KAWASAKI DYNAMICS IN THE CONTINUUM VIA GENERATING FUNCTIONALS EVOLUTION

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Abstract. We construct the time evolution of Kawasaki dynamics for a spatial infinite particle system in terms of generating functionals. This is carried out by an Ovsjannikov-type result in a scale of Banach spaces, which leads to a local (in time) solution. An application of this approach to Vlasov-type scaling in terms of generating functionals is considered as well.

1. Introduction

Originally, Bogoliubov generating functionals (GF for short) were introduced by N. N. Bogoliubov in [2] to define correlation functions for statistical mechanics systems. Apart from this specific application, and many others, GF are, by themselves, a subject of interest in infinite dimensional analysis. This is partially due to the fact that to a probability measure μ defined on the space Γ of locally finite configurations $\gamma \subset \mathbb{R}^d$ one may associate a GF

$$B_\mu(\theta) := \int_\Gamma d\mu(\gamma) \prod_{x\in\gamma}(1+\theta(x)),$$

yielding an alternative method to study the stochastic dynamics of an infinite particle system in the continuum by exploiting the close relation between measures and GF [4, 9].

Existence and uniqueness results for the Kawasaki dynamics through GF arise naturally from Picard-type approximations and a method suggested in [6, Appendix 2, A2.1] in a scale of Banach spaces (see e.g. [5, Theorem 2.5]). This method, originally presented for equations with coefficients time independent, has been extended to an abstract and general framework by T. Yamanaka in [12] and L. V. Ovsjannikov in [10] in the linear case, and many applications were exposed by F. Treves in [11]. As an aside, within an analytical framework outside of our setting, all these statements are very closely related to variants of the abstract Cauchy-Kovalevskaya theorem. However, all these abstract forms only yield a local solution, that is, a solution which is defined on a finite time interval. Moreover, starting with an initial condition from a certain Banach space, in general the solution evolves on larger Banach spaces.

As a particular application, this work concludes with the study of the Vlasov-type scaling proposed in [3] for general continuous particle systems and accomplished in [1] for the Kawasaki dynamics. The general scheme proposed in [3] for correlation functions yields a limiting hierarchy which possesses a chaos preservation property, namely, starting with a Poissonian (non-homogeneous) initial state this structural property is preserved during the time evolution. In Section 4 the same problem is formulated in terms of GF

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and its analysis is carried out by the general Ovsjannikov-type result in a scale of Banach spaces presented in [5, Theorem 4.3].

2. General Framework

In this section we briefly recall the concepts and results of combinatorial harmonic analysis on configuration spaces and Bogoliubov generating functionals needed throughout this work (for a detailed explanation see [7, 9]).

2.1. **Harmonic analysis on configuration spaces.** Let $\Gamma := \Gamma_{\mathbb{R}^d}$ be the configuration space over $\mathbb{R}^d$, $d \in \mathbb{N}$,

$$\Gamma := \left\{\gamma \subset \mathbb{R}^d : |\gamma \cap \Lambda| < \infty \text{ for every compact } \Lambda \subset \mathbb{R}^d\right\},$$

where $|\cdot|$ denotes the cardinality of a set. We identify each $\gamma \in \Gamma$ with the non-negative Radon measure $\sum_{x\in\gamma} \delta_x$ on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$, where δ_x is the Dirac measure with mass at x, which allows to endow Γ with the vague topology and the corresponding Borel σ-algebra $\mathcal{B}(\Gamma)$.

For any $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ let

$$\Gamma^{(n)} := \{\gamma \in \Gamma : |\gamma| = n\}, \ n \in \mathbb{N}, \quad \Gamma^{(0)} := \{\emptyset\}.$$

Clearly, each $\Gamma^{(n)}$, $n \in \mathbb{N}$, can be identify with the symmetrization of the set $\{(x_1, ..., x_n) \in (\mathbb{R}^d)^n : x_i \neq x_j \text{ if } i \neq j\}$, which induces a natural (metrizable) topology on $\Gamma^{(n)}$ and the corresponding Borel σ-algebra $\mathcal{B}(\Gamma^{(n)})$. In particular, for the Lebesgue product measure $(dx)^{\otimes n}$ fixed on $(\mathbb{R}^d)^n$, this identification yields a measure $m^{(n)}$ on $(\Gamma^{(n)}, \mathcal{B}(\Gamma^{(n)}))$. For $n = 0$ we set $m^{(0)}(\{\emptyset\}) := 1$. This leads to the definition of the space of finite configurations

$$\Gamma_0 := \bigsqcup_{n=0}^{\infty} \Gamma^{(n)}$$

endowed with the topology of disjoint union of topological spaces and the corresponding Borel σ-algebra $\mathcal{B}(\Gamma_0)$, and to the so-called Lebesgue-Poisson measure on $(\Gamma_0, \mathcal{B}(\Gamma_0))$,

$$\lambda := \lambda_{dx} := \sum_{n=0}^{\infty} \frac{1}{n!} m^{(n)}. \tag{2.1}$$

Let $\mathcal{B}_c(\mathbb{R}^d)$ be the set of all bounded Borel sets in $\mathbb{R}^d$ and, for each $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, let $\Gamma_\Lambda := \{\eta \in \Gamma : \eta \subset \Lambda\}$. Evidently $\Gamma_\Lambda = \bigsqcup_{n=0}^{\infty} \Gamma_\Lambda^{(n)}$, where $\Gamma_\Lambda^{(n)} := \Gamma_\Lambda \cap \Gamma^{(n)}$, $n \in \mathbb{N}_0$. Given a complex-valued $\mathcal{B}(\Gamma_0)$-measurable function G such that $G\restriction_{\Gamma\setminus\Gamma_\Lambda} \equiv 0$ for some $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, the K-transform of G is a mapping $KG : \Gamma \to \mathbb{C}$ defined at each $\gamma \in \Gamma$ by

$$(KG)(\gamma) := \sum_{\substack{\eta\subset\gamma \\ |\eta|<\infty}} G(\eta). \tag{2.2}$$

It has been shown in [7] that the K-transform is a linear and invertible mapping.

Let $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be the set of all probability measures μ on $(\Gamma, \mathcal{B}(\Gamma))$ with finite local moments of all orders, i.e.,

$$\int_\Gamma d\mu(\gamma)\, |\gamma \cap \Lambda|^n < \infty \quad \text{for all } n \in \mathbb{N} \text{ and all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d),$$

and let $B_{\mathrm{bs}}(\Gamma_0)$ be the set of all complex-valued bounded $\mathcal{B}(\Gamma_0)$-measurable functions with bounded support, i.e., $G\restriction_{\Gamma_0\setminus\left(\bigsqcup_{n=0}^{N} \Gamma_\Lambda^{(n)}\right)} \equiv 0$ for some $N \in \mathbb{N}_0, \Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. Given a $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, the so-called correlation measure ρ_μ corresponding to μ is a measure on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ defined for all $G \in B_{\mathrm{bs}}(\Gamma_0)$ by

$$\int_{\Gamma_0} d\rho_\mu(\eta)\, G(\eta) = \int_\Gamma d\mu(\gamma)\ (KG)(\gamma). \tag{2.3}$$

This definition implies, in particular, that $B_{\mathrm{bs}}(\Gamma_0) \subset L^1(\Gamma_0, \rho_\mu)$.[1] Moreover, still by (2.3), on $B_{\mathrm{bs}}(\Gamma_0)$ the inequality $\|KG\|_{L^1(\Gamma,\mu)} \le \|G\|_{L^1(\Gamma_0,\rho_\mu)}$ holds, allowing an extension of the K-transform to a bounded operator $K : L^1(\Gamma_0, \rho_\mu) \to L^1(\Gamma, \mu)$ in such a way that equality (2.3) still holds for any $G \in L^1(\Gamma_0, \rho_\mu)$. For the extended operator the explicit form (2.2) still holds, now μ-a.e. In particular, for coherent states $e_\lambda(f)$ of complex-valued $\mathcal{B}(\mathbb{R}^d)$-measurable functions f,

$$e_\lambda(f, \eta) := \prod_{x\in\eta} f(x)\,,\ \eta \in \Gamma_0\backslash\{\emptyset\}, \quad e_\lambda(f, \emptyset) := 1. \tag{2.4}$$

Additionally, if f has compact support we have

$$(Ke_\lambda(f))(\gamma) = \prod_{x\in\gamma}(1 + f(x)) \tag{2.5}$$

for all $\gamma \in \Gamma$, while for functions f such that $e_\lambda(f) \in L^1(\Gamma_0, \rho_\mu)$ equality (2.5) holds, but only for μ-a.a. $\gamma \in \Gamma$. Concerning the Lebesgue-Poisson measure (2.1), we observe that $e_\lambda(f) \in L^p(\Gamma_0, \lambda)$ whenever $f \in L^p := L^p(\mathbb{R}^d, dx)$ for some $p \ge 1$. In this case, $\|e_\lambda(f)\|^p_{L^p} = \exp(\|f\|^p_{L^p})$. In particular, for $p = 1$, in addition we have

$$\int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(f, \eta) = \exp\left(\int_{\mathbb{R}^d} dx\, f(x)\right),$$

for all $f \in L^1$. For more details see [8].

2.2. **Bogoliubov generating functionals.** Given a probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ the so-called Bogoliubov generating functional (GF for short) B_μ corresponding to μ is the functional defined at each $\mathcal{B}(\mathbb{R}^d)$-measurable function θ by

$$B_\mu(\theta) := \int_\Gamma d\mu(\gamma) \prod_{x\in\gamma}(1 + \theta(x)), \tag{2.6}$$

provided the right-hand side exists. It is clear from (2.6) that the domain of a GF B_μ depends on the underlying measure μ and, conversely, the domain of B_μ reflects special properties over the measure μ. Throughout this work we will consider GF defined on the whole complex L^1 space. This implies, in particular, that the underlying measure μ has finite local exponential moments, i.e.,

$$\int_\Gamma d\mu(\gamma)\, e^{\alpha|\gamma\cap\Lambda|} < \infty \quad \text{for all } \alpha > 0 \text{ and all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d),$$

and thus $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$. According to the previous subsection, this implies that to such a measure μ one may associate the correlation measure ρ_μ, which leads to a description of the functional B_μ in terms of either the measure ρ_μ:

$$B_\mu(\theta) = \int_\Gamma d\mu(\gamma)\ (Ke_\lambda(\theta))(\gamma) = \int_{\Gamma_0} d\rho_\mu(\eta)\, e_\lambda(\theta, \eta),$$

or the so-called correlation function $k_\mu := \frac{d\rho_\mu}{d\lambda}$ corresponding to the measure μ, if ρ_μ is absolutely continuous with respect to the Lebesgue–Poisson measure λ:

$$B_\mu(\theta) = \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta, \eta) k_\mu(\eta). \tag{2.7}$$

[1]Throughout this work all L^p-spaces, $p \ge 1$, consist of complex-valued functions.

Throughout this work we will assume, in addition, that GF are entire on the L^1 space [9], which is a natural environment, namely, to recover the notion of correlation function. For a generic entire functional B on L^1, this assumption implies that B has a representation in terms of its Taylor expansion,

$$B(\theta_0 + z\theta) = \sum_{n=0}^{\infty} \frac{z^n}{n!} d^n B(\theta_0; \theta, ..., \theta), \quad z \in \mathbb{C}, \theta \in L^1,$$

being each differential $d^n B(\theta_0; \cdot), n \in \mathbb{N}, \theta_0 \in L^1$ defined by a symmetric kernel

$$\delta^n B(\theta_0; \cdot) \in L^\infty(\mathbb{R}^{dn}) := L^\infty\big((\mathbb{R}^d)^n, (dx)^{\otimes n}\big),$$

called the variational derivative of n-th order of B at the point θ_0. That is,

$$\begin{aligned} d^n B(\theta_0; \theta_1, ..., \theta_n) &:= \frac{\partial^n}{\partial z_1 ... \partial z_n} B\left(\theta_0 + \sum_{i=1}^n z_i \theta_i\right)\Bigg|_{z_1 = ... = z_n = 0} \\ &=: \int_{(\mathbb{R}^d)^n} dx_1 \dots dx_n \, \delta^n B(\theta_0; x_1, \dots, x_n) \prod_{i=1}^n \theta_i(x_i) \end{aligned} \tag{2.8}$$

for all $\theta_1, ..., \theta_n \in L^1$. Moreover, the operator norm of the bounded n-linear functional $d^n B(\theta_0; \cdot)$ is equal to $\|\delta^n B(\theta_0; \cdot)\|_{L^\infty(\mathbb{R}^{dn})}$ and for all $r > 0$ one has

$$\|\delta B(\theta_0; \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{r} \sup_{\|\theta'\|_{L^1} \leq r} |B(\theta_0 + \theta')| \tag{2.9}$$

and, for $n \geq 2$,

$$\|\delta^n B(\theta_0; \cdot)\|_{L^\infty(\mathbb{R}^{dn})} \leq n! \left(\frac{e}{r}\right)^n \sup_{\|\theta'\|_{L^1} \leq r} |B(\theta_0 + \theta')|. \tag{2.10}$$

In particular, if B is an entire GF B_μ on L^1 then, in terms of the underlying measure μ, the entireness property of B_μ implies that the correlation measure ρ_μ is absolutely continuous with respect to the Lebesgue-Poisson measure λ and the Radon-Nykodim derivative $k_\mu = \dfrac{d\rho_\mu}{d\lambda}$ is given by

$$k_\mu(\eta) = \delta^{|\eta|} B_\mu(0; \eta) \quad \text{for } \lambda\text{-a.a. } \eta \in \Gamma_0.$$

In what follows, for each $\alpha > 0$, we consider the Banach space $\mathcal{E}_\alpha$ of all entire functionals B on L^1 such that

$$\|B\|_\alpha := \sup_{\theta \in L^1} \left(|B(\theta)| \, e^{-\frac{1}{\alpha}\|\theta\|_{L^1}}\right) < \infty,$$

see [9]. This class of Banach spaces has the particularity that, for each $\alpha_0 > 0$, the family $\{\mathcal{E}_\alpha : 0 < \alpha \leq \alpha_0\}$ is a scale of Banach spaces, that is,

$$\mathcal{E}_{\alpha''} \subseteq \mathcal{E}_{\alpha'}, \quad \|\cdot\|_{\alpha'} \leq \|\cdot\|_{\alpha''}$$

for any pair α', α'' such that $0 < \alpha' < \alpha'' \leq \alpha_0$.

3. The Kawasaki dynamics

The Kawasaki dynamics is an example of a hopping particle model where, in this case, particles randomly hop over the space $\mathbb{R}^d$ according to a rate depending on the interaction between particles. More precisely, let $a : \mathbb{R}^d \to [0, +\infty)$ be an even and integrable function and let $\phi : \mathbb{R}^d \to [0, +\infty]$ be a pair potential, that is, a $\mathcal{B}(\mathbb{R}^d)$-measurable function such that $\phi(-x) = \phi(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d \setminus \{0\}$, which we will assume to be integrable. A particle located at a site x in a given configuration $\gamma \in \Gamma$

hops to a site y according to a rate given by $a(x-y)\exp(-E(y,\gamma))$, where $E(y,\gamma)$ is a relative energy of interaction between the site y and the configuration γ defined by

$$E(y,\gamma) := \sum_{x\in\gamma}\phi(x-y) \in [0,+\infty].$$

Informally, the behavior of such an infinite particle system is described by

$$(LF)(\gamma) = \sum_{x\in\gamma}\int_{\mathbb{R}^d} dy\, a(x-y)e^{-E(y,\gamma)}\left(F(\gamma\setminus\{x\}\cup\{y\}) - F(\gamma)\right). \tag{3.1}$$

Given an infinite particle system, as the Kawasaki dynamics, its time evolution in terms of states is informally given by the so-called Fokker-Planck equation,

$$\frac{d\mu_t}{dt} = L^*\mu_t, \qquad \mu_t\Big|_{t=0} = \mu_0, \tag{3.2}$$

where L^* is the dual operator of L. Technically, the use of definition (2.3) allows an alternative approach to the study of (3.2) through the corresponding correlation functions $k_t := k_{\mu_t}$, $t \geq 0$, provided they exist. This leads to the Cauchy problem

$$\frac{\partial}{\partial t}k_t = \hat{L}^* k_t, \quad k_{t|t=0} = k_0,$$

where k_0 is the correlation function corresponding to the initial distribution μ_0 and $\hat{L}^*$ is the dual operator of $\hat{L} := K^{-1}LK$ in the sense

$$\int_{\Gamma_0} d\lambda(\eta)\,(\hat{L}G)(\eta)k(\eta) = \int_{\Gamma_0} d\lambda(\eta)\,G(\eta)(\hat{L}^*k)(\eta).$$

Through the representation (2.7), this gives us a way to express the dynamics also in terms of the GF B_t corresponding to μ_t, i.e., informally,

$$\begin{aligned}\frac{\partial}{\partial t}B_t(\theta) &= \int_{\Gamma_0} d\lambda(\eta)\,e_\lambda(\theta,\eta)\left(\frac{\partial}{\partial t}k_t(\eta)\right) = \int_{\Gamma_0} d\lambda(\eta)\,e_\lambda(\theta,\eta)(\hat{L}^*k_t)(\eta)\\ &= \int_{\Gamma_0} d\lambda(\eta)\,(\hat{L}e_\lambda(\theta))(\eta)k_t(\eta) =: (\tilde{L}B_t)(\theta).\end{aligned} \tag{3.3}$$

This leads to the time evolution equation

$$\frac{\partial B_t}{\partial t} = \tilde{L}B_t, \tag{3.4}$$

where, in the case of the Kawasaki dynamics, $\tilde{L}$ is given cf. [4] by

$$\begin{aligned}&(\tilde{L}B)(\theta)\\ &= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, a(x-y)e^{-\phi(x-y)}(\theta(y)-\theta(x))\delta B(\theta e^{-\phi(y-\cdot)} + e^{-\phi(y-\cdot)} - 1; x).\end{aligned} \tag{3.5}$$

Theorem 3.1. *Given an $\alpha_0 > 0$, let $B_0 \in \mathcal{E}_{\alpha_0}$. For each $\alpha \in (0,\alpha_0)$ there is a $T > 0$ (which depends on α, α_0) such that there is a unique solution B_t, $t \in [0,T)$, to the initial value problem* (3.4), (3.5), *$B_{t|t=0} = B_0$ in the space $\mathcal{E}_\alpha$.*

This theorem follows as a particular application of an abstract Ovsjannikov-type result in a scale of Banach spaces which can be found e.g. in [5, Theorem 2.5], and the following estimate of norms.

Proposition 3.2. *Let $0 < \alpha < \alpha_0$ be given. If $B \in \mathcal{E}_{\alpha''}$ for some $\alpha'' \in (\alpha,\alpha_0]$, then $\tilde{L}B \in \mathcal{E}_{\alpha'}$ for all $\alpha \leq \alpha' < \alpha''$, and we have*

$$\|\tilde{L}B\|_{\alpha'} \leq 2e^{\frac{\|\phi\|_{L^1}}{\alpha}}\|a\|_{L^1}\frac{\alpha_0}{\alpha''-\alpha'}\|B\|_{\alpha''}.$$

To prove this result as well as other forthcoming ones the next lemma shows to be useful.

Lemma 3.3. *Let $\varphi, \psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be such that, for a.a. $y \in \mathbb{R}^d$, $\varphi(y,\cdot) \in L^\infty := L^\infty(\mathbb{R}^d)$, $\psi(y,\cdot) \in L^1$ and $\|\varphi(y,\cdot)\|_{L^\infty} \le c_0$, $\|\psi(y,\cdot)\|_{L^1} \le c_1$ for some constants $c_0, c_1 > 0$ independent of y. For each $\alpha > 0$ and all $B \in \mathcal{E}_\alpha$ let*

$$(L_0 B)(\theta) := \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, a(x-y) e^{-k\phi(x-y)} \left(\theta(y) - \theta(x)\right) \delta B(\varphi(y,\cdot)\theta + \psi(y,\cdot); x),$$

$\theta \in L^1$. Here a and ϕ are defined as before and $k \ge 0$ is a constant. Then, for all $\alpha' > 0$ such that $c_0\alpha' < \alpha$, we have $L_0 B \in \mathcal{E}_{\alpha'}$ and

$$\|L_0 B\|_{\alpha'} \le 2 e^{\frac{c_1}{\alpha}} \|a\|_{L^1} \frac{\alpha'}{\alpha - c_0\alpha'} \|B\|_\alpha.$$

Proof. First we observe that from the considerations done in Subsection 2.2 it follows that $L_0 B$ is an entire functional on L^1 and, in addition, that for all $r > 0$, $\theta \in L^1$, and a.a. $x, y \in \mathbb{R}^d$,

$$\begin{aligned} |\delta B(\varphi(y,\cdot)\theta + \psi(y,\cdot); x)| &\le \|\delta B(\varphi(y,\cdot)\theta + \psi(y,\cdot); \cdot)\|_{L^\infty} \\ &\le \frac{1}{r} \sup_{\|\theta_0\|_{L^1} \le r} |B(\varphi(y,\cdot)\theta + \psi(y,\cdot) + \theta_0)|, \end{aligned}$$

where, for all $\theta_0 \in L^1$ such that $\|\theta_0\|_{L^1} \le r$,

$$|B(\varphi(y,\cdot)\theta + \psi(y,\cdot) + \theta_0)| \le \|B\|_\alpha e^{\frac{\|\varphi(y,\cdot)\theta + \psi(y,\cdot)\|_{L^1}}{\alpha} + \frac{r}{\alpha}} \le \|B\|_\alpha e^{\frac{c_0\|\theta\|_{L^1} + c_1 + r}{\alpha}}.$$

As a result, due to the positiveness of ϕ and to the fact that a is an even function, for all $\theta \in L^1$ one has

$$\begin{aligned} |(L_0 B)(\theta)| &\le \frac{1}{r} e^{\frac{c_0\|\theta\|_{L^1} + c_1 + r}{\alpha}} \|B\|_\alpha \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, a(x-y) e^{-k\phi(x-y)} |\theta(y) - \theta(x)| \\ &\le \frac{2}{r} e^{\frac{c_1 + r}{\alpha}} \|a\|_{L^1} \|\theta\|_{L^1} e^{\frac{c_0\|\theta\|_{L^1}}{\alpha}} \|B\|_\alpha. \end{aligned}$$

Thus,

$$\begin{aligned} \|L_0 B\|_{\alpha'} &= \sup_{\theta \in L^1} \left(e^{-\frac{1}{\alpha'}\|\theta\|_{L^1}} |(L_0 B)(\theta)| \right) \\ &\le \frac{2}{r} e^{\frac{c_1 + r}{\alpha}} \|a\|_{L^1} \|B\|_\alpha \sup_{\theta \in L^1} \left(e^{-\left(\frac{1}{\alpha'} - \frac{c_0}{\alpha}\right)\|\theta\|_{L^1}} \|\theta\|_{L^1} \right), \end{aligned}$$

where the supremum is finite provided $\frac{1}{\alpha'} - \frac{c_0}{\alpha} > 0$. In such a situation, the use of the inequality $xe^{-mx} \le \frac{1}{em}$, $x \ge 0$, $m > 0$ leads for each $r > 0$ to

$$\|L_0 B\|_{\alpha'} \le \frac{2}{r} \|a\|_{L^1} e^{\frac{c_1 + r}{\alpha}} \frac{\alpha\alpha'}{e(\alpha - c_0\alpha')} \|B\|_\alpha.$$

The required estimate of norms follows by minimizing the expression $\frac{1}{r} e^{\frac{c_1 + r}{\alpha}}$ in the parameter r, that is, $r = \alpha$. □

Proof of Proposition 3.2. In Lemma 3.3 replace φ by $e^{-\phi}$ and ψ by $e^{-\phi} - 1$, and consider $k = 1$. Due to the positiveness and integrability properties of ϕ one has $e^{-\phi} \le 1$ and $|e^{-\phi} - 1| = 1 - e^{-\phi} \le \phi \in L^1$, ensuring the conditions to apply Lemma 3.3. □

Remark 3.4. *Concerning the initial conditions considered in Theorem 3.1, observe that, in particular, B_0 can be an entire GF B_{μ_0} on L^1 such that, for some constants $\alpha_0, C > 0$, $|B_{\mu_0}(\theta)| \le C \exp(\frac{\|\theta\|_{L^1}}{\alpha_0})$ for all $\theta \in L^1$. In such a situation an additional analysis is need in order to guarantee that for each t the local solution B_t given by Theorem 3.1 is a GF (corresponding to some measure). For more details see e.g.* [5, 9] *and references therein.*

4. Vlasov scaling

We proceed to investigate the Vlasov-type scaling proposed in [3] for generic continuous particle systems and accomplished in [1] for the Kawasaki dynamics. As explained in both references, we start with a rescaling of an initial correlation function k_0, denoted by $k_0^{(\varepsilon)}$, $\varepsilon > 0$, which has a singularity with respect to ε of the type $k_0^{(\varepsilon)}(\eta) \sim \varepsilon^{-|\eta|} r_0(\eta)$, $\eta \in \Gamma_0$, being r_0 a function independent of ε. The aim is to construct a scaling of the operator L defined in (3.1), L_ε, $\varepsilon > 0$, in such a way that the following two conditions are fulfilled. The first one is that under the scaling $L \mapsto L_\varepsilon$ the solution $k_t^{(\varepsilon)}$, $t \geq 0$, to

$$\frac{\partial}{\partial t} k_t^{(\varepsilon)} = \hat{L}_\varepsilon^* k_t^{(\varepsilon)}, \quad k_t^{(\varepsilon)}{}_{|t=0} = k_0^{(\varepsilon)}$$

preserves the order of the singularity with respect to ε, that is, $k_t^{(\varepsilon)}(\eta) \sim \varepsilon^{-|\eta|} r_t(\eta)$, $\eta \in \Gamma_0$. The second condition is that the dynamics $r_0 \mapsto r_t$ preserves the Lebesgue-Poisson exponents, that is, if r_0 is of the form $r_0 = e_\lambda(\rho_0)$, then each r_t, $t > 0$, is of the same type, i.e., $r_t = e_\lambda(\rho_t)$, where ρ_t is a solution to a non-linear equation (called a Vlasov-type equation).

The previous scheme was accomplished in [1] through the scale transformation $\phi \mapsto \varepsilon\phi$ of the operator L, that is,

$$(L_\varepsilon F)(\gamma) := \sum_{x \in \gamma} \int_{\mathbb{R}^d} dy\, a(x-y) e^{-\varepsilon E(y,\gamma)} \left(F(\gamma \setminus \{x\} \cup \{y\}) - F(\gamma)\right).$$

As shown in [3, Example 12], [1], the corresponding Vlasov-type equation is given by

$$\frac{\partial}{\partial t} \rho_t(x) = (\rho_t * a)(x) e^{-(\rho_t * \phi)(x)} - \rho_t(x)(a * e^{-(\rho_t * \phi)})(x), \quad x \in \mathbb{R}^d, \tag{4.1}$$

where $*$ denotes the usual convolution of functions. Existence of classical solutions $0 \leq \rho_t \in L^\infty$ to (4.1) has been discussed in [1]. Therefore, it is natural to consider the same scaling, but in GF.

To proceed towards GF, we consider $k_t^{(\varepsilon)}$ defined as before and $k_{t,\mathrm{ren}}^{(\varepsilon)}(\eta) := \varepsilon^{|\eta|} k_t^{(\varepsilon)}(\eta)$. In terms of GF, these yield

$$B_t^{(\varepsilon)}(\theta) := \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) k_t^{(\varepsilon)}(\eta),$$

and

$$B_{t,\mathrm{ren}}^{(\varepsilon)}(\theta) := \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta,\eta) k_{t,\mathrm{ren}}^{(\varepsilon)}(\eta) = \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\varepsilon\theta,\eta) k_t^{(\varepsilon)}(\eta) = B_t^{(\varepsilon)}(\varepsilon\theta),$$

leading, as in (3.3), to the initial value problem

$$\frac{\partial}{\partial t} B_{t,\mathrm{ren}}^{(\varepsilon)} = \tilde{L}_{\varepsilon,\mathrm{ren}} B_{t,\mathrm{ren}}^{(\varepsilon)}, \quad B_{t,\mathrm{ren}|t=0}^{(\varepsilon)} = B_{0,\mathrm{ren}}^{(\varepsilon)}. \tag{4.2}$$

Proposition 4.1. *For all $\varepsilon > 0$ and all $\theta \in L^1$, we have*

$$\begin{aligned}(\tilde{L}_{\varepsilon,\mathrm{ren}} B)(\theta) = &\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, a(x-y) e^{-\varepsilon\phi(x-y)} (\theta(y) - \theta(x)) \\ &\times \delta B\left(\theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}; x\right).\end{aligned} \tag{4.3}$$

Proof. Since

$$(\tilde{L}_{\varepsilon,\mathrm{ren}} B)(\theta) = \int_{\Gamma_0} d\lambda(\eta)\, (\hat{L}_{\varepsilon,\mathrm{ren}} e_\lambda(\theta))(\eta) k(\eta),$$

first we have to calculate $(\hat{L}_{\varepsilon,\mathrm{ren}} e_\lambda(\theta))(\eta) := \varepsilon^{-|\eta|}\hat{L}_\varepsilon(e_\lambda(\varepsilon\theta,\eta))$, $\hat{L}_\varepsilon = K^{-1}L_\varepsilon K$ cf. [3]. Similar calculations done in [4, Subsection 4.2.1] show

$$(\hat{L}_{\varepsilon,\mathrm{ren}} e_\lambda(\theta))(\eta) = \sum_{x\in\eta}\int_{\mathbb{R}^d} dy\, a(x-y)e^{-\varepsilon\phi(x-y)}(\theta(y)-\theta(x)) \times e_\lambda\left(\theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon}, \eta\setminus\{x\}\right),$$

and thus, using the relation between variational derivatives derived in [9, Proposition 11], one finds

$$\begin{aligned}(\tilde{L}_{\varepsilon,\mathrm{ren}} B)(\theta) &= \int_{\Gamma_0} d\lambda(\eta)\, k(\eta)\sum_{x\in\eta}\int_{\mathbb{R}^d} dy\, a(x-y)e^{-\varepsilon\phi(x-y)}(\theta(y)-\theta(x)) \\ &\qquad\times e_\lambda\left(\theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon}, \eta\setminus\{x\}\right) \\ &= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, a(x-y)e^{-\varepsilon\phi(x-y)}(\theta(y)-\theta(x)) \\ &\qquad \int_{\Gamma_0} d\lambda(\eta)\, k(\eta\cup\{x\}) e_\lambda\left(\theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon}, \eta\right) \\ &= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, a(x-y)e^{-\varepsilon\phi(x-y)}(\theta(y)-\theta(x)) \\ &\qquad\times \delta B\left(\theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon}; x\right).\end{aligned}$$

□

Proposition 4.2. *(i) If $B\in\mathcal{E}_\alpha$ for some $\alpha>0$, then, for all $\theta\in L^1$, $(\tilde{L}_{\varepsilon,\mathrm{ren}}B)(\theta)$ converges as ε tends to zero to*

$$(\tilde{L}_V B)(\theta) := \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, a(x-y)(\theta(y)-\theta(x))\delta B(\theta-\phi(y-\cdot);x).$$

(ii) Let $\alpha_0>\alpha>0$ be given. If $B\in\mathcal{E}_{\alpha''}$ for some $\alpha''\in(\alpha,\alpha_0]$, then $\{\tilde{L}_{\varepsilon,\mathrm{ren}}B, \tilde{L}_V B\}\subset \mathcal{E}_{\alpha'}$ for all $\alpha\le\alpha'<\alpha''$, and we have

$$\|\tilde{L}_\# B\|_{\alpha'} \le 2\|a\|_{L^1}\frac{\alpha_0}{(\alpha''-\alpha')}e^{\frac{\|\phi\|_{L^1}}{\alpha}}\|B\|_{\alpha''}$$

where $\tilde{L}_\# = \tilde{L}_{\varepsilon,\mathrm{ren}}$ or $\tilde{L}_\# = \tilde{L}_V$.

Proof. (i) To prove this result we first analyze the pointwise convergence of the variational derivative (4.3) appearing in $\tilde{L}_{\varepsilon,\mathrm{ren}}$. For this purpose we will use the relation between variational derivatives derived in [9, Proposition 11], i.e.,

$$\delta B(\theta_1+\theta_2;x) = \int_{\Gamma_0} d\lambda(\eta)\,\delta^{|\eta|+1}B(\theta_1;\eta\cup\{x\})e_\lambda(\theta_2,\eta),\quad a.a.\, x\in\mathbb{R}^d, \theta_1,\theta_2\in L^1,$$

which allows to rewrite (4.3) as

$$\begin{aligned}&\delta B\left(\theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon}; x\right) \\ &= \int_{\Gamma_0} d\lambda(\eta)\,\delta^{|\eta|+1}B(\theta-\phi(y-\cdot);\eta\cup\{x\}) \\ &\qquad\times e_\lambda\left(\theta\left(e^{-\varepsilon\phi(y-\cdot)}-1\right) + \frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon} + \phi(y-\cdot), \eta\right),\end{aligned} \tag{4.4}$$

for a.a. $x, y \in \mathbb{R}^d$. Concerning the function

$$f_\varepsilon := f_\varepsilon(\theta, \phi, y) := \theta\left(e^{-\varepsilon\phi(y-\cdot)} - 1\right) + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon} + \phi(y - \cdot)$$

which appears in (4.4), for a.a. $y \in \mathbb{R}^d$, one clearly has $\lim_{\varepsilon\to 0} f_\varepsilon = 0$ a.e. in $\mathbb{R}^d$. By definition (2.4), the latter implies that $e_\lambda(f_\varepsilon)$ converges λ-a.e. to $e_\lambda(0)$. Moreover, for the whole integrand function in (4.4), estimates (2.9), (2.10) yield for any $r > 0$ and λ-a.a. $\eta \in \Gamma_0$,

$$\begin{aligned}
&\left|\delta^{|\eta|+1} B(\theta - \phi(y - \cdot); \eta \cup \{x\}) e_\lambda(f_\varepsilon, \eta)\right| \\
&\leq \left\|\delta^{|\eta|+1} B(\theta - \phi(y - \cdot); \cdot)\right\|_{L^\infty(\mathbb{R}^{d(|\eta|+1)})} e_\lambda(|f_\varepsilon|, \eta) \\
&\leq (|\eta| + 1)! \left(\frac{e}{r}\right)^{|\eta|+1} e_\lambda(|f_\varepsilon|, \eta) \sup_{\|\theta_0\|_{L^1} \leq r} |B(\theta - \phi(y - \cdot) + \theta_0)| \\
&\leq (|\eta| + 1)! \left(\frac{e}{r}\right)^{|\eta|+1} e_\lambda(|\theta| + 2|\phi(y - \cdot)|, \eta) e^{\frac{\|\theta-\phi(y-\cdot)\|_{L^1}+r}{\alpha}} \|B\|_\alpha
\end{aligned}$$

with

$$\int_{\Gamma_0} d\lambda(\eta)\, (|\eta|+1)! \left(\frac{e}{r}\right)^{|\eta|+1} e_\lambda(|\theta|+2|\phi(y-\cdot)|, \eta) = \sum_{n=0}^{\infty}(n+1) \left(\frac{e}{r}\right)^{n+1} \left(\|\theta\|_{L^1}+2\|\phi\|_{L^1}\right)^n$$

being finite for any $r > e(\|\theta\|_{L^1} + 2\|\phi\|_{L^1})$.

As a result, by an application of the Lebesgue dominated convergence theorem we have proved that, for a.a. $x, y \in \mathbb{R}^d$, (4.4) converges as ε tends to zero to

$$\int_{\Gamma_0} d\lambda(\eta)\, \delta^{|\eta|+1} B(\theta - \phi(y - \cdot); \eta \cup \{x\}) e_\lambda(0, \eta) = \delta B(\theta - \phi(y - \cdot); x).$$

In addition, for the integrand function which appears in $(\tilde{L}_{\varepsilon,\mathrm{ren}} B)(\theta)$ we have

$$\begin{aligned}
&\left|a(x - y)e^{-\varepsilon\phi(x-y)}(\theta(y) - \theta(x))\delta B\left(\theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}; x\right)\right| \\
&\leq \frac{e}{\alpha} a(x - y)|\theta(y) - \theta(x)|\|B\|_\alpha \exp\left(\frac{1}{\alpha}\|\theta\|_{L^1} + \frac{1}{\alpha}\|\phi\|_{L^1}\right),
\end{aligned}$$

for all $\varepsilon > 0$ and a.a. $x, y \in \mathbb{R}^d$, leading through a second application of the Lebesgue dominated convergence theorem to the required limit.

(ii) In Lemma 3.3 replace φ by $e^{-\varepsilon\phi}$, ψ by $\frac{e^{-\varepsilon\phi}-1}{\varepsilon}$, and k by ε. Arguments similar to prove Proposition 3.2 complete the proof for $\tilde{L}_{\varepsilon,\mathrm{ren}}$. A similar proof holds for $\tilde{L}_V$. □

Proposition 4.2 (ii) provides similar estimate of norms for $\tilde{L}_{\varepsilon,\mathrm{ren}}$, $\varepsilon > 0$, and the limiting mapping $\tilde{L}_V$. According to the Ovsjannikov-type result used to prove Theorem 3.1, this means that given any $B_{0,V}, B_{0,\mathrm{ren}}^{(\varepsilon)} \in \mathcal{E}_{\alpha_0}$, $\varepsilon > 0$, for each $\alpha \in (0, \alpha_0)$ there is a $T > 0$ such that there is a unique solution $B_{t,\mathrm{ren}}^{(\varepsilon)} : [0, T) \to \mathcal{E}_\alpha$, $\varepsilon > 0$, to each initial value problem (4.2) and a unique solution $B_{t,V} : [0, T) \to \mathcal{E}_\alpha$ to the initial value problem

$$\frac{\partial}{\partial t} B_{t,V} = \tilde{L}_V B_{t,V}, \quad B_{t,V|t=0} = B_{0,V}. \tag{4.5}$$

In other words, independent of the initial value problem under consideration, the solutions obtained are defined on the same time-interval and with values in the same Banach space. For more details see e.g. Theorem 2.5 and its proof in [5]. Therefore, it is natural to analyze under which conditions the solutions to (4.2) converge to the solution to (4.5). This follows from a general result presented in [5] (Theorem 4.3). However, to proceed to an application of this general result one needs the following estimate of norms.

Proposition 4.3. *Assume that $0 \le \phi \in L^1 \cap L^\infty$ and let $\alpha_0 > \alpha > 0$ be given. Then, for all $B \in \mathcal{E}_{\alpha''}$, $\alpha'' \in (\alpha, \alpha_0]$, the following estimate holds*

$$\begin{aligned}
&\|\tilde{L}_{\varepsilon,\mathrm{ren}}B - \tilde{L}_V B\|_{\alpha'} \\
&\le 2\varepsilon \|a\|_{L^1}\|\phi\|_{L^\infty}\frac{e\alpha_0}{\alpha}\|B\|_{\alpha''}e^{\frac{\|\phi\|_{L^1}}{\alpha}}\left(\left(2e\|\phi\|_{L^1}+\frac{\alpha_0}{e}\right)\frac{1}{\alpha''-\alpha'}+\frac{8\alpha_0^2}{(\alpha''-\alpha')^2}\right)
\end{aligned}$$

for all α' such that $\alpha \le \alpha' < \alpha''$ and all $\varepsilon > 0$.

Proof. First we observe that

$$\begin{aligned}
&\left|(\tilde{L}_{\varepsilon,\mathrm{ren}}B)(\theta) - (\tilde{L}_V B)(\theta)\right| \le \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, a(x-y)\,|\theta(y)-\theta(x)| \\
&\times \left| e^{-\varepsilon\phi(x-y)}\delta B\left(\theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon};x\right) - \delta B\left(\theta - \phi(y-\cdot);x\right)\right|
\end{aligned}$$

with

$$\begin{aligned}
&\left| e^{-\varepsilon\phi(x-y)}\delta B\left(\theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon};x\right) - \delta B\left(\theta - \phi(y-\cdot);x\right)\right| \\
&\le \left| \delta B\left(\theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon};x\right) - \delta B\left(\theta - \phi(y-\cdot);x\right)\right| \qquad (4.6)\\
&+ \left(1 - e^{-\varepsilon\phi(x-y)}\right)\left|\delta B\left(\theta-\phi(y-\cdot);x\right)\right|.
\end{aligned}$$

In order to estimate (4.6), given any $\theta_0, \theta_1, \theta_2 \in L^1$, let us consider the function $C_{\theta_0,\theta_1,\theta_2}(t) = dB\,(t\theta_1 + (1-t)\theta_2;\theta_0)$, $t \in [0,1]$, where dB is the first order differential of B, defined in (2.8). One has

$$\begin{aligned}
\frac{\partial}{\partial t}C_{\theta_0,\theta_1,\theta_2}(t) &= \frac{\partial}{\partial s}C_{\theta_0,\theta_1,\theta_2}(t+s)\Big|_{s=0} \\
&= \frac{\partial}{\partial s}dB\big(\theta_2 + t(\theta_1-\theta_2) + s(\theta_1-\theta_2);\theta_0\big)\Big|_{s=0} \\
&= \frac{\partial^2}{\partial s_1 \partial s_2}B\big(\theta_2 + t(\theta_1-\theta_2) + s_1(\theta_1-\theta_2) + s_2\theta_0\big)\Big|_{s_1=s_2=0} \\
&= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, (\theta_1(x)-\theta_2(x))\theta_0(y)\,\delta^2 B(\theta_2 + t(\theta_1-\theta_2);x,y),
\end{aligned}$$

leading to

$$\begin{aligned}
&\left|dB(\theta_1;\theta_0) - dB(\theta_2;\theta_0)\right| \\
&= \left|C_{\theta_0,\theta_1,\theta_2}(1) - C_{\theta_0,\theta_1,\theta_2}(0)\right| \\
&\le \max_{t\in[0,1]}\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\ |\theta_1(x)-\theta_2(x)|\,|\theta_0(y)|\,\left|\delta^2 B(\theta_2 + t(\theta_1-\theta_2);x,y)\right| \\
&\le \|\theta_1-\theta_2\|_{L^1}\|\theta_0\|_{L^1}\max_{t\in[0,1]}\|\delta^2 B(\theta_2+t(\theta_1-\theta_2);\cdot)\|_{L^\infty(\mathbb{R}^{2d})},
\end{aligned}$$

where, through estimate (2.10) with $r = \alpha''$,

$$\|\delta^2 B(\theta_2+t(\theta_1-\theta_2);\cdot)\|_{L^\infty(\mathbb{R}^{2d})} \le 2\frac{e^3}{\alpha''^2}\|B\|_{\alpha''}\exp\left(\frac{\|\theta_2+t(\theta_1-\theta_2)\|_{L^1}}{\alpha''}\right).$$

As a result,

$$\begin{aligned}
&\left|dB(\theta_1;\theta_0) - dB(\theta_2;\theta_0)\right| \\
&\le 2\frac{e^3}{\alpha''^2}\|\theta_1-\theta_2\|_{L^1}\|\theta_0\|_{L^1}\|B\|_{\alpha''}\max_{t\in[0,1]}\exp\left(\frac{t\|\theta_1\|_{L^1}+(1-t)\|\theta_2\|_{L^1}}{\alpha''}\right),
\end{aligned}$$

for all $\theta_0, \theta_1, \theta_2 \in L^1$. In particular, this shows that for all $\theta_0 \in L^1$,

$$\begin{aligned}
&\left| dB\left(\theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon};\theta_0\right) - dB\left(\theta-\phi(y-\cdot);\theta_0\right)\right| \\
&\leq 2\varepsilon\frac{e^3}{\alpha''^2}\|\phi\|_{L^\infty}\|B\|_{\alpha''}\left(\|\theta\|_{L^1}+\|\phi\|_{L^1}\right)\|\theta_0\|_{L^1} \\
&\qquad\times\max_{t\in[0,1]}\exp\left(\frac{1}{\alpha''}\left(t\left(\|\theta\|_{L^1}+\|\phi\|_{L^1}\right)+(1-t)\left(\|\theta\|_{L^1}+\|\phi\|_{L^1}\right)\right)\right) \\
&= 2\varepsilon\frac{e^3}{\alpha''^2}\|\phi\|_{L^\infty}\|B\|_{\alpha''}\left(\|\theta\|_{L^1}+\|\phi\|_{L^1}\right)\exp\left(\frac{1}{\alpha''}\left(\|\theta\|_{L^1}+\|\phi\|_{L^1}\right)\right)\|\theta_0\|_{L^1},
\end{aligned}$$

where we have used the inequalities

$$\begin{aligned}
\|\theta e^{-\varepsilon\phi(y-\cdot)}-\theta\|_{L^1} &\leq \varepsilon\|\phi\|_{L^\infty}\|\theta\|_{L^1}, \\
\left\|\frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon}+\phi(y-\cdot)\right\|_{L^1} &\leq \varepsilon\|\phi\|_{L^\infty}\|\phi\|_{L^1}, \\
\left\|\theta e^{-\varepsilon\phi(y-\cdot)}+\frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon}\right\|_{L^1} &\leq \|\theta\|_{L^1}+\|\phi\|_{L^1}.
\end{aligned}$$

In other words, we have shown that the norm of the bounded linear functional on L^1

$$L^1 \ni \theta_0 \mapsto dB\left(\theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon};\theta_0\right) - dB\left(\theta-\phi(y-\cdot);\theta_0\right)$$

is bounded by

$$Q := 2\varepsilon\frac{e^3}{\alpha''^2}\|\phi\|_{L^\infty}\|B\|_{\alpha''}\left(\|\theta\|_{L^1}+\|\phi\|_{L^1}\right)\exp\left(\frac{1}{\alpha''}\left(\|\theta\|_{L^1}+\|\phi\|_{L^1}\right)\right).$$

Since this operator norm is given by

$$\left\|\delta B\left(\theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon};\cdot\right) - \delta B\left(\theta-\phi(y-\cdot);\cdot\right)\right\|_{L^\infty}$$

cf. Subsection 2.2, this means that

$$\left\|\delta B\left(\theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon};\cdot\right) - \delta B\left(\theta-\phi(y-\cdot);\cdot\right)\right\|_{L^\infty} \leq Q.$$

In this way we obtain

$$\begin{aligned}
&\left|(\tilde{L}_{\varepsilon,\mathrm{ren}}B)(\theta)-(\tilde{L}_V B)(\theta)\right| \\
&\leq \int_{\mathbb{R}^d}dx\int_{\mathbb{R}^d}dy\,a(x-y)\,|\theta(y)-\theta(x)| \\
&\quad\times\left\{\left\|\delta B\left(\theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon};\cdot\right) - \delta B\left(\theta-\phi(y-\cdot);\cdot\right)\right\|_{L^\infty}\right. \\
&\qquad\left.+\varepsilon\|\phi\|_{L^\infty}\left\|\delta B\left(\theta-\phi(y-\cdot);\cdot\right)\right\|_{L^\infty}\right\} \\
&\leq 2\varepsilon\|\phi\|_{L^\infty}\|a\|_{L^1}\frac{e}{\alpha''}\exp\left(\frac{1}{\alpha''}\left(\|\theta\|_{L^1}+\|\phi\|_{L^1}\right)\right)\|\theta\|_{L^1} \\
&\qquad\times\left\{2\frac{e^2}{\alpha''}\left(\|\theta\|_{L^1}+\|\phi\|_{L^1}\right)+1\right\}\|B\|_{\alpha''},
\end{aligned}$$

and thus

$$\begin{aligned}
&\|\tilde{L}_{\varepsilon,\mathrm{ren}}B - \tilde{L}_V B\|_{\alpha'} \\
&\leq 2\varepsilon\|\phi\|_{L^\infty}\|a\|_{L^1}\frac{e}{\alpha''}e^{\frac{\|\phi\|_{L^1}}{\alpha''}}\left\{2\frac{e^2}{\alpha''}\sup_{\theta\in L^1}\left(\|\theta\|_{L^1}^2\exp\left(\|\theta\|_{L^1}\left(\frac{1}{\alpha''}-\frac{1}{\alpha'}\right)\right)\right)\right. \\
&\quad\left.+\left(2\frac{e^2}{\alpha''}\|\phi\|_{L^1}+1\right)\sup_{\theta\in L^1}\left(\|\theta\|_{L^1}\exp\left(\|\theta\|_{L^1}\left(\frac{1}{\alpha''}-\frac{1}{\alpha'}\right)\right)\right)\right\}\|B\|_{\alpha''},
\end{aligned}$$

and the proof follows using the inequalities $xe^{-mx} \leq \frac{1}{me}$ and $x^2e^{-mx} \leq \frac{4}{m^2e^2}$ for $x \geq 0$, $m > 0$. □

We are now in conditions to state the following result.

Theorem 4.4. *Given an $0 < \alpha < \alpha_0$, let $B_{t,\mathrm{ren}}^{(\varepsilon)}, B_{t,V}$, $t \in [0,T)$, be the local solutions in $\mathcal{E}_\alpha$ to the initial value problems (4.2), (4.5) with $B_{0,\mathrm{ren}}^{(\varepsilon)}, B_{0,V} \in \mathcal{E}_{\alpha_0}$. If $0 \leq \phi \in L^1 \cap L^\infty$ and $\lim_{\varepsilon\to 0}\|B_{0,\mathrm{ren}}^{(\varepsilon)} - B_{0,V}\|_{\alpha_0} = 0$, then, for each $t \in [0,T)$,*

$$\lim_{\varepsilon\to 0}\|B_{t,\mathrm{ren}}^{(\varepsilon)} - B_{t,V}\|_\alpha = 0.$$

Moreover, if $B_{0,V}(\theta) = \exp\left(\int_{\mathbb{R}^d}dx\,\rho_0(x)\theta(x)\right)$, $\theta \in L^1$, for some function $0 \leq \rho_0 \in L^\infty$ such that $\|\rho_0\|_{L^\infty} \leq \frac{1}{\alpha_0}$, then for each $t \in [0,T)$,

$$B_{t,V}(\theta) = \exp\left(\int_{\mathbb{R}^d}dx\,\rho_t(x)\theta(x)\right), \quad \theta \in L^1, \tag{4.7}$$

where $0 \leq \rho_t \in L^\infty$ is a classical solution to the equation (4.1).

Proof. The first part follows directly from Proposition 4.3 and [5, Theorem 4.3], taking in [5, Theorem 4.3] $p = 2$ and

$$N_\varepsilon = 2\varepsilon\|a\|_{L^1}\|\phi\|_{L^\infty}\frac{e\alpha_0}{\alpha}e^{\frac{\|\phi\|_{L^1}}{\alpha}}\max\left\{2e\|\phi\|_{L^1} + \frac{\alpha_0}{e}, 8\alpha_0^2\right\}.$$

Concerning the last part, we begin by observing that it has been shown in [1, Subsection 4.2] that given a $0 \leq \rho_0 \in L^\infty$ such that $\|\rho_0\|_{L^\infty} \leq \frac{1}{\alpha_0}$, there is a solution $0 \leq \rho_t \in L^\infty$ to (4.1) such that $\|\rho_t\|_{L^\infty} \leq \frac{1}{\alpha_0}$. This implies that $B_{t,V}$, given by (4.7), does not leave the initial Banach space $\mathcal{E}_{\alpha_0} \subset \mathcal{E}_\alpha$. Then, by an argument of uniqueness, to prove the last assertion amounts to show that $B_{t,V}$ solves equation (4.5). For this purpose we note that for any $\theta, \theta_1 \in L^1$ we have

$$\left.\frac{\partial}{\partial z_1}B_{t,V}(\theta + z_1\theta_1)\right|_{z_1=0} = B_{t,V}(\theta)\int_{\mathbb{R}^d}dx\rho_t(x)\theta_1(x),$$

and thus $\delta B_{t,V}(\theta;x) = B_{t,V}(\theta)\rho_t(x)$. Hence, for all $\theta \in L^1$,

$$\begin{aligned}
(\tilde{L}_V B_{t,V})(\theta) &= B_{t,V}(\theta)\left(\int_{\mathbb{R}^d}dx\int_{\mathbb{R}^d}dy\,a(x-y)\left(\theta(y)-\theta(x)\right)\rho_t(x)e^{-(\rho_t*\phi)(y)}\right) \\
&= B_{t,V}(\theta)\left(\int_{\mathbb{R}^d}dy\,\theta(y)\left(a*\rho_t\right)(y)e^{-(\rho_t*\phi)(y)}\right. \\
&\qquad\left.-\int_{\mathbb{R}^d}dx\,\theta(x)\left(a*e^{-(\rho_t*\phi)(y)}\right)(x)\rho_t(x)\right).
\end{aligned}$$

Since ρ_t is a classical solution to (4.1), ρ_t solves a weak form of equation (4.1), that is, the right-hand side of the latter equality is equal to

$$B_{t,V}(\theta)\frac{d}{dt}\int_{\mathbb{R}^d}dx\,\rho_t(x)\theta(x) = \frac{\partial}{\partial t}B_{t,V}(\theta).$$

□

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