ON A RANDOM SCALED POROUS MEDIA EQUATION

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Abstract

It is shown that a random scaled porous media equation arising from a stochastic porous media equation with linear multiplicative noise through a random transformation is well-posed in L^∞. In the fast diffusion case we show existence in L^p.

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1 Introduction

In recent years, there has been a lot of interest in stochastic porous media and fast diffusion equations (see, e.g., [3], [4],[5], [6], [7], [8], [10], [11], [13], [18], [20], [21], [26], [27], [29], [30]). In this paper, we analyze a deterministic nonlinear partial differential equation (PDE) with random coefficients, which arises from a class of stochastic porous media equations (SPME) through a random transformation (cf. [8]). First, let us introduce this class of SPME, describe the random transformation and the resulting random PDE.

Consider the following SPME

$$\begin{aligned} dX(t) &= \Delta(X(t)|X(t)|^{m-1})dt + \sigma(X(t))dW(t) \\ X(0) &= x \in H^{-1}(\mathcal{O}), \end{aligned} \tag{1.1}$$

on $H^{-1}(\mathcal{O})$, i.e., the dual space of the Dirichlet–Sobolev space $H_0^1(\mathcal{O})$ of order 1 in $L^2(\mathcal{O})$, where $\mathcal{O} \subset \mathbb{R}^d$, $\mathcal{O}$ bounded, open, and $d = 1, 2, 3$. Here, $m \in (0, \infty)$ (hence stochastic fast diffusion equations are included) and $W(t)$, $t \in [0,T]$, is a cylindrical Wiener process on $L^2(\mathcal{O})$ over a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Furthermore, Δ is the Dirichlet Laplacian in $L^2(\mathcal{O})$, which in turn is equipped with the usual inner product $\langle \cdot, \cdot \rangle$, and for $x \in H^{-1}(\mathcal{O})$, $\sigma(x) : L^2(\mathcal{O}) \to H^{-1}(\mathcal{O})$ is defined by

$$\sigma(x)y := \sum_{k=1}^{N} \mu_k e_k \langle e_k, y \rangle x, \quad y \in L^2(\mathcal{O}), \tag{1.2}$$

where $e_1, ..., e_N$ are the first N eigenvectors of Δ, normalized in $L^2(\mathcal{O})$ and $\mu_1, ..., \mu_N \in \mathbb{R}$. Clearly, $\sigma(x) \in L_2(L^2(\mathcal{O}); H^{-1}(\mathcal{O}))$, i.e., $\sigma(x)$ is a Hilbert–Schmidt operator, and $x \mapsto \sigma(x)$ is Lipschitz from $H^{-1}(\mathcal{O})$ to $L_2(L^2(\mathcal{O}); H^{-1}(\mathcal{O}))$.

We recall the following result which is a special case of [26, Theorem 3.9], (see, also, [27, Theorem 2.1] for the special formulation given here).

Theorem 1.1 *Consider the situation described above. Then, for every $x \in H^{-1}(\mathcal{O})$, there exists a unique $X \in L^{m+1}([0,T] \times \Omega, dt \times \mathbb{P}; L^{m+1}(\mathcal{O}))$ such that $X(t)$, $t \in [0,T]$, is a continuous adapted process in $H^{-1}(\mathcal{O})$, $\int_0^t X(s)|X(s)|^{m-1} ds$, $t \in [0,T]$, is a continuous process in $H_0^1(\mathcal{O})$ and $\mathbb{P}$-a.s.*

$$X(t) = x + \Delta \left(\int_0^t X(s)|X(s)|^{m-1} ds \right) + \int_0^t \sigma(X(s)) dW(s), \quad t \in [0,T]. \tag{1.3}$$

For the general theory of stochastic PDE with monotone coefficients, we refer to the seminal papers [24], [19] as well as to the monograph [25].

For $k \in \{1, ..., N\}$ and $t \in [0,T]$, set

$$\beta_k(t) := \langle e_k, W(t) \rangle .$$

Then, $\beta_1, ..., \beta_N$ are independent $\mathbb{R}$-valued Brownian motions. Set

$$\mu(t) := - \sum_{k=1}^{N} \mu_k e_k \beta_k(t), \quad t \in [0,T], \tag{1.4}$$

and define

$$Y(t) := e^{\mu(t)} X(t), \quad t \in [0,T], \tag{1.5}$$

where X is the solution to (1.1) from Theorem 1.1.

The following was proved in [8, Lemma 4.1].

Theorem 1.2 *Consider the situation described above and let Y be given as in* (1.5)*. Then, $\mathbb{P}$-a.s. Y is a weak solution* (*i.e., in the sense of Schwarz distributions*) *to the following random PDE*

$$\begin{aligned} &\frac{\partial Y}{\partial t}(t) = e^{\mu(t)} \Delta (e^{-m\mu(t)} Y(t)|Y(t)|^{m-1}) - \frac{1}{2}\, \widetilde{\mu} Y(t), \quad t \in [0,T], \\ &Y(0) = x, \end{aligned} \tag{1.6}$$

where $\widetilde{\mu} := \sum_{k=1}^N \mu_k^2 e_k^2$.

The purpose of this paper is to prove that, for every $\omega \in \Omega$ fixed, equation (1.6) has in fact a unique strong solution, at least for a large class of initial conditions (see the next section for precise formulations of the results).

The motivation for fulfilling this task comes from several directions:

(1) There is strong interest in the study of random attractors for stochastic PDE (see, e.g., [12],[14], [15], [28]). Existence of a random attractor for SPME with multiplicative noise is an open problem. One main obstacle is to show the cocycle property for the corresponding random dynamical systems with the exceptional set of $\omega \in \Omega$ being the same for all times and guaranteeing continuity in the initial condition (see, e.g., [12, Definition 1.7] and [22], [23] for the case of mild solutions). The only rigorous method to achieve this for stochastic PDE known so far, is to transform it to a random PDE as above and prove the existence and uniqueness of solutions for the latter.

(2) If one can solve (1.6) for every $\omega \in \Omega$ strongly, one gets more precise information about the solution for (1.1). E.g., from Theorems 2.1 and 2.2 below, it follows immediately by transforming back and using that $e^{\mu(t,\omega)}$ is a multiplier in $H_0^1(\mathcal{O})$, that $X(t)|X(t)|^{m-1} \in H_0^1(\mathcal{O})$ and that hence Δ interchanges with the integral in (1.3). A much more general result of this type was, however, obtained independently by B. Gess in [16, Theorem 5.3].

(3) ω-wise solutions of (1.6) and corresponding ω-wise inequalities (see Section 2) are quite important instruments for establishing convergence of numerical methods for stochastic equations with nonglobally Lipschitz coefficients (see, e.g., [17] for an example in finite-dimensions).

(4) Last, but not least, equation (1.6), with fixed $\omega \in \Omega$, is a kind of scaled porous media equation with time-dependent coefficients and it is thus a type of nonlinear PDE for which there is no standard theory which can be applied. The reason is that the nonlinear diffusion operator is not dissipative in the standard spaces, where the porous media equation is treated, that is, in $H^{-1}(\mathcal{O})$ and $L^1(\mathcal{O})$. (See, e.g., [2].) Therefore, it is not only an important prototype of PDE directly related to stochastics, but is of its own interest from a purely analytical point of view.

The paper is organized as follows. In Section 2, we introduce our framework and formulate our main results (i.e., Theorems 2.1 and 2.2). Section 3 contains the proofs.

2 Framework and main results

Let $\mathcal{O} \subset \mathbb{R}^d$, be a bounded open set with smooth boundary. Let $L^p(\mathcal{O})$, $p \in [1, \infty]$, denote the usual real L^p-space with respect to Lebesgue measure and norm $\|\cdot\|_p$. Let Δ be the Dirichlet Laplacian on $\mathcal{O}$ modelled on $L^2(\mathcal{O})$, i.e. $\Delta : D(\Delta) \subset L^2(\mathcal{O}) \to L^2(\mathcal{O})$ with $D(\Delta) := H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$. Here $L^p(\mathcal{O})$, $p \geq 1$, denote the usual L^p-spaces with respect to

Lebesgue measure dx on $\mathcal{O}$ with norms $\|\cdot\|_p$ and $H^2(\mathcal{O})$, $H^1_0(\mathcal{O})$, the usual Sobolev space in $L^2(\mathcal{O})$, with subindex "0" referring to Dirichlet (i.e., zero) boundary conditions. By $W^{k,p}(\mathcal{O})$, respectively $W^{k,p}_0(\mathcal{O})$, we denote the corresponding Sobolev spaces in $L^p(\mathcal{O})$. We also set $W^{-k,p'}(\mathcal{O}) = (W^{k,p}_0(\mathcal{O}))'$, $\frac{1}{p} + \frac{1}{p'} = 1$. Fix the first N normalized eigenvectors of Δ, $e_1, ..., e_N$ say, and let $\mu_1, ..., \mu_N \in \mathbb{R}$; $\beta_1, ..., \beta_N \in C([0,T];\mathbb{R})$ for some fixed $T > 0$. Define $\mu : [0,T] \times \mathcal{O} \to \mathbb{R}$ by

$$\mu(t,\xi) := -\sum_{k=1}^{N} \mu_k e_k(\xi)\beta_k(t), \quad (t,\xi) \in [0,T] \times \mathcal{O},$$

and for fixed $m \in (0,\infty)$ consider the following partial differential equation in $H^{-1}(\mathcal{O})$

$$\left\{\begin{array}{ll} \dfrac{\partial Y}{\partial t} = e^{\mu}\Delta(e^{-m\mu}\,Y|Y|^{m-1}) - \dfrac{1}{2}\widetilde{\mu}Y & \text{on } (0,T)\times\mathcal{O} \\ Y(0,\xi) = Y_0(\xi) & \text{a.e. } \xi \in \mathcal{O}, \\ Y = 0 & \text{on } (0,T)\times\partial\mathcal{O}. \end{array}\right. \tag{2.1}$$

Here, $\widetilde{\mu} \in C([0,T];\, H^1_0(\mathcal{O})) \cap C([0,T] \times \overline{\mathcal{O}})$, $\widetilde{\mu} \geq 0$, and $H^{-1}(\mathcal{O})$ denotes the dual of $H^1_0(\mathcal{O})$. The space $H^{-1}(\mathcal{O})$ will be also denoted by H^{-1} and its norm by $\|\cdot\|_{-1}$. We note that $\widetilde{\mu}$ is a multiplier in $H^{-1}(\mathcal{O})$. By $\langle\cdot,\cdot\rangle_{-1}$, we denote the scalar product of H^{-1}.

For our motivation to study (2.1), we refer to the introduction.

Let us recall the definition of the nonlinear operator $A : D(A) \subset H^{-1}(\mathcal{O}) \to H^{-1}(\mathcal{O})$ ("porous medium operator") in (2.1) (cf., e.g., [2, p. 228])

$$D(A) := \{Y \in H^{-1}(\mathcal{O}) \cap L^1(\mathcal{O}) : Y|Y|^{m-1} \in H^1_0(\mathcal{O})\}$$

and, for $Y \in D(A)$,

$$AY := \Delta(Y|Y|^{m-1})(\in H^{-1}(\mathcal{O})).$$

Our main results are the following two theorems:

Theorem 2.1 *Assume* $1 \leq d \leq 3$, $m \in (1,5]$ *and* $Y_0 \in L^\infty(\mathcal{O})$. *Then, for almost all* $\omega \in \Omega$, *equation* (2.1) *has a unique solution* $Y = Y(t,Y_0)$ *on* $[0,T]$ *satisfying*

(i) $Y \in L^\infty((0,T)\times\mathcal{O}) \cap C([0,T];\; H^{-1}(\mathcal{O}))$;

(ii) $Y|Y|^{m-1} \in L^2(0,T;H^1_0(\mathcal{O}))$; $\dfrac{dY}{dt} \in L^2(0,T;H^{-1}(\mathcal{O}))$.

Moreover, if $Y_0 \geq 0$ *on* $\mathcal{O}$, *then* $Y \geq 0$ *on* $(0,T)\times\mathcal{O}$.

By a solution to (2.1) we mean a function Y which satisfies (ii) above and such that

$$\begin{array}{l} \dfrac{d}{dt}Y(t) = e^{\mu(t)}\Delta(e^{-m\mu(t)}Y(t)|Y(t)|^{m-1}) - \dfrac{1}{2}\widetilde{\mu}(t)Y(t), \qquad \text{a.e. } t \in (0,T), \\ Y(0) = Y_0, \end{array} \tag{2.2}$$

where $\frac{d}{dt}$ is the strong derivative of $Y : [0,T] \to H^{-1}(\mathcal{O})$. As usual, we set

$$W^{1,2}([0,T];\ H^{-1}(\mathcal{O})) := \left\{ Y \in L^2(0,T;H^{-1}(\mathcal{O}));\ \frac{dY}{dt} \in L^2(0,T;H^{-1}(\mathcal{O})) \right\},$$

where $\frac{dY}{dt}$ is taken in the sense of (vector-valued) distributions.

In fact (see, e.g., [2, p. 22]), $W^{1,2}([0,T];H^{-1}(\mathcal{O}))$ coincides with the space of all absolutely continuous functions $Y : [0,T] \to H^{-1}(\mathcal{O})$ which have a strong derivative in $L^2(0,T;H^{-1}(\mathcal{O}))$.

More generally, $W^{1,p}([0,T];X)$, where $p \geq 1$ and X is a Banach space, is the space of all X-valued absolutely continuous functions u which are a.e. differentiable in $(0,T)$ and $\frac{du}{dt} \in L^p(0,T;X)$.

In the case $m \in (0,1]$, which corresponds to the fast diffusion porous media equation, we do not need to restrict ourselves to bounded initial conditions, if we merely want to prove existence.

Theorem 2.2 *Assume that* $1 \leq d \leq 3$*,* $0 < m \leq 1$ *and* $m \geq \frac{1}{5}$ *if* $d = 3$*. Then, for each* $Y_0 \in L^{m+1}(\mathcal{O})$*, equation* (2.1) *has a solution* Y *in* $C([0,T];H^{-1}(\mathcal{O}))$ *satisfying* (ii) *in Theorem* 2.1 *and also the last assertion in Theorem* 2.1 *holds. If* $Y_0 \in L^\infty(\mathcal{O})$*, then* Y *also satisfies (i) in Theorem* 2.1*.*

3 Proofs of the main results

3.1 Existence of solutions in Theorem 2.1

Let $\beta_k^\varepsilon \in C^1([0,T];\mathbb{R})$, $k = 1,...,N$, be such that $\beta_k^\varepsilon \to \beta_k$ uniformly on $[0,T]$ as $\varepsilon \to 0$ for all $k = 1,...,N$, and set

$$\mu_\varepsilon := -\sum_{k=1}^N \mu_k\, e_k\, \beta_k^\varepsilon.$$

For $\varepsilon \in (0,1)$ consider the approximating equation

$$\text{(3.1)}\qquad \begin{cases} \dfrac{\partial Y_\varepsilon}{\partial t} = e^{\mu_\varepsilon}\Delta(e^{-m\mu_\varepsilon}\, Y_\varepsilon |Y_\varepsilon|^{m-1} + \varepsilon e^{-\mu_\varepsilon} Y_\varepsilon) - \dfrac{1}{2}\widetilde{\mu} Y_\varepsilon & \text{on } (0,T)\times\mathcal{O}, \\ Y_\varepsilon(0,\xi) = Y_0(\xi) & \text{a.e. } \xi \in \mathcal{O}, \\ Y_\varepsilon = 0 & \text{on } (0,T)\times\partial\mathcal{O}. \end{cases}$$

Now, setting $Z_\varepsilon := e^{-\mu_\varepsilon} Y_\varepsilon$ and changing variables in (3.1), we obtain that Z_ε solves

$$\text{(3.2)}\qquad \begin{cases} \dfrac{\partial}{\partial t} Z_\varepsilon = \Delta(Z_\varepsilon |Z_\varepsilon|^{m-1} + \varepsilon Z_\varepsilon) - \left(\dfrac{\partial}{\partial t}\mu^\varepsilon + \dfrac{1}{2}\widetilde{\mu}\right) Z_\varepsilon & \text{on } (0,T)\times\mathcal{O}, \\ Z_\varepsilon(0) = Y_0 & \text{on } \mathcal{O}, \\ Z_\varepsilon = 0 & \text{on } (0,T)\times\partial\mathcal{O}. \end{cases}$$

Since equation (3.2) is of the form

$$\text{(3.3)}\qquad \begin{cases} \dfrac{\partial Z}{\partial t} - \Delta\beta(Z) - \varepsilon\Delta Z + aZ = 0 & \text{on } (0,T)\times\mathcal{O}, \\ Z(0,\xi) = Y_0(\xi) & \text{on } \mathcal{O}, \\ Z = 0 & \text{on } (0,T)\times\partial\mathcal{O}, \end{cases}$$

where β is a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ and $a \in L^\infty((0,T)\times\mathcal{O})$, $\beta(\mathbb{R}) = (-\infty,+\infty)$, it follows by general existence theory for equations of type (3.3), which is essentially due to H. Brezis (see, e.g., [2, Theorem 5.3, p. 229]), that for $Y_0 \in H^{-1}(\mathcal{O}) \cap L^1(\mathcal{O})$ there is a unique solution $Z \in C([0,T]; H^{-1}(\mathcal{O}))$ of (3.3) such that

$$\text{(3.4)}\qquad Z(t)|Z|^{m-1}(t) \in H_0^1(\mathcal{O}) \quad \text{for a.e. } t \in [0,T],$$

$$\text{(3.5)}\qquad t^{\frac{1}{2}}\,\frac{dZ}{dt} \in L^2(0,T;H^{-1}(\mathcal{O})),\ t^{\frac{1}{2}}Z|Z|^{m-1} \in L^2(0,T;\ H_0^1(\mathcal{O})).$$

Moreover, if we set

$$j(r) := \int_0^r \beta(s)ds,\ r \in \mathbb{R},$$

and if $j(Y_0) \in L^1(\mathcal{O})$, then

$$\frac{dZ}{dt} \in L^2(0,T;H^{-1}(\mathcal{O})),\ Z|Z|^{m-1} \in L^2(0,T;H_0^1(\mathcal{O})).$$

If $Y_0 \in D(A)$, then

$$\frac{dZ}{dt} \in L^\infty(0,T;H^{-1}(\mathcal{O})),\ Z|Z|^{m-1} \in L^\infty(0,T;H_0^1(\mathcal{O})).$$

This means that in our case, for any $Y_0 \in L^{m+1}(\mathcal{O})$ and $\varepsilon > 0$, the equation

$$\text{(3.6)}\qquad \begin{cases} \dfrac{\partial Y_\varepsilon}{\partial t} - e^{\mu_\varepsilon}\Delta(e^{-m\mu_\varepsilon}Y_\varepsilon|Y_\varepsilon|^{m-1} + \varepsilon Y_\varepsilon e^{-\mu_\varepsilon}) + \dfrac{1}{2}\widetilde{\mu}Y_\varepsilon = 0 & \text{on } (0,T)\times\mathcal{O} \\ Y_\varepsilon(0) = Y_0 & \text{on } \mathcal{O} \\ Y_\varepsilon = 0 & \text{on } (0,T)\times\partial\mathcal{O} \end{cases}$$

has a unique solution Y_ε such that

$$\text{(3.7)}\qquad Y_\varepsilon \in C\left([0,T]; H^{-1}(\mathcal{O})\right)$$

$$\text{(3.8)}\qquad Y_\varepsilon|Y_\varepsilon|^{m-1} \in L^2(0,T;H_0^1(\mathcal{O})),\ \frac{dY_\varepsilon}{dt} \in L^2(0,T;H^{-1}(\mathcal{O})).$$

Since $x \mapsto x|x|^{m-1} + \varepsilon x$ has a Lipschitz inverse, we also have that

$$\text{(3.9)}\qquad Y_\varepsilon \in L^2(0,T;H_0^1(\mathcal{O})).$$

We are going to prove that, for $\varepsilon \to 0$, $\{Y_\varepsilon\}$ is convergent to a solution Y to equation (2.1). To this end, we need some preliminary results and a priori estimates on the solutions Y_ε to equation (3.6). We begin with the following lemma:

Lemma 3.1 *For each* $\varepsilon > 0$, $Y_\varepsilon \in L^\infty((0,T)\times\mathcal{O})$.

Proof. Below, for a real valued function f we set $f^+ := \sup\{f, 0\}$. Set $Z := Z_\varepsilon$ and fix $r \in [0,\infty)$ such that $\|Z(r)\|_\infty < \infty$. Set

$$K_{a,r}(t) := K(t) := a(t-r) + \|Z(r)\|_\infty,\ t \geq r, \tag{3.10}$$

with $a \in (0,\infty)$ to be chosen later. Then, (3.2) implies that

$$\begin{aligned}\frac{d}{dt}(Z-K)(t) - \Delta(Z(t)|Z|^{m-1}(t) + \varepsilon Z(t)) + \left(\frac{\partial}{\partial t}\mu_\varepsilon(t) + \frac{1}{2}\widetilde{\mu}\right)(Z-K)(t) \\ = -a - \left(\frac{\partial}{\partial t}\mu_\varepsilon(t) + \frac{1}{2}\widetilde{\mu}\right)K(t),\end{aligned} \tag{3.11}$$

where here and below $\frac{d}{dt}$ denotes the derivative of $Z - K : [0,T] \to H^{-1}(\mathcal{O})$.

Applying $(1-\eta\Delta)^{-1}$ for $\eta > 0$, multiplying by $(Z-K)^+$ and integrating over $(r,t)\times\mathcal{O}$ from (3.11), we obtain that

$$\begin{aligned}&\int_r^t \int_{\mathcal{O}} (Z-K)^+(s)(1-\eta\Delta)^{-1}\frac{d}{ds}(Z-K)(s)\,d\xi\,ds \\ &\quad - \int_r^t \int_{\mathcal{O}} (Z-K)^+(s)(1-\eta\Delta)^{-1}\Delta(Z(s)|Z|^{m-1}(s) + \varepsilon Z(s))\,d\xi ds \\ &\quad + \int_r^t \int_{\mathcal{O}} (Z-K)^+(s)(1-\eta\Delta)^{-1}\left(\left(\frac{\partial}{\partial s}\mu_\varepsilon(s) + \frac{1}{2}\widetilde{\mu}\right)(Z-K)(s)\right)d\xi ds \\ &\quad = -\int_r^t \int_{\mathcal{O}} (Z-K)^+(s)(1-\eta\Delta)^{-1}\left(a + \left(\frac{\partial}{\partial s}\mu_\varepsilon(s) + \frac{1}{2}\widetilde{\mu}\right)K(s)\right)d\xi ds.\end{aligned} \tag{3.12}$$

Setting

$$T_0 := \left(\sup_{[0,T]\times\mathcal{O}}\left|\frac{\partial}{\partial s}\mu_\varepsilon\right| + 1\right)^{-1},$$

we can choose a $(= a(\varepsilon, r))$ so large that

$$a + \left(\frac{\partial}{\partial s}\mu_\varepsilon(s) + \frac{1}{2}\widetilde{\mu}\right)K(s) \geq 0 \ \text{ for all } s \in [r, r+T_0].$$

Recalling that $(1-\eta\Delta)^{-1}$ is positivity preserving, we see that for this choice of a, the right-hand side of (3.12) is negative for all $t \in [r, r+T_0]$.

Next, we note that the second summand on the left-hand side of (3.12) is equal to

$$\frac{1}{\eta}\int_r^t\int_{\mathcal{O}}(Z-K)^+(s)\left[1-(1-\eta\Delta)^{-1}\right](Z(s)|Z(s)|^{m-1}+\varepsilon Z(s))\,d\xi ds. \tag{3.13}$$

Recall that $(1-\eta\Delta)^{-1}$ has an integral kernel $g_\eta:\mathcal{O}\times\mathcal{O}\to[0,\infty)$, i.e., for $f\in L^1(\mathcal{O})$,

$$(1-\eta\Delta)^{-1}\,f(\xi)=\int f(\widetilde{\xi})\;g_\eta(\xi,\widetilde{\xi})\;d\widetilde{\xi},\quad \xi\in\mathcal{O}.$$

Furthermore,

$$\int_{\mathcal{O}} I_{\mathcal{O}}(\widetilde{\xi})g_\eta(\xi,\widetilde{\xi})d\widetilde{\xi}\le 1\ \text{ for all } \xi\in\mathcal{O},$$

where $I_{\mathcal{O}}$ denotes the constant function equal to 1 on $\mathcal{O}$.

Plugging this into (3.13), by an elementary computation we obtain that the term in (3.13) is equal to

$$\begin{aligned}&\frac{1}{2\eta}\int_r^t\int_{\mathcal{O}}\int_{\mathcal{O}}((Z-K)^+(s,\widetilde{\xi})-(Z-K)^+(s,\xi))(Z(s,\widetilde{\xi})|Z(s,\widetilde{\xi})|^{m-1}+\varepsilon Z(s,\widetilde{\xi})\\ &\qquad\qquad -(Z(s,\xi)|Z(s,\xi)|^{m-1}+\varepsilon Z(s,\xi)))g_\eta(\xi,\widetilde{\xi})\,d\xi\,d\widetilde{\xi}\,ds\\ &+\int_r^t\int_{\mathcal{O}}(I_{\mathcal{O}}-(1-\eta\Delta)^{-1}I_{\mathcal{O}})((Z-K)^+(s))(Z(s)|Z(s)|^{m-1}+\varepsilon Z(s))\,d\xi\,ds.\end{aligned} \tag{3.14}$$

Since for $k\in[0,\infty)$ the maps $x\mapsto(x-k)^+$ and $x\mapsto x|x|^{m-1}+\varepsilon x$ are increasing on $\mathbb{R}$ and zero at zero, it follows that the sum in (3.14) is nonnegative. Altogether, for every $\eta>0$, we obtain

$$\begin{aligned}&\int_r^t\int_{\mathcal{O}}(Z-K)^+(s)(1-\eta\Delta)^{-1}\frac{d}{ds}(Z-K)(s)\,d\xi\,ds\\ &\quad\le-\int_r^t\int_{\mathcal{O}}(Z-K)^+(s)(1-\eta\Delta)^{-1}\left(\left(\frac{\partial}{\partial s}\mu_\varepsilon(s)+\frac{1}{2}\widetilde{\mu}\right)(Z-K)(s)\right)\,d\xi ds.\end{aligned} \tag{3.15}$$

Letting $\eta\to 0$, for $t\in[r,r+T_0]$ we obtain that

$$\int_r^t {}_{H_0^1}\!\left\langle(Z-K)^+(s),\frac{d}{ds}\,(Z-K)(s)\right\rangle_{H^{-1}}ds\le\sup_{[0,T]\times\mathcal{O}}\left|\frac{\partial}{\partial s}\mu_\varepsilon\right|\int_r^t\left\|(Z-K)^+(s)\right\|_2^2\,ds. \tag{3.16}$$

We claim that

$$\int_r^t {}_{H_0^1}\!\left\langle(Z-K)^+(s),\frac{d}{ds}\,(Z-K)(s)\right\rangle_{H^{-1}}ds=2\|(Z-K)^+(t)\|_2^2. \tag{3.17}$$

Recall that $\frac{d}{ds}$ denotes the differential for a map from (r,T) to H^{-1}, so to prove (3.17) we have to regularize again by applying $(1-\eta\Delta)^{-1}$, $\eta>0$. Then, $s\mapsto(1-\eta\Delta)^{-1}(Z-K)(s)$ is differentiable in $H_0^1(\mathcal{O})$, so the left-hand side of (3.17) by Lebesgue's dominated convergence theorem is equal to

$$\begin{aligned}\lim_{\eta\to0}&\int_r^t\int_{\mathcal{O}}((1-\eta\Delta)^{-1}(Z-K)(s))^+\frac{d}{ds}\,(1-\eta\Delta)^{-1}(Z-K)(s)d\xi\\&=\lim_{\eta\to0}\int_r^t\frac{d}{ds}\,\|((1-\eta\Delta)^{-1}(Z-K)(s))^+\|_2^2ds=2\|(Z-K)^+(t)\|_2^2,\end{aligned}$$

and (3.17) is proved. (See also Lemma 3.3 below for another argument to prove (3.17).) Now, by Gronwall's Lemma, (3.16) and (3.17) imply that

$$Z\le K\quad\text{ on } [r,r+T_0]\times\mathcal{O}.$$

Since $(-Z)$ also solves (3.2), but with initial condition $-Y_0$, the above proof also yields

$$-Z\le K\quad\text{ on } [r,r+T_0]\times\mathcal{O}.$$

Since $\|Z(0)\|_\infty<\infty$, the assertion follows by iteration and because $Z=Z_\varepsilon=e^{-\mu_\varepsilon}Y_\varepsilon$. ■

Lemma 3.2 *Suppose $Y_0\in L^{m+1}(\mathcal{O})$ such that $Y_0\ge0$ on $\mathcal{O}$ and let Y_ε be the corresponding solution to* (3.6)*. Then,*

$$Y_\varepsilon\ge0\quad\text{on } (0,T)\times\mathcal{O}. \tag{3.18}$$

Proof. Setting $Z:=-e^{-\mu_\varepsilon}Y_\varepsilon$, it suffices to prove

$$Z\le0\quad\text{on } (0,T)\times\mathcal{O}. \tag{3.19}$$

We have

$$\begin{cases}\dfrac{d}{dt}\,Z-\Delta(Z|Z|^{m-1}+\varepsilon Z)+\left(\dfrac{d}{dt}\,\mu_\varepsilon+\dfrac12\,\widetilde{\mu}\right)Z=0 & \text{on } (0,T)\times\mathcal{O}\\ Z(0)\le0 & \text{on } \mathcal{O},\\ Z=0 & \text{on } (0,T)\times\partial\mathcal{O}.\end{cases} \tag{3.20}$$

Since $\frac{dZ}{dt}\in L^2(0,T;H^{-1}(\mathcal{O}))$ and $Z\in L^2(0,T;H_0^1(\mathcal{O}))$, we have

$${}_{H^{-1}(\mathcal{O})}\left\langle\frac{d}{dt}\,Z(t),Z^+(t)\right\rangle_{H_0^1(\mathcal{O})}=\frac12\,\frac{d}{dt}\,\|Z^+(t)\|_2^2,\quad\text{a.e. } t\in(0,T).$$

Indeed, for $A=-\Delta$, $D(A)=H_0^1(\mathcal{O})\cap H^2(\mathcal{O})$ and $\eta>0$, we have

$$\begin{aligned}&{}_{H^{-1}(\mathcal{O})}\left\langle\frac{d}{dt}\,(1+\eta A)^{-1}Z(t),((1+\eta A)^{-1}Z(t))^+\right\rangle_{H_0^1(\mathcal{O})}\\&\qquad=\frac12\,\frac{d}{dt}\,\|((1+\eta A)^{-1}Z(t))^+\|_2^2,\quad\text{a.e. } t\in(0,T).\end{aligned}$$

This yields

$$\begin{aligned}\frac{1}{2}\ &\|((1+\eta A)^{-1}Z(t))^+\|_2^2-\frac{1}{2}\ \|((1+\eta A)^{-1}Z(s))^+\|_2^2\\ &=\int_s^t\ {}_{H^{-1}(\mathcal{O})}\left\langle\frac{d}{d\tau}(1+\eta A)^{-1}Z(\tau),((1+\eta A)^{-1}Z(\tau))^+\right\rangle_{H_0^1(\mathcal{O})}d\tau.\end{aligned}$$

Letting $\eta\to 0$, we get

$$\frac{1}{2}\ \|Z^+(t)\|_2^2-\frac{1}{2}\ \|Z^+(s)\|_2^2=\int_0^t\ {}_{H^{-1}(\mathcal{O})}\left\langle\frac{dZ}{d\tau},Z^+\right\rangle_{H_0^1(\mathcal{O})}d\tau,$$

for all $s\le t\le T$ which implies the desired formula.

Then, applying ${}_{H_0^1(\mathcal{O})}\langle Z^+(t),\cdot\rangle_{H^{-1}(\mathcal{O})}$ to (3.20), we obtain

$$\begin{aligned}\frac{1}{2}\ \|Z^+(t)\|_2^2\ &+\ \int_0^t\int_{\mathcal{O}}(\nabla(Z^+|Z^+|^{m-1})\cdot\nabla Z^++\varepsilon|\nabla Z^+|^2)d\xi\,ds\\ &+\ \int_0^t\int_{\mathcal{O}}|Z^+|^2\left(\frac{d}{ds}\,\mu_\varepsilon+\frac{1}{2}\,\widetilde{\mu}\right)d\xi\,ds=0.\end{aligned}\tag{3.21}$$

We conclude that $Z^+\equiv 0$ by Gronwall's lemma. ■

Lemma 3.3 *Let $Y_0\in L^\infty(\mathcal{O})$. Then there exists $\varepsilon_0\in(0,1]$ such that*

$$\sup_{\varepsilon\in(0,\varepsilon_0]}\|Y_\varepsilon\|_\infty<\infty.$$

Proof. Consider the solution $\varphi_1\in C^2(\overline{\mathcal{O}})$ to the Dirichlet problem

$$\Delta\varphi_1=-1\ \text{ in }\mathcal{O},\qquad \varphi_1=1\ \text{ on }\partial\mathcal{O}.\tag{3.22}$$

Note that $\varphi_1\ge 1$ by the maximum principle.

Now, take a partition $0=\tau_0<\tau_1<\cdots<\tau_n=T$ such that, for all $0\le i\le n-1$,

$$\begin{aligned}\sup_{t\in[\tau_i,\tau_{i+1}]}\ &\max\big(\|\nabla(\mu(t)-\mu(\tau_i))\|_\infty,\|\nabla(\mu(t)-\mu(\tau_i))\|_\infty^2,\|\Delta(\mu(t)-\mu(\tau_i))\|_\infty\big)\\ &<\frac{1}{4(m+1)^2(1+\|\varphi_1\|_\infty+\|\nabla\varphi_1\|_\infty)},\end{aligned}\tag{3.23}$$

where $\|\cdot\|_\infty$ denotes supnorm over $\mathcal{O}$. Let $\varepsilon_0\in(0,1]$ such that

$$\begin{aligned}\sup_{\varepsilon\in(0,\varepsilon_0]}\sup_{t\in[\tau_i,\tau_{i+1}]}\ &\max\big(\|\nabla(\mu_\varepsilon(t)-\mu_\varepsilon(\tau_i))\|_\infty,\|\nabla(\mu_\varepsilon(t)-\mu_\varepsilon(\tau_i))\|_\infty^2,\|\Delta(\mu_\varepsilon(t)-\mu_\varepsilon(\tau_i))\|_\infty\big)\\ &<\frac{1}{4(m+1)^2(1+\|\varphi_1\|_\infty+\|\nabla\varphi_1\|_\infty)}.\end{aligned}\tag{3.24}$$

Define the step function

$$K^\varepsilon := \sum_{i=0}^{n-1} 1_{[\tau_i,\tau_{i+1}]} K_i^\varepsilon, \tag{3.25}$$

where

$$K_i^\varepsilon := (1 + \|Y_\varepsilon(\tau_i)\|_\infty)\varphi_1^{\frac{1}{m}} e^{\|\mu_\varepsilon(\tau_i)\|_\infty + \mu_\varepsilon(\tau_i)}. \tag{3.26}$$

We shall prove by induction that for all $i \in \{0, ..., n-1\}$

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \|Y_\varepsilon(\tau_i)\|_\infty < \infty \tag{3.27}$$

by showing that

$$Y_\varepsilon \le K^\varepsilon \quad \text{on } (\tau_i, \tau_{i+1}) \times \mathcal{O} \quad \text{for } \varepsilon \in (0, \varepsilon_0]. \tag{3.28}$$

Clearly, (3.27) implies that

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \|K^\varepsilon\|_\infty < \infty,$$

and the assertion of the lemma follows (since, as in the proof of Lemma 3.1, we also get $-Y_\varepsilon \le K^\varepsilon$).

So, to prove (3.27), which holds by assumption for $i = 0$, we assume that, for some $i \in \{0, ..., n-1\}$,

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \|Y_\varepsilon(\tau_i)\|_\infty < \infty,$$

hence

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \|K_i^\varepsilon\|_\infty < \infty.$$

Fix $\varepsilon \in (0, \varepsilon_0]$ below and, for simplicity, set $K := K_i^\varepsilon$. Then, by (3.1), we have

$$\begin{aligned}
&\frac{d}{dt}(Y_\varepsilon - K) - e^{\mu_\varepsilon}\Delta(e^{-m\mu_\varepsilon}[Y_\varepsilon|Y_\varepsilon|^{m-1} + \varepsilon e^{(m-1)\mu_\varepsilon} Y_\varepsilon - (K^m + \varepsilon e^{(m-1)\mu_\varepsilon} K)]) \\
&\qquad\qquad + \frac{1}{2}\,\widetilde{\mu}(Y_\varepsilon - K) = F_\varepsilon \text{ on } (\tau_i, \tau_{i+1}) \times \mathcal{O}, \\
&(Y_\varepsilon - K)(\tau_i) \le 0 \ \text{ in } \mathcal{O}, \\
&Y_\varepsilon - K \le 0 \quad \text{on } (\tau_i, \tau_{i+1}) \times \partial\mathcal{O},
\end{aligned} \tag{3.29}$$

where, for $t \in [\tau_i, \tau_{i+1}]$,

$$\begin{aligned}
F_\varepsilon(t) &= e^{\mu_\varepsilon(t)}(1 + \|Y_\varepsilon(\tau_i)\|_\infty)^m e^{m\|\mu_\varepsilon(\tau_i)\|_\infty} \Delta(e^{-m(\mu_\varepsilon(t)-\mu_\varepsilon(\tau_i))}\varphi_1) \\
&+ \varepsilon e^{\mu_\varepsilon(t)}(1 + \|Y_\varepsilon(\tau_i)\|_\infty) e^{\|\mu_\varepsilon(\tau_i)\|_\infty} \Delta(e^{-(\mu_\varepsilon(t)-\mu_\varepsilon(\tau_i))}\varphi_1^{\frac{1}{m}}) - \frac{1}{2}\,\widetilde{\mu}\, K.
\end{aligned} \tag{3.30}$$

We note that (3.29) is, of course, meant in the sense of distributions, i.e., as an equation in $\mathcal{D}'(\mathcal{O})$ (the dual of $\mathcal{D}(\mathcal{O}) := C_0^\infty(\mathcal{O})$), since K and K^m are not in the domain of the Dirichlet Laplacian Δ. So, in (3.29) we use the symbol Δ, also to denote the usual Laplacian acting by duality on distributions. More precisely, we have

$$\begin{aligned}&\int_{\mathcal{O}} \frac{d}{dt}(Y_\varepsilon - K)\psi\, d\xi + \int_{\mathcal{O}} \nabla(e^{-m\mu_\varepsilon} Y_\varepsilon |Y_\varepsilon|^{m-1} + \varepsilon e^{-\mu_\varepsilon} Y_\varepsilon)\cdot\nabla(e^{\mu_\varepsilon}\psi)d\xi \\ &\quad + \int_{\mathcal{O}} e^{\mu_\varepsilon}\Delta(e^{-m\mu_\varepsilon}K^m + \varepsilon e^{-\mu_\varepsilon}K)\psi\, d\xi + \int_{\mathcal{O}} \frac{1}{2}\,\widetilde{\mu}(Y_\varepsilon - K)\psi\, d\xi = \int_{\mathcal{O}} F_\varepsilon \psi\, d\xi, \\ &\qquad\qquad \forall \psi \in H_0^1(\mathcal{O}),\ t\in(\tau_i,\tau_{i+1}).\end{aligned}\tag{3.31}$$

Furthermore, obviously, for $t \in [\tau_i, \tau_{i+1}]$,

$$\begin{aligned}F_\varepsilon(t) &= e^{(1-m)\mu_\varepsilon(t)} e^{m(\|\mu_\varepsilon(\tau_i)\|_\infty - \mu_\varepsilon(\tau_i))}(1+\|Y_\varepsilon(\tau_i)\|_\infty)^m \\ &\quad \cdot[-1 - 2m\nabla(\mu_\varepsilon(t)-\mu_\varepsilon(\tau_i))\cdot\nabla\varphi_1 \\ &\quad -m\varphi_1(\Delta(\mu_\varepsilon(t)-\mu_\varepsilon(\tau_i)) - m|\nabla(\mu_\varepsilon(t)-\mu_\varepsilon(\tau_i))|^2)] \\ &\quad + e^{\|\mu_\varepsilon(\tau_i)\|_\infty - \mu_\varepsilon(\tau_i)}(1+\|Y_\varepsilon(\tau_i)\|_\infty)\frac{1}{m}\varphi_1^{\frac{1}{m}-1} \\ &\quad \cdot\left[-1 + \left(\tfrac{1}{m}-1\right)\varphi_1^{-1}|\nabla\varphi_1|^2 - 2\nabla(\mu_\varepsilon(t)-\mu_\varepsilon(\tau_i))\cdot\nabla\varphi_1\right. \\ &\quad -m\varphi_1(\Delta(\mu_\varepsilon(t)-\mu_\varepsilon(\tau_i)) - |\nabla(\mu_\varepsilon(t)-\mu_\varepsilon(\tau_i))|^2)] - \frac{1}{2}\,\widetilde{\mu}\,K \\ &\le -\frac{1}{4}\, e^{(1-m)\mu_\varepsilon(t)} e^{m(\|\mu_\varepsilon(\tau_i)\|_\infty - \mu_\varepsilon(\tau_i))}(1+\|Y_\varepsilon(\tau_i)\|_\infty)^m \\ &\quad -\frac{1}{4m}\, e^{\|\mu_\varepsilon(\tau_i)\|_\infty - \mu_\varepsilon(\tau_i)}(1+\|Y_\varepsilon(\tau_i)\|_\infty)\varphi_1^{\frac{1}{m}-1} \\ &\le 0.\end{aligned}$$

Here we used that $\frac{1}{m} - 1 \le 0$, since $m \ge 1$, in the previous to last step. Hence,

$$F_\varepsilon \le 0 \ \text{ in } (\tau_i, \tau_{i+1})\times\mathcal{O}.\tag{3.32}$$

Now, we come back to (3.29) and rewrite it as

$$\begin{aligned}&\frac{dZ}{dt} - e^{\mu_\varepsilon}\Delta\widetilde{\varphi}(Z) + \frac{1}{2}\,\widetilde{\mu}Z = F_\varepsilon && \text{on } (\tau_i,\tau_{i+1})\times\mathcal{O}, \\ &Z(0) \le 0 && \text{on } \mathcal{O}, \\ &Z \le 0 && \text{on } (\tau_i,\tau_{i+1})\times\mathcal{O},\end{aligned}\tag{3.33}$$

where $Z := Y_\varepsilon - K$ and the maps $\varphi, \widetilde{\varphi} : \mathbb{R}\times\mathcal{O} \to \mathbb{R}$ are defined by

$$\begin{aligned}\varphi(r,\xi) &:= (r + K(\xi))|r+K(\xi)|^{m-1} - K^m(\xi), \\ \widetilde{\varphi}(t,r,\xi) &:= e^{-m\mu_\varepsilon(t,\xi)}\varphi(r,\xi) + \varepsilon e^{-\mu_\varepsilon(t,\xi)} r,\ (t,r,\xi)\in[0,T]\times\mathbb{R}\times\mathcal{O}.\end{aligned}$$

For simplicity of notation, here and below, for $z \in L^1_{\text{loc}}(\mathcal{O})$ we set $\varphi(z)(\xi) = \varphi(z(\xi), \xi)$, $\xi \in \mathcal{O}$. Likewise we use $\widetilde{\varphi}(z)$ (and $j(z)$ below). We note that $\varphi(\cdot, \xi)$ is monotonically increasing in r, $\varphi(0, \xi) = 0$ and so, $\varphi(r^+, \xi) = (\varphi(r, \xi))^+$. In particular, for $z \in H^1_0(\mathcal{O})$ it follows that $\nabla\varphi(z) = \nabla\varphi^+(z) = \nabla\varphi(z^+)$ on $\{z \geq 0\}$. We shall use this fact several times below. We can write (3.33), equivalently,

$$\frac{dZ}{dt} - \Delta(e^{\mu_\varepsilon}\widetilde{\varphi}(Z)) + \frac{1}{2}\,\widetilde{\mu}Z = -\Delta(e^{\mu_\varepsilon})\widetilde{\varphi}(Z) - 2\nabla(e^{\mu_\varepsilon}) \cdot \nabla\widetilde{\varphi}(Z) + F_\varepsilon. \tag{3.34}$$

We set for $(r, \xi) \in \mathbb{R} \times \mathcal{O}$

$$j(r, \xi) = \int_0^r \varphi^+(s, \xi)ds = \frac{I_{[0,\infty)}(r)}{m+1}\,((r + K(\xi))^{m+1} - K^{m+1}(\xi)) - K^m(\xi)r.$$

Define the convex function $\phi : H^{-1}(\mathcal{O}) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ by

$$\phi(z) = \begin{cases} \displaystyle\int_{\mathcal{O}} j(z(\xi), \xi)d\xi & \text{if } z \in L^1_{\text{loc}}(\mathcal{O}), \\ +\infty & \text{otherwise.} \end{cases}$$

If $\varphi^+(z) \in H^1_0(\mathcal{O})$, we have $\partial\phi(z) = -\Delta\varphi^+(z)$ in $H^{-1}(\mathcal{O})$ and, by the standard chain differentiation rule, we have (see, e.g., [9, p. 73], [2, p. 68])

$$\frac{d}{dt}\,\phi(z(t)) = \left\langle \partial\phi(z(t)), \frac{dz}{dt}(t) \right\rangle_{H^{-1}} = {}_{H^{-1}}\!\left\langle \frac{dz}{dt}\,(t), \varphi^+(z(t)) \right\rangle_{H^1_0} \quad \text{if } \varphi^+(z) \in L^2(0, T; H^1_0(\mathcal{O})).$$

We note that, if $\varphi(z) \in H^1_0(\mathcal{O})$, we have $(\varphi(z))^+ = \varphi(z^+) \in H^1_0(\mathcal{O})$ and, therefore, applying ${}_{H^1_0}\langle \varphi(Z^+), \cdot \rangle_{H^{-1}}$ to both sides of (3.34), we get

$$\begin{aligned} &\frac{d}{dt} \int_{\mathcal{O}} j(Z^+(t, \xi), \xi)d\xi + \int_{\mathcal{O}} \nabla(e^{\mu_\varepsilon}\widetilde{\varphi}(Z^+)) \cdot \nabla\varphi(Z^+)d\xi \\ &\quad + \frac{1}{2} \int_{\mathcal{O}} \widetilde{\mu}Z^+\varphi(Z^+)d\xi = -\int_{\mathcal{O}} \Delta(e^{\mu_\varepsilon})\widetilde{\varphi}(Z^+)\varphi(Z^+)d\xi \\ &\quad -2 \int_{\mathcal{O}} \nabla(e^{\mu_\varepsilon}) \cdot \nabla(\widetilde{\varphi}(Z^+))\varphi(Z^+)d\xi + \int_{\mathcal{O}} F_\varepsilon\varphi(Z^+)d\xi. \end{aligned} \tag{3.35}$$

The right-hand side of (3.35) is dominated by

$$C\left[\int_{\mathcal{O}} \varphi^2(Z^+)d\xi + (m+1)\int_{\mathcal{O}} j(Z^+)d\xi + \left(\int_{\mathcal{O}} |\nabla(e^{\mu_\varepsilon}\widetilde{\varphi}(Z^+))|^2 d\xi\right)^{\frac{1}{2}} \left(\int_{\mathcal{O}} \varphi^2(Z^+)d\xi\right)^{\frac{1}{2}}\right],$$

where C is independent of ε and we used the estimate that $r\varphi(r) \leq (m+1)j(r)$, $r \geq 0$. Furthermore, the third integral on the left-hand side of (3.35) is positive, while the integrand of the second, because

$$\nabla\varphi(Z^+) = \nabla(e^{(m-1)\mu_\varepsilon})e^{(1-m)\mu_\varepsilon}\varphi(Z^+) + e^{(m-1)\mu_\varepsilon}\nabla(e^{(1-m)\mu_\varepsilon}\varphi(Z^+))$$

and

$$e^{(1-m)\mu_\varepsilon}\varphi(Z^+) = e^{\mu_\varepsilon}\widetilde{\varphi}(Z^+) - \varepsilon Z^+,$$

can be rewritten as

$$\begin{aligned}
&\nabla(e^{\mu_\varepsilon}\widetilde{\varphi}(Z^+)) \cdot \nabla(e^{(m-1)\mu_\varepsilon})e^{(1-m)\mu_\varepsilon}\varphi(Z^+) \\
&\quad + [|\nabla(e^{\mu_\varepsilon}\widetilde{\varphi}(Z^+))|^2 - \varepsilon\nabla(e^{\mu_\varepsilon}\widetilde{\varphi}(Z^+)) \cdot \nabla Z^+]e^{(m-1)\mu_\varepsilon} \\
&\quad = \nabla(e^{\mu_\varepsilon}\widetilde{\varphi}(Z^+)) \cdot \nabla(e^{(m-1)\mu_\varepsilon})e^{(1-m)\mu_\varepsilon}\varphi(Z^+) + \frac{1}{2}\,|\nabla(e^{\mu_\varepsilon}\widetilde{\varphi}(Z^+))|^2 e^{(m-1)\mu_\varepsilon} \\
&\quad + \frac{1}{2}\,(|\nabla(e^{1-m)\mu_\varepsilon}\varphi(Z^+))|^2 - \varepsilon^2|\nabla Z^+|^2)e^{(m-1)\mu_\varepsilon}.
\end{aligned}$$

Plugging this and the previous into (3.35) and applying Young's inequality, we obtain for some (other) constant $C > 0$ independent of ε, after integrating (since $Z^+(\tau_i) = 0$), that for a.e. $t \in [\tau_i, \tau_{i+1}]$

$$\begin{aligned}
&\int_{\mathcal{O}} j(Z^+(t,\xi),\xi)d\xi + \int_{\tau_i}^t \int_{\mathcal{O}} (|\nabla(e^{\mu_\varepsilon}\widetilde{\varphi}(Z^+))|^2 + |\nabla(e^{(1-m)\mu_\varepsilon}\varphi(Z^+))|^2)d\xi\,ds \\
&\quad \leq C\int_{\tau_i}^t \int_{\mathcal{O}} (\varphi^2(Z^+(t,\xi),\xi) + j(Z^+(t,\xi),\xi))d\xi\,ds + C\varepsilon^2 \int_{\tau_i}^t \int_{\mathcal{O}} |\nabla Z^+|^2 d\xi\,ds.
\end{aligned} \tag{3.36}$$

We proceed, similarly, by applying ${}_{H_0^1}\langle Z^+, \cdot\rangle_{H^{-1}}$ to both sides of (3.34) and, after integration, get (with C independent of ε)

$$\begin{aligned}
&\int_{\mathcal{O}} |Z^+(t,\xi)|^2 d\xi + \int_{\tau_i}^t \int_{\mathcal{O}} \nabla(e^{(1-m)\mu_\varepsilon}\varphi(Z)) \cdot \nabla Z^+ d\xi\,ds + \varepsilon\int_{\tau_i}^t \int_{\mathcal{O}} |\nabla Z^+|^2 d\xi\,ds \\
&\quad \leq C\int_{\tau_i}^t \int_{\mathcal{O}} |\widetilde{\varphi}(Z^+)||Z^+|d\xi\,ds + C\int_{\tau_i}^t \int_{\mathcal{O}} |\nabla(e^{\mu_\varepsilon}\widetilde{\varphi}(Z))||Z^+|d\xi\,ds \\
&\quad \leq \int_{\tau_i}^t \int_{\mathcal{O}} (C\varphi^2(Z^+) + C|Z^+|^2)d\xi\,ds + \frac{\varepsilon}{4}\int_{\tau_i}^t \int_{\mathcal{O}} |\nabla Z^+|^2 d\xi\,ds \\
&\quad + C\int_{\tau_i}^t \int_{\mathcal{O}} |\nabla(e^{(1-m)\mu_\varepsilon}\varphi(Z^+))|^2)d\xi\,ds.
\end{aligned} \tag{3.37}$$

We have

$$\begin{aligned}
\nabla(e^{(1-m)\mu_\varepsilon}\varphi(Z,\xi)) \cdot \nabla Z^+ &= (\nabla(e^{(1-m)\mu_\varepsilon}) \cdot \nabla Z^+)\varphi(Z,\xi) + e^{(1-m)\mu_\varepsilon}\varphi'(Z,\xi)|\nabla Z^+|^2 \\
&\quad + e^{(1-m)\mu_\varepsilon}\varphi_\xi(Z,\xi) \cdot \nabla Z^+,
\end{aligned}$$

where $\varphi'(r,\xi) = \frac{\partial\varphi}{\partial r}$, $\varphi_\xi = \frac{\partial\varphi}{\partial\xi}$. This yields, since $\varphi'(r,\xi) \geq 0$,

$$\nabla(e^{(1-m)\mu_\varepsilon}\varphi(Z)) \cdot \nabla Z^+ \geq -C(|\nabla Z^+|\varphi(Z^+,\xi) + |\nabla Z^+ \cdot \varphi_\xi(Z,\xi)|).$$

On the other hand, we have

$$\begin{aligned}|\nabla Z^+ \cdot \varphi_\xi(Z,\xi)| &= m|\nabla K \cdot \nabla Z^+||(Z^+ + K)^{m-1} - K^{m-1}|\\ &\le CZ^+|\nabla Z^+|\sup\{|r+K|^{m-2};\ 0 \le r \le Z^+\}\\ &\le C\max\{K^{m-2}, \|Z^+ + K\|_\infty^{m-2}\}Z^+|\nabla Z^+| \le \frac{\varepsilon}{4}\ |\nabla Z^+|^2 + C_\varepsilon |Z^+|^2.\end{aligned}$$

Then, by (3.36) and (3.37), we obtain for ε sufficiently small

$$\int_{\mathcal{O}} (j(Z^+(t,\xi)) + |Z^+(t,\xi)|^2)d\xi \le C_\varepsilon \int_{\tau_i}^t \int_{\mathcal{O}} (|\varphi(Z^+(s,\xi))|^2 + |Z^+(s,\xi)|^2)d\xi\, ds,\ \forall t \in [\tau_i, \tau_{i+1}].$$

Taking into account that

$$(\varphi(s))^2 \le Cj(s)(1+s^{m-1}),\ \forall s,$$

and that $Z^+ \in L^\infty(\mathcal{O})$, we get by Gronwall's lemma that $j(Z^+) = 0$ and, therefore, $Z^+ = 0$ on $[\tau_i, \tau_{i+1}]$. Hence, $Y_\varepsilon \le K_i^\varepsilon$ on $[\tau_i, \tau_{i+1}]$ and (3.27), (3.28) follow by induction. ■

Now, we can complete the proof of the existence part of Theorem 2.1. We shall use the following identity which is easy to check for all $u \in H_0^1(\mathcal{O})$:

$$\begin{aligned} e^{\mu_\varepsilon}\Delta(e^{-m\mu_\varepsilon}u) &= e^{\frac{1}{2}(1-m)\mu_\varepsilon}\Delta(e^{\frac{1}{2}(1-m)\mu_\varepsilon}u)\\ &+ \frac{1}{2}(m+1)\left[\frac{1}{2}(m+1)|\nabla\mu_\varepsilon|^2 - \Delta\mu_\varepsilon\right]e^{(1-m)\mu_\varepsilon}u\\ &- (m+1)e^{\frac{1}{2}(1-m)\mu_\varepsilon}\nabla\mu_\varepsilon \cdot \nabla(e^{\frac{1}{2}(1-m)\mu_\varepsilon}u).\end{aligned} \tag{3.38}$$

Using this in (3.1) for $u := Y_\varepsilon(s)|Y_\varepsilon(s)|^{m-1}$, $s \in [0,T]$, and for $m = 1$ and $u := Y_\varepsilon(s)$ for the term with ε in front, and applying subsequently ${}_{H_0^1}\langle Y_\varepsilon|Y_\varepsilon|^{m-1}, \cdot\rangle_{H^{-1}}$ to the resulting equation, we find (applying the same arguments to the left-hand side as in Lemma 3.3) that

$$\begin{aligned}&\frac{1}{m+1}\ \frac{d}{dt}\int_{\mathcal{O}} |Y_\varepsilon|^{m+1}d\xi + \int_{\mathcal{O}} |\nabla(Y_\varepsilon|Y_\varepsilon|^{m-1}e^{\frac{1}{2}(1-m)\mu_\varepsilon})|^2 d\xi + \varepsilon m\int_{\mathcal{O}} |Y_\varepsilon|^{m-1}|\nabla Y_\varepsilon|^2 d\xi\\ &\quad +(m+1)\int_{\mathcal{O}} (e^{\frac{1}{2}(1-m)\mu_\varepsilon}Y_\varepsilon|Y_\varepsilon|^{m-1})\nabla\mu_\varepsilon \cdot \nabla(e^{\frac{1}{2}(1-m)\mu_\varepsilon}Y_\varepsilon|Y_\varepsilon|^{m-1})d\xi\\ &\quad +2\varepsilon\int_{\mathcal{O}} Y_\varepsilon|Y_\varepsilon|^{m-1}\nabla\mu_\varepsilon \cdot \nabla Y_\varepsilon d\xi\\ &\quad = \frac{1}{2}(m+1)\int_{\mathcal{O}} e^{(1-m)\mu_\varepsilon}|Y_\varepsilon|^{2m}\left[\frac{1}{2}(m+1)|\nabla\mu_\varepsilon|^2 - \Delta\mu_\varepsilon\right]d\xi\\ &\quad +\varepsilon\int_{\mathcal{O}} [|\nabla\mu_\varepsilon|^2 - \Delta\mu_\varepsilon]|Y_\varepsilon|^{m+1}\,d\xi,\ \text{a.e. } t \in (0,T),\end{aligned}$$

consequently, since

$$\sup_{\substack{[0,T]\times\mathcal{O}\\ \varepsilon>0}} (|\nabla\mu_\varepsilon| + |\Delta\mu_\varepsilon|) < \infty,$$

there exists $C \in (0,\infty)$ independent of $\varepsilon > 0$ such that for a.e. $t \in [0,T]$, we get

$$\int_{\mathcal{O}} |Y_\varepsilon(t,\xi)|^{m+1} d\xi + \int_0^t \int_{\mathcal{O}} |\nabla (e^{\frac{1}{2}(1-m)\mu_\varepsilon} |Y_\varepsilon|^{m-1} Y_\varepsilon)|^2 d\xi\, ds$$
$$\leq C \left(\int_{\mathcal{O}} |Y_0(\xi)|^{m+1} d\xi + \int_0^t \int_{\mathcal{O}} |Y_\varepsilon(s,\xi)|^{2m} d\xi\, ds \right),\ \forall \varepsilon > 0,\ t \in [0,\tau].$$

Recalling that, by Lemma 3.3, $\{Y_\varepsilon\}$ is bounded in $L^\infty((0,\tau)\times\mathcal{O})$, we obtain that

$$\int_{\mathcal{O}} |Y_\varepsilon(t,\xi)|^{m+1} d\xi + \int_0^t \int_{\mathcal{O}} |\nabla (|Y_\varepsilon(t,\xi)|^{m-1} Y_\varepsilon(t,\xi))|^2 d\xi\, ds \leq C, \quad \text{a.e. } t \in (0,\tau),\ \forall \varepsilon \in (0,\varepsilon_0]. \tag{3.39}$$

We note that, from here, all arguments below also work for $m \in (0,1]$ with $m \geq \frac{1}{5}$, if $d=3$, which we shall refer to in Subsection 3.3 below. (3.39) is equivalent to

$$\sup_{t\in(0,T)} \|Y_\varepsilon(t)\|_{m+1} + \int_0^T \||Y_\varepsilon(s)|^{m-1} Y_\varepsilon(s)\|^2_{H^1_0(\mathcal{O})} ds \leq C, \quad \forall \varepsilon \in (0,\varepsilon_0]. \tag{3.40}$$

Hence, along a subsequence $\varepsilon \to 0$, we have

$$Y_\varepsilon \ \to\ Y \qquad \text{weak-star in } L^\infty(0,T;L^{m+1}(\mathcal{O})), \tag{3.41}$$

$$|Y_\varepsilon|^{m-1}Y_\varepsilon \ \to\ \eta \qquad \text{weakly in } L^2(0,T;H^1_0(\mathcal{O})). \tag{3.42}$$

Hence,

$$\frac{dY_\varepsilon}{dt} \ \to\ \frac{dY}{dt} \qquad \text{weakly in } L^2(0,T;H^{-1}(\mathcal{O})), \tag{3.43}$$

since, by (3.42), also,

$$e^{\mu_\varepsilon}\Delta(e^{-m\mu_\varepsilon}|Y_\varepsilon|^{m-1}Y_\varepsilon) \ \to\ e^{\mu}\Delta(e^{-m\mu}\eta) \quad \text{weakly in } L^2(0,T;H^{-1}(\mathcal{O})), \tag{3.44}$$

and, since, by (3.40) and the Sobolev embedding theorem, $\{Y_\varepsilon\}$ is bounded in $L^2(0,T;L^{6m}(\mathcal{O}))$, and, therefore,

$$Y_\varepsilon \ \to\ Y \qquad \text{weakly in } L^2(0,T;H^{-1}(\mathcal{O})), \tag{3.45}$$

hence

$$\widetilde{\mu}Y_\varepsilon \ \to\ \widetilde{\mu}Y \qquad \text{weakly in } L^2(0,T;H^{-1}(\mathcal{O})), \tag{3.46}$$

because $m \geq \frac{1}{5}$ if $d=3$. Hence, we may pass to the limit in (3.1) and obtain

$$\begin{cases} \dfrac{dY}{dt} - e^{\mu}\Delta(e^{-m\mu}\eta) + \dfrac{1}{2}\widetilde{\mu}Y = 0 & \text{a.e. } t \in (0,T), \\ Y(0) = Y_0 & \text{in } \mathcal{O}, \\ \eta = 0 & \text{on } (0,T)\times\partial\mathcal{O}. \end{cases} \tag{3.47}$$

(We note that $\varepsilon e^{\mu_\varepsilon}\Delta(e^{-\mu_\varepsilon}Y_\varepsilon) \to 0$ in $\mathcal{D}'((0,T)\times\mathcal{O})$ by virtue of (3.45).)

In order to complete the proof of existence, we must show that

$$\eta(t,\xi) = |Y(t,\xi)|^{m-1}Y(t,\xi) \ \text{ a.e. } (t,\xi)\in(0,T)\times\mathcal{O}. \tag{3.48}$$

To prove (3.48), it suffices to check the inequality

$$\limsup_{\varepsilon\to 0}\int_0^T\int_{\mathcal{O}} e^{(1-m)\mu_\varepsilon}|Y_\varepsilon|^{m+1}d\xi\,dt \le \int_0^T\int_{\mathcal{O}} e^{(1-m)\mu}\eta Y\,d\xi\,dt. \tag{3.49}$$

Indeed, if (3.49) holds, then by the inequality

$$\int_0^T\int_{\mathcal{O}} e^{(1-m)\mu_\varepsilon}(|Y_\varepsilon|^{m-1}Y_\varepsilon - |z|^{m-1}z)(Y_\varepsilon - z)d\xi\,dt \ge 0,\ \forall z\in L^{m+1}((0,T)\times\mathcal{O}),$$

we get by (3.49) and (3.42)

$$\int_0^T\int_{\mathcal{O}} e^{(1-m)\mu}(\eta - |z|^{m-1}z)(Y-z)d\xi\,dt \ge 0,\ \forall z\in L^{m+1}((0,T)\times\mathcal{O}), \tag{3.50}$$

and, if we take $z=z^*$, the solution to the equation $|z|^{m-1}z + z = \eta + Y$ a.e. in $(0,T)\times\mathcal{O}$, we obtain that $z^* = Y$ and $\eta = |Y|^{m-1}Y$, as claimed.

We note that, since $x\mapsto |x|^{m-1}x + x$ has a Lipschitz inverse, we conclude that z^* has the same integrability property as $\eta + Y$, i.e., $z^*\in L^\infty((0,T)\times\mathcal{O})$ in this case, where $m>1$. But, if $m\in(0,1]$, we still have $z^*\in L^{m+1}((0,T)\times\mathcal{O})$, since $\eta\in L^2(0,T;L^6(\mathcal{O}))$ and $Y\in L^{m+1}((0,T)\times\mathcal{O})$. This is used in the proof of Theorem 2.2 below.

To prove (3.49), we apply $\langle Y_\varepsilon,\cdot\rangle_{-1}$ to (3.1) and integrate over $(0,T)$ to obtain

$$\begin{aligned}
&\frac{1}{2}\left(\|Y_\varepsilon(\tau)\|_{-1}^2 - \|Y_0\|_{-1}^2\right) + \int_0^T\int_{\mathcal{O}} e^{(1-m)\mu_\varepsilon}|Y_\varepsilon|^{m+1}d\xi\,dt + \varepsilon\int_0^T\int_{\mathcal{O}}|Y_\varepsilon|^2 d\xi\,dt\\
&\quad = -\int_0^T\int_{\mathcal{O}}(-\Delta)^{-1}(Y_\varepsilon)(e^{-m\mu_\varepsilon}|Y_\varepsilon|^{m-1}Y_\varepsilon + \varepsilon e^{-\mu_\varepsilon}Y_\varepsilon)\Delta(e^{\mu_\varepsilon})d\xi\,dt\\
&\quad -2\int_0^T\int_{\mathcal{O}}(-\Delta)^{-1}(Y_\varepsilon)\nabla(e^{\mu_\varepsilon})\cdot\nabla(e^{-m\mu_\varepsilon}|Y_\varepsilon|^{m-1}Y_\varepsilon + \varepsilon e^{-\mu_\varepsilon}Y_\varepsilon)d\xi\,dt\\
&\quad -\frac{1}{2}\int_0^T\int_{\mathcal{O}}(-\Delta)^{-1}(Y_\varepsilon)Y_\varepsilon\widetilde{\mu}\,d\xi\,dt.
\end{aligned} \tag{3.51}$$

We also have that

$$\begin{aligned}
&\varepsilon\left|\int_0^T\int_{\mathcal{O}}(-\Delta)^{-1}(Y_\varepsilon)\nabla(e^{\mu_\varepsilon})\cdot\nabla(e^{\mu_\varepsilon}Y_\varepsilon)d\xi\,dt\right|\\
&\quad \le C\varepsilon\int_0^T\|Y_\varepsilon\|_{-1}\|Y_\varepsilon\|_2 dt \le \frac{\varepsilon}{4}\int_0^T\|Y_\varepsilon\|_2^2 dt + C_1\varepsilon\int_0^T|Y_\varepsilon(t)|_{-1}^2 dt.
\end{aligned}$$

On the other hand, we see by (3.41), (3.43) that $\{(-\Delta)^{-1}Y_\varepsilon\}$ is bounded in $L^\infty((0,T);W^{2,m+1}(\mathcal{O}))$ (because $\{Y_\varepsilon\}$ is bounded in $L^\infty((0,T);L^{m+1}(\mathcal{O}))$ and $\{\frac{d}{dt}(-\Delta)^{-1}Y_\varepsilon\}$ is bounded in $L^2(0,T;H_0^1(\mathcal{O}))$). Then, by the Aubin–Lions compactness theorem (see [5, Theorem 5.1 in Chap. 1]), and since $W^{2,m+1}(\mathcal{O}) \subset L^6(\mathcal{O})$ compactly if $m \geq \frac{1}{5}$ and $d=3$, and for all $m \in (0,\infty)$ if $d=1,2$, we may pass to the limit in (3.51) along a subsequence and get

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \int_0^T \int_{\mathcal{O}} e^{(1-m)\mu_\varepsilon}|Y_\varepsilon|^{m+1} d\xi\, dt &\leq -\frac{1}{2}\left(|Y(\tau)|_{-1}^2 - |Y_0|_{-1}^2\right) \\
&- \int_0^T \int_{\mathcal{O}} (-\Delta)^{-1}(Y) e^{-m\mu}|Y|^{m-1} Y \Delta(e^{-\mu}) d\xi\, dt \\
&-2 \int_0^T \int_{\mathcal{O}} (-\Delta)^{-1} Y \nabla(e^{\mu}) \cdot \nabla(e^{-m\mu}|Y|^{m-1}Y) d\xi\, dt \\
&-\frac{1}{2}\int_0^T \int_{\mathcal{O}} (-\Delta)^{-1}(Y) Y \widetilde{\mu}\, d\xi\, dt = I.
\end{aligned} \tag{3.52}$$

On the other hand, by (3.47) we obtain by a similar computation that

$$\int_0^T \int_{\mathcal{O}} e^{(1-m)\mu} \eta Y\, dt\, d\xi = I,$$

which proves (3.49).

Remark 3.4 *If $\beta_1,\dots,\beta_N$ in the definition of μ are independent Brownian motions on a stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P})$, then clearly the solution Z_ε to (3.2) and hence the solution Y_ε to (3.6) are $(\mathcal{F}_{t+\sigma(\varepsilon)})$-adapted with $\sigma(\varepsilon)\to 0$ as $\varepsilon \to 0$. Hence as a limit of these the solution Y to (2.1) constructed in the previous proof is $(\mathcal{F}_t)$-adapted.*

3.2 Uniqueness of solutions in Theorem 2.1

Let Y_1, Y_2 be two solutions to equation (2.1) satisfying conditions (i) and (ii). We set

$$\chi = \begin{cases} \dfrac{Y_1|Y_1|^{m-1} - |Y_2|^{m-1}Y_2}{Y_1 - Y_2} & \text{on } [Y_1 \neq Y_2], \\ 0 & \text{on } [Y_1 = Y_2]. \end{cases}$$

We set also $z = Y_1 - Y_2$. Then, by (2.1), we have

$$\begin{aligned}
&\frac{dz}{dt} - \Delta(e^{(1-m)\mu}\chi z) + e^{-m\mu}\chi z \Delta(e^{\mu}) + 2\nabla(e^{\mu})\cdot\nabla(e^{-m\mu}\chi z) + \frac{1}{2}\widetilde{\mu} z = 0 \text{ in } (0,T)\times\mathcal{O}, \\
&z(0,\xi) = 0, \qquad \text{a.e. } \xi \in \mathcal{O}, \\
&z = 0 \qquad \text{on } (0,T)\times\partial\mathcal{O}.
\end{aligned}$$

Equivalently,

$$(-\Delta)^{-1}\left(\frac{dz}{dt}\right)+e^{(1-m)\mu}\chi z+(-\Delta)^{-1}(e^{-m\mu}\chi z\Delta(e^{\mu})+2\nabla(e^{\mu})\cdot\nabla(e^{-m\mu}\chi z))$$
$$+\frac{1}{2}(-\Delta)^{-1}(\widetilde{\mu}z)=0.$$

We multiply the latter by z and integrate on $(0,T)\times\mathcal{O}$. We get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|z(t)\|_{-1}^2+\int_{\mathcal{O}}e^{(1-m)\mu}\chi z^2d\xi\\ &\quad=-\int_{\mathcal{O}}(-\Delta)^{-1}(e^{-m\mu}\chi z\Delta(e^{\mu}))z\,d\xi-2\int_{\mathcal{O}}(-\Delta)^{-1}(\nabla(e^{\mu})\cdot\nabla(e^{-m\mu}\chi z))z\,d\xi\\ &\quad-\frac{1}{2}\int_{\mathcal{O}}(-\Delta)^{-1}(\widetilde{\mu}z)z\,d\xi=I_1(t)+I_2(t)+I_3(t),\ \text{a.e. } t\in(0,T).\end{aligned}\tag{3.53}$$

We have via Sobolev embedding (see, e.g., [1, p .217])

$$\begin{aligned}|I_1(t)|&\le C\|z(t)\|_{-1}\|(\chi z)(t)\|_{-1}\\ &\le C\|z(t)\|_{-1}\|(\chi z)(t)\|_{\frac{m+1}{m}}\\ &\le C\|z(t)\|_{-1}\left(\int_{\mathcal{O}}e^{(1-m)\mu(t)}\chi(t)z^2(t)d\xi\right)^{\frac{1}{2}}\left(\int_{\mathcal{O}}(\chi(t))^{\frac{m+1}{m-1}}d\xi\right)^{\frac{m-1}{2(m+1)}}\\ &\le C\|z(t)\|_{-1}^2+\frac{1}{4}\int_{\mathcal{O}}e^{(1-m)\mu(t)}(\chi z^2)(t)d\xi\end{aligned}\tag{3.54}$$

because $|\chi(t)|\le C(|Y_1|^{m-1}+|Y_2|^{m-1})\in L^{\frac{m+1}{m-1}}((0,T)\times\mathcal{O})$.

Similarly, we have

$$\begin{aligned}|I_2(t)|&\le C\|z(t)\|_{-1}\|\nabla(e^{-m\mu(t)}\chi(t)z(t))\|_{-1}\\ &\le C\|z(t)\|_{-1}\|\chi(t)z(t)\|_2\le C\|z(t)\|_{-1}^2+\frac{1}{4}\int_{\mathcal{O}}\chi(t)|z(t)|^2e^{(1-m)\mu(t)}d\xi,\end{aligned}\tag{3.55}$$

because $\chi\in L^{\infty}((0,T)\times\mathcal{O})$, and

$$|I_3(t)|\le C\|z(t)\|_{-1}^2.\tag{3.56}$$

Substituting (3.54)–(3.56) into (3.53), we obtain

$$\begin{aligned}&\frac{d}{dt}\|z(t)\|_{-1}^2+\int_{\mathcal{O}}e^{(1-m)\mu(t)}\chi(t)|z(t)|^2d\xi\le C\|z(t)\|_{-1}^2,\\ &z(0)=0,\end{aligned}\tag{3.57}$$

which, clearly, implies $z(t)\equiv 0$, as claimed.

3.3 Proof of Theorem 2.2

The proof of existence for $Y_0 \in L^{m+1}(\mathcal{O})$ is exactly the same as that of Theorem 2.1, with the observation that for the solution Y_ϵ to equation (3.1), by (3.39) we get in the case $m \in (0,1]$, with $m \geq \frac{1}{5}$ if $d = 3$, the estimate

$$\|Y_\epsilon(t)\|_{m+1}^{m+1} + \int_0^t \int_{\mathcal{O}} |\nabla(e^{\frac{1}{2}(1-m)\mu_\varepsilon} |Y_\varepsilon|^{m-1} Y_\varepsilon)|^2 d\xi\, dt \leq C\|Y_0\|_{m+1}^{m+1}, \ \forall \epsilon > 0.$$

Then, we can proceed as in the proof of Theorem 2.1 to get the existence of a solution on the interval $[0,T]$, which satisfies (ii) in Theorem 2.1.

We furthermore emphasize that in the proof of Lemma 3.2 we did not use that $m > 1$. So, the last assertion of Theorem 2.1 also holds here. To prove boundedness of Y, if $Y_0 \in L^\infty(\mathcal{O})$, one just replaces $\varphi^{\frac{1}{m}}$ in the definition of K_i^ε (see (3.26)) by φ. Then again $F_\varepsilon \leq 0$ and one can prove Lemma 3.3 for $m \in (0,1]$, with $m \geq \frac{1}{5}$ if $d = 3$, exactly analogously as in Subsection 3.1. Then boundedness of Y follows as in the proof of Theorem 2.1.

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