Markov evolutions and hierarchical equations in the continuum: II. Multicomponent systems

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Abstract

General birth-and-death as well as hopping stochastic dynamics of infinite multicomponent particle systems in the continuum are considered. We derive the corresponding evolution equations for quasi-observables and correlation functions. We also present sufficient conditions that allows us to consider these equations on suitable Banach spaces.

Keywords: Continuous system; Markov generator; Markov process; Stochastic dynamics; Configuration spaces; Birth-and-death process; Hopping particles

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1 Introduction

Spatial Markov processes in $\mathbb{R}^d$ can be described as stochastic evolutions of locally finite configurations. From this standpoint, two important classes of stochastic dynamics are represented by birth-and-death and hopping Markov processes on the configuration space Γ over $\mathbb{R}^d$,

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d : |\gamma \cap \Lambda| < \infty, \text{ for every compact } \Lambda \subset \mathbb{R}^d \right\}.$$

These are processes where randomly, at each random moment of time, particles (or individuals) disappear and new particles appear or, in the case of hopping particle systems, particles hop over the space $\mathbb{R}^d$, according to rates which in both cases depend on the configuration of the whole system at that time. However, both cases concern only one type of particles.

Motivated by concrete ecological models [CFM08, DM10, FFK08], socio-economics models or even mathematical physics problems, e.g., the Potts model [GH96, GMSRZ06, KZ07], in this work we extend these two classes of stochastic dynamics to Markov stochastic evolutions of different particle types. For simplicity of notation, we just present this extension for two particle types. A similar procedure applies to $n > 2$ particle types, but with a more cumbersome notation.

Since two particles cannot be located at the same position, the natural phase space is a subset of the direct product of two copies of the space Γ, Γ^+ and Γ^-, namely,

$$\Gamma^2 := \left\{ (\gamma^+, \gamma^-) \in \Gamma^+ \times \Gamma^- : \gamma^+ \cap \gamma^- = \emptyset \right\}.$$

Given a configuration $(\gamma^+, \gamma^-) \in \Gamma^2$, the aforementioned fields of applications suggest that, according to certain rates of probability, at each random moment of time several random phenomena may occur:[1]

Death of a +-particle: $(\gamma^+, \gamma^-) \longmapsto (\gamma^+ \setminus x, \gamma^-)$, $x \in \gamma^+$;

Birth of a new +-particle: $(\gamma^+, \gamma^-) \longmapsto (\gamma^+ \cup x, \gamma^-)$, $x \in (\mathbb{R}^d \setminus \gamma^+) \setminus \gamma^-$;

Hop of a +-particle to a free site:

$$(\gamma^+, \gamma^-) \longmapsto (\gamma^+ \setminus x \cup y, \gamma^-), \qquad x \in \gamma^+,\ y \in (\mathbb{R}^d \setminus \gamma^+) \setminus \gamma^-;$$

Hop of a +-particle flipping the mark to $-$:

$$(\gamma^+, \gamma^-) \longmapsto (\gamma^+ \setminus x, \gamma^- \cup y), \qquad x \in \gamma^+,\ y \in (\mathbb{R}^d \setminus \gamma^+) \setminus \gamma^-;$$

[1] Here and below, for simplicity of notation, we have just written x, y instead of $\{x\}, \{y\}$, respectively.

Flip the mark + to −, keeping the site:

$$(\gamma^+, \gamma^-) \longmapsto (\gamma^+ \setminus x, \gamma^- \cup x), \qquad x \in \gamma^+.$$

Similar events naturally may occur with −-particles. In other words, besides the natural complexity imposed by the existence of different particle types, the treatment of multicomponent particle systems also deals with a higher number of possible random phenomena.

Heuristically, the stochastic dynamics of a multicomponent particle system is described through a Markov generator L defined according to the aforementioned elementary random phenomena and corresponding rates. The time evolution of states (that is, probability measures on Γ^2) in the weak form may be formulated by means of the following initial value problems

$$\frac{d}{dt}\langle F, \mu_t\rangle = \langle LF, \mu_t\rangle, \qquad \mu_t\big|_{t=0} = \mu_0, \tag{1.1}$$

for a wide class of functions F on Γ^2 (where $\langle\cdot,\cdot\rangle$ is the usual dual pairing between functions and measures on Γ^2). For the study of (1.1), we may consider the corresponding time evolution equations for correlation functionals (factorial moments) k_t corresponding to the measures μ_t. These are equations having a hierarchical structure similar to the well-known BBGKY-hierarchy for the Hamiltonian dynamics. However, in applications, frequently correlation functionals are not integrable, being a technical difficulty to proceed this study, even in a weak sense (corresponding to (1.1)). Having in mind the construction of a weak solution, we then analyze the (pre-)dual problem, that is, the so-called time evolution of quasi-observables. These are functions which naturally can be considered in proper spaces of integrable functions, allowing then to overtake the technical difficulties pointed out. Furthermore, the evolution equation for quasi-observables still has hierarchical structure.

For further developments and applications, in this work explicit formulas for the aforementioned hierarchical equations of general birth-and-death, hopping, and flipping multicomponent particle systems are derived. For the one-component case, a similar scheme has been proposed in [FKO09] and explicit forms for corresponding hierarchical equations have been presented therein. Within this setting, problems concerning one-component hierarchies and many applications were exposed, e.g., in [FKK11a, FKK09a, FKK11c, FKK10b, FKK10a, FKKZ09, KKM08, KKP08, KKZ06]. Naturally, due to the complexity mentioned above, one cannot infer from the one-component case corresponding results for multicomponent systems. Motivated by recent applications, in this work we slightly change the procedure used in [FKO09], which for one-component birth-and-death models is used in [FKK11c]. This change allows, in particular, to wide the class of rates. Sufficient conditions on the rates to give rise to linear operators on suitable Banach spaces and concrete examples of rates are analyzed as well.

2 Markov evolutions in multicomponent configuration spaces

2.1 One-component configuration spaces

The configuration space $\Gamma := \Gamma_{\mathbb{R}^d}$ over $\mathbb{R}^d$, $d \in \mathbb{N}$, is defined as the set of all locally finite subsets of $\mathbb{R}^d$ (that is, configurations),

$$\Gamma := \left\{\gamma \subset \mathbb{R}^d : |\gamma_\Lambda| < \infty, \text{ for every compact } \Lambda \subset \mathbb{R}^d\right\},$$

where $|\cdot|$ denotes the cardinality of a set and $\gamma_\Lambda := \gamma \cap \Lambda$. We identify each $\gamma \in \Gamma$ with the non-negative Radon measure $\sum_{x\in\gamma} \delta_x \in \mathcal{M}(\mathbb{R}^d)$, where δ_x is the Dirac measure with unit mass at x, $\sum_{x\in\emptyset} \delta_x$ is, by definition, the zero measure, and $\mathcal{M}(\mathbb{R}^d)$ denotes the space of all non-negative Radon measures on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. This identification allows to endow Γ with the topology induced by the vague topology on $\mathcal{M}(\mathbb{R}^d)$, that is, the weakest topology on Γ with respect to which all mappings $\Gamma \ni \gamma \mapsto \sum_{x\in\gamma} f(x)$, $f \in C_c(\mathbb{R}^d)$, are continuous. Here $C_c(\mathbb{R}^d)$ denotes the set of all continuous functions on $\mathbb{R}^d$ with compact support. We denote by $\mathcal{B}(\Gamma)$ the corresponding Borel σ-algebra on Γ.

Let us now consider the space of finite configurations

$$\Gamma_0 := \bigsqcup_{n=0}^{\infty} \Gamma^{(n)},$$

where $\Gamma^{(n)} := \{\gamma \in \Gamma : |\gamma| = n\}$ for $n \in \mathbb{N}$ and $\Gamma^{(0)} := \{\emptyset\}$. For $n \in \mathbb{N}$, there is a natural bijection between the space $\Gamma^{(n)}$ and the symmetrization $\widetilde{(\mathbb{R}^d)^n}/S_n$ of the set $\widetilde{(\mathbb{R}^d)^n} := \{(x_1, ..., x_n) \in (\mathbb{R}^d)^n : x_i \neq x_j \text{ if } i \neq j\}$ under the permutation group S_n over $\{1, ..., n\}$ acting on $\widetilde{(\mathbb{R}^d)^n}$ by permuting the coordinate indexes. This bijection induces a metrizable topology on $\Gamma^{(n)}$, and we endow Γ_0 with the metrizable topology of disjoint union of topological spaces. We denote the corresponding Borel σ-algebras on $\Gamma^{(n)}$ and Γ_0 by $\mathcal{B}(\Gamma^{(n)})$ and $\mathcal{B}(\Gamma_0)$, respectively.

We proceed to consider the K-transform [Len73, Len75a, Len75b, KK02]. Let $\mathcal{B}_c(\mathbb{R}^d)$ denote the set of all bounded Borel sets in $\mathbb{R}^d$, and for each $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ let $\Gamma_\Lambda := \{\eta \in \Gamma : \eta \subset \Lambda\}$. Evidently $\Gamma_\Lambda = \bigsqcup_{n=0}^{\infty} \Gamma_\Lambda^{(n)}$, where

$\Gamma_\Lambda^{(n)} := \Gamma_\Lambda \cap \Gamma^{(n)}$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, leading to a situation similar to the one for Γ_0, described above. We endow Γ_Λ with the topology of the disjoint union of topological spaces and with the corresponding Borel σ-algebra $\mathcal{B}(\Gamma_\Lambda)$. To define the K-transform, among the functions defined on Γ_0 we distinguish the bounded $\mathcal{B}(\Gamma_0)$-measurable functions G with bounded support, i.e., $G\restriction_{\Gamma_0 \setminus \left(\bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}\right)} \equiv 0$ for some $N \in \mathbb{N}_0$, $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. We denote the space of all such functions G by $B_{\mathrm{bs}}(\Gamma_0)$. Given a $G \in B_{\mathrm{bs}}(\Gamma_0)$, the K-transform of G is a mapping $KG : \Gamma \to \mathbb{R}$ defined at each $\gamma \in \Gamma$ by

$$(KG)(\gamma) := \sum_{\substack{\eta \subset \gamma \\ |\eta| < \infty}} G(\eta). \tag{2.1}$$

Note that for each function $G \in B_{\mathrm{bs}}(\Gamma_0)$ the sum in (2.1) has only a finite number of summands different from zero, and thus KG is a well-defined function on Γ. Moreover, if G has support described as before, then the restriction $(KG)\restriction_{\Gamma_\Lambda}$ is a $\mathcal{B}(\Gamma_\Lambda)$-measurable function and $(KG)(\gamma) = (KG)\restriction_{\Gamma_\Lambda} (\gamma_\Lambda)$ for all $\gamma \in \Gamma$. That is, KG is a cylinder function. In addition, for each constant $C \geq |G|$ one finds $|(KG)(\gamma)| \leq C(1 + |\gamma_\Lambda|)^N$ for all $\gamma \in \Gamma$. As a result, besides the cylindricity property, KG is also polynomially bounded.

It has been shown in [KK02] that $K : B_{\mathrm{bs}}(\Gamma_0) \to K(B_{\mathrm{bs}}(\Gamma_0))$ is a linear isomorphism whose inverse mapping is defined by

$$\left(K^{-1}F\right)(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \quad \eta \in \Gamma_0.$$

2.2 Multicomponent configuration spaces

The previous definitions naturally extend to any n-component configuration spaces. For simplicity of notation, we just present the extension for $n = 2$. A similar procedure is used for $n > 2$, but with a more cumbersome notation.

Given two copies of the space Γ, denoted by Γ^+ and Γ^-, let

$$\Gamma^2 := \left\{ (\gamma^+, \gamma^-) \in \Gamma^+ \times \Gamma^- : \gamma^+ \cap \gamma^- = \emptyset \right\}.$$

Concerning the elements in Γ^2, we observe they may be regarded as marked one-configurations for the space of marks $\{+, -\}$ (spins). Similarly, given two copies of the space Γ_0, Γ_0^+ and Γ_0^-, we consider the space

$$\Gamma_0^2 := \left\{ (\eta^+, \eta^-) \in \Gamma_0^+ \times \Gamma_0^- : \eta^+ \cap \eta^- = \emptyset \right\}.$$

We endow Γ^2 and Γ_0^2 with the topology induced by the product of the topological spaces $\Gamma^+ \times \Gamma^-$ and $\Gamma_0^+ \times \Gamma_0^-$, respectively, and with the corresponding Borel σ-algebras, denoted by $\mathcal{B}(\Gamma^2)$ and $\mathcal{B}(\Gamma_0^2)$. Thus, a bounded

$\mathcal{B}(\Gamma_0^2)$-measurable function $G : \Gamma_0^2 \to \mathbb{R}$ has bounded support ($G \in B_{\mathrm{bs}}(\Gamma_0^2)$, for short) whenever $G \restriction_{\Gamma_0^2 \setminus \left(\bigsqcup_{n=0}^{N^+} \Gamma_{\Lambda^+}^{(n)} \times \bigsqcup_{n=0}^{N^-} \Gamma_{\Lambda^-}^{(n)}\right)} \equiv 0$ for some $N^+, N^- \in \mathbb{N}_0$, $\Lambda^+, \Lambda^- \in \mathcal{B}_c(\mathbb{R}^d)$. In this way, given a function $G \in B_{\mathrm{bs}}(\Gamma_0^2)$, the mapping $\mathrm{K}G$ defined at each $\gamma = (\gamma^+, \gamma^-) \in \Gamma^2$ by

$$(\mathrm{K}G)(\gamma) := \sum_{\substack{\eta^+ \subset \gamma^+ \\ |\eta^+| < \infty}} \sum_{\substack{\eta^- \subset \gamma^- \\ |\eta^-| < \infty}} G(\eta^+, \eta^-) \tag{2.2}$$

is a well-defined function on Γ^2. For this verification, as well as for other forthcoming ones, let us observe that given the unit operator $I^\pm$ on functions on $\Gamma^\pm$ (and thus, on $\Gamma_0^\pm$) and the operators defined on functions on Γ_0^2 by $K^+ := K \otimes I^-$, $K^- := I^+ \otimes K$ one may write, equivalently to (2.2),

$$\mathrm{K} = K^+ K^- = K^- K^+. \tag{2.3}$$

We call the mapping $\mathrm{K}G : \Gamma^2 \to \mathbb{R}$ the K-transform of G.

Either directly from definition (2.2) or from (2.3), it is clear that given a $G \in B_{\mathrm{bs}}(\Gamma_0^2)$ described as before, the $\mathrm{K}G$ is a polynomially bounded cylinder function such that $(\mathrm{K}G)(\gamma^+, \gamma^-) = (\mathrm{K}G)(\gamma^+_{\Lambda^+}, \gamma^-_{\Lambda^-})$ for all $(\gamma^+, \gamma^-) \in \Gamma^2$ and, for each constant $C \geq |G|$,

$$|(\mathrm{K}G)(\gamma^+, \gamma^-)| \leq C(1 + |\gamma^+_{\Lambda^+}|)^{N^+} (1 + |\gamma^-_{\Lambda^-}|)^{N^-}, \quad (\gamma^+, \gamma^-) \in \Gamma^2.$$

Moreover, $\mathrm{K} : B_{\mathrm{bs}}(\Gamma_0^2) \to \mathcal{FP}(\Gamma^2) := \mathrm{K}(B_{\mathrm{bs}}(\Gamma_0^2))$ is a linear and positivity preserving isomorphism whose inverse mapping is defined by

$$\left(\mathrm{K}^{-1} F\right)(\eta^+, \eta^-) := \sum_{\xi^+ \subset \eta^+} \sum_{\xi^- \subset \eta^-} (-1)^{|\eta^+ \setminus \xi^+| + |\eta^- \setminus \xi^-|} F(\xi^+, \xi^-), \tag{2.4}$$

for all $(\eta^+, \eta^-) \in \Gamma_0^2$.

Remark 2.1. *Given any $\mathcal{B}(\Gamma^2)$-measurable function F, observe that the right-hand side of* (2.4) *is also well-defined for $F \restriction_{\Gamma_0^2}$. In this case, since there will be no risk of confusion, we will denote the right-hand side of* (2.4) *by $\mathrm{K}^{-1} F$.*

Let $\mathcal{M}^1_{\mathrm{fm}}(\Gamma^2)$ denote the set of all probability measures μ on $(\Gamma^2, \mathcal{B}(\Gamma^2))$ with finite local moments of all orders, i.e.,

$$\int_{\Gamma^2} d\mu(\gamma^+, \gamma^-) \, |\gamma^+_\Lambda|^n |\gamma^-_\Lambda|^n < \infty \quad \text{for all } n \in \mathbb{N} \text{ and all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d). \tag{2.5}$$

Given a $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma^2)$, the so-called correlation measure ρ_μ corresponding to μ is a measure on $(\Gamma^2_0, \mathcal{B}(\Gamma^2_0))$ defined for all $G \in B_{\mathrm{bs}}(\Gamma^2_0)$ by

$$\int_{\Gamma^2_0} d\rho_\mu(\eta^+, \eta^-)\, G(\eta^+, \eta^-) = \int_{\Gamma^2} d\mu(\gamma^+, \gamma^-)\ (\mathrm{K}G)\,(\gamma^+, \gamma^-). \tag{2.6}$$

Note that under these assumptions $\mathrm{K}\,|G|$ is μ-integrable, and thus, (2.6) is well-defined. In terms of correlation measures, this means that $B_{\mathrm{bs}}(\Gamma^2_0) \subset L^1(\Gamma^2_0, \rho_\mu)$. Actually, $B_{\mathrm{bs}}(\Gamma^2_0)$ is dense in $L^1(\Gamma^2_0, \rho_\mu)$. Moreover, still by (2.6), on $B_{\mathrm{bs}}(\Gamma^2_0)$ the inequality $\|\mathrm{K}G\|_{L^1(\Gamma^2,\mu)} \leq \|G\|_{L^1(\Gamma^2_0,\rho_\mu)}$ holds, allowing an extension of the K-transform to a bounded linear operator $\mathrm{K} : L^1(\Gamma^2_0, \rho_\mu) \to L^1(\Gamma^2, \mu)$ in such a way that equality (2.6) still holds for any $G \in L^1(\Gamma^2_0, \rho_\mu)$. For the extended operator the explicit form (2.1) still holds, now μ-a.e.

Just to conclude this part, let us observe that in terms of correlation measures property (2.5) means that ρ_μ is locally finite, that is, $\rho_\mu((\Gamma^{(n)}_\Lambda \times \Gamma^{(m)}_\Lambda) \cap \Gamma^2_0) < \infty$ for all $n, m \in \mathbb{N}_0$ and all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$.

Poisson and Lebesgue-Poisson measures. Given a constant $z > 0$, let λ_z be the Lebesgue–Poisson measure on $(\Gamma_0, \mathcal{B}(\Gamma_0))$,

$$\lambda_z := \sum_{n=0}^{\infty} \frac{z^n}{n!} m^{(n)}, \tag{2.7}$$

where each $m^{(n)}$, $n \in \mathbb{N}$, is the image measure on $\Gamma^{(n)}$ of the product measure $dx_1...dx_n$ under the mapping $\widetilde{(\mathbb{R}^d)^n} \ni (x_1, ..., x_n) \mapsto \{x_1, ..., x_n\} \in \Gamma^{(n)}$. For $n = 0$ one sets $m^{(0)}(\{\emptyset\}) := 1$. The product measure $\lambda^2_z := \lambda_z \otimes \lambda_z$ on $(\Gamma^2_0, \mathcal{B}(\Gamma^2_0))$ is the correlation measure corresponding to the product measure $\pi_z \otimes \pi_z$ of the Poisson measure π_z on $(\Gamma, \mathcal{B}(\Gamma))$ with intensity $z dx$, that is, the probability measure defined on $(\Gamma, \mathcal{B}(\Gamma))$ by

$$\int_\Gamma d\pi_z(\gamma)\, \exp\left(\sum_{x \in \gamma} \varphi(x) \right) = \exp\left(z \int_{\mathbb{R}^d} dx\ \left(e^{\varphi(x)} - 1 \right) \right)$$

for all smooth functions φ on $\mathbb{R}^d$ with compact support.

If a correlation measure ρ_μ is absolutely continuous with respect to the Lebesgue–Poisson measure $\lambda^2 := \lambda^2_1$, the Radon–Nikodym derivative $k_\mu := \dfrac{d\rho_\mu}{d\lambda^2}$ is called the correlation functional corresponding to μ. Sufficient conditions for the existence of correlation functionals may be found e.g. in [Fin09].

Technically, the next statement will be useful. It is an extension to the multicomponent case of an integration result over Γ_0 (see e.g. [FF91, KMZ04, Rue69]).

Lemma 2.2. *The following equality holds*

$$\int_{\Gamma_0^2} d\lambda^2(\eta^+,\eta^-) \sum_{\substack{\xi^+\subset\eta^+\\ \xi^-\subset\eta^-}} H(\eta^+,\eta^-,\xi^+,\xi^-) \tag{2.8}$$
$$= \int_{\Gamma_0^2} d\lambda^2(\eta^+,\eta^-) \int_{\Gamma_0^2} d\lambda^2(\xi^+,\xi^-)\, H(\eta^+\cup\xi^+,\eta^-\cup\xi^-,\xi^+,\xi^-)$$

for all measurable functions $H : \Gamma_0^2 \times \Gamma_0^2 \to \mathbb{R}$ *with respect to which at least one side of equality* (2.8) *is finite for* $|H|$.

Algebraic properties. The extension to functions defined on Γ_0^2 of the $\star$-convolution introduced in [KK02] for functions defined on Γ_0 has very similar properties. Given G_1 and G_2 two $\mathcal{B}(\Gamma_0^2)$-measurable functions we define the $\circledast$-convolution between G_1 and G_2 by

$$\begin{aligned}
&(G_1 \circledast G_2)(\eta^+,\eta^-)\\
&:= \sum_{\substack{(\eta_1^+,\eta_2^+,\eta_3^+)\in\mathcal{P}_3(\eta^+)\\ (\eta_1^-,\eta_2^-,\eta_3^-)\in\mathcal{P}_3(\eta^-)}} G_1(\eta_1^+\cup\eta_2^+,\eta_1^-\cup\eta_2^-)G_2(\eta_2^+\cup\eta_3^+,\eta_2^-\cup\eta_3^-)\\
&= \sum_{\substack{\xi^+\subset\eta^+\\ \xi^-\subset\eta^-}} G_1(\xi^+,\xi^-) \sum_{\substack{\zeta^+\subset\xi^+\\ \zeta^-\subset\xi^-}} G_2((\eta^+\setminus\xi^+)\cup\zeta^+,(\eta^-\setminus\xi^-)\cup\zeta^-),
\end{aligned} \tag{2.9}$$

where $\mathcal{P}_3(\eta^\pm)$ denotes the set of all partitions of $\eta^\pm$ in three parts which may be empty. It is straightforward to verify that the space of all $\mathcal{B}(\Gamma_0^2)$-measurable functions endowed with this product has the structure of a commutative algebra with unit element $0^{|\eta^+|}0^{|\eta^-|}$. Furthermore, for each $G_1, G_2 \in B_{\mathrm{bs}}(\Gamma_0^2)$ we have $G_1 \circledast G_2 \in B_{\mathrm{bs}}(\Gamma_0^2)$, and

$$\mathrm{K}\,(G_1 \circledast G_2) = (\mathrm{K}G_1)\cdot(\mathrm{K}G_2)\,.$$

From definition (2.9) it follows that for any $\mathcal{B}(\Gamma^2)$-measurable functions F_1, F_2 such that $F_1\restriction_{\Gamma_0^2}, F_2\restriction_{\Gamma_0^2}$ are $\mathcal{B}(\Gamma_0^2)$-measurable we have (cf. Remark 2.1)

$$(\mathrm{K}^{-1}F_1)\circledast(\mathrm{K}^{-1}F_2) = \mathrm{K}^{-1}(F_1F_2). \tag{2.10}$$

2.3 Markov generators and related evolution equations

Heuristically, the stochastic evolution of an infinite two-component particle system is described by a Markov process on Γ^2, which is determined by a Markov generator L defined on a proper space of functions on Γ^2. If such a Markov process exists, then it provides a solution to the (backward) Kolmogorov equation

$$\frac{d}{dt}F_t = LF_t, \qquad F_t\big|_{t=0} = F_0.$$

However, the construction of a generic Markov process, either on Γ^2 or Γ, is essentially an open problem (for some particular cases on Γ see e.g. [GK06, GK08]).

In spite of this technical difficulty, in applications it turns out that we need a knowledge on certain characteristics of the stochastic evolution in terms of mean values rather than pointwise. These characteristics concern e.g. observables, that is, functions defined on Γ^2, which expected values are given by

$$\langle F, \mu\rangle := \int_{\Gamma^2} d\mu(\gamma^+,\gamma^-)F(\gamma^+,\gamma^-),$$

being μ a probability measure on Γ^2, that is, a state of the system. This leads to the following time evolution problem on states,

$$\frac{d}{dt}\langle F, \mu_t\rangle = \langle LF, \mu_t\rangle, \qquad \mu_t\big|_{t=0} = \mu_0. \tag{2.11}$$

For F being of the type $F = \mathrm{K}G$, $G \in B_{\mathrm{bs}}(\Gamma_0^2)$, (2.11) may be rewritten in terms of the correlation functionals $k_t = k_{\mu_t}$ corresponding to the measures μ_t, provided these functionals exist (or, more generally, in terms of correlation measures $\rho_t = \rho_{\mu_t}$), yielding

$$\frac{d}{dt}\langle\!\langle G, k_t\rangle\!\rangle = \langle\!\langle \hat{L}G, k_t\rangle\!\rangle, \qquad k_t\big|_{t=0} = k_0, \tag{2.12}$$

where $\hat{L} := \mathrm{K}^{-1}L\mathrm{K}$ (cf. Remark 2.1) and $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ is the usual pairing

$$\langle\!\langle G, k\rangle\!\rangle := \int_{\Gamma_0^2} d\lambda^2(\eta^+,\eta^-)\, G(\eta^+,\eta^-)\, k(\eta^+,\eta^-). \tag{2.13}$$

Of course, a strong version of equation (2.12) is

$$\frac{d}{dt}k_t = \hat{L}^* k_t, \qquad k_t\big|_{t=0} = k_0, \tag{2.14}$$

for $\hat{L}^*$ being the dual operator of $\hat{L}$ in the sense defined in (2.13). One may associate to any function k on Γ_0^2 a double sequence $\left\{k^{(n,m)}\right\}_{n,m\in\mathbb{N}_0}$, where $k^{(n,m)} := k\restriction_{\{(\eta^+,\eta^-)\in\Gamma_0^2:|\eta^+|=n,|\eta^-|=m\}}$ is a symmetric function on $(\mathbb{R}^d)^n \times (\mathbb{R}^d)^m$.

This means that related to (2.14) one has a countable infinite number of equations having an hierarchical structure,

$$\frac{d}{dt}k_t^{(n,m)} = (\hat{L}^* k_t)^{(n,m)}, \qquad k_t^{(n,m)}\big|_{t=0} = k_0^{(n,m)} \quad n,m \in \mathbb{N}_0, \tag{2.15}$$

where each equation only depends on a finite number of coordinates. As a result, we have reduced the infinite-dimensional problem (2.11) to the infinite system of equations (2.15). However, it is convenient to recall here that, due to (2.12), we are only interesting in weak solutions to (2.15).

Evolutions (2.12), (2.14) are obviously connected with an initial value problem on quasi-observables, that is, functions defined on Γ_0^2, namely,

$$\frac{d}{dt}G_t = \hat{L}G_t, \qquad G_t\big|_{t=0} = G_0. \tag{2.16}$$

As explained before, one may also associate to (2.16) a double sequence, and thus, a countable infinite number of equations having also an hierarchical structure. In concrete cases, sometimes equation (2.16) appears easier to be analyzed in a suitable space. Having a solution to (2.16), by duality (2.13), one might find a solution to (2.12). For instance, for birth-and-death systems on Γ, this scheme has been accomplished in [FKK11c] through the derivation of semigroup evolutions for quasi-observables and correlation functions. Those results can be naturally extended to the multicomponent case. However, on each concrete application of other multicomponent models, namely, the conservative models considered below, the explicit form of the rates determines specific assumptions, and thus a specific analysis, which only hold for that concrete application.

According to the considerations above, there is a close connection between the Markov evolution (2.11) and the hierarchical equations (2.14) and (2.16). Of course, to derive solutions to (2.11) from solutions to (2.12) an additional analysis is needed, namely, to distinguish the correlation functionals from the set of solutions to (2.12).

In what follows we derive explicit formulas for $\hat{L}, \hat{L}^*$ of general birth-and-death, hopping and flipping particle systems. For each case, explicit expressions are first derived on the space $B_{\mathrm{bs}}(\Gamma_0^2)$, and then extended to linear operators on suitable Banach spaces.

3 Birth-and-death dynamics

3.1 Hierarchical equations

In a birth-and-death dynamics of a stochastic spatial type model, at each random moment of time, particles randomly appear or disappear according to birth and death rates which depend on the configuration of the whole system at that time. As each particle is of one of the two possible types, $+$ and $-$, generators for such systems are informally described as the sum of birth-and-death generators L_+ and L_- of the $+$-system and the $-$-system of particles involved. That is,

$$L = L_+ + L_-, \tag{3.1}$$

where

$$\begin{aligned}(L_+F)(\gamma^+,\gamma^-) := &\sum_{x\in\gamma^+} d^+(x,\gamma^+\setminus x,\gamma^-)\left(F(\gamma^+\setminus x,\gamma^-)-F(\gamma^+,\gamma^-)\right) \\ &+\int_{\mathbb{R}^d} dx\, b^+(x,\gamma^+,\gamma^-)\left(F(\gamma^+\cup x,\gamma^-)-F(\gamma^+,\gamma^-)\right)\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}(L_-F)(\gamma^+,\gamma^-) := &\sum_{y\in\gamma^-} d^-(y,\gamma^+,\gamma^-\setminus y)\left(F(\gamma^+,\gamma^-\setminus y)-F(\gamma^+,\gamma^-)\right) \\ &+\int_{\mathbb{R}^d} dy\, b^-(y,\gamma^+,\gamma^-)\left(F(\gamma^+,\gamma^-\cup y)-F(\gamma^+,\gamma^-)\right).\end{aligned} \tag{3.3}$$

We observe that in (3.2) the coefficient $d^+(x,\gamma^+,\gamma^-)\geq 0$ indicates the rate at which a $+$ particle located at $x\in\gamma^+$ dies or disappears, while $b^+(x,\gamma^+,\gamma^-)\geq 0$ indicates the rate at which, given a configuration (γ^+,γ^-), a new $+$ particle is born or appears at a site x. A similar interpretation holds for the rates d^- and b^- appearing in (3.3).

In order to give a meaning to (3.2), (3.3), in what follows we assume that $d^\pm, b^\pm\geq 0$ are measurable functions such that, for a.a. $x\in\mathbb{R}^d$, $d^\pm(x,\cdot,\cdot), b^\pm(x,\cdot,\cdot)$ are $\mathcal{B}(\Gamma_0^2)$-measurable functions and, for $(\eta^+,\eta^-)\in\Gamma_0^2$, $d^\pm(\cdot,\eta^+,\eta^-)$, $b^\pm(\cdot,\eta^+,\eta^-)\in L^1_{\mathrm{loc}}(\mathbb{R}^d,dx)$. These conditions are sufficient to ensure that for any $F\in\mathcal{FP}(\Gamma^2)=\mathrm{K}(B_{\mathrm{bs}}(\Gamma_0^2))$ the expression for LF, defined above, is well-defined at least on Γ_0^2, which allows to define $\mathrm{K}^{-1}L\mathrm{K}G$ (Remark 2.1). This means, in particular, that for functions $G\in B_{\mathrm{bs}}(\Gamma_0^2)$,

$$(\hat{L}G)(\eta^+,\eta^-) = (\mathrm{K}^{-1}L\mathrm{K}G)(\eta^+,\eta^-)$$

is well-defined on Γ_0^2. In addition, the previous conditions allow to introduce the functions

$$D^\pm(x,\xi^+,\xi^-,\eta^+,\eta^-) := \left(\mathrm{K}^{-1}d^\pm(x,\cdot\cup\xi^+,\cdot\cup\xi^-)\right)(\eta^+,\eta^-), \tag{3.4}$$
$$B^\pm(x,\xi^+,\xi^-,\eta^+,\eta^-) := \left(\mathrm{K}^{-1}b^\pm(x,\cdot\cup\xi^+,\cdot\cup\xi^-)\right)(\eta^+,\eta^-), \tag{3.5}$$

for a.a. $x\in\mathbb{R}^d$, $(\eta^+,\eta^-),(\xi^+,\xi^-)\in\Gamma_0^2$ such that $\eta^\pm\cap\xi^\pm=\emptyset$. We set

$$D_x^\pm(\eta^+,\eta^-):=D^\pm(x,\emptyset,\emptyset,\eta^+,\eta^-),$$
$$B_x^\pm(\eta^+,\eta^-):=B^\pm(x,\emptyset,\emptyset,\eta^+,\eta^-).$$

Proposition 3.1. *The action of $\hat{L}$ on functions $G\in B_{\mathrm{bs}}(\Gamma_0^2)$ is given for any $(\eta^+,\eta^-)\in\Gamma_0^2$ by*

$$\begin{aligned}
&(\hat{L}G)(\eta^+,\eta^-) \qquad (3.6)\\
&=-\sum_{\substack{\xi^+\subset\eta^+\\ \xi^-\subset\eta^-}} G(\xi^+,\xi^-)\sum_{x\in\xi^+} D^+\big(x,\xi^+\setminus x,\xi^-,\eta^+\setminus\xi^+,\eta^-\setminus\xi^-\big)\\
&\quad+\sum_{\substack{\xi^+\subset\eta^+\\ \xi^-\subset\eta^-}}\int_{\mathbb{R}^d} dx\, G(\xi^+\cup x,\xi^-)B^+\big(x,\xi^+,\xi^-,\eta^+\setminus\xi^+,\eta^-\setminus\xi^-\big)\\
&\quad-\sum_{\substack{\xi^+\subset\eta^+\\ \xi^-\subset\eta^-}} G(\xi^+,\xi^-)\sum_{y\in\xi^-} D^-\big(y,\xi^+,\xi^-\setminus y,\eta^+\setminus\xi^+,\eta^-\setminus\xi^-\big)\\
&\quad+\sum_{\substack{\xi^+\subset\eta^+\\ \xi^-\subset\eta^-}}\int_{\mathbb{R}^d} dy\, G(\xi^+,\xi^-\cup y)B^-\big(y,\xi^+,\xi^-,\eta^+\setminus\xi^+,\eta^-\setminus\xi^-\big).
\end{aligned}$$

Proof. We begin by observing that the integrability property of $b^\pm,d^\pm$ implies that $B^\pm,D^\pm$ are locally integrable on $\mathbb{R}^d$, and thus, for $G\in B_{\mathrm{bs}}(\Gamma_0^2)$, both integrals appearing in (3.6) are finite.

Since L is of the form (3.1), the proof of this result reduces to show the statement for L_+ and L_-. For this purpose, first we observe that from definition (2.2) of the K-transform, for any $(\gamma^+,\gamma^-)\in\Gamma_0^2$ we have

$$(\mathrm{K}G)(\gamma^+\setminus x,\gamma^-)-(\mathrm{K}G)(\gamma^+,\gamma^-)=-\sum_{\eta^+\subset\gamma^+\setminus x}\sum_{\eta^-\subset\gamma^-}G(\eta^+\cup x,\eta^-),$$
$$(\mathrm{K}G)(\gamma^+\cup x,\gamma^-)-(\mathrm{K}G)(\gamma^+,\gamma^-)=\sum_{\eta^+\subset\gamma^+}\sum_{\eta^-\subset\gamma^-}G(\eta^+\cup x,\eta^-),\ x\notin\gamma^+.$$

We observe, in addition, that given a function H of the form

$$H(\gamma^+,\gamma^-):=\sum_{x\in\gamma^+}h(x,\gamma^+\setminus x,\gamma^-),$$

for some suitable $h:\mathbb{R}^d\times\Gamma^2\to\mathbb{R}$, it follows from definition (2.4) of K^{-1} that

$$(\mathrm{K}^{-1}H)(\eta^+,\eta^-)=\sum_{x\in\eta^+}(\mathrm{K}^{-1}h)(x,\eta^+\setminus x,\eta^-). \qquad (3.7)$$

As a result, using definitions (3.4), (3.5) of B^+, D^+ and the algebraic property (2.10) of the $\circledstar$-convolution, we obtain the following expression for $\hat{L}_+G := \mathrm{K}^{-1}L_+\mathrm{K}G$, $G \in B_{\mathrm{bs}}(\Gamma_0^2)$,

$$(\hat{L}_+G)(\eta^+,\eta^-) = -\sum_{x\in\eta^+}\left(D_x^+ \circledstar G(\cdot\cup x,\cdot)\right)(\eta^+\setminus x,\eta^-) + \int_{\mathbb{R}^d} dx\,\left(B_x^+ \circledstar G(\cdot\cup x,\cdot)\right)(\eta^+,\eta^-),$$

which, by definition (2.9) of the $\circledstar$-convolution, is equivalent to

$$\begin{aligned}&(\hat{L}_+G)(\eta^+,\eta^-)\\ &= -\sum_{x\in\eta^+}\sum_{\substack{\xi^+\subset\eta^+\setminus x\\ \xi^-\subset\eta^-}} G(\xi^+\cup x,\xi^-)\sum_{\substack{\zeta^+\subset\xi^+\\ \zeta^-\subset\xi^-}} D_x^+(((\eta^+\setminus x)\setminus\xi^+)\cup\zeta^+,(\eta^-\setminus\xi^-)\cup\zeta^-)\\ &\quad+\int_{\mathbb{R}^d} dx\sum_{\substack{\xi^+\subset\eta^+\\ \xi^-\subset\eta^-}} G(\xi^+\cup x,\xi^-)\sum_{\substack{\zeta^+\subset\xi^+\\ \zeta^-\subset\xi^-}} B_x^+((\eta^+\setminus\xi^+)\cup\zeta^+,(\eta^-\setminus\xi^-)\cup\zeta^-).\end{aligned}$$

Given a $\mathcal{B}(\Gamma_0^2)$-measurable function G' and (η_1^+,η_1^-), $(\eta_2^+,\eta_2^-)\in\Gamma_0^2$, from the equality

$$(\mathrm{K}G')(\eta_1^+\cup\eta_2^+,\eta_1^-\cup\eta_2^-) = \sum_{\xi_1^+\subset\eta_1^+}\sum_{\xi_2^+\subset\eta_2^+}\sum_{\xi_1^-\subset\eta_1^-}\sum_{\xi_2^-\subset\eta_2^-} G'(\xi_1^+\cup\xi_2^+,\xi_1^-\cup\xi_2^-)$$

it follows that, for $F'(\eta^+,\eta^-) := (\mathrm{K}G')(\eta^+,\eta^-)$, we have

$$\left(\mathrm{K}^{-1}F'(\cdot\cup\xi^+,\cdot\cup\xi^-)\right)(\eta^+,\eta^-) = \left(\mathrm{K}G'(\eta^+\cup\cdot,\eta^-\cup\cdot)\right)(\xi^+,\xi^-).$$

This applies, in particular, to $G' = D_x^+$, $F' = d^+(x,\cdot,\cdot)$ as well as to $G' = B_x^+$, $F' = b^+(x,\cdot,\cdot)$, yielding

$$\begin{aligned}&(\hat{L}_+G)(\eta^+,\eta^-)\\ &= -\sum_{x\in\eta^+}\sum_{\substack{\xi^+\subset\eta^+\setminus x\\ \xi^-\subset\eta^-}} G(\xi^+\cup x,\xi^-)\left(\mathrm{K}^{-1}d^+(x,\cdot\cup\xi^+,\cdot\cup\xi^-)\right)((\eta^+\setminus x)\setminus\xi^+,\eta^-\setminus\xi^-)\\ &\quad+\int_{\mathbb{R}^d} dx\sum_{\substack{\xi^+\subset\eta^+\\ \xi^-\subset\eta^-}} G(\xi^+\cup x,\xi^-)\left(\mathrm{K}^{-1}b^+(x,\cdot\cup\xi^+,\cdot\cup\xi^-)\right)(\eta^+\setminus\xi^+,\eta^-\setminus\xi^-).\end{aligned}$$

The required expression for $\hat{L}_+$ then follows by interchanging the two sums appearing in the first summand and using (3.4), (3.5). Similar arguments applied to L_- complete the proof. □

As we have mentioned in Subsection 2.3, $\hat{L}^*$ is defined on any $\mathcal{B}(\Gamma_0^2)$-measurable function k with respect to which the following equality holds

$$\int_{\Gamma_0^2} d\lambda^2\, \hat{L}G\, k = \int_{\Gamma_0^2} d\lambda^2\, G\, \hat{L}^* k$$

for all $G \in B_{\mathrm{bs}}(\Gamma_0^2)$. In the next subsection we will give a meaning to $\hat{L}^*$ as an operator defined on a proper space of functions on Γ_0^2. Before that, we derive an explicit expression for $\hat{L}^*k$, $k \in B_{\mathrm{bs}}(\Gamma_0^2)$.

Proposition 3.2. *Assume that for all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and all $n, m \in \mathbb{N}_0$,*

$$A^+_{\Lambda,m,n} := \int_{\Gamma_\Lambda^{(n,m)}} d\lambda^2(\eta^+,\eta^-) \sum_{\substack{\xi^+\subset\eta^+\\ \xi^-\subset\eta^-}} \left(\sum_{x\in\xi^+} \left| D^+(x, \xi^+\setminus x, \xi^-, \eta^+\setminus\xi^+, \eta^-\setminus\xi^-)\right| + \int_\Lambda dx \left| B^+(x,\xi^+,\xi^-,\eta^+\setminus\xi^+,\eta^-\setminus\xi^-)\right| \right) < \infty$$

and

$$A^-_{\Lambda,m,n} := \int_{\Gamma_\Lambda^{(n,m)}} d\lambda^2(\eta^+,\eta^-) \sum_{\substack{\xi^+\subset\eta^+\\ \xi^-\subset\eta^-}} \left(\sum_{y\in\xi^-} \left| D^-(y, \xi^+, \xi^-\setminus y, \eta^+\setminus\xi^+, \eta^-\setminus\xi^-)\right| + \int_\Lambda dy \left| B^-(y,\xi^+,\xi^-,\eta^+\setminus\xi^+,\eta^-\setminus\xi^-)\right| \right) < \infty,$$

where $\Gamma_\Lambda^{(n,m)} := \left(\Gamma_\Lambda^{(n)} \times \Gamma_\Lambda^{(m)}\right) \cap \Gamma_0^2$. Then, for each $k \in B_{\mathrm{bs}}(\Gamma_0^2)$,

$$\begin{aligned}
&(\hat{L}^*k)(\eta^+,\eta^-) \\
&= -\sum_{x\in\eta^+} \int_{\Gamma_0^2} d\lambda^2(\xi^+,\xi^-) k(\eta^+\cup\xi^+, \eta^-\cup\xi^-) D^+\big(x, \eta^+\setminus x, \eta^-, \xi^+, \xi^-\big) \\
&\quad + \sum_{x\in\eta^+} \int_{\Gamma_0^2} d\lambda^2(\xi^+,\xi^-) k((\eta^+\setminus x)\cup\xi^+, \eta^-\cup\xi^-) B^+\big(x, \eta^+\setminus x, \eta^-, \xi^+, \xi^-\big) \\
&\quad - \sum_{y\in\eta^-} \int_{\Gamma_0^2} d\lambda^2(\xi^+,\xi^-) k(\eta^+\cup\xi^+, \eta^-\cup\xi^-) D^-\big(y, \eta^+, \eta^-\setminus y, \xi^+, \xi^-\big) \\
&\quad + \sum_{y\in\eta^-} \int_{\Gamma_0^2} d\lambda^2(\xi^+,\xi^-) k(\eta^+\cup\xi^+, (\eta^-\setminus y)\cup\xi^-) B^-\big(y, \eta^+, \eta^-\setminus y, \xi^+, \xi^-\big),
\end{aligned} \tag{3.8}$$

for λ^2-a.a. $(\eta^+,\eta^-) \in \Gamma_0^2$.

Proof. By the definition of the space $B_{\mathrm{bs}}(\Gamma_0^2)$, given $G, k \in B_{\mathrm{bs}}(\Gamma_0^2)$ there are $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $N \in \mathbb{N}$, $C > 0$ such that

$$|G|, |k| \leq C \mathbb{1}_{\left(\bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)} \times \bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}\right) \cap \Gamma_0^2},$$

where $\mathbb{1}_\cdot$ denotes the indicator function of a set. Therefore,

$$\begin{aligned}
&\int_{\Gamma_0^2} d\lambda^2(\eta^+, \eta^-) \sum_{\substack{\xi^+ \subset \eta^+ \\ \xi^- \subset \eta^-}} \Bigg(\left|G(\xi^+, \xi^-)\right| \sum_{x \in \xi^+} \left|D^+\big(x, \xi^+ \setminus x, \xi^-, \eta^+ \setminus \xi^+, \eta^- \setminus \xi^-\big)\right| \\
&\qquad + \int_{\mathbb{R}^d} dx \left|G(\xi^+ \cup x, \xi^-)\right| \left|B^+\big(x, \xi^+, \xi^-, \eta^+ \setminus \xi^+, \eta^- \setminus \xi^-\big)\right| \Bigg) \left|k(\eta^+, \eta^-)\right| \\
&\leq C^2 \sum_{m,n=0}^{N} A^+_{\Lambda,m,n} < \infty.
\end{aligned}$$

This shows that the product $(\hat{L}_+ G)k$ is integrable over Γ_0^2 with respect to the measure λ^2. Moreover, using the expression for $\hat{L}_+ G$ (derive in Proposition 3.1 and its proof) and Lemma 2.2 we obtain

$$\begin{aligned}
&\int_{\Gamma_0^2} d\lambda^2(\eta^+, \eta^-)(\hat{L}_+ G)(\eta^+, \eta^-)\, k(\eta^+, \eta^-) \\
&= -\int_{\Gamma_0^2} d\lambda^2(\eta^+, \eta^-) \int_{\Gamma_0^2} d\lambda^2(\xi^+, \xi^-) k(\eta^+ \cup \xi^+, \eta^- \cup \xi^-) \\
&\qquad \times G(\xi^+, \xi^-) \sum_{x \in \xi^+} D^+\big(x, \xi^+ \setminus x, \xi^-, \eta^+, \eta^-\big) \\
&\quad + \int_{\Gamma_0^2} d\lambda^2(\eta^+, \eta^-) \int_{\Gamma_0^2} d\lambda^2(\xi^+, \xi^-) k(\eta^+ \cup \xi^+, \eta^- \cup \xi^-) \\
&\qquad \times \int_{\mathbb{R}^d} dx\, G(\xi^+ \cup x, \xi^-) B^+\big(x, \xi^+, \xi^-, \eta^+, \eta^-\big),
\end{aligned}$$

where a second application of Lemma 2.2 to the latter summand leads to the expression for $\hat{L}_+^*$. Similar considerations yield an expression for $\hat{L}_-^*$. □

3.2 Definition of operators

For each $C > 0$, let us consider the Banach space

$$\mathscr{L}_C := L^1(\Gamma_0^2, \lambda_C^2) \tag{3.9}$$

with the usual norm

$$\|G\|_{\mathscr{L}_C} := \int_{\Gamma_0^2} d\lambda^2(\eta^+,\eta^-)\,|G(\eta^+,\eta^-)|\,C^{|\eta^+|+|\eta^-|}.$$

Assume that there is a function $N : \Gamma_0^2 \to \mathbb{R}$ such that

$$\int_{\Gamma_\Lambda^{(n,m)}} d\lambda^2(\eta^+,\eta^-)\,N(\eta^+,\eta^-) < \infty \quad \text{for all } n,m \in \mathbb{N} \text{ and all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \tag{3.10}$$

and, for λ^2-a.a. $(\eta^+,\eta^-) \in \Gamma_0^2$,

$$\begin{aligned}
&\sum_{x\in\eta^+} \left\| D^+\big(x,\eta^+\setminus x,\eta^-,\cdot,\cdot\big)\right\|_{\mathscr{L}_C} + \frac{1}{C}\sum_{x\in\eta^+} \left\| B^+\big(x,\eta^+\setminus x,\eta^-,\cdot,\cdot\big)\right\|_{\mathscr{L}_C} \\
&\quad + \sum_{y\in\eta^-} \left\| D^-\big(y,\eta^+,\eta^-\setminus y,\cdot,\cdot\big)\right\|_{\mathscr{L}_C} + \frac{1}{C}\sum_{y\in\eta^-} \left\| B^-\big(y,\eta^+,\eta^-\setminus y,\cdot,\cdot\big)\right\|_{\mathscr{L}_C} \\
&\le N(\eta^+,\eta^-) < \infty.
\end{aligned} \tag{3.11}$$

This allows to define the set

$$\mathcal{D} := \mathcal{D}_{N,C} := \big\{ G \in \mathscr{L}_C \;\big|\; NG \in \mathscr{L}_C \big\}.$$

It is clear that $B_{\mathrm{bs}}(\Gamma_0^2) \subset \mathcal{D}$, which implies that also $\mathcal{D}$ is dense in $\mathscr{L}_C$.

Proposition 3.3. *Assume that integrability conditions* (3.10), (3.11) *hold. Then, equality* (3.6) *provides a densely defined linear operator $\hat{L}$ in $\mathscr{L}_C$ with domain $\mathcal{D}$. In particular, for any $G \in \mathcal{D}$, the right-hand side of* (3.6) *is λ^2-a.e. well-defined on Γ_0^2.*

Proof. Given a $G \in \mathcal{D}$, an application of Lemma 2.2 to the expression corresponding to $\hat{L}_+$ (derived in Proposition 3.1 and its proof) yields

$$\begin{aligned}
&\big\|\hat{L}_+ G\big\|_{\mathscr{L}_C} \\
&\le \int_{\Gamma_0^2} d\lambda^2(\eta^+,\eta^-)\,C^{|\eta^+|+|\eta^-|} \int_{\Gamma_0^2} d\lambda^2(\xi^+,\xi^-)\,C^{|\xi^+|+|\xi^-|}\big|G(\xi^+,\xi^-)\big| \\
&\qquad\qquad \times \sum_{x\in\xi^+} \big|D^+\big(x,\xi^+\setminus x,\xi^-,\eta^+,\eta^-\big)\big| \\
&\quad + \int_{\Gamma_0^2} d\lambda^2(\eta^+,\eta^-)\,C^{|\eta^+|+|\eta^-|} \int_{\Gamma_0^2} d\lambda^2(\xi^+,\xi^-)\,C^{|\xi^+|+|\xi^-|} \\
&\qquad\qquad \times \int_{\mathbb{R}^d} dx\,\big|G(\xi^+\cup x,\xi^-)\big|\big|B^+\big(x,\xi^+,\xi^-,\eta^+,\eta^-\big)\big|,
\end{aligned}$$

and a similar estimate holds for $\|\hat{L}_-G\|_{\mathscr{L}_C}$. As a result,

$$\big\|\hat{L}G\big\|_{\mathscr{L}_C} \le \big\|NG\big\|_{\mathscr{L}_C} < \infty. \qquad \square$$

Let us consider the dual space $(\mathscr{L}_C)'$, which can be realized by the Banach space

$$\mathscr{K}_C := \Big\{k : \Gamma_0^2 \to \mathbb{R} \;\Big|\; k \cdot C^{-|\cdot^+|-|\cdot^-|} \in L^\infty(\Gamma_0^2, \lambda^2)\Big\}$$

with the norm

$$\|k\|_{\mathscr{K}_C} := \|C^{-|\cdot^+|-|\cdot^-|}k\|_{L^\infty(\Gamma_0^2,\lambda^2)}.$$

The duality between the Banach spaces $\mathscr{L}_C$ and $\mathscr{K}_C$ is given by (2.13) with $|\langle\!\langle G, k\rangle\!\rangle| \le \|G\|_{\mathscr{L}_C} \cdot \|k\|_{\mathscr{K}_C}$. We observe that if $k \in \mathscr{K}_C$, then

$$|k(\eta^+, \eta^-)| \le \|k\|_{\mathscr{K}_C}\, C^{|\eta^+|+|\eta^-|} \tag{3.12}$$

for λ^2-a.a. $(\eta^+, \eta^-) \in \Gamma_0^2$.

Proposition 3.4. *Assume that integrability conditions* (3.10), (3.11) *hold. In addition, assume that there are constants $A > 0$, $M \in \mathbb{N}$, $\nu \ge 1$ such that*

$$N(\eta^+, \eta^-) \le A\big(1 + |\eta^+| + |\eta^-|\big)^M \nu^{|\eta^+|+|\eta^-|}. \tag{3.13}$$

Then, equality (3.8) *provides a linear operator $\hat{L}^*$ in $\mathscr{K}_C$ with domain $\mathscr{K}_{\alpha C}$, $\alpha \in \big(0, \frac{1}{\nu}\big)$. In particular, given a $k \in \mathscr{K}_{\alpha C}$ for some $\alpha \in \big(0, \frac{1}{\nu}\big)$, the right-hand side of* (3.8) *is λ^2-a.e. well-defined on Γ_0^2.*

Proof. For some $\alpha \in \big(0, \frac{1}{\nu}\big)$, let $k \in \mathscr{K}_{\alpha C}$. Then, using the expression corresponding to $\hat{L}^*_+$, defined in Proposition 3.2 and its proof, for λ^2-a.a. $(\eta^+, \eta^-) \in \Gamma_0^2$ we obtain

$$\begin{aligned}
&C^{-|\eta^+|-|\eta^-|}\big|(\hat{L}^*_+k)(\eta^+, \eta^-)\big| \\
&\le \|k\|_{\mathscr{K}_{\alpha C}}\alpha^{|\eta^+|+|\eta^-|} \sum_{x\in\eta^+} \int_{\Gamma_0^2} d\lambda^2(\xi^+, \xi^-)(\alpha C)^{|\xi^+|+|\xi^-|} \\
&\qquad \times \big|D^+\big(x, \eta^+ \setminus x, \eta^-, \xi^+, \xi^-\big)\big| \\
&\quad + \|k\|_{\mathscr{K}_{\alpha C}}(\alpha C)^{-1}\alpha^{|\eta^+|+|\eta^-|} \sum_{x\in\eta^+} \int_{\Gamma_0^2} d\lambda^2(\xi^+, \xi^-)(\alpha C)^{|\xi^+|+|\xi^-|} \\
&\qquad \times \big|B^+\big(x, \eta^+ \setminus x, \eta^-, \xi^+, \xi^-\big)\big|,
\end{aligned}$$

where we have used inequality (3.12). A similar estimate holds for $C^{-|\eta^+|-|\eta^-|}\cdot \big|(\hat{L}^*_-k)(\eta^+, \eta^-)\big|$. Both estimates combined with (3.13) lead to

$$\begin{aligned}
C^{-|\eta^+|-|\eta^-|}\big|(\hat{L}^*k)(\eta^+, \eta^-)\big| &\le \frac{\|k\|_{\mathscr{K}_{\alpha C}}}{\alpha}\alpha^{|\eta^+|+|\eta^-|}N(\eta^+, \eta^-) \\
&\le \frac{A\|k\|_{\mathscr{K}_{\alpha C}}}{\alpha}(\alpha\nu)^{|\eta^+|+|\eta^-|}\big(1 + |\eta^+| + |\eta^-|\big)^M.
\end{aligned}$$

Since $\alpha < 1$, and thus $\alpha\nu < 1$, an application of inequality

$$(1+t)^b a^t \leq \frac{1}{a}\left(\frac{b}{-e\ln a}\right)^b, \qquad b \geq 1,\ a \in (0,1)\,,\ t \geq 0,$$

yields

$$\left\|\hat{L}^* k\right\|_{\mathscr{K}_C} \leq \frac{A\|k\|_{\mathscr{K}_{\alpha C}}}{\alpha}\frac{1}{\alpha\nu}\Big(\frac{M}{-e\ln(\alpha\nu)}\Big)^M < \infty,$$

completing the proof. □

Remark 3.5. *Since the space $\mathscr{L}_C$ is not reflexive, a priori we cannot expect that the domain of $\hat{L}^*$ is dense in $\mathscr{K}_C$.*

4 Conservative dynamics

In contrast to the birth-and-death dynamics, in the following dynamics there is conservation on the total number of particles involved.

4.1 Hopping particles: hierarchical equations

Dynamically, in a hopping particle system, at each random moment of time particles randomly hop from one site to another according to a rate depending on the configuration of the whole system at that time. Since the particles are of two types, two situations may occur. The $\pm$ particles located in $\gamma^\pm$ hop over $\gamma^\pm$, or hop to sites in $\gamma^\mp$, thus changing its mark. In terms of generators these two different behaviors are informally described by

$$\begin{aligned}
&(L_1F)(\gamma^+,\gamma^-)\\
&:= \sum_{x\in\gamma^+}\int_{\mathbb{R}^d} dx'\, c_1^+(x,x',\gamma^+\setminus x,\gamma^-)\left(F(\gamma^+\setminus x\cup x',\gamma^-) - F(\gamma^+,\gamma^-)\right)\\
&\quad + \sum_{y\in\gamma^-}\int_{\mathbb{R}^d} dy'\, c_1^-(y,y',\gamma^+,\gamma^-\setminus y)\left(F(\gamma^+,\gamma^-\setminus y\cup y') - F(\gamma^+,\gamma^-)\right)
\end{aligned}$$

and

$$\begin{aligned}
&(L_2F)(\gamma^+,\gamma^-) \qquad (4.1)\\
&:= \sum_{x\in\gamma^+}\int_{\mathbb{R}^d} dy\, c_2^+\left(x,y,\gamma^+\setminus x,\gamma^-\right)\left(F\left(\gamma^+\setminus x,\gamma^-\cup y\right) - F\left(\gamma^+,\gamma^-\right)\right)\\
&\quad + \sum_{y\in\gamma^-}\int_{\mathbb{R}^d} dx\, c_2^-\left(x,y,\gamma^+,\gamma^-\setminus y\right)\left(F\left(\gamma^+\cup x,\gamma^-\setminus y\right) - F\left(\gamma^+,\gamma^-\right)\right),
\end{aligned}$$

respectively. Here the coefficient $c_1^+(x, x', \gamma^+, \gamma^-) \geq 0$ indicates the rate at which a + particle located at a position x in a configuration γ^+ hops to a free site x' keeping its mark, and $c_2^+(x, y, \gamma^+, \gamma^-) \geq 0$ indicates the rate at which, given a configuration (γ^+, γ^-), a + particle located at a site $x \in \gamma^+$ hops to a free site y and changes its mark to $-$. A similar interpretation holds for the rates $c_i^- \geq 0$, $i = 1, 2$.

In what follows we assume that $c_i^\pm$, $i = 1, 2$, are measurable functions such that, for a.a. x, y, $c_i^\pm(x, y, \cdot, \cdot)$ are $\mathcal{B}(\Gamma_0^2)$-measurable functions and, for $(\eta^+, \eta^-) \in \Gamma_0^2$, $c_i^\pm(\cdot, \cdot, \eta^+, \eta^-) \in L^1_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}^d, dx \otimes dy)$. Under these conditions, for each $F \in \mathcal{FP}(\Gamma^2) = \mathrm{K}(B_{\mathrm{bs}}(\Gamma_0^2))$, the expression for $L_i F$, $i = 1, 2$, is well-defined at least on Γ_0^2, ensuring that for any $G \in B_{\mathrm{bs}}(\Gamma_0^2)$

$$\hat{L}_i G = \mathrm{K}^{-1} L_i \mathrm{K} G$$

is well-defined on Γ_0^2 (Remark 2.1). Moreover, the above conditions allow to define the functions

$$C_i^\pm(x, y, \xi^+, \xi^-, \eta^+, \eta^-) := \left(\mathrm{K}^{-1} c_i^\pm(x, y, \cdot \cup \xi^+, \cdot \cup \xi^-)\right)(\eta^+, \eta^-), \quad i = 1, 2,$$

for a.a. $x, y \in \mathbb{R}^d$, $(\eta^+, \eta^-), (\xi^+, \xi^-) \in \Gamma_0^2$ such that $\eta^\pm \cap \xi^\pm = \emptyset$. We set

$$C_{i,x,y}^\pm(\eta^+, \eta^-) := C_i^\pm(x, y, \emptyset, \emptyset, \eta^+, \eta^-), \quad i = 1, 2.$$

Proposition 4.1. *The action of $\hat{L}_i$, $i = 1, 2$, on functions $G \in B_{\mathrm{bs}}(\Gamma_0^2)$ is given for any $(\eta^+, \eta^-) \in \Gamma_0^2$ by*

$$\begin{aligned}(\hat{L}_1 G)(\eta^+, \eta^-) = &\sum_{\substack{\xi^+ \subset \eta^+ \\ \xi^- \subset \eta^-}} \sum_{x \in \xi^+} \int_{\mathbb{R}^d} dx' \left(G(\xi^+ \cup x' \setminus x, \xi^-) - G(\xi^+, \xi^-)\right) \\ &\qquad \times C_1^+\left(x, x', \xi^+ \setminus x, \xi^-, \eta^+ \setminus \xi^+, \eta^- \setminus \xi^-\right) \\ &+ \sum_{\substack{\xi^+ \subset \eta^+ \\ \xi^- \subset \eta^-}} \sum_{y \in \xi^-} \int_{\mathbb{R}^d} dy' \left(G(\xi^+, \xi^- \cup y' \setminus y) - G(\xi^+, \xi^-)\right) \\ &\qquad \times C_1^-\left(y, y', \xi^+, \xi^- \setminus y, \eta^+ \setminus \xi^+, \eta^- \setminus \xi^-\right),\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}(\hat{L}_2 G)(\eta^+, \eta^-) = &\sum_{\substack{\xi^+ \subset \eta^+ \\ \xi^- \subset \eta^-}} \sum_{x \in \xi^+} \int_{\mathbb{R}^d} dy \left(G(\xi^+ \setminus x, \xi^- \cup y) - G(\xi^+, \xi^-)\right) \\ &\qquad \times C_2^+\left(x, y, \xi^+ \setminus x, \xi^-, \eta^+ \setminus \xi^+, \eta^- \setminus \xi^-\right)\end{aligned} \tag{4.3}$$

$$+\sum_{\substack{\xi^+\subset\eta^+\\ \xi^-\subset\eta^-}}\sum_{y\in\xi^-}\int_{\mathbb{R}^d}dx\big(G(\xi^+\cup x,\xi^-\setminus y)-G(\xi^+,\xi^-)\big)$$
$$\times C_2^-\left(x,y,\xi^+,\xi^-\setminus y,\eta^+\setminus\xi^+,\eta^-\setminus\xi^-\right).$$

Proof. We begin by observing that, similarly to the proof of Proposition 3.1, the integrability property of $c_i^\pm$, $i=1,2$, on $\mathbb{R}^d$ is sufficient to ensure that, for any $G\in B_{\mathrm{bs}}(\Gamma_0^2)$, all integrals appearing in (4.2), (4.3) are finite.

Since each L_i, $i=1,2$, is of the form $L_i=L_i^+ + L_i^-$, with L_i^+ concerning the $+$-system and L_i^- the $-$-system, the proof reduces to prove the statement for each summand L_i^+, L_i^-, $i=1,2$. We will do it for L_i^+, $i=1,2$, being the proof for L_i^-, $i=1,2$, similar. For this purpose, first we observe that from definition (2.2) of the K-transform, for any $(\gamma^+,\gamma^-)\in\Gamma_0^2$ one has

$$\begin{aligned}
&(\mathrm{K}G)(\gamma^+\setminus x\cup x',\gamma^-)-(\mathrm{K}G)(\gamma^+,\gamma^-)\\
&\qquad=\big(\mathrm{K}G(\cdot\cup x',\cdot)\big)(\gamma^+\setminus x,\gamma^-)-\big(\mathrm{K}G(\cdot\cup x,\cdot)\big)(\gamma^+\setminus x,\gamma^-),\\
&(\mathrm{K}G)(\gamma^+\setminus x,\gamma^-\cup y)-(\mathrm{K}G)(\gamma^+,\gamma^-)\\
&\qquad=\big(\mathrm{K}G(\cdot,\cdot\cup y)\big)(\gamma^+\setminus x,\gamma^-)-\big(\mathrm{K}G(\cdot\cup x,\cdot)\big)(\gamma^+\setminus x,\gamma^-).
\end{aligned}$$

This leads to

$$(\hat L_1^+G)(\eta^+,\eta^-)=\sum_{x\in\eta^+}\int_{\mathbb{R}^d}dx'\ \left(C_{1,x,x'}^+\circledast\left(G(\cdot\cup x',\cdot)-G(\cdot\cup x,\cdot)\right)\right)(\eta^+\backslash x,\eta^-),$$
$$(\hat L_2^+G)(\eta^+,\eta^-)=\sum_{x\in\eta^+}\int_{\mathbb{R}^d}dy\ \left(C_{2,x,y}^+\circledast\left(G(\cdot,\cdot\cup y)-G(\cdot\cup x,\cdot)\right)\right)(\eta^+\backslash x,\eta^-),$$

where we have used equality (3.7). Similar arguments used to prove Proposition 3.1 complete the proof for L_i^+, $i=1,2$. □

Concerning $\hat L_i^*$, $i=1,2$, one has the following explicit expressions.

Proposition 4.2. *Assume that for all $\Lambda\in\mathcal{B}_c(\mathbb{R}^d)$ and all $n,m\in\mathbb{N}_0$,*

$$\begin{aligned}
C_{1,\Lambda,m,n}:=&\int_{\Gamma_\Lambda^{(n,m)}}d\lambda^2(\eta^+,\eta^-)\int_\Lambda dx'\\
&\times\sum_{\substack{\xi^+\subset\eta^+\\ \xi^-\subset\eta^-}}\Bigg(\sum_{x\in\xi^+}\left|C_1^+\left(x,x',\xi^+\setminus x,\xi^-,\eta^+\setminus\xi^+,\eta^-\setminus\xi^-\right)\right|\\
&\qquad+\sum_{y\in\xi^-}\left|C_1^-\left(y,x',\xi^+,\xi^-\setminus y,\eta^+\setminus\xi^+,\eta^-\setminus\xi^-\right)\right|\Bigg)<\infty
\end{aligned}\tag{4.4}$$

and

$$C_{2,\Lambda,m,n} := \int_{\Gamma_\Lambda^{(n,m)}} d\lambda^2(\eta^+,\eta^-) \int_\Lambda dx' \times \sum_{\substack{\xi^+\subset\eta^+\\ \xi^-\subset\eta^-}} \Bigg(\sum_{x\in\xi^+} \big| C_2^+\big(x,x',\xi^+\setminus x,\xi^-,\eta^+\setminus\xi^+,\eta^-\setminus\xi^-\big)\big| + \sum_{y\in\xi^-} \big| C_2^-\big(x',y,\xi^+,\xi^-\setminus y,\eta^+\setminus\xi^+,\eta^-\setminus\xi^-\big)\big| \Bigg) < \infty, \quad (4.5)$$

where, as before, $\Gamma_\Lambda^{(n,m)} = \big(\Gamma_\Lambda^{(n)} \times \Gamma_\Lambda^{(m)}\big) \cap \Gamma_0^2$*. Then, for each* $k \in B_{\rm bs}(\Gamma_0^2)$*,*

$$\begin{aligned}
&(\hat{L}_1^* k)(\eta^+,\eta^-) \qquad (4.6)\\
&= \sum_{x\in\eta^+} \int_{\Gamma_0^2} d\lambda^2(\xi^+,\xi^-) \int_{\mathbb{R}^d} dx'\, k(\xi^+\cup\eta^+\cup x'\setminus x, \xi^-\cup\eta^-) \\
&\qquad \times C_1^+\big(x',x,\eta^+\setminus x,\eta^-,\xi^+,\xi^-\big) \\
&\quad - \sum_{x\in\eta^+} \int_{\Gamma_0^2} d\lambda^2(\xi^+,\xi^-)\, k(\xi^+\cup\eta^+,\xi^-\cup\eta^-) \\
&\qquad \times \int_{\mathbb{R}^d} dx' C_1^+\big(x,x',\eta^+\setminus x,\eta^-,\xi^+,\xi^-\big) \\
&\quad + \sum_{y\in\eta^-} \int_{\Gamma_0^2} d\lambda^2(\xi^+,\xi^-) \int_{\mathbb{R}^d} dy'\, k(\xi^+\cup\eta^+,\xi^-\cup\eta^-\cup y'\setminus y) \\
&\qquad \times C_1^-\big(y',y,\eta^+,\eta^-\setminus y,\xi^+,\xi^-\big) \\
&\quad - \sum_{y\in\eta^-} \int_{\Gamma_0^2} d\lambda^2(\xi^+,\xi^-) k(\xi^+\cup\eta^+,\xi^-\cup\eta^-) \\
&\qquad \times \int_{\mathbb{R}^d} dy'\, C_1^-\big(y,y',\eta^+,\eta^-\setminus y,\xi^+,\xi^-\big),
\end{aligned}$$

and

$$\begin{aligned}
&(\hat{L}_2^* k)(\eta^+,\eta^-) \qquad (4.7)\\
&= \sum_{y\in\eta^-} \int_{\Gamma_0^2} d\lambda^2(\xi^+,\xi^-) \int_{\mathbb{R}^d} dx\, k(\xi^+\cup\eta^+\cup x,\xi^-\cup\eta^-\setminus y) \\
&\qquad \times C_2^+\big(x,y,\eta^+,\eta^-\setminus y,\xi^+,\xi^-\big) \\
&\quad - \sum_{x\in\eta^+} \int_{\Gamma_0^2} d\lambda^2(\xi^+,\xi^-) k(\xi^+\cup\eta^+,\xi^-\cup\eta^-)
\end{aligned}$$

$$\times \int_{\mathbb{R}^d} dy\, C_2^+\big(x, y, \eta^+ \setminus x, \eta^-, \xi^+, \xi^-\big)$$

$$+ \sum_{x\in\eta^+} \int_{\Gamma_0^2} d\lambda^2(\xi^+,\xi^-) \int_{\mathbb{R}^d} dy\, k(\xi^+ \cup \eta^+ \setminus x, \xi^- \cup \eta^- \cup y)$$

$$\times C_2^-\big(x, y, \eta^+ \setminus x, \eta^-, \xi^+, \xi^-\big)$$

$$- \sum_{y\in\eta^-} \int_{\Gamma_0^2} d\lambda^2(\xi^+,\xi^-) \int_{\mathbb{R}^d} dx\, k(\xi^+ \cup \eta^+, \xi^- \cup \eta^-)$$

$$\times C_2^-\big(x, y, \eta^+, \eta^- \setminus y, \xi^+, \xi^-\big),$$

for λ^2*-a.a.* $(\eta^+,\eta^-) \in \Gamma_0^2$.

Proof. Similarly to the proof of Proposition 3.2, conditions (4.4), (4.5) ensure that for any $G, k \in B_{\mathrm{bs}}(\Gamma_0^2)$, one has $(\hat{L}_i^{\pm} G)k \in L^1(\Gamma_0^2, \lambda^2)$, $i = 1, 2$. Moreover, for $\hat{L}_1^+$, the use of its expression, derived in Proposition 4.1 and its proof, leads through an application of Lemma 2.2 to

$$\int_{\Gamma_0^2} d\lambda^2(\eta^+,\eta^-)\, (\hat{L}_1^+ G)(\eta^+,\eta^-)\, k(\eta^+,\eta^-)$$

$$= \int_{\Gamma_0^2} d\lambda^2(\eta^+,\eta^-) \int_{\Gamma_0^2} d\lambda^2(\xi^+,\xi^-)\, k(\eta^+ \cup \xi^+, \eta^- \cup \xi^-) \sum_{x\in\xi^+} \int_{\mathbb{R}^d} dx'$$

$$\times \big(G(\xi^+ \cup x' \setminus x, \xi^-) - G(\xi^+,\xi^-)\big) C_1^+\big(x, x', \xi^+ \setminus x, \xi^-, \eta^+, \eta^-\big)$$

$$= \int_{\Gamma_0^2} d\lambda^2(\xi^+,\xi^-)\, G(\xi^+,\xi^-) \int_{\Gamma_0^2} d\lambda^2(\eta^+,\eta^-)$$

$$\times \sum_{x'\in\xi^+} \int_{\mathbb{R}^d} dx\, k(\eta^+ \cup \xi^+ \cup x \setminus x', \eta^- \cup \xi^-) C_1^+\big(x, x', \xi^+ \setminus x', \xi^-, \eta^+, \eta^-\big)$$

$$- \int_{\Gamma_0^2} d\lambda^2(\xi^+,\xi^-)\, G(\xi^+,\xi^-) \sum_{x\in\xi^+} \int_{\Gamma_0^2} d\lambda^2(\eta^+,\eta^-)\, k(\eta^+ \cup \xi^+, \eta^- \cup \xi^-)$$

$$\times \int_{\mathbb{R}^d} dx'\, C_1^+\big(x, x', \xi^+ \setminus x, \xi^-, \eta^+, \eta^-\big).$$

Similarly, for $\hat{L}_2^+$, we obtain

$$\int_{\Gamma_0^2} d\lambda^2(\eta^+,\eta^-)\, (\hat{L}_2^+ G)(\eta^+,\eta^-)\, k(\eta^+,\eta^-)$$

$$= \int_{\Gamma_0^2} d\lambda^2(\eta^+,\eta^-) \int_{\Gamma_0^2} d\lambda^2(\xi^+,\xi^-)\, k(\eta^+ \cup \xi^+, \eta^- \cup \xi^-)$$

$$\times \sum_{x\in\xi^+} \int_{\mathbb{R}^d} dy\, \big(G(\xi^+ \setminus x, \xi^- \cup y) - G(\xi^+,\xi^-)\big)$$

$$\times\, C_2^+\big(x, y, \xi^+ \setminus x, \xi^-, \eta^+, \eta^-\big).$$

The rest of the proof follows now straightforwardly. □

4.2 Hopping particles: definition of operators

Assume that for each $i = 1, 2$ there is a function $N_i : \Gamma_0^2 \to \mathbb{R}$ such that

$$\int_{\Gamma_\Lambda^{(n,m)}} d\lambda^2(\eta^+, \eta^-)\, N_i(\eta^+, \eta^-) < \infty \quad \text{for all } n, m \in \mathbb{N} \text{ and all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \tag{4.8}$$

and, for λ^2-a.a. $(\eta^+, \eta^-) \in \Gamma_0^2$,

$$\begin{aligned}
&\sum_{x\in\eta^+}\bigg(\Big\|\int_{\mathbb{R}^d} dy\, C_1^+\big(x, y, \eta^+ \setminus x, \eta^-, \cdot, \cdot\big)\Big\|_{\mathscr{L}_C} \\
&\qquad\qquad + \Big\|\int_{\mathbb{R}^d} dy\, C_1^+\big(y, x, \eta^+ \setminus x, \eta^-, \cdot, \cdot\big)\Big\|_{\mathscr{L}_C}\bigg) \\
&+ \sum_{y\in\eta^-}\bigg(\Big\|\int_{\mathbb{R}^d} dx\, C_1^-\big(x, y, \eta^+, \eta^- \setminus y, \cdot, \cdot\big)\Big\|_{\mathscr{L}_C} \\
&\qquad\qquad + \Big\|\int_{\mathbb{R}^d} dx\, C_1^-\big(y, x, \eta^+, \eta^- \setminus y, \cdot, \cdot\big)\Big\|_{\mathscr{L}_C}\bigg) \\
&\le N_1(\eta^+, \eta^-) < \infty,
\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}
&\sum_{x\in\eta^+}\bigg(\Big\|\int_{\mathbb{R}^d} dy\, C_2^+\big(x, y, \eta^+ \setminus x, \eta^-, \cdot, \cdot\big)\Big\|_{\mathscr{L}_C} \\
&\qquad\qquad + \Big\|\int_{\mathbb{R}^d} dy\, C_2^-\big(x, y, \eta^+ \setminus x, \eta^-, \cdot, \cdot\big)\Big\|_{\mathscr{L}_C}\bigg) \\
&+ \sum_{y\in\eta^-}\bigg(\Big\|\int_{\mathbb{R}^d} dx\, C_2^+\big(x, y, \eta^+, \eta^- \setminus y, \cdot, \cdot\big)\Big\|_{\mathscr{L}_C} \\
&\qquad\qquad + \Big\|\int_{\mathbb{R}^d} dx\, C_2^-\big(x, y, \eta^+, \eta^- \setminus y, \cdot, \cdot\big)\Big\|_{\mathscr{L}_C}\bigg) \\
&\le N_2(\eta^+, \eta^-) < \infty.
\end{aligned} \tag{4.10}$$

Under these conditions, let us consider the sets

$$\mathcal{D}_i := \mathcal{D}_i(N_i, C) := \big\{G \in \mathscr{L}_C \;\big|\; N_i G \in \mathscr{L}_C\big\}, \quad i = 1, 2,$$

where $\mathscr{L}_C$ is the Banach space defined in (3.9). Of course, $B_{\mathrm{bs}}(\Gamma_0^2) \subset \mathcal{D}_1 \cap \mathcal{D}_2$, which implies that both $\mathcal{D}_1$ and $\mathcal{D}_2$ are dense in $\mathscr{L}_C$.

Proposition 4.3. *Assume that integrability conditions* (4.8), (4.9), (4.10) *hold. Then, equality* (4.2) *(resp.,* (4.3)*) provides a densely defined linear operator* $\hat{L}_1$ *(resp.,* $\hat{L}_2$*) in* $\mathscr{L}_C$ *with domain* $\mathcal{D}_1$ *(resp.,* $\mathcal{D}_2$*). In particular, for any* $G \in \mathcal{D}_1$ *(resp.,* $G \in \mathcal{D}_2$*), the right-hand side of* (4.2) *(resp.,* (4.3)*) is* λ^2*-a.e. well-defined on* Γ_0^2.

Proof. We just estimate $\|\hat{L}_1^+ G\|_{\mathscr{L}_C}$, being similar the estimate for $\hat{L}_1^-$. Given a $G \in \mathcal{D}_1$, an application of Lemma 2.2 to the expression corresponding to $\hat{L}_1^+$ (derived in Proposition 4.1 and its proof) yields

$$\begin{aligned}\|\hat{L}_1^+ G\|_{\mathscr{L}_C} \leq \int_{\Gamma_0^2} d\lambda^2(\xi^+, \xi^-) C^{|\xi^+|+|\xi^-|} \\ \times \sum_{x \in \xi^+} \int_{\mathbb{R}^d} dx' \left(|G(\xi^+ \cup x' \setminus x, \xi^-)| + |G(\xi^+, \xi^-)|\right) \\ \times \left\|C_1^+\big(x, x', \xi^+ \setminus x, \xi^-, \cdot, \cdot\big)\right\|_{\mathscr{L}_C}.\end{aligned}$$

This shows that $\|\hat{L}_1 G\|_{\mathscr{L}_C} \leq \|\hat{L}_1^+ G\|_{\mathscr{L}_C} + \|\hat{L}_1^- G\|_{\mathscr{L}_C} \leq \|N_1 G\|_{\mathscr{L}_C} < \infty$. The proof for $\hat{L}_2$ is analogous. □

Similar arguments used to prove Proposition 3.4 lead to the next result.

Proposition 4.4. *Assume that integrability conditions* (4.8), (4.9), (4.10) *hold. In addition, assume that there are constants* $A > 0$*,* $M \in \mathbb{N}$*,* $\nu \geq 1$ *such that*

$$N_i(\eta^+, \eta^-) \leq A\big(1 + |\eta^+| + |\eta^-|\big)^M \nu^{|\eta^+|+|\eta^-|}, \quad i = 1, 2.$$

Then, equality (4.6) *(resp.,* (4.7)*) provides a linear operator* $\hat{L}_1^*$ *(resp.,* $\hat{L}_2^*$*) in* $\mathscr{K}_C$ *with domain* $\mathscr{K}_{\alpha C}$*,* $\alpha \in \big(0, \frac{1}{\nu}\big)$*. In particular, given a* $k \in \mathscr{K}_{\alpha C}$ *for some* $\alpha \in \big(0, \frac{1}{\nu}\big)$*, the right-hand side of* (4.6) *(resp.,* (4.7)*) is* λ^2*-a.e. well-defined on* Γ_0^2.

4.3 Flipping particles

Dynamically, in a flipping particle system, at each random moment of time particles randomly flip marks keeping their sites. In terms of generators this behavior is informally described by

$$\begin{aligned}(L_0 F)(\gamma^+, \gamma^-) = \sum_{x \in \gamma^+} a^+(x, \gamma^+ \setminus x, \gamma^-)\big(F(\gamma^+ \setminus x, \gamma^- \cup x) - F(\gamma^+, \gamma^-)\big) \\ + \sum_{y \in \gamma^-} a^-(x, \gamma^+, \gamma^- \setminus y)\big(F(\gamma^+ \cup y, \gamma^- \setminus y) - F(\gamma^+, \gamma^-)\big),\end{aligned} \tag{4.11}$$

where $a^+(x,\gamma^+,\gamma^-)\geq 0$ indicates the rate at which a +-particle located at $x\in\gamma^+$ flips the mark to "−". A similar interpretation holds for the rate $a^-\geq 0$ appearing in (4.11). We observe that, formally, L_0 is a particular case of the mapping L_2 defined in (4.1) with

$$c_2^{\pm}(x,y,\gamma^+,\gamma^-)=\delta(x-y)a^{\pm}(x,\gamma^+,\gamma^-).$$

Therefore, the results obtained therein justify the results for L_0. The proof of Proposition 4.5 below is then fully similar.

In what follows we assume that $a^{\pm}$ are measurable functions such that, for a.a. $x\in\mathbb{R}^d$, $a^{\pm}(x,\cdot,\cdot)$ are $\mathcal{B}(\Gamma_0^2)$-measurable functions and, for $(\eta^+,\eta^-)\in\Gamma_0^2$, $a^{\pm}(\cdot,\eta^+,\eta^-)\in L^1_{\mathrm{loc}}(\mathbb{R}^d,dx)$. We set

$$A^{\pm}(x,\xi^+,\xi^-,\eta^+,\eta^-):=\big(\mathrm{K}^{-1}a^{\pm}(x,\cdot\cup\xi^+,\cdot\cup\xi^-)\big)(\eta^+,\eta^-),$$

for a.a. $x\in\mathbb{R}^d$ and $(\eta^+,\eta^-),(\xi^+,\xi^-)\in\Gamma_0^2$ such that $\eta^{\pm}\cap\xi^{\pm}=\emptyset$.

Proposition 4.5. *If $G\in B_{\mathrm{bs}}(\Gamma_0^2)$, then for any $(\eta^+,\eta^-)\in\Gamma_0^2$*

$$\begin{aligned}&(\hat{L}_0G)(\eta^+,\eta^-)\\&=\sum_{\substack{\xi^+\subset\eta^+\\ \xi^-\subset\eta^-}}\sum_{x\in\xi^+}\big(G(\xi^+\setminus x,\xi^-\cup x)-G(\xi^+,\xi^-)\big)A^+\big(x,\xi^+\setminus x,\xi^-,\eta^+\setminus\xi^+,\eta^-\setminus\xi^-\big)\\&\quad+\sum_{\substack{\xi^+\subset\eta^+\\ \xi^-\subset\eta^-}}\sum_{y\in\xi^-}\big(G(\xi^+\cup y,\xi^-\setminus y)-G(\xi^+,\xi^-)\big)A^-\big(y,\xi^+,\xi^-\setminus y,\eta^+\setminus\xi^+,\eta^-\setminus\xi^-\big).\end{aligned}$$

If, in addition, there is a function $N_0:\Gamma_0^2\to\mathbb{R}$ such that

$$\int_{\Gamma_\Lambda^{(n,m)}}d\lambda^2(\eta^+,\eta^-)\,N_0(\eta^+,\eta^-)<\infty\quad\text{for all } n,m\in\mathbb{N}\text{ and all }\Lambda\in\mathcal{B}_c(\mathbb{R}^d)$$

and, for λ^2-a.a. $(\eta^+,\eta^-)\in\Gamma_0^2$,

$$\begin{aligned}&\sum_{x\in\eta^+}\Big(\big\|A^+\big(x,\eta^+\setminus x,\eta^-,\cdot,\cdot\big)\big\|_{\mathscr{L}_C}+\big\|A^-\big(x,\eta^+\setminus x,\eta^-,\cdot,\cdot\big)\big\|_{\mathscr{L}_C}\Big)\\&\quad+\sum_{y\in\eta^-}\Big(\big\|A^+\big(y,\eta^+,\eta^-\setminus y,\cdot,\cdot\big)\big\|_{\mathscr{L}_C}+\big\|A^-\big(y,\eta^+,\eta^-\setminus y,\cdot,\cdot\big)\big\|_{\mathscr{L}_C}\Big)\\&\leq N_0(\eta^+,\eta^-)<\infty,\end{aligned}$$

then, for each $G\in\mathscr{L}_C$ such that $N_0G\in\mathscr{L}_C$, we have $\hat{L}_0G\in\mathscr{L}_C$. Moreover, if there are $A>0$, $M\in\mathbb{N}$, $\nu\geq 1$ such that

$$N_0(\eta^+,\eta^-)\leq A\big(1+|\eta^+|+|\eta^-|\big)^M\nu^{|\eta^+|+|\eta^-|},$$

then, $\hat{L}_0^* k \in \mathscr{K}_C$ *for any* $k \in \mathscr{K}_{\alpha C}$*,* $\alpha \in \left(0, \frac{1}{\nu}\right)$*, and*

$$
\begin{aligned}
&(\hat{L}_0^* k)(\eta^+, \eta^-) \\
=& \sum_{y \in \eta^-} \int_{\Gamma_0^2} d\lambda^2(\xi^+, \xi^-) k(\xi^+ \cup \eta^+ \cup y, \xi^- \cup \eta^- \setminus y) A^+\big(y, \eta^+, \eta^- \setminus y, \xi^+, \xi^-\big) \\
&- \sum_{x \in \eta^+} \int_{\Gamma_0^2} d\lambda^2(\xi^+, \xi^-) k(\xi^+ \cup \eta^+, \xi^- \cup \eta^-) A^+\big(x, \eta^+ \setminus x, \eta^-, \xi^+, \xi^-\big) \\
&+ \sum_{x \in \eta^+} \int_{\Gamma_0^2} d\lambda^2(\xi^+, \xi^-)\, k(\xi^+ \cup \eta^+ \setminus x, \xi^- \cup \eta^- \cup x) A^-\big(x, \eta^+ \setminus x, \eta^-, \xi^+, \xi^-\big) \\
&- \sum_{y \in \eta^-} \int_{\Gamma_0^2} d\lambda^2(\xi^+, \xi^-) k(\xi^+ \cup \eta^+, \xi^- \cup \eta^-) A^-\big(y, \eta^+, \eta^- \setminus y, \xi^+, \xi^-\big),
\end{aligned}
$$

for λ^2*-a.a.* $(\eta^+, \eta^-) \in \Gamma_0^2$*.*

5 Examples of rates

For one-component systems there are many examples of birth-and-death dynamics (e.g. Glauber-type dynamics in mathematical physics, Bolker–Dieckmann–Law–Pacala dynamics in mathematical biology) as well as of hopping dynamics (e.g. Kawasaki-type dynamics). These dynamics have been studied, in particular, in [FK09,FKK09b,FKL07,KKL08,KKZ06,KLR07, KL05].

From the point of view of applications, multicomponent systems lead naturally to a richer situation due to many different possibilities for concrete models and corresponding rates $b^\pm, d^\pm, c_i^\pm$, discussed in the previous sections. For instance, one may consider (birth-and-death) predator-prey models in which the death rate of preys (representing e.g. the $+$-system) is higher due to the presence of a higher number of predators (representing the $-$-system) in a close neighborhood, while the birth rate of predators is higher if there is a higher number of preys nearby. For simplicity, assuming that there is no competition between predators as well as between preys, typical rates are of the type

$$\begin{aligned} d^+(x,\gamma^+,\gamma^-) &= m^+ + \sum_{y\in\gamma^-} a_1(x-y),\\ d^-(y,\gamma^+,\gamma^-) &\equiv m^-,\\ b^+(x,\gamma^+,\gamma^-) &= \sum_{x'\in\gamma^+} a_2(x-x'),\\ b^-(y,\gamma^+,\gamma^-) &= \sum_{y'\in\gamma^-} a_3(y-y')\left(\kappa + \sum_{x\in\gamma^+} a_4(x-y')\right), \end{aligned} \tag{5.1}$$

for $m^\pm, \kappa > 0$ and for even functions $0 \leq a_i \in L^1(\mathbb{R}^d, dx)$, $i = 1,2,3,4$. A similar situation occurs in other biological systems such as host-parasite or age-structured dynamics. On the other hand, on mathematical physics models, variants of the continuous Ising model [GH96, GMSRZ06, KZ07] (an analog of the Glauber dynamics) concern birth and death rates of a different type. The simplest variant is $d^\pm(x,\gamma^+,\gamma^-) \equiv m^\pm > 0$ and

$$b^\pm(x,\gamma^+,\gamma^-) = b^\pm(x,\gamma^\mp) = \exp\left(-\sum_{y\in\gamma^\mp}\phi(x-y)\right), \tag{5.2}$$

with $\phi : \mathbb{R}^d \to \mathbb{R}\cup\{\infty\}$ being a pair-potential in $\mathbb{R}^d$.

These examples of rates are natural and quite general. Indeed, applications deal with rates which are either "linear" functions

$$\langle a_x, \gamma^\pm\rangle := \sum_{y\in\gamma^\pm} a_x(y),$$

with $a_x(y) = a(x-y)$ for some even function a, products of such linear functions on different variables γ^+, γ^- (in particular, of polynomial type), or exponentials of these linear functions. For instance, in biological models concerning the so-called establishment and fecundity, rates are naturally defined by products or superpositions of linear functions and their exponentials (for the one-component case see [FKK11b]).

The results of the previous sections have shown that to derive explicit expressions for the mappings $\hat{L}$, $\hat{L}^*$ and to define sufficient conditions allowing an extension of $\hat{L}$, $\hat{L}^*$ to linear operators one only has to study $A^\pm, B^\pm, C^\pm, D^\pm$. We explain now how to proceed for linear and exponential rates.

Let $b^\pm$, $d^\pm$ be defined as in (5.1). Then, for example for d^+,

$$d^+(x,\eta^+\cup\gamma^+,\eta^-\cup\gamma^-) = m^+ + \sum_{y\in\eta^-} a_1(x-y) + \sum_{y\in\gamma^-} a_1(x-y).$$

By definitions (3.4) of D^+ and (2.4) of K^{-1}, a simple calculation yields

$$D^+(x,\eta^+,\eta^-,\xi^+,\xi^-) = \Big(m^+ + \sum_{y\in\eta^-} a_1(x-y)\Big)0^{|\xi^+|}0^{|\xi^-|} + 0^{|\xi^+|}1\!\!1_{\{\xi^-=\{y\}\}}a_1(x-y),$$

being easy to show that for each $C>0$,

$$\sum_{x\in\eta^+}\big\|D^+(x,\eta^+\backslash x,\eta^-,\cdot,\cdot)\big\|_{\mathscr{L}_C} \le m|\eta^+| + \sum_{x\in\eta^+}\sum_{y\in\eta^-} a_1(x-y) + C|\eta^+|\int_{\mathbb{R}^d} dx\, a_1(x).$$

Similar estimates naturally hold for d^- and $b^\pm$. All together, these estimates yield an explicit form for the function N introduced in (3.11).

Let us now assume that $b^\pm$ are defined as in (5.2) with $d^\pm$ being constants. Then,

$$b^+(x,\eta^+\cup\gamma^+,\eta^-\cup\gamma^-) = \exp\left(-\sum_{y\in\eta^-}\phi(x-y)\right)\exp\left(-\sum_{y\in\gamma^-}\phi(x-y)\right),$$

and again the use of definitions (3.4) and (2.4) leads to

$$B^+(x,\eta^+,\eta^-,\xi^+,\xi^-) = 0^{|\xi^+|}\exp\left(-\sum_{y\in\eta^-}\phi(x-y)\right)\prod_{y\in\xi^-}\left(e^{-\phi(x-y)}-1\right).$$

Assuming that $\phi(x)\ge -\upsilon$, $x\in\mathbb{R}^d$, for some $\upsilon\ge 0$, and $\beta := \int_{\mathbb{R}^d} dx\,\big|e^{-\phi(x)} - 1\big| < \infty$, we then obtain

$$\sum_{x\in\eta^+}\big\|B^+(x,\eta^+\setminus x,\eta^-,\cdot,\cdot)\big\|_{\mathscr{L}_C} \le |\eta^+|e^{\upsilon|\eta^-|}e^{C\beta},$$

where we have used the following equality which follows from definition (2.7) of the measure λ,

$$\int_{\Gamma_0} d\lambda(\xi^-)\prod_{y\in\xi^-}|f(y)| = \exp\left(\|f\|_{L^1(\mathbb{R}^d,dx)}\right), \quad f\in L^1(\mathbb{R}^d,dx).$$

Similar estimates naturally hold for b^-, allowing at the end to derive an explicit form for the function N, introduced in (3.11).

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