

This contradiction proves (3.25).

To prove (3.27) we consider a compact $K \subset \mathscr{O}$ and $K' \subset \mathscr{O}$ a compact neighborhood of K. Choose a function $\mu^\alpha \in C_0^\infty(\mathscr{O})$ such that $0 \le \mu^\alpha \le 1$, $\mu^\alpha \le 1$ on K and $\mu^\alpha = 0$ on $\mathscr{O} \setminus K$. We set $C_K = \inf_{K'} \widetilde{\mu}$.

Multiplying (3.31) by $\mu^\alpha Y_\lambda$ and integrating over $\mathscr{O}$, we obtain after some calculation, that

$$(\mu^\alpha)^{\frac{1}{2}} |Y_\lambda(t)|_2^2 \le |(\mu^\alpha)^{\frac{1}{2}} x|_2^2 e^{-C_K t} + \lambda \int_0^t e^{-C_K(t-s)} \eta_\lambda(s) ds, \quad \forall\, t \ge 0 \text{ } \mathbb{P}\text{-a.s.}. \tag{3.34}$$

Then letting $\lambda \to 0$ in (3.34) we get

$$|(\mu^\alpha)^{\frac{1}{2}} Y_\lambda(t)|_2^2 \le e^{-C_K t} \mu^\alpha |x|_2^2 \le e^{-C_K t} |x|_2^2, \quad \forall\, t \ge 0 \text{ } \mathbb{P}\text{-a.s.}. \tag{3.35}$$

Taking into account that

$$\int_K X(t,\xi) d\xi = \int_K Y(t,\xi) e^{-W(t)} d\xi, \quad \forall\, t \ge 0 \text{ } \mathbb{P}\text{-a.s.},$$

by (3.33) we obtain the desired estimate (3.27) as claimed. □

Remark 3.6. It should be noted that if W is the of form (3.23), but $e_k \in C^2(\overline{\mathscr{O}})$ are such that $|e_k|_\infty > 0$, then $\inf\{\widetilde{\mu}(\xi) :\ \xi \in \mathscr{O}\} > 0$ and so in (3.28) we may replace K by $\overline{\mathscr{O}}$ and so (3.28) implies that

$$\lim_{t \to \infty} \int_{\mathscr{O}} X(t,\xi) d\xi = 0, \quad \mathbb{P}\text{-a.s.}$$

and in particular

$$\lim_{t \to \infty} X(t,\xi) d\xi = 0 \quad \text{a.e. in } \ \mathscr{O} \times \Omega.$$

3.2 Self-organized criticality and convergence to equilibrium

Self-organized criticality (SOC) is the property of dynamical systems which have a critical point as attractor and converges spontaneously to this point. The standard model for SOC is the sand-pile model [2], [3], which is at the origin of a large class of other models. If $X(t,\xi)$, $\xi \in \mathscr{O}$, is the state of a system at time t and $X_c = X_c(\xi)$ is the critical state, $\mathscr{O}$ can be separated in the following three regions:

- *critical region* $\mathscr{O}_0^t := \{\xi \in \mathscr{O} :\ X(t,\xi) = X_c(\xi\}$,

- *sub critical region* $\mathscr{O}_-^t := \{\xi \in \mathscr{O}: \ X(t,\xi) < X_c(\xi\}$,
- *super critical region* $\mathscr{O}_+^t := \{\xi \in \mathscr{O}: \ X(t,\xi) > X_c(\xi\}$,

The sub critical and super critical regions are unstable and absorbed in time by the critical zone.

The standard sand-pile SOC is best described via cellular automaton formalism (see [2], [3], [4] and [5]) by the equation

$$\begin{cases} dX(t) - \rho\Delta H(X(t)-X_c)dt = (X(t)-X_c)dW(t), \quad \text{in } (0,\infty)\times\mathscr{O}, \\ \\ X(0)=x, \quad \text{in } \mathscr{O}, \end{cases} \tag{3.36}$$

where H is the Heaviside function.

We have assumed here that the SOC process is perturbed by a noise $(X(t)-X_c)W(t)$ which is proportional to the deviation of $X(t)$ from the critical state X_c. (There are other SOC models described by fast or superfast diffusion equations of the form encountered above.) It should be said that (3.36) is more appropriate to describe the SOC processes which are always in critical or super critical phase and so in general SOC is better described by the equation

$$\begin{cases} dX(t) - \rho\Delta \text{sign } (X(t)-X_c)dt \ni (X(t)-X_c)dW(t), \quad \text{in } (0,\infty)\times\mathscr{O}, \\ \\ \text{sign } (X(t)-X_c) = 0, \quad \text{on } (0,\infty)\times\partial\mathscr{O}, \\ \\ X(0)=x, \quad \text{in } \mathscr{O}, \end{cases} \tag{3.37}$$

to which the above asymptotic results apply neatly. In fact, by Theorem 3.1 we have

Theorem 3.7. *Assume $d=1$ and $x, X_c \in L^4(\mathscr{O})$ such that $x \geq X_C$ a.e. on $\mathscr{O}$. Let $\tau_c = \inf\{t \geq 0: \ |X(t)-X_c|_{-1} = 0\}$. Then for each $t>0$ we have*

$$\mathbb{P}(\tau_c \leq t) \geq 1 - |x - X_c|_{-1}\left(\rho\gamma\int_0^t e^{-C^* s}ds\right)^{-1}. \tag{3.38}$$

Similarly by Theorem 3.3 we have.

Theorem 3.8. *Let $x, X_c \in L^4(\mathscr{O})$, $d=1,2,3$, $x \geq X_c$ a.e. on $\mathscr{O}$. Then we have*

$$\lim_{t\to\infty}\int_{\mathscr{O}} X(t,\xi)d\xi = l < \infty, \quad \mathbb{P}\text{-}a.s.$$

and

$$\int_0^\infty m(\mathscr{O} \setminus \mathscr{O}_0^t)dt < \infty, \quad \mathbb{P}\text{-}a.s. \tag{3.39}$$

As mentioned earlier, (3.39) amounts to saying that there is a sequence $\{t_n\} \to \infty$ such that with probability 1 the whole domain $\mathscr{O}$ is absorbed at moment t_n with the exception of a subset (supercritical) $\mathscr{O}_+^{t_n}$ of Lebesgue measure lesser than $\delta_n \to 0$ as $n \to \infty$.

Theorem 3.5 has a similar interpretation in terms of SOC processes $X(t)$ given by (3.37). In particular, by (3.25), (3.26) we have

$$\lim_{t\to\infty} e^{-W(t)}(X(t) - X_c) = 0 \quad \text{in } L^1(\mathscr{O}),\ \mathbb{P}\text{-a.s.} \tag{3.40}$$

and

$$\lim_{t\to\infty} (X(t) - X_c) = 0 \quad \text{in } L^1_{loc}(\mathscr{O}),\ \mathbb{P}\text{-a.s.} \tag{3.41}$$

Remark 3.9. The stochastic SOC model described here by equation (3.37) can be realized experimentally by adding grains of sand to random locations with Gaussian distributions. The result is a process obtained from the standard sand-pile dynamics described by the cellular automata formalism perturbed by a stochastic process $\int_0^t (X(s) - X_c)dW(s)$. The effect of this fluctuation is described in Theorems 3.1–3.7.

As in the case of fast diffusions (see Remark 2.2) the process X reaches the critical state X_c in time t if the loss of mass

$$R_1(t) = -\rho \int_0^t \int_{\partial\mathscr{O}} \frac{\partial}{\partial\nu} \text{ sign } (X(s) - X_c)d\sigma\, ds + \int_0^t \langle (X(s) - X_c), dW(s)\rangle_2,$$

is greater than $|(X(s) - X_c)|_1$. Formula (3.38) estimates the probability that this happens before time t.

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