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ON UNIQUENESS PROBLEMS RELATED TO ELLIPTIC EQUATIONS FOR MEASURES

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Dedicated to Professor V. V. Zhikov
on occasion of his 70th birthday

We consider equations of the form $L^*\mu = 0$ *for bounded measures on* $\mathbb{R}^d$*, where* L *is a second order elliptic operator, for example,* $Lu = \Delta u + (b, \nabla u)$*, and the equation is understood as the identity*

$$\int Lu\,d\mu = 0$$

for all compactly supported smooth functions u*. Stationary Kolmogorov equations for invariant measures of diffusion processes belong to this type. Solutions are considered in the class of probability measures and in the class of signed measures with integrable densities. We discuss the following problems: When is a probability solution to this equation unique? When does a given probability solution have the property that any integrable solution is a multiple of it? Which dimension can have the simplex of probability solutions? We present some recent positive results, give counterexamples and formulate open problems. Bibliography*: 19 *titles.*

This paper gives a survey of recent results related to various uniqueness problems for elliptic equations for measures. In particular, it includes a number of new results not covered by the survey [1]. However, for completeness and reader's convenience we also briefly recall some results presented in [1]. Let us consider a second order elliptic operator

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$$Lu(x) = \sum_{i,j\leqslant d} a^{ij}(x)\partial_{x_i}\partial_{x_j}u(x) + \sum_{i\leqslant d} b^i(x)\partial_{x_i}u(x)$$

on smooth compactly supported functions on $\mathbb{R}^d$. We say that a locally finite Borel measure μ on $\mathbb{R}^d$ (possibly signed) satisfies the elliptic equation

$$L^*\mu = 0 \tag{1}$$

if a^{ij}, b^i are $|\mu|$-integrable on every ball $U \subset \mathbb{R}^d$ (notation: $a^{ij}, b^i \in L^1(U, |\mu|)$) and

$$\int_{\mathbb{R}^d} Lu\, d\mu = 0 \quad \forall u \in C_0^\infty(\mathbb{R}^d).$$

If $\mu \geqslant 0$ and $\mu(\mathbb{R}^d) = 1$, then μ is called a probability solution. Apart probability solutions we also consider signed solutions given by densities in $L^1(\mathbb{R}^d)$. Such solutions are called integrable.

We discuss the following problems:

1. When is a probability solution to Equation (1) unique?
2. When does a given probability solution have the property that any integrable solution is a multiple of it?
3. Which dimension can have the simplex of probability solutions $\mathscr{P}$?

These problems have been actively investigated in [1]–[9]. Close problems related, however, to the uniqueness for the Dirichlet problem for elliptic equations in bounded domains and in the whole space were studied in [10]–[13].

Let us introduce some notation and our basic assumptions.

Let $p > d$ and let $W^{p,1}_{\text{loc}}(\mathbb{R}^d)$ denote the Sobolev class of functions on $\mathbb{R}^d$ that are locally Lebesgue integrable to power p along with their generalized first order partial derivatives.

Assume that the matrix $A = (a^{ij})_{1\leqslant i,j\leqslant d}$ is symmetric and satisfies the following conditions:

(C1) the functions a^{ij} belong to the class $W^{p,1}_{\text{loc}}(\mathbb{R}^d)$ and for every ball $U \subset \mathbb{R}^d$ there exist numbers $m = m(U) > 0$ and $M = M(U) > 0$ such that for all $x \in U$ and $y \in \mathbb{R}^d$ one has

$$m|y|^2 \leqslant \sum_{1\leqslant i,j\leqslant d} a^{ij}(x)y_iy_j \leqslant M|y|^2.$$

It was shown in [14] that the condition $\det A > 0$ implies the absolute continuity of the measure μ with respect to the Lebesgue measure. Equation (1) can be written as an equation for the density ϱ of the measure μ as follows:

$$\partial_{x_i}\partial_{x_j}\big(a^{ij}\varrho\big) - \partial_{x_i}\big(b^i\varrho\big) = 0.$$

Certainly, the latter is also understood in the sense of distributions.

Moreover, if, in addition to (C1), the drift coefficient $b = (b^i)$ satisfies the condition

(C2) $\quad |b| \in L^p_{\text{loc}}(\mathbb{R}^d)$,

then the measure μ possesses a continuous density $\varrho \in W^{p,1}_{\text{loc}}(\mathbb{R}^d)$.

If conditions (C1) and (C2) are fulfilled, then, for any nonnegative solution $\mu = \varrho\, dx$ and every ball U, one has the Harnack inequality

$$\sup_{x\in U} \varrho(x) \leqslant C(U) \inf_{x\in U} \varrho(x),$$

which yields that the continuous version of ϱ is strictly positive.

Certainly, in general, our equation may fail to have nonzero integrable solutions at all. For example, this happens if $A = I$ and $b = 0$. The following existence theorem was proved in [15].

Theorem 1. *Let conditions* (C1) *and* (C2) *be fulfilled. Suppose that there exists a function* $V \in C^2(\mathbb{R}^d)$ *such that*

$$\lim_{|x|\to\infty} V(x) = +\infty \quad \text{and} \quad \lim_{|x|\to\infty} LV(x) = -\infty.$$

Then there exists a unique probability solution μ *to Equation* (1).

In particular, the hypotheses of the theorem are fulfilled if A is unit matrix, b is locally bounded and $(b(x), x) \to -\infty$ as $|x| \to \infty$. Indeed, one can take $V(x) = |x|^2$.

If one does not assume conditions (C1) and (C2), then it is easy to give examples where there are several probability solutions.

Example 1. Let $d = 1$, and let $A = 1$. We set

$$\varrho(x) = ce^{-x^{-2}-x^2}, \quad b(x) = \frac{\nabla \varrho(x)}{\varrho(x)} = \frac{2}{x^3} - 2x$$

if $x \neq 0$ and $b(0) = 0$, $\varrho(0) = 0$, where a number $c > 0$ is such that the measure $\mu = \varrho\, dx$ is a probability. It is readily verified that, in addition to the measure μ, Equation (1) is also satisfied by the measures $\mu^1 = c_1\chi^+\varrho\, dx$ and $\mu^2 = c_2\chi^-\varrho\, dx$, where χ^+ and χ^- are the indicator functions of $[0, +\infty)$ and $(-\infty, 0]$, respectively, and c_1 and c_2 are normalizing constants.

In this example, the coefficient b is not integrable with respect to the Lebesgue measure in a neighborhood of the origin. But if in the one-dimensional case we have $b \in L^1_{\mathrm{loc}}(\mathbb{R})$ and A is an absolutely continuous and positive function such that

$$\int_0^{+\infty} \frac{1}{\sqrt{A(s)}}\, ds = \int_{-\infty}^{0} \frac{1}{\sqrt{A(s)}}\, ds = \infty,$$

then, as shown in [5], the set $\mathscr{P}$ of probability solutions consists of at most one element. However, even in this case Equation (1) can have integrable solutions not proportional to a unique probability solution.

Example 2. Let $d = 1$. We set

$$\varrho(x) = \frac{c_1}{1 + 4x^4} e^{-x^4}, \quad b(x) = \frac{\varrho'(x)}{\varrho(x)}, \quad A(x) = 1.$$

Here, c_1 is a positive number such that the measure $\mu = \varrho\, dx$ is a probability. Then μ is a probability solution to Equation (1). It is readily verified that the measure ν with density $u(x) = x(1 + 4x^4)^{-1}$ is an integrable solution and μ, ν are linearly independent. However, a probability solution is unique since $d = 1$ and b is locally integrable with respect to the Lebesgue measure.

It is worth noting that if $d = 1$, $A = 1$, and b is locally Lebesgue integrable, then Equation (1) can be solved explicitly since we have $(\varrho' - b\varrho)' = 0$, so that the space of all solutions (not necessarily integrable on $\mathbb{R}$) is two-dimensional. One can show (cf. [9]) that if there is a nonzero integrable solution, then there is also a probability solution. In the previous example, all solutions are integrable.

In the case $d \geqslant 2$, the situation is much more complicated. For example, even in the case of unit matrix A and infinitely differentiable vector field b Equation (1) can have several probability solutions. The first example of such an equation was constructed in [5]. We recall that example.

Example 3. Let $d \geqslant 2$, let A be unit matrix, and let

$$b^i(x) = \frac{f''(x_i)}{f'(x_i)} + 2\frac{f''(x_{\sigma(i)})}{f'(x_i)}, \quad x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d,$$

where $\sigma\colon \{1, 2, \dots, d\} \to \{1, 2, \dots, d\}$ is one-to-one and $\sigma(i) \neq i$, the function $f \in C^\infty(\mathbb{R}^1)$ is bounded, $f, f' > 0$, and $f' \in L^1(\mathbb{R}^1)$. Then our equation has at least two probability solutions:

$$\mu_1 = c_1 \prod_{i=1}^{d} f'(x_i)\, dx, \quad \mu_2 = c_2 \sum_{i=1}^{d} f(x_i)\, \mu_1.$$

There are deep connections between the uniqueness of a probability solution and the properties of a special sub-Markov semigroup whose generator coincides with the operator L on $C_0^\infty(\mathbb{R}^d)$. These connections were studied in [2, 5], and [3].

Let μ be a probability solution to Equation (1). It was shown in [5] that, under conditions (C1) and (C2), there exists a sub-Markov contracting C_0-semigroup of operators T_t^μ on $L^1(\mu)$ whose generator coincides with L on $C_0^\infty(\mathbb{R}^d)$. In general, the measure μ is just sub-invariant for the semigroup $\{T_t^\mu\}_{t \geqslant 0}$, i.e., for every nonnegative function $f \in L^\infty(\mu)$, one has the inequality

$$\int_{\mathbb{R}^d} T_t^\mu f\, d\mu \leqslant \int_{\mathbb{R}^d} f\, d\mu.$$

If in the latter relation the inequality is replaced by equality, then the measure μ is called an invariant measure for the semigroup $\{T_t^\mu\}_{t \geqslant 0}$. It was also shown in [5] that if μ is an invariant measure for $\{T_t^\mu\}_{t \geqslant 0}$, then μ is a unique probability solution to Equation (1) and that the invariance of μ is equivalent to the essential m-dissipativity of the operator $(L, C_0^\infty(\mathbb{R}^d))$ in $L^1(\mu)$, which means that

$$\overline{(L - I)(C_0^\infty(\mathbb{R}^d))} = L^1(\mu).$$

Therefore, any sufficient condition for m-dissipativity of L is also sufficient for the uniqueness of a probability solution μ to Equation (1). Such conditions were obtained in [5] and [3]. For example, for the essential m-dissipativity of L, hence for the uniqueness of a probability solution, it suffices to have at least one of the following conditions in addition to (C1) and (C2):

(i) the matrix-valued mappings A and A^{-1} are uniformly bounded, A is globally Lipschitzian, and $b \in L^2(\mu)$ for at least one measure $\mu \in \mathscr{P}$,

(ii) $a^{ij}, b^i - \beta_\mu^i \in L^1(\mu)$ for at least one measure $\mu \in \mathscr{P}$, where

$$\beta_\mu^i = \sum_{j=1}^{d} \Big(\partial_{x_j} a^{ij} + a^{ij} \frac{\partial_{x_j} \varrho}{\varrho}\Big),$$

(iii) there is a function $V \in C^2(\mathbb{R}^d)$ such that $V > 0$, $\lim\limits_{|x|\to\infty} V(x) = +\infty$, and $LV(x) \leqslant CV(x)$ for all $x \in \mathbb{R}^d$ and some number $C > 0$.

Since under our standing assumptions the continuous density of a nonnegative solution has no zeros, a sufficient condition for the uniqueness of a probability solution (as well as for the absence of a signed integrable solution) is the property that the absolute value of a solution is again a solution. In general, one can show that the absolute value of a solution is a super-solution: $L^*|\mu| \geqslant 0$, which means that

$$\int Lu\, d|\mu| \geqslant 0 \quad \forall u \in C_0^\infty(\mathbb{R}^d),\ u \geqslant 0.$$

Thus, the problem is to ensure that $|\mu|$ is a sub-solution.

Here, we discuss another approach to the uniqueness problem for probability solutions; its main idea is a reduction of the problem to Liouville type theorems for certain special differential equations. This approach was suggested in [6, 7] and developed in [8, 9].

Thus, let conditions (C1) and (C2) be fulfilled, and let Equation (1) have a probability solution μ, which, as noted above, possesses a positive continuous density ϱ in the class $W^{p,1}_{\rm loc}(\mathbb{R}^d)$. Suppose that a locally finite measure ν (not necessarily a probability) satisfies Equation (1). Then ν is also given by a continuous density ζ with respect to the Lebesgue measure and $\zeta \in W^{p,1}_{\rm loc}(\mathbb{R}^d)$. We set $v = \zeta/\varrho$. The function v satisfies the equation

$$\operatorname{div}\bigl(\varrho A\nabla v - hv\bigr) = 0, \tag{2}$$

where

$$h^i = b^i - \beta^i_\mu, \quad \beta^i_\mu = \sum_{j=1}^d \Bigl(\partial_{x_j} a^{ij} + a^{ij}\frac{\partial_{x_j}\varrho}{\varrho}\Bigr).$$

We observe that $\operatorname{div} h = 0$ in the sense that

$$\int_{\mathbb{R}^d} (h(x), \nabla\psi(x))\, dx = 0 \quad \forall\, \psi \in C_0^\infty(\mathbb{R}^d).$$

Let $L^1_+(\mu)$ denote the convex subset of $L^1(\mu)$ consisting of nonnegative functions. Let us formulate our questions in terms of the function v.

1. When is any solution v from the class $L^1_+(\mu)$ constant?

2. When is any solution v from class $L^1(\mu)$ constant?

3. Which dimension can have the simplex of solutions v to Equation (2) belonging to the set $L^1_+(\mu)$?

Therefore, our uniqueness problems for Equation (1) are reduced to proving Liouvill type theorems for Equation (2) in the classes $L^1(\mu)$ and $L^1_+(\mu)$. In our exposition we follow [8, 9].

To answer the first two questions, we need an auxiliary lemma. Let f be a smooth bounded function on $[0, +\infty)$.

Lemma 1. *Suppose that v is a solution to Equation* (2). *Let $p = 2$ if $d = 1$. Then for every function $\psi \in C_0^\infty(\mathbb{R}^d)$ one has the following equality*:

$$\int_{\mathbb{R}^d} |\sqrt{A}\nabla v^+|^2 f''(v^+)\psi\, d\mu = \int_{\mathbb{R}^d} f(v^+)L\psi\, d\mu + f'(0)\int_{\mathbb{R}^d} v^- L\psi\, d\mu.$$

Proof. Since v satisfies Equation (2), we have

$$\int_{\mathbb{R}^d} (\varrho A\nabla v - hv, \nabla\varphi)\, dx = 0 \quad \forall \varphi \in C_0^\infty(\mathbb{R}^d).$$

Clearly, this identity extends to functions $\varphi \in W^{2,1}_{\rm loc}(\mathbb{R}^d)$. Taking $\varphi = f'(v^+)\psi$, we obtain

$$\int_{\mathbb{R}^d} \varrho|\sqrt{A}\nabla v^+|^2 f''(v^+)\psi\, dx = -\int_{\mathbb{R}^d} \varrho(A\nabla v, \nabla\psi) f'(v^+)\, dx - \int_{\mathbb{R}^d} (h, \nabla v) f'(v^+)\psi\, dx.$$

Let us consider the first summand on the right-hand side of this equality. Writing the function v as $v = v^+ + v^-$ and integrating by parts, we find that

$$-\int_{\mathbb{R}^d} \varrho(A\nabla v, \nabla\psi) f'(v^+)\, dx = \int_{\mathbb{R}^d} \varrho\, {\rm div}\,(A\nabla\psi) f(v^+)\, dx + \int_{\mathbb{R}^d} (A\nabla\varrho, \nabla\psi) f(v^+)\, dx$$

$$+ \int_{\mathbb{R}^d} \varrho v^- f'(v^+)\, {\rm div}\,(A\nabla\psi)\, dx + \int_{\mathbb{R}^d} v^-(A\nabla\varrho, \nabla\psi) f'(v^+)\, dx.$$

Acting similarly with the second term, we obtain

$$-\int_{\mathbb{R}^d} (h, \nabla v) f'(v^+)\psi\, dx = \int_{\mathbb{R}^d} (h, \nabla\psi) f(v^+)\, dx + \int_{\mathbb{R}^d} (h, \nabla\psi) v^- f'(v^+)\, dx.$$

Let us sum these equalities and use the identity

$$h^i = (b^i - \beta^i_\mu)\varrho = b^i\varrho - \sum_{j=1}^d \partial_{x_j}(a^{ij}\varrho).$$

This yields

$$\int_{\mathbb{R}^d} \varrho|\sqrt{A}\nabla v^+|^2 f''(v^+)\psi\, dx = \int_{\mathbb{R}^d} \varrho f(v^+) L\psi\, dx + \int_{\mathbb{R}^d} \varrho v^- f'(v^+) L\psi\, dx.$$

It remains to observe that $v^- f'(v^+) = v^- f'(0)$ almost everywhere. □

Remark 1. In the case $v \geqslant 0$, we obtain $v^+ = v$, $v^- = 0$, and

$$\int \varrho|\sqrt{A}\nabla v|^2 f''(v)\psi\, dx = \int \varrho f(v) L\psi\, dx.$$

The next result was obtained in [9].

Theorem 2. *Let conditions* (C1) *and* (C2) *be fulfilled, and let a function* $v \in L^1_+(\mu)$ *be a solution to Equation* (2). *Suppose also that at least one of the following conditions is satisfied*:

(i) $a^{ij}, b^i \in L^1(\mu)$,

(ii) $a^{ij}, (b^i - \beta^i_\mu) \in L^1(\mu)$,

(iii) *there exists a positive function* $V \in C^2(\mathbb{R}^d)$ *such that* $V(x) \to +\infty$ *as* $|x| \to +\infty$ *and for some number* $C > 0$ *and all* x *we have the estimates*

$$LV(x) \leqslant C, \quad |\sqrt{A(x)}\nabla V(x)| \leqslant C.$$

Then the function v *is constant. In particular, the solution* μ *is the only probability solution to Equation* (1).

Proof. We consider the case where condition (iii) is fulfilled. The proof for other cases is similar. We set $f(t) = -\operatorname{arc\,tg} t$ and $\psi(x) = \zeta(N^{-1}V(x))$, where a nonnegative function ζ is such that $\zeta(x) = 1$ if $|x| \leqslant 1$, $\zeta(x) = 0$ if $|x| > 2$, and $\zeta' \leqslant 0$ if $x > 0$, and there exists a number $M > 0$ such that for all x the following estimates are fulfilled:

$$|\zeta(x)| \leqslant M, \quad |\zeta'(x)| \leqslant M, \quad |\zeta''(x)| \leqslant M.$$

Lemma 1 yields the inequality

$$2\int_{\mathbb{R}^d} \varrho|\nabla v^+|^2 v(1+v^2)^{-2}\psi\,dx \leqslant 2^{-1}\pi(MCN^{-1} + MC^2N^{-2})\int_{V>N} \varrho\,dx.$$

Letting $N \to +\infty$ we obtain $v = \mathrm{const}$. □

Remark 2. 1. One can relax condition (i) on the coefficients. Namely, in place of the inclusions $a^{ij}, b^i \in L^1(\mu)$ it suffices to require the inclusions

$$\frac{|a^{ij}|}{1+|x|^2}, \quad \frac{|b^i|}{1+|x|} \in L^1(\mu).$$

2. Passing from the function V to $\ln V$, one can easily show that in condition (iii) the inequalities

$$LV \leqslant C \quad \text{and} \quad |\sqrt{A}\nabla V| \leqslant C$$

can be replaced with the following weaker bounds:

$$LV \leqslant CV \quad \text{and} \quad |\sqrt{A}\nabla V| \leqslant CV.$$

3. We observe that we never used in the proof that $v \in L^1(\mu)$, only the estimate $v \geqslant 0$ was employed. Therefore, if either of conditions (i), (ii), and (iii) holds, we find that any nonnegative solution to Equation (2) is constant, in particular, any nonnegative solution to Equation (1) is proportional to the probability solution μ.

Example 4. Let $V(x) = \ln(\ln(1+|x|))$ if $|x| > 1$. Then, whenever $|x| > 1$, we have

$$|\sqrt{A(x)}\nabla V(x)|^2 = (A(x)\nabla V(x), \nabla V(x)) = \frac{(A(x)x, x)}{|x|^2(|x|+1)^2\ln^2(|x|+1)}.$$

Let us calculate $LV(x)$ if $|x| > 1$:

$$LV(x) = -\frac{(A(x)x,x)}{|x|^2(1+|x|)^2\ln(1+|x|)}\Big(1 + \frac{1}{\ln(1+|x|)} + \frac{1+|x|}{|x|}\Big)$$
$$+ \frac{\operatorname{trace} A(x)}{|x|(1+|x|)\ln(1+|x|)} + \frac{(b(x),x)}{|x|(|x|+1)\ln(|x|+1)}.$$

Therefore, in order to ensure condition (iii) in Theorem 2, it suffices to have the estimates

$$(A(x)x,x) \leqslant C + C|x|^4\ln^2(1+|x|), \quad (b(x),x) \leqslant -C|x|^2\ln(1+|x|) - C$$

for all $x \in \mathbb{R}^d$ and some number $C > 0$.

Theorem 3. *Let conditions* (C1) *and* (C2) *be fulfilled, and let a function $v \in L^1(\mu)$ be a solution to Equation* (2)*. Let $p = 2$ if $d = 1$. Suppose that there exists a function $V \in C^2(\mathbb{R}^d)$ such that $V(x) \to +\infty$ as $|x| \to +\infty$ and for some number $C > 0$ and all x*

$$LV(x) \geqslant -C, \quad |\sqrt{A(x)}\nabla V(x)| \leqslant C.$$

Then, the function v is constant, in particular, any integrable solution to Equation (1) *is proportional to the probability solution μ.*

Proof. Let $v \in L^1(\mu)$ be a solution to Equation (2). We set $f(t) = (1+t)^{-1}$ and $\psi(x) = \zeta(N^{-1}V(x))$, where a nonnegative function ζ is such that $\zeta(x) = 1$ if $|x| \leqslant 1$, $\zeta(x) = 0$ if $|x| > 2$, $\zeta' \leqslant 0$ if $x > 0$, and there exists a number $M > 0$ such that for all x one has the estimates

$$|\zeta(x)| \leqslant M, \quad |\zeta'(x)| \leqslant M, \quad |\zeta''(x)| \leqslant M.$$

Applying Lemma 1 and taking into account that $(1+v^+)^{-1} \leqslant 1$ and $\zeta' \leqslant 0$, we obtain

$$2\int \varrho|\sqrt{A}\nabla v^+|^2(1+v^+)^{-3}\psi\, dx \leqslant (MCN^{-1} + MC^2N^{-2})\Bigl(\int\limits_{V>N} \varrho\, dx + \int\limits_{V>N} |\sigma|\, dx\Bigr).$$

Letting $N \to +\infty$ we obtain $v^+ = \text{const}$. Similarly, we consider $-\sigma$ and conclude that $v^- = \text{const}$. Therefore, we have $v = \text{const}$ and $\sigma = \text{const}\cdot\varrho$. □

Example 5. Let $V(x) = \ln(\ln(1+|x|))$ if $|x| > 1$. Then, whenever $|x| > 1$, we have

$$|\sqrt{A(x)}\nabla V(x)|^2 = (A(x)\nabla V(x), \nabla V(x)) = \frac{(A(x)x, x)}{|x|^2(|x|+1)^2\ln^2(|x|+1)}.$$

Let us calculate $LV(x)$ for $|x| > 1$:

$$LV(x) = -\frac{(A(x)x,x)}{|x|^2(1+|x|)^2\ln(1+|x|)}\Bigl(1 + \frac{1}{\ln(1+|x|)} + \frac{1+|x|}{|x|}\Bigr) + \frac{\operatorname{trace} A(x)}{|x|(1+|x|)\ln(1+|x|)} + \frac{(b(x),x)}{|x|(|x|+1)\ln(|x|+1)}.$$

Now, in order to ensure the hypotheses in Theorem 3, it suffices to have the estimates

$$(A(x)x, x) \leqslant C + C|x|^4\ln(1+|x|),$$
$$(b(x), x) \geqslant -C|x|^2\ln(1+|x|) - C$$

for all $x \in \mathbb{R}^d$ and some number $C > 0$.

The results presented in Theorems 2 and 3 give sufficient conditions for the absence of nontrivial solutions to Equation (2) in the classes of nonnegative or μ-integrable functions. However, if we want to construct an example of Equation (1) possessing at least two different probability solutions, we should go in the opposite direction and prove a theorem of existence of a nontrivial solution to Equation (2) in the indicated classes of functions. Now, we follow [6, 7].

Let A be unit matrix, and let $b \in C^\infty(\mathbb{R}^d)$. Similarly to [12], we introduce the following bilinear skew-symmetric form on the space $C_0^\infty(\mathbb{R}^d)$:

$$[f, g] := \int\limits_{\mathbb{R}^d} (h, \nabla f) g\, dx.$$

The next theorem proved in [7] gives sufficient conditions for the existence of a nonconstant bounded positive solution to Equation (2).

Theorem 4. *Suppose that there exists a function $\varphi \in C_b^2(\mathbb{R}^d)$ such that $(h, \nabla\varphi) \in L^1(\mathbb{R}^d)$,*

$$[\varphi, 1] = 0 \quad \textit{and} \quad [\varphi, \varphi] < 0. \tag{3}$$

Then there exists a nonconstant bounded positive solution to Equation (2).

Note some similarity between the hypotheses in this theorem and in Lemma 2.1 in [12].

Remark 3. In order to give an example of Equation (1) possessing at least two different probability solutions, it suffices to find a vector field h with zero divergence and a function φ satisfying the hypotheses in Theorem 4. For obtaining a coefficient b we have to take an arbitrary smooth positive probability density ϱ and define b by the formula

$$b = \frac{\nabla\varrho}{\varrho} + \frac{h}{\varrho}. \tag{4}$$

Then one solution to Equation (1) is the measure $\mu = \varrho\, dx$ and another one is the measure $v\mu$, where the function v is a nonconstant solution to Equation (2).

Therefore, given an arbitrary smooth positive probability density ϱ on $\mathbb{R}^d$ with $d > 1$, we can fabricate a smooth vector field b such that our equation with $A = I$ and this b has several probability solutions, one of which is the measure $\mu = \varrho\, dx$.

Example 6. Let $d \geqslant 2$. Suppose that functions $q \in C^\infty(\mathbb{R}^{d-1})$, $\psi \in C_b^2(\mathbb{R}^{d-1})$, and $\sigma \in C_b^2(\mathbb{R}^1)$ are not identically zero. Assume also that $q \in L^1(\mathbb{R}^{d-1})$, $\sigma' \in L^1(\mathbb{R}^1)$, $q > 0$, $\lim\limits_{n\to\infty} \sigma(n) = 1$, and $\lim\limits_{n\to\infty} \sigma(-n) = 0$. We set $h^i(x) = 0$ if $1 \leqslant i \leqslant d-1$ and $h^d(x) = -q(x')$, $\varphi(x) = \psi(x')\sigma(x_d)$, where $x' = (x_1, x_2, \ldots, x_{d-1})$. Then

$$[\varphi, \varphi] = -\frac{1}{2} \int\limits_{\mathbb{R}^{d-1}} q(x')\psi^2(x')\, dx', \quad [\varphi, 1] = -\int\limits_{\mathbb{R}^{d-1}} q(x')\psi(x')\, dx'.$$

In order to ensure the hypotheses in Theorem 4, it suffices to have the orthogonality of the functions ψ and 1 in the space $L^2(\mathbb{R}^{d-1}, q\, dx')$, which can be easily achieved.

The next result gives a sufficient condition for linear independence of solutions constructed in Theorem 4.

Theorem 5. *Let $n \geqslant 1$. Assume that there exist functions φ_1, φ_2, ..., φ_{n+1} in the class $C_b^2(\mathbb{R}^d)$ such that each of them satisfies the hypotheses in Theorem 4, and let v_1, v_2, ..., v_{n+1} be solutions to equation (2) associated to these functions according to Theorem 4. Suppose also that the functions 1, v_1, ..., v_n are linearly independent and that for all $\alpha = (\alpha_k)_{1\leqslant k\leqslant n}$ one has the inequality*

$$\Big[\varphi_{n+1} - \sum_{k=1}^{n} \alpha_k\varphi_k, \varphi_{n+1} - \sum_{k=1}^{n} \alpha_k\varphi_k\Big] < 0. \tag{5}$$

Then the functions $1, v_1, \ldots, v_n$ and v_{n+1} are linearly independent.

Remark 4. Let h be a vector field with div $h = 0$. Assume that there exist functions φ_i, $1 \leqslant i \leqslant n+1$, satisfying the hypotheses in Theorem 5. Let ϱ be a positive infinitely differentiable probability density. Then Equation (1) with the coefficient b expressed by means of h and ϱ by formula (4) has at least $n+1$ linearly independent probability solutions, one of which is the measure $\mu = \varrho\, dx$, and n other ones are the measures $\nu_i = c_i v_i \mu$, where c_i is a normalizing constant and the function v_i is the solution to Equation (2) associated with the function φ_i.

Example 7. Let $x' = (x_1, x_2, \ldots, x_{d-1})$. Assume that $q \in C^\infty(\mathbb{R}^{d-1})$, $\psi_1, \psi_2 \in C_b^2(\mathbb{R}^{d-1})$, $\sigma \in C_b^2(\mathbb{R}^1)$ are not identically zero. Assume also that $q \in L^1(\mathbb{R}^{d-1})$, $q > 0$, $\sigma' \in L^1(\mathbb{R}^1)$, $\lim\limits_{n\to\infty} \sigma(n) = 1$, and $\lim\limits_{n\to\infty} \sigma(-n) = 0$. We set $h^k(x) = 0$ if $1 \leqslant k \leqslant d-1$ and $h^d(x) = -q(x')$, $\varphi_1(x) = \psi_1(x')\sigma(x_d)$, $\varphi_2(x) = \psi_2(x')\sigma(x_d)$. Then, whenever $1 \leqslant i,\ j \leqslant 2$, we have

$$[\varphi_i, \varphi_j] + [\varphi_j, \varphi_i] = -\int\limits_{\mathbb{R}^{d-1}} \psi_i(x')\psi_j(x')q(x')\, dx', \quad [\varphi_i, 1] = -\int\limits_{\mathbb{R}^{d-1}} \psi_i(x')q(x')\, dx'.$$

In order to ensure the hypotheses in Theorem 5, it suffices to have the orthogonality of the functions 1, ψ_1, ψ_2 in the space $L^2(\mathbb{R}^{d-1}, q\, dx')$, which can be easily achieved.

This example can be easily extended to an arbitrary number of functions φ_i. Moreover, one can find an example of Equation (1) that possesses a countable sequence of linearly independent probability solutions, in particular, the space of bounded solutions to this equation is infinite-dimensional. To this end, it suffices to find a sequence of nonnegative bounded solutions $\{v_i\}_{i\geqslant 1}$ to Equation (2) such that the functions 1, $\{v_i\}_{i\geqslant 1}$ are linearly independent. According to what has been said above (cf. Theorem 5 and Remark 4), we have to take a vector field h with div $h = 0$ and a family of functions $\{\varphi_i\}_{i\in\mathbb{N}}$ satisfying the hypotheses in Theorem 4 such that for every n the functions $\varphi_1, \varphi_2, \ldots, \varphi_{n+1}$ satisfy the condition (5).

Example 8. As above, let $x' = (x_1, x_2, \ldots, x_{d-1})$, and let functions $q \in C^\infty(\mathbb{R}^{d-1})$, $\psi_i \in C_b^2(\mathbb{R}^{d-1})$, and $\sigma \in C_b^2(\mathbb{R}^1)$ do not vanish identically. Assume also that $q \in L^1(\mathbb{R}^{d-1})$, $q > 0$, $\sigma' \in L^1(\mathbb{R}^1)$, $\lim\limits_{n\to\infty} \sigma(n) = 1$, and $\lim\limits_{n\to\infty} \sigma(-n) = 0$. We set $h^k(x) = 0$ if $1 \leqslant k \leqslant d-1$ and $h^d(x) = -q(x')$, $\varphi_i(x) = \psi_i(x')\sigma(x_d)$ if $i \geqslant 1$. Then for all $i, j \geqslant 1$ we have

$$[\varphi_i, \varphi_j] + [\varphi_j, \varphi_i] = -\int\limits_{\mathbb{R}^{d-1}} \psi_i(x')\psi_j(x')q(x')\, dx', \quad [\varphi_i, 1] = -\int\limits_{\mathbb{R}^{d-1}} \psi_i(x')q(x')\, dx'.$$

Let $1, \{\psi_i\}_{i\in\mathbb{N}}$ be an orthonormal system in $L^2(\mathbb{R}^{d-1}, q\, dx')$. Then for every n the functions $1, \varphi_1, \ldots, \varphi_{n+1}$ satisfy the condition (5). Indeed, $[\varphi_i, \varphi_j] = -\delta_{ij}$ and

$$\Big[\varphi_{n+1} - \sum_{k=1}^n \alpha_k\varphi_k, \varphi_{n+1} - \sum_{k=1}^n \alpha_k\varphi_k\Big] = -1 - \sum_{k=1}^n \alpha_k^2 < 0 \quad \text{for every } n \geqslant 1.$$

Example 9. Let us return to Example 3, where Equation (1) has at least two different probability solutions. Actually, it possesses countably many linearly independent probability solutions if f'' is bounded. In the notation of that example, we have

$$h^i(x) = 2c_1 f''(x_{\sigma(i)}) \prod_{j\neq i} f'(x_j).$$

We observe that h^d depends only on $x' = (x_1, x_2, \dots, x_{d-1})$. In addition, there exists a ball $U \subset \mathbb{R}^{d-1}$ on which $h^d < 0$ since otherwise $f'' \geqslant 0$ and f' does not increase, which contradicts the conditions t hat $f' > 0$ and $f' \in L^1(\mathbb{R}^1)$. Let us define a sequence of functions $\omega, \psi_i^{**} \in C_0^\infty(U)$ with pairwise disjoint supports such that

$$\int_U \omega(x')h^d(x')\,dx' = -1.$$

We set

$$\psi_i^*(x') = \psi_i^{**}(x') + \omega(x') \int_U \psi_i^{**}(x')h^d(x')\,dx'$$

and note that

$$\int_U \psi_i^*(x')h^d(x')\,dx' = 0 \quad \text{for each } i \geqslant 1.$$

It is clear that the functions ψ_i^* are linearly independent. Let us apply the orthogonalization procedure to the functions $\{\psi_i^*\}_{i\geqslant 1}$ in the space $L^2(U, -h^d(x')\,dx')$ and obtain functions $\{\psi_i\}_{i\geqslant 1}$. We observe that the constructed functions possess the following properties: $\psi_i \in C_0^\infty(U)$ and

$$\int_U \psi_i(x')h^d(x')\,dx' = 0, \quad \int_U \psi_i(x')\psi_j(x')h^d(x')\,dx' = \delta_{ij} \quad \forall\, j \geqslant 1.$$

We extend ψ_i by zero to all of $\mathbb{R}^{d-1}$. Let η be a smooth function such that $\lim\limits_{n\to\infty} \eta(n) = 1$ and $\lim\limits_{n\to\infty} \eta(-n) = 0$. Let also $\eta' \in L^1(\mathbb{R}^1)$. We set $\varphi_i(x) = \psi_i(x')\eta(x_d)$. Using Theorem 5, we construct the solutions corresponding to the functions φ_i. As in the previous example, we obtain a sequence of linearly independent (along with the function 1) solutions.

In the case $d \geqslant 2$, Equation (1) with $A = I$ can have infinitely many probability solutions even if b is a smooth gradient. Let us consider an example (cf. [8]).

Example 10. Let φ, ψ be smooth functions such that e^φ, e^ψ are integrable on $\mathbb{R}^1$. We set

$$\varrho(x,y) = \exp(\varphi(x) + \psi(y)), \quad v(x,y) = \varphi(x) + \psi(y) - \int_0^x e^{-\varphi(s)}\,ds - \int_0^y e^{-\psi(s)}\,ds.$$

Then $\varrho(x,y)\partial_x v(x,y) = \varphi'(x)\varrho(x,y) - e^{\psi(y)}$ and $\varrho(x,y)\partial_y v(x,y) = \psi'(y)\varrho(x,y) - e^{\varphi(x)}$. Therefore, $\operatorname{div}(\nabla\varrho - \varrho\nabla v) = \partial_x e^{\psi(y)} + \partial_y e^{\varphi(x)} = 0$. In addition,

$$\exp(v(x,y)) = \varrho(x,y)\exp\left(-\int_0^x e^{-\varphi(s)}\,ds - \int_0^y e^{-\psi(s)}\,ds\right).$$

For this v Equation (1) has infinitely many linearly independent probability solutions. We verify this by using Theorem 5. We have $h(x,y) = (-e^{\psi(y)}, -e^{\varphi(x)})$. As noted above, it suffices to construct a system of functions $\varphi_i \in C_b^2(\mathbb{R}^2)$, $i = 1, 2, \dots$, such that $[\varphi_i, 1] = 0$, and $[\varphi_i, \varphi_j] + [\varphi_j, \varphi_i] = -\delta_{ij}$. We set $\varphi_i(x,y) = \tau_i(x)\sigma(y)$, where $\tau_i \in C_0^\infty(\mathbb{R}^1)$, $\sigma \in C^\infty(\mathbb{R}^1)$, $\lim\limits_{y\to+\infty} \sigma(y) = 1$ and $\lim\limits_{y\to-\infty} \sigma(y) = 0$. Taking into account that

$$\int_{-\infty}^{+\infty} \tau_i'(x)\,dx = 0,$$

we obtain

$$[\varphi_i, 1] = -\int \tau_i(x) e^{\varphi(x)}\, dx, \quad [\varphi_i, \varphi_j] + [\varphi_j, \varphi_i] = -\int \tau_i(x)\tau_j(x) e^{\varphi(x)}\, dx.$$

Now we choose functions τ_i such that $\{1, \tau_i\}$ is an orthonormal system in the space $L^2(e^{\varphi}\, dx)$. This gives the required relations.

The previous examples aim at finding as many linearly independent solutions as possible. However, the following problem remains open: suppose that $A = I$ and b is infinitely differentiable on $\mathbb{R}^d$.

(Q1) Can it happen that our equation has only finitely many linearly independent probability solutions (but more than one)?

In dimension $d > 1$, there is another open question with a very simple formulation:

(Q2) Can it happen that our equation has a signed integrable solution, but has no probability solutions?

Now we consider the case of a unit diffusion matrix and a gradient drift coefficient. Equation (1) in this case takes the form

$$\Delta\mu - \operatorname{div}(b\mu) = 0,$$

where $b = \nabla\Phi$ for some function Φ.

Our question is this: when does the probability solution μ have the form $Ce^{\Phi}\, dx$?

This problem is closely related to the problem of constructing a probability measure $\mu = \varrho\, dx$ such that $b = \nabla\varrho/\varrho$, where b is a given vector field. A possible approach to this problem is to construct such a measure as a stationary distribution of the diffusion process with the drift coefficient b (cf., for example, [16]). Another approach is to construct such a measure μ as a solution to the stationary Kolmogorov equation.

It should be noted that even in the one-dimensional case a probability solution may fail to have such a form. Indeed, let

$$\Phi(x) = -\frac{x^2}{2} + \int_0^x e^{s^2/2}\, ds.$$

Then our equation has a probability solution $(2\pi)^{-1/2} e^{-x^2/2}\, dx$, but e^{Φ} is not integrable.

We give a sufficient condition guaranteeing $\mu = Ce^{\Phi}\, dx$. This result was obtained in [8].

Theorem 6. *Let $\Phi \in W^{p,1}_{\mathrm{loc}}(\mathbb{R}^d)$, where $p > d$. Suppose that a probability measure $\mu = \varrho\, dx$ satisfies equation* (1)*. If $e^{\Phi} \in L^1(\mathbb{R}^d)$, then $\varrho = Ce^{\Phi}$ for some constant C.*

Proof. We set $w = \varrho e^{-\Phi}$. Then $\operatorname{div}(e^{\Phi}\nabla w) = 0$, which can be written as

$$\int_{\mathbb{R}^d} e^{\Phi}(\nabla w, \nabla\varphi)\, dx = 0, \quad \varphi \in C_0^{\infty}(\mathbb{R}^d). \tag{6}$$

We consider the case $d > 1$ since, in the case $d = 1$, Equation (1) has a unique probability solution and e^{Φ} is a nonnegative integrable solution. Thus, $p > d \geqslant 2$. Hence (6) remains valid for all $\varphi \in W_0^{2,1}(\mathbb{R}^d)$. We apply this integral identity to $\varphi = w^{-1}\zeta_k^2$, where $\zeta_k(x) = \zeta(x/k)$, $\zeta \in C_0^\infty(\mathbb{R}^d)$, $\zeta \geqslant 0$, $\zeta(x) = 1$ if $|x| < 1$. We obtain

$$\int\limits_{\mathbb{R}^d} e^\Phi |\nabla w|^2 w^{-2}\zeta_k^2\,dx = 2\int\limits_{\mathbb{R}^d} e^v w^{-1}(\nabla w, \nabla\zeta_k)\zeta_k\,dx.$$

Applying the Cauchy inequality and taking into account that $|\nabla\zeta_k| \leqslant Ck^{-1}$, we find that

$$\int\limits_{\mathbb{R}^d} e^\Phi |\nabla w|^2 w^{-2}\zeta_k^2\,dx \leqslant 4k^{-2}\int\limits_{\mathbb{R}^d} e^v\,dx.$$

Letting $k \to +\infty$, we arrive at the equality

$$\int\limits_{\mathbb{R}^d} e^\Phi |\nabla w|^2 w^{-2}\,dx = 0,$$

which yields that $|\nabla w| = 0$. Thus, the function w is constant. Therefore, $\varrho = Ce^\Phi$. □

Corollary 1. *Under the hypotheses of Theorem 6, the measure $\mu = e^\Phi\,dx$ is the only probability solution.*

Applying Theorem 2 and Remark 2, we arrive at the following result.

Theorem 7. *Let $\Phi \in W_{\text{loc}}^{p,1}(\mathbb{R}^d)$, where $p > d$. Suppose that a probability measure $\mu = \varrho\,dx$ satisfies Equation (1) and at least one of the following conditions is fulfilled:*

(i) $|\nabla\Phi| \in L^1(\mu)$,

(ii) *there exists a positive function $V \in C^2(\mathbb{R}^d)$ such that $V(x) \to +\infty$ as $|x| \to +\infty$ and for some number $C > 0$ and all x one has the estimates*

$$LV(x) \leqslant C, \quad |\nabla V(x)| \leqslant C$$

or even weaker estimates

$$LV(x) \leqslant CV, \quad |\nabla V(x)| \leqslant CV.$$

Then, there exists a constant C such that $\varrho = Ce^\Phi$.

Proof. The case $d = 1$ is considered in the same manner as above. In the case $d \geqslant 2$, it suffices to observe that $e^\Phi\,dx$ is a nonnegative solution and apply Remark 2. □

Let us conclude with a short summary for the case $A = I$.

- If $d = 1$ and we allow b that is not locally Lebesgue integrable, then Equation (1) can have infinitely many linearly independent probability solutions. If b is locally Lebesgue integrable, then there is at most one probability solution and the existence of a nonzero integrable solution yields the existence of a probability solution.

- If $d > 1$, then, even for smooth gradient type b, one can have infinitely many linearly independent probability solutions; however, there are sufficient conditions in terms of b under which there is only one probability solution or there are no signed solutions.

- In the case $d > 1$, there are open problems with simple formulations.

Similar problems have been studied for measures on Riemannian manifolds, see [17], [18], [19]. They also have parabolic analogues, which will be discussed in another paper.

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