Stochastic porous media equations and self-organized criticality: convergence to the critical state in all dimensions

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Abstract

If $X = X(t, \xi)$ is the solution to the stochastic porous media equation in $\mathcal{O} \subset \mathbb{R}^d$, $1 \leq d \leq 3$, modelling the self-organized criticality [5] and X_c is the critical state, then it is proved that $\int_0^\infty m(\mathcal{O} \setminus \mathcal{O}_0^t)dt < \infty$, $\mathbb{P}$-a.s. and $\lim_{t\to\infty} \int_{\mathcal{O}} |X(t) - X_c|d\xi = \ell < \infty$, $\mathbb{P}$-a.s. Here, m is the Lebesgue measure and $\mathcal{O}_c^t$ is the critical region $\{\xi \in \mathcal{O};\ X(t, \xi) = X_c(\xi)\}$ and $X_c(\xi) \leq X(0, \xi)$ a.e. $\xi \in \mathcal{O}$. If the stochastic Gaussian perturbation has only finitely many modes (but is still function-valued), $\lim_{t\to\infty} \int_K |X(t) - X_c|d\xi = 0$ exponentially fast for all compact $K \subset \mathcal{O}$ with probability one, if the noise is sufficiently strong. We also recover that in the deterministic case $\ell = 0$.

Key words and phrases: porous media equation, multiplicative noise, self-organized criticality, Ito formula.

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1 Introduction

The self-organized criticality is a property of dynamical systems which have a critical point as an attractor and which emerges spontaneously to this attractor. If $X = X(t, \xi)$, $t \geq 0$, $\xi \in \mathcal{O} \subset \mathbb{R}^d$, $d = 1, 2, 3$, is the state of the system distributed in the spatial domain $\mathcal{O}$ and if $X_c = X_c(\xi)$ is a critical state, then $X(t, \cdot)$ divides the space into the following three spatial regions:

$$
\begin{array}{rlcl}
\textit{critical region} & \mathcal{O}_c^t & = & \{\xi \in \mathcal{O};\ X(t,\xi) = X_c(\xi)\}, \\
\textit{subcritical region} & \mathcal{O}_-^t & = & \{\xi \in \mathcal{O};\ X(t,\xi) < X_c(\xi)\}, \\
\textit{supercritical region} & \mathcal{O}_+^t & = & \{\xi \in \mathcal{O};\ X(t,\xi) > X_c(\xi)\},
\end{array}
$$

The main feature of the self-criticality phenomena is that the subcritical and supercritical regions are unstable and absorbed in time by the critical region via an autonomous mechanism. The standard model of self-organized criticality is the celebrated *sand-pile* model introduced by Bak, Tang and Wiesenfeld [1], which is formalized via automation theory ([2]) and leads to parabolic nonlinear equations of porous media type

$$
\frac{\partial X}{\partial t} = a\Delta H(X - X_c) \ \text{ in } \ (0,\infty) \times \mathcal{O}, \tag{1.1}
$$

where $a > 0$ and H is the Heaviside functions. (See, also, [3] for a complete description of this model.) In the presence of a stochastic Gaussian perturbation, the model is best described by the stochastic (porous media) equation

$$
\begin{aligned}
& dX(t) - a\Delta H(X(t) - X_c) = \sigma(X(t) - X_c)dW_t \ \text{ in } (0,\infty) \times \mathcal{O}, \\
& X(0) = x \ \text{ in } \mathcal{O}.
\end{aligned} \tag{1.2}
$$

In [5], existence and uniqueness of solutions to (1.2) are shown and it is also proved that we have finite-time extinction of $t \to X(t) - X_c$ with positive probability in $1 - D$. In terms of self-organized criticality behavior, this means that the subcritical and supercritical regions are absorbed in finite-time with positive probability by the critical region $\mathcal{O}_c^t$. Our aim here is to establish a similar result in dimensions $d = 2, 3$, at least asymptotically. The first main result, Theorem 2.2 below, amounts to saying that "for almost all $\{t_n\} \to \infty$" we have

$$
\lim_{n\to\infty} m(\mathcal{O} \setminus \mathcal{O}_c^{t_n}) = 0, \ \mathbb{P}\text{-a.s.}, \tag{1.3}
$$

where m is the Lebesgue measure on $\mathcal{O}$ and we assume for the initial state that $x \geq X_c$ a.e. in $\mathcal{O}$. The second main result, Theorem 2.3 below, says that $X(t)$, multiplied by the exponential of the function-valued noise, converges to X_c in $L^1(\mathcal{O})$ asymptotically and that, if the noise is nondegenerate away from the boundary of $\mathcal{O}$ (see (2.10) below), then $X(t)$ itself converges asymptotically to X_c locally in $L^1(\mathcal{O})$ exponentially fast.

Notation. In the following, $\mathcal{O}$ is a bounded and open subset of $\mathbb{R}^d$, $d \geq 1$, with smooth boundary, and $L^p(\mathcal{O})$, $1 \leq p \leq \infty$, is the space of all p-integrable functions in$\mathcal{O}$ with the usual norm denoted by $|\cdot|_p$. For $k = 1, 2$, $H^k(\mathcal{O})$, $H_0^1(\mathcal{O})$ and $H^{-1}(\mathcal{O})$ are standard Sobolev spaces on $\mathcal{O}$. More precisely, $H_0^1(\mathcal{O})$ is the subspace of functions $u \in H^1(\mathcal{O})$ with zero trace on the boundary $\partial\mathcal{O}$ of $\mathcal{O}$ and $H = H^{-1}(\mathcal{O})$ is the dual of $H_0^1(\mathcal{O})$ with the norm

$$|u|_{-1} = \langle A^{-1}u, u\rangle_2^{\frac{1}{2}}.$$

Here, $A = -\Delta$, $D(A) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$ and $\langle \cdot, \cdot \rangle_2$ is the scalar product of $L^2(\mathcal{O})$.

Everywhere in the following, $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}$ is a stochastic basis and $\{\beta_j\}_{j=1}^\infty$ is a sequence of mutually independent Brownian motions which induces the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. By $L^q(0, T; L^p(\Omega, Y))$, where Y is a Hilbert space, we denote the space of all q-integrable processes $u : (0, T) \to L^p(\Omega, Y)$. By $C([0, T]; L^2(\Omega, Y))$ we denote the space of all Y-valued processes which are mean-square continuous on $[0, T]$.

2 Hypotheses and the main result

Consider the equation

$$(2.1) \qquad \begin{aligned} & dX(t) - a\Delta\psi(X(t))dt \ni \sigma(X(t))dW_t \ \text{ in } (0, \infty) \times \mathcal{O}, \\ & X(0, \xi) = x(\xi), \quad \xi \in \mathcal{O}, \\ & \psi(X(t, \xi)) \ni 0, \quad \text{on } (0, \infty) \times \partial\mathcal{O}. \end{aligned}$$

Here, a is a positive constant and

(H1) $\psi(r) = \text{sign } r$,

where sign $r = r|r|^{-1}$ if $r \neq 0$, sign $0 = [-1, 1]$, (H2) $\sigma(X)dW = \sum_{k=1}^{\infty} \mu_k X e_k d\beta_k$,

where $\{\mu_k\}$ is a sequence of real numbers such that

$$\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^2 < \infty \tag{2.2}$$

and $\{e_k\}$ is the orthonormal basis in $L^2(\mathcal{O})$ consisting of eigenvectors of A with eigenvalues $\{\lambda_k\}$, that is, $Ae_k = \lambda_k e_k$, $k = 1,\dots.$ Here $\{\lambda_j\}_{j=1}^{\infty}$ is taken in increasing order.

Definition 2.1 Let $x \in H = H^{-1}(\mathcal{O})$. An H-valued continuous $\mathcal{F}_t$ adapted process $X = X(t)$ is said to be a solution to equation (2.1) if, on every interval $(0,T)$, $T > 0$,

$$X \in L^1(\Omega \times (0,T) \times \mathcal{O}) \cap L^2(0,T; L^2(\mathcal{O}, H)) \tag{2.3}$$

and there is $\eta \in L^1(\Omega \times (0,T) \times \mathcal{O})$ such that

$$\begin{aligned} \langle X(t), e_j \rangle_2 = \langle x, e_j \rangle_2 &+ \int_0^t \int_{\mathcal{O}} \eta(s,\xi) \Delta e_j(\xi) d\xi\, ds \\ &+ \sum_{k=1}^{\infty} \mu_k \int_0^t \langle X(s) e_k, e_j \rangle_2\, d\beta_k(s),\ \forall j \in \mathbb{N},\ t \in [0,T], \end{aligned} \tag{2.4}$$

$$\eta \in \psi(X) \ \text{ a.e. on } (0,T) \times \mathcal{O} \times \Omega. \tag{2.5}$$

One of the main results established in [5] (see, also, [4]) is that, for each $x \in L^p(\mathcal{O})$, $p \geq 4$, *there is a unique solution* $X \in L^\infty(0,T; L^p(\Omega, \mathcal{O})) \cap L^2(\Omega, C([0,T]; H^{-1}(\mathcal{O})))$ *to equation* (2.1). *Moreover, if* $x \geq 0$ *a.e. in* $\mathcal{O}$, *then* $X \geq 0$ *a.e.,* $\mathbb{P}$*-a.s.* (See [5, Theorem 2.2].) Other existence results for the stochastic porous media equation (2.1) for general maximal monotone functions ψ with the range all of R were established in [6], [9].

In this paper we prove the following asymptotic results for solutions to equation (2.1).

Theorem 2.2 *Assume that* (H1) *and* (H2) *hold and that* $x \in L^4(\mathcal{O})$*,* $x \geq 0$*, on* $\mathcal{O}$*. Then*

$$\lim_{t\to\infty} \int_{\mathcal{O}} X(t,\xi)d\xi = \ell < \infty, \ \mathbb{P}\text{-}a.s. \tag{2.6}$$

and

$$\int_0^\infty m(\mathcal{O} \setminus \mathcal{O}_0^t)dt < \infty, \ \mathbb{P}\text{-}a.s., \tag{2.7}$$

where m *is the Lebesgue measure and*

$$\mathcal{O}_0^t = \{\xi \in \mathcal{O}; \ X(t,\xi) = 0\}, \quad t \geq 0.$$

As mentioned earlier, Theorem 2.2 applies to the self-organized stochastic model (1.2), that is,

$$\begin{aligned} &dX(t) - a\Delta \operatorname{sign}(X(t) - X_c)dt = \sigma(X(t) - X_c)dW_t, \\ &X(0) = x - X_c \ \text{ in } \mathcal{O}. \end{aligned} \tag{2.8}$$

If $x - X_c \geq 0$ a.e. in $\mathcal{O}$, then $X(t) - X_c \geq 0$ a.e. on $\mathcal{O}$ for all $t \geq 0$, $\mathbb{P}$-a.s. and so, by Theorem 2.2, it follows that $m(\mathcal{O} \setminus \mathcal{O}_0^t) \in L^1(0,\infty)$, $\mathbb{P}$-a.s., which roughly speaking means that, "for almost all sequences" $\{t_n\} \to \infty$, we have $m(\mathcal{O} \setminus \mathcal{O}_0^t) \to 0$, $\mathbb{P}$-a.s.

As regards the asymptotic result (2.6), one might expect that $\ell = 0$, $\mathbb{P}$-a.s. Indeed, this is the case in the deterministic case (see [3]). For equation (2.1), we have

$$\lim_{t\to\infty} X(t) = 0 \quad \text{in } L^1_{\text{loc}}(\mathcal{O}),$$

if the Gaussian noise $\sigma(X)W$ has a finite number of modes, that is,

$$\sigma(X)W(t) = \sum_{k=1}^N \mu_k e_k X(t)\beta_k(t) \text{ on } (0,\infty) \times \mathcal{O} \tag{2.9}$$

and

$$\widetilde{\mu}(\xi) = \sum_{k=1}^N \mu_k^2 e_k(\xi) > 0, \quad \forall \xi \in \mathcal{O}. \tag{2.10}$$

More precisely,

Theorem 2.3 *Consider the situation of Theorem 2.2. In addition, assume that* (2.9) *holds and set*

$$\mu(t) = -\sum_{k=1}^{N} \mu_k e_k \beta_k(t), \quad t \geq 0.$$

Then:

(i) $\lim_{t\to\infty} e^{\mu(t)} X(t) = 0$ *in* $L^1(\mathcal{O})$, $\mathbb{P}$-*a.s. In particular,* $\ell = 0$ *in the deterministic case* (*cf.* [3]).

(ii) *If, additionally,* (2.10) *holds, then*

$$\lim_{t\to\infty} X(t) = 0 \quad \text{in } L^1_{\text{loc}}(\mathcal{O}) \quad \mathbb{P}\text{-a.s.}.$$

Moreover, for each compact set $K \subset \mathcal{O}$,

(2.11)
$$\int_K X(t,\xi)d\xi \leq |x|_2 m(K)^{1/2} \exp\left(\sup_K(\widetilde{\mu})^{1/2} \left(\sum_{k=1}^{N} \beta_k(t)\right)^{1/2}\right) e^{-\frac{t}{2}\inf_{K'}\widetilde{\mu}},$$
$$t \geq 0,\ \mathbb{P}\text{-a.s.},$$

where $K' \subset \mathcal{O}$ *is any compact neighborhood of* K. *In particular* (*by the law of the iterated logarithm for Brownian motion*), *there exists a constant* $\rho_K > 0$ *such that, for* $\mathbb{P}$-*a.e.* $\omega \in \Omega$

(2.12)
$$\int_K X(t,\xi,\omega)d\xi \leq |x|_2 m(K)^{1/2} e^{-\rho_K t}, \quad \forall t \geq t_0(\omega).$$

We note that, if $|\mu_1| > 0$, then assumption (2.10) holds, because the first eigenfunction of the Laplace operator is strictly positive on $\mathcal{O}$ (see, e.g., [7, p. 340]).

The proofs of Theorems 2.2 and 2.3 are given in Sections 3 and 4, respectively. For simplicity, we take $a = 1$ in (2.1).

3 Proof of Theorem 2.2

Consider the approximating equation

$$
\begin{aligned}
& dX_\lambda(t) - \Delta(\psi_\lambda(X_\lambda(t)) + \lambda X_\lambda(t))dt = \sigma(X_\lambda(t))dW_t \ \text{ in } (0,\infty)\times\mathcal{O},\\
& X_\lambda(0) = x \ \text{ on } \mathcal{O}, \\
& X_\lambda = 0 \ \text{ on } (0,\infty)\times\partial\mathcal{O},
\end{aligned} \tag{3.1}
$$

where $\lambda \in (0,1)$ and

$$
\psi_\lambda(r) = \frac{1}{r}\,(1-(1+\lambda\psi)^{-1}(r)) = \begin{cases} \dfrac{r}{\lambda} & \text{if } |r| \le \lambda, \\ 1 & \text{if } r > \lambda, \\ -1 & \text{if } r < -\lambda. \end{cases}
$$

(Here 1 is the identity map.) As shown in [4, Theorem 2.2] and [5, Proposition 3.5], equation (3.1) has a unique solution X_λ in the sense of Definition 2.1 and

$$
X_\lambda \in L^2(\Omega, C([0,T];H)) \cap L^2(0,T;L^2(\Omega,H_0^1(\mathcal{O}))).
$$

Moreover, since $x\in L^4(\mathcal{O})$ and $x \ge 0$, also $X_\lambda \ge 0$ on $(0,\infty)\times\mathcal{O}\times\Omega$ and, as proved in [5], for $\lambda\to 0$, we have for all $T>0$,

$$
\begin{aligned}
X_\lambda \ \ \to \ \ X \ \ & \text{weakly in } L^2(\Omega\times(0,T)\times\mathcal{O}),\\
& \text{weak}^* \text{ in } L^\infty(0,T;L^2(\Omega,L^2(\mathcal{O})) \text{ and}\\
& \text{strongly in } L^2(\Omega;C([0,T];H)),
\end{aligned} \tag{3.2}
$$

and

$$
\psi_\lambda(X_\lambda)) + \lambda X_\lambda \ \ \to \ \ \eta \ \ \text{ weakly in } L^2(\Omega\times(0,T)\times\mathcal{O}) \tag{3.3}
$$

$$
\eta \in \psi(X) \ \text{ a.e. on } \Omega\times(0,\infty)\times\mathcal{O}. \tag{3.4}
$$

By Ito's formula and the monotonicity of ψ_λ, we have (cf. [9, Proof of Theorem 2.8 and Remark 2.9(iii)])

$$
\begin{aligned}
\frac{1}{2}\,|X_\lambda(t)|_2^2 \ &+ \ \int_0^t\int_{\mathcal{O}} \nabla(\psi_\lambda(X_\lambda)+\lambda X_\lambda)\cdot\nabla X_\lambda d\xi\,ds \\
&= \ \frac{1}{2}\,|x|_2^2 + \frac{1}{2}\sum_{k=1}^\infty \mu_k^2 \int_0^t\int_{\mathcal{O}} |X_\lambda(s)e_k|^2 d\xi\,ds \\
&+ \ \sum_{k=1}^\infty \mu_k \int_0^t\int_{\mathcal{O}} X_\lambda^2(s) e_k d\xi d\beta_k(s), \ t\in[0,\infty), \ \mathbb{P}\text{-a.s.},
\end{aligned}
$$

and $t \to X_\lambda(t) \in L^2(\mathcal{O})$ is continuous $\mathbb{P}$-a.s.

This yields, since $\|e_k\|_\infty \le C\lambda_k$, because $d \le 3$,

$$
\begin{aligned}
|X_\lambda(t)|_2^2 \;&+\; 2\lambda \int_0^t |\nabla X_\lambda(s)|_2^2 ds \le |x|_2^2 + C \sum_{k=1}^\infty \mu_k^2 \lambda_k^2 \int_0^t |X_\lambda(s)|_2^2 ds \\
&+\; \sum_{k=1}^\infty \mu_k \int_0^t \int_{\mathcal{O}} X_\lambda^2(s) e_k d\xi \, d\beta_k(s).
\end{aligned}
\tag{3.5}
$$

Then, by (2.2) and by the Burkholder–Davis–Gundy inequality, we obtain the estimate that for some constant $C_T > 0$ and all $\lambda \in (0,1)$

$$
\mathbb{E} \sup_{t\in[0,T]} |X_\lambda(t)|_2^2 + \lambda \mathbb{E} \int_0^T |\nabla X_\lambda(s)|_2^2 ds \le C_T |x|_2^2. \tag{3.6}
$$

To see this we consider the real-valued martingale

$$
N_t := \sum_{k=1}^\infty \mu_k \int_0^t \int_{\mathcal{O}} X_\lambda^2(s) e_k d\xi d\beta_k(s),\; t \in [0,T].
$$

We recall that since $x \in L^4(\mathcal{O})$, by [5, Lemma 3.1] the above sum converges in $L^2(\Omega;\; C([0,T];\; \mathbb{R}))$ by (2.2). Then (3.5), the Burkholder-Davis-Gundy inequality for $p = 1$ and Fubini's theorem imply that for some constants $C_1, C_2 > 0$

$$
\begin{aligned}
\mathbb{E} \sup_{t\in[0,T]} |X_\lambda(t)|_2^2 &\le |x|_2^2 + C_1 \int_0^T \mathbb{E}|X_\lambda(s)|_2^2 \, ds \\
&\quad + 3\mathbb{E}\left[\left\{ \sum_{k=1}^\infty \mu_k^2 \int_0^T \left(\int_{\mathcal{O}} X_\lambda^2(s) e_k d\xi \right)^2 ds \right\}^{\frac{1}{2}} \right] \\
&\le |x|_2^2 + C_1 \int_0^T \mathbb{E}|X_\lambda(s)|_2^2 \, ds \\
&\quad + \left(\sum_{k=1}^\infty \mu_k^2 ||e_k||_\infty^2 \right)^{\frac{1}{2}} \mathbb{E}\left[\sup_{s\in[0,T]} |X_\lambda(s)|_2 \left(\int_0^T X_\lambda(s)|_2^2 ds \right)^{\frac{1}{2}} \right] \\
&\le |x|_2^2 + C_1 \int_0^T \mathbb{E}|X_\lambda(s)|_2^2 \, ds \\
&\quad + C_2 \left(\mathbb{E} \sup_{t\in[0,T]} |X_\lambda(t)|_2^2 \right)^{\frac{1}{2}} \left(\int_0^T \mathbb{E}|X_\lambda(s)|_2^2 \, ds \right)^{\frac{1}{2}}.
\end{aligned}
\tag{3.7}
$$

Hence Young's inequality and Gronwall's Lemma imply that

$$\mathbb{E}\sup_{t\in[0,T]}|X_\lambda(t)|_2^2\le C_t|x|_2^2 \tag{3.8}$$

for some $c_t\in(0,\infty)$. But since the right hand side of (3.7) also dominates the second term in the left hand side of (3.5), by (3.8) we deduce (3.6).

Now, arguing as in Proposition 3.5 in [5], we consider a function $\varphi_\lambda\in C_b^3(\mathbb{R})$ such that $\varphi_\lambda(0)=0$ and

$$\begin{aligned}&\varphi'_\lambda(r)=\frac{r}{\lambda}\text{ for }|r|\le\lambda,\quad \varphi'_\lambda(r)=1+\lambda\text{ for }r\ge 2\lambda,\\ &\varphi'_\lambda(r)=-1-\lambda\text{ for }r\le-2\lambda\text{ and }0\le\varphi''_\lambda(r)\le\frac{C}{\lambda},\end{aligned} \tag{3.9}$$

for all $r\in\mathbb{R}$ and some $C>0$.

This is a smooth approximation of the function $r\to|r|$ and it is easily seen that

$$|\varphi'_\lambda(r)-\psi_\lambda(r)|\le C\lambda,\quad\forall r\in\mathbb{R},\ \lambda>0. \tag{3.10}$$

We set $Y_\lambda^\varepsilon=(1+\varepsilon A)^{-1}X_\lambda$ and note that

$$\begin{aligned}&dY_\lambda^\varepsilon(t)+A(1+\varepsilon A)^{-1}(\psi_\lambda(X_\lambda(t))+\lambda X_\lambda(t))dt=(1+\varepsilon A)^{-1}\sigma(X_\lambda(t))dW_t\\ &Y_\lambda^\varepsilon(0)=(1+\varepsilon A)^{-1}X_\lambda(0),\quad\varepsilon>0.\end{aligned}$$

Also, the process $t\to Y_\lambda^\varepsilon(t)$ is continuous $H_0^1(\mathcal{O})$-valued on $[0,T]$. Then, by Ito's formula applied to the H_0^1-valued process Y_λ^ε, we have

$$\begin{aligned}&\int_{\mathcal{O}}\varphi_\lambda(Y_\lambda^\varepsilon(t,\xi))d\xi\\ &\quad+\int_0^t\int_{\mathcal{O}}\nabla((1+\varepsilon A)^{-1}(\psi_\lambda(X_\lambda(s,\xi)+\lambda X_\lambda(s,\xi))\cdot\nabla\varphi'_\lambda(Y_\lambda^\varepsilon(s,\xi))d\xi\,ds\\ &\quad=\int_{\mathcal{O}}\varphi_\lambda((1+\varepsilon A)^{-1}x)d\xi\\ &\quad+\sum_{k=1}^\infty\mu_k^2\int_0^t\int_{\mathcal{O}}\varphi''_\lambda(Y_\lambda^\varepsilon(s,\xi))|(1+\varepsilon A)^{-1}(X_\lambda e_k)(s,\xi)|^2ds\,d\xi\\ &\quad+\sum_{k=1}^\infty\mu_k\int_0^t\left\langle\varphi'_\lambda(Y_\lambda^\varepsilon(s,\xi)),(1+\varepsilon A)^{-1}(X_\lambda e_k)(s,\xi))\right\rangle_2 d\beta_k(s),\\ &\qquad\qquad\qquad\qquad\forall t\ge0,\ \mathbb{P}\text{-a.s.}\end{aligned} \tag{3.11}$$

Now, recalling that $X_\lambda \in L^2(0,T;L^2(\Omega,H_0^1(\mathcal{O}))$ for all $\lambda>0$, we have that $Y_\lambda^\varepsilon \to X_\lambda$ strongly in $L^2(0,T;H_0^1(\mathcal{O})), \mathbb{P}$-a.s. as $\varepsilon \to 0$. Similarly, for all $T>0$, we have

$$\begin{aligned}\nabla((1+\varepsilon A)^{-1}(\psi_\lambda(X_\lambda)+\lambda X_\lambda)) &\to \nabla(\psi_\lambda(X_\lambda)+\lambda X_\lambda)\\ (1+\varepsilon A)^{-1}(X_\lambda e_k) &\to X_\lambda e_k\end{aligned}$$

strongly in $L^2(0,T;L^2\mathcal{O}))$, $\mathbb{P}$-a.s. as $\varepsilon\to 0$.

Furthermore, it is easy to see that by (3.6) and the Burkholder–Davis–Gundy inequality for $p=1$, the stochastic term converges in $L^1(\Omega;C([0,T],L^2(\mathcal{O}))$, as $\varepsilon\to 0$. Also, the first term in (3.11) converges for a.e. $t\in[0,T]$ after passing to a subsequence $\varepsilon_n\to 0$. So, altogether, we obtain

$$\begin{aligned}&\int_{\mathcal{O}} \varphi_\lambda(X_\lambda(t))d\xi\\ &\quad+\int_0^t ds\int_{\mathcal{O}}\nabla(\psi_\lambda(X_\lambda(s))+\lambda X_\lambda(s))\cdot\nabla\varphi'_\lambda(X_\lambda(s))d\xi\\ &\quad=\int_{\mathcal{O}}\varphi_\lambda(x)d\xi+\sum_{k=1}^\infty \mu_k^2\int_0^t\int_{\mathcal{O}}\varphi''_\lambda(X_\lambda(s))|X_\lambda(s)e_k|^2 d\xi\,ds\\ &\quad+\sum_{k=1}^\infty\mu_k\int_0^t\left\langle X_\lambda(s)e_k,\varphi'_\lambda(X_\lambda(s))\right\rangle_2 d\beta_k(s)\text{ for a.e. } t>0,\ \mathbb{P}\text{-a.s.}\end{aligned}\tag{3.12}$$

On the other hand, by the $L^2(\mathcal{O})$-continuity of X_λ it follows that the first term in (3.12) is continuous, as are all the other terms in (3.12). Hence, (3.12) holds for all $t\geq 0$, $\mathbb{P}$-a.s.

On the other hand, by (3.9) we have the following estimate

$$\begin{aligned}&\sum_{k=1}^\infty\mu_k^2\int_0^t\int_{\mathcal{O}}\varphi''_\lambda(X_\lambda)|X_\lambda e_k|^2 d\xi\,ds\\ &\qquad\leq 4\lambda C\sum_{k=1}^\infty\mu_k^2\lambda_k^2\int_0^t\int_{\mathcal{O}}\mathbb{1}_\lambda(s,\xi)d\xi\,ds,\ \mathbb{P}\text{-a.s.},\end{aligned}\tag{3.13}$$

where $\mathbb{1}_\lambda$ is the characteristic function of the set

$$\{(s,\xi,\omega)\in(0,\infty)\times\mathcal{O}\times\Omega;\ 0\leq X_\lambda(s,\xi,\omega)\leq 2\lambda\}.$$

Now, we prove

$$(3.14)\qquad \lim_{\lambda\to 0}\int_{\mathcal{O}}\varphi_\lambda(X_\lambda(t,\xi))d\xi=\int_{\mathcal{O}}X(t,\xi)d\xi,\quad \forall t\ge 0 \text{ weakly in } L^2(\Omega).$$

Indeed, by (3.9) we have for fixed $t\ge 0$

$$\begin{aligned}\int_{\mathcal{O}}\varphi_\lambda(X_\lambda(t,\xi))d\xi &= (1+\lambda)\int_{[X_\lambda(t,\xi)\ge 2\lambda]}X_\lambda(t,\xi)d\xi\\ &\quad+\frac{1}{2\lambda}\int_{[X_\lambda\le\lambda]}X_\lambda^2(t,\xi)d\xi+\int_{[\lambda\le X_\lambda\le 2\lambda]}\varphi_\lambda(X_\lambda(t,\xi))d\xi.\end{aligned}$$

Taking into account (3.2) and that $\varphi_\lambda(r)\le C\lambda$ for $r\in[\lambda,2\lambda]$, this yields

$$(3.15)\qquad \int_{\mathcal{O}}\varphi_\lambda(X_\lambda(t,\xi))d\xi=\int_{\mathcal{O}}X_\lambda(t,\xi)d\xi+o(\lambda),\ \text{a.e. } \forall t\ge 0,\ \mathbb{P}\text{-a.s.}$$

We also note that, by (3.2), we have

$$(3.16)\qquad X_\lambda(t)\ \to\ X(t)\quad \text{weakly in } L^2(\Omega\times\mathcal{O}),\quad \text{for all } t>0.$$

(Indeed, $\{X_\lambda(t)\}$ is strongly convergent to $X(t)$ in $L^2(\Omega;H)$ for each $t\in[0,\infty)$ and is bounded in $L^2(\Omega\times\mathcal{O})$ for all $t\in[0,\infty)$.)

Then, by (3.15) and (3.16) we find that

$$\lim_{\lambda\to 0}\int_{\mathcal{O}}\varphi_\lambda(X_\lambda(t,\xi))d\xi=\int_{\mathcal{O}}X(t,\xi)d\xi,\quad \forall t\ge 0 \text{ weakly in } L^2(\Omega),$$

as claimed.

Now, we set

$$(3.17)\qquad I_\lambda(t)=\int_0^t\int_{\mathcal{O}}(\nabla\psi_\lambda(X_\lambda)+\lambda\nabla X_\lambda)\cdot\nabla\varphi'_\lambda(X_\lambda)d\xi\,ds,\ t\ge 0,$$

$$(3.18)\qquad M_\lambda(t)=\sum_{k=1}^\infty\mu_k\int_0^t\langle X_\lambda e_k,\varphi'_\lambda(X_\lambda)\rangle_2\,d\beta_k(s),\ t\ge 0.$$

We recall that, by (H2),

$$\begin{aligned}M_\lambda(t) &= \int_0^t\langle\varphi'_\lambda(X_\lambda(s)),\sigma(X_\lambda(s))dW(s)\rangle_2\\ &= \int_0^t\langle\sigma(X_\lambda(s))^*\varphi'(X_\lambda(s)),dW(s)\rangle_2\,.\end{aligned}$$

where, for $h \in L^2(\mathcal{O})$,

$$\sigma(X_\lambda(s))h = \sum_{k=1}^{\infty} \mu_k \langle e_k, h \rangle_2 X_\lambda(s) e_k.$$

We shall prove below that, for the adjoint operators $\sigma(X_\lambda(s))^*$ on $L^2(\mathcal{O})$ we have, for all $T > 0$,

$$\text{(3.19)} \quad \sigma(X_\lambda)^* \varphi'_\lambda(X_\lambda) \to \sigma(X)^* \eta \text{ weakly in } L^2((0,T) \times \mathcal{O} \times \Omega) \text{ as } \lambda \to 0.$$

This implies that

$$\text{(3.20)} \qquad \begin{aligned} \lim_{\lambda \to 0} M_\lambda(t) = M(t) = \sum_{k=1}^{\infty} \mu_k \int_0^t \langle X(s) e_k, \eta \rangle_2 \, d\beta_k(s) \\ \text{weakly in } L^2(\Omega), \ \forall t \geq 0. \end{aligned}$$

Now, let us prove (3.19). First, we note that by (3.3), (3.4), (3.6) and (3.10) as $\lambda \to 0$

$$\text{(3.21)} \qquad X_\lambda \to X \text{ and } \varphi'_\lambda(X_\lambda) \to \eta \text{ weakly in } L^2((0,T) \times \mathcal{O} \times \Omega),$$

and that, by (3.9) and (3.4),

$$\text{(3.22)} \qquad |\varphi'_\lambda(X_\lambda)|_\infty, |\eta|_\infty \leq 2,$$

where the norm refers to $L^\infty((0,T) \times \mathcal{O} \times \Omega)$. (3.22) implies that, for some constant $C = C(T, \mathcal{O}) > 0$

$$\begin{aligned} &\mathbb{E} \int_0^T |\sigma(X_\lambda(s))^* \varphi'(X_\lambda(s))|_2^2 ds \\ &\quad = \mathbb{E} \int_0^T \sup_{|h|_2 \leq 1} \left\langle \varphi'(X_\lambda(s)), \sum_{k=1}^{\infty} \mu_k \langle e_k, h \rangle_2 X_\lambda(s) e_k \right\rangle_2^2 ds \\ &\quad \leq C \sum_{k=1}^{\infty} \mu_k^2 \lambda_\lambda^2 \mathbb{E} \int_0^T |X_\lambda(s)|^2 ds, \end{aligned}$$

which, by (3.21) is uniformly bounded for $\lambda \in (0,1)$. Hence

$$\text{(3.23)} \qquad \{\sigma(X_\lambda)^* \varphi'(X_\lambda)\}_{\lambda \in (0,1]} \text{ is bounded in } L^2((0,T) \times \mathcal{O} \times \Omega).$$

Now, let $F \in L^\infty((0,T)\times\Omega; H_0^1(\mathcal{O}))$. Then

$$
\begin{aligned}
&\left|\mathbb{E}\int_0^T \langle F(s), \sigma(X(s))^*\eta(s) - \sigma(X_\lambda(s))^*\varphi'_\lambda(X_\lambda(s))\rangle_2\, ds\right| \\
&\quad\le \left|\mathbb{E}\int_0^T \langle F(s), \sigma(X(s))(\eta(s)-\varphi'_\lambda(X_\lambda(s)))\rangle_2\, ds\right| \\
&\quad + \left|\mathbb{E}\int_0^T \langle F(s), \sigma(X(s)-X_\lambda(s))\varphi'_\lambda(X_\lambda(s))\rangle_2\, ds\right| \\
&\quad\le \left|\sum_{k=1}^N \mu_k \mathbb{E}\int_0^T \langle \langle F(s), X(s)e_k\rangle_2\, e_k, \eta(s)-\varphi'_\lambda(X_\lambda(s))\rangle_2\, ds\right| \\
&\quad + \mathbb{E}\int_0^T \left(\sum_{k=N+1}^\infty \mu_k^2 \langle e_k, \eta(s)-\varphi'_\lambda(X_\lambda(s))\rangle_2^2\right)^{\frac{1}{2}} |F(s)X(s)|_2 ds \\
&\quad + \left|\sum_{k=1}^N \mu_k \mathbb{E}\int_0^T \langle e_k, \varphi'_\lambda(X_\lambda(s))\rangle_2 \langle F(s)e_k, X(s)-X_\lambda(s)\rangle_2\, ds\right| \\
&\quad + \mathbb{E}\int_0^T \left(\sum_{k=N+1}^\infty \mu_k^2 \langle e_k, \varphi'_\lambda(X_\lambda(s))\rangle_2^2\right)^{\frac{1}{2}} |F(s)(X(s)-X_\lambda(s))|_2 ds
\end{aligned}
$$

of which the second and fourth term by (3.21), (3.22) and the boundedness of F converge to zero uniformly in $\lambda \in (0,1)$ as $N \to \infty$. By (3.21), the same is true for the first term for each fixed N as $\lambda \to 0$. Furthermore, the third term is up to a constant $C(T,\mathcal{O}) > 0$ dominated by

$$
|F|_{L^\infty((0,T)\times\mathcal{O};H_0^1)} \sum_{k=1}^N \mu_k \mathbb{E}\int_0^T |X(s)-X_\lambda(s)|_{-1}^2 ds,
$$

which, for each fixed N as $\lambda \to 0$, also converges to zero by (3.2). Hence, first letting $\lambda \to 0$ and then $N \to \infty$ and, using (3.23), we obtain (3.19).

Then, by (3.12), (3.13), (3.14) and (3.20), we have

$$
\int_{\mathcal{O}} X(t,\xi)d\xi + \widetilde{I}(t) = \int_{\mathcal{O}} x(\xi)d\xi + M(t),\ \forall t \ge 0,\ \mathbb{P}\text{-a.s.}, \tag{3.24}
$$

where

$$
\widetilde{I}(t) = w - \lim_{\lambda\to 0} I_\lambda(t), \quad t \ge 0, \tag{3.25}
$$

and $w-\lim_{\lambda\to 0}$ denotes weak limit in $L^2(\mathcal{O})$.

We set

$$Z(t)=\int_{\mathcal{O}} X(t,\xi)d\xi,\ t\geq 0.$$

We see that Z is a nonnegative semimartingale with $EZ(t)<\infty$, $\forall t\geq 0$.

By (3.6) and (3.2) and lower-semicontinuity, it follows that, for all $T>0$,

$$\mathbb{E}\left[\sup_{t\in[0,T]}|X(t)|_2^2\right]<\infty, \tag{3.26}$$

where we note that $\sup_{t\in[0,T]}|X(t)|_2^2 = \operatorname{ess\,sup}_{t\in[0,T]}|X(t)|_2^2$ since $\mathbb{P}$-a.s. $t\mapsto|X(t)|_2^2$ is lower-semicontinuous by Definition 2.1. The latter then together with (3.26) implies that $\mathbb{P}$-a.s. the function $t\to X(t)$ is weakly continuous in $L^2(\mathcal{O})$ on $[0,\infty)$ and so the function $t\to Z(t)$ is $\mathbb{P}$-a.s. continuous on $[0,\infty)$. Define

$$I(t):=Z(0)-Z(t)+M(t),\quad t\geq 0,$$

then I is a continuous version of $\widetilde{I}$. We note that, clearly, by (3.25) for all $0\leq s\leq t$

$$\widetilde{I}(s)\leq\widetilde{I}(t),\ \mathbb{P}\text{-a.s.}$$

with the $\mathbb{P}$-exceptional set depending on s,t.

Hence (first considering all rational $s,t\in[0,\infty)$, $0\leq s\leq t$), we conclude by continuity that

$$I(s)\leq I(t),\quad\forall\ 0\leq s\leq t,\ \mathbb{P}\text{-a.s.},$$

i.e. I is a $\mathbb{P}$-a.s. nondecreasing process.

Hence, altogether we have

$$Z(t)+I(t)=Z(0)+M(t),\quad\forall t\geq 0,$$

where M is a continuous local martingale and I is an a.s. nondecreasing process. Then, by [8, p. 139] we may conclude that

$$\exists\lim_{t\to\infty}Z(t)<\infty,\ I(\infty)<\infty,\ \mathbb{P}\text{-a.s.} \tag{3.27}$$

It follows therefore that there exists

$$\ell=\lim_{t\to\infty}\int_{\mathcal{O}} X(t,\xi)d\xi,\ \mathbb{P}\text{-a.s.} \tag{3.28}$$

Fix $t \geq 0$. Noting that $\mathbb{P}$-a.s.

$$
\begin{aligned}
I_\lambda(t) &\geq \int_0^t \nabla\psi_\lambda(X_\lambda(s))\cdot\nabla\varphi'_\lambda(X_\lambda(s))ds \\
&= \int_0^t \nabla\psi_\lambda(X_\lambda(s))\cdot\nabla\psi_\lambda(X_\lambda(s))ds,
\end{aligned} \tag{3.29}
$$

it follows by (3.3), (3.6) and (3.25) that, as $\lambda \to 0$,

$$\psi_\lambda(X_\lambda) \to \eta \text{ weakly in } L^2((0,T)\times\Omega; H_0^1(\mathcal{O})). \tag{3.30}$$

This, as well as (3.25), remains true if $\mathbb{P}$ is replaced by $\rho\cdot\mathbb{P}$ for every $\rho \in L^\infty(\Omega)$, $\rho \geq 0$. Hence (3.29) and (3.25) imply

$$\mathbb{E}\left[\int_0^t |\nabla\eta|_2^2 ds\ \rho\right] \leq \liminf_{\lambda\to 0}\mathbb{E}[I_\lambda(t)\rho] = \mathbb{E}[\widetilde{I}(t)\rho].$$

Since $\rho \in L^\infty(\Omega)$, $\rho \geq 0$, was arbitrary, this implies that

$$\int_0^t |\nabla\eta|_2^2 ds \leq \widetilde{I}(t),\ \mathbb{P}\text{-a.s.}$$

Hence, by continuity,

$$\int_0^t |\nabla\eta|_2^2 ds \leq I(t),\quad \forall\ t \geq 0,\ \mathbb{P}\text{-a.s.}$$

and, consequently, by (3.27),

$$\lim_{t\to 0}\int_0^t |\nabla\eta|_2^2 ds \leq I(\infty) < \infty,\ \mathbb{P}\text{-a.s.} \tag{3.31}$$

Now, by the Sobolev embedding theorem, we have by (3.31) that

$$\int_0^\infty dt \left(\int_{\mathcal{O}} |\eta|^{p^*} d\xi\right)^{\frac{2}{p^*}} < \infty,\ \mathbb{P}\text{-a.s.}, \tag{3.32}$$

where $\frac{1}{p^*} = \frac{1}{2} - \frac{1}{d}$ for $d > 2$, $p^* \in [2,\infty)$ for $d = 2$ and $p^* = \infty$ for $d = 1$.

Recalling that $\eta \in \text{sign}\ X = 1$ on $[X \neq 0]$, a.e. on $(0,\infty)\times\mathcal{O}\times\Omega$, it follows by (3.32) that

$$\int_0^\infty (m(\mathcal{O}\setminus\mathcal{O}_0^t))^{\frac{2}{p^*}} dt < \infty,\ \mathbb{P}\text{-a.s.},$$

which implies (2.7), as claimed. This completes the proof of Theorem 2.2.

4 Proof of Theorem 2.3

Assume in this section that (2.9) holds. We recall that

$$\mu = -\sum_{k=1}^{N} \mu_k e_k \beta_k, \quad \widetilde{\mu} = \sum_{k=1}^{N} \mu_k^2 e_k^2 \tag{4.1}$$

and that the initial datum x belongs to $L^4(\mathcal{O})$.

Take

$$Y(t) = e^{\mu(t)} X(t), \quad \forall\, t \geq 0. \tag{4.2}$$

Then we have (see [5, Lemma 4.1])

$$\begin{aligned}
&\frac{d}{dt}\, Y(t) = e^{\mu} \Delta \psi(e^{-\mu} Y) - \frac{1}{2}\, \widetilde{\mu} Y, && \forall t \geq 0,\ \mathbb{P}\text{-a.s.},\\
&Y(0) = x && \text{on } \mathcal{O},\\
&\psi(e^{-\mu} Y) \in H_0^1(\mathcal{O}), && \forall t \geq 0,\ \mathbb{P}\text{-a.s.},
\end{aligned} \tag{4.3}$$

where the derivative $\frac{d}{dt}$ is taken in $H^{-1}(\mathcal{O})$. (Recall that $\psi(r) = \operatorname{sign} r$ and in (4.3), by Definition 2.1, there arises a section η of $\operatorname{sign}(e^{-\mu} Y)$.)

First, we shall establish a few estimates on the solution Y to (4.3), which have also an interest in themselves.

Lemma 4.1 *We have*

$$|Y(t)|_2 \leq |x|_2, \quad \forall t \geq 0, \mathbb{P}\text{-}a.s. \tag{4.4}$$

Proof. Consider the solution Y_λ to the approximating equation

$$\begin{aligned}
&\frac{dY_\lambda}{dt} = e^{\mu} \Delta(\psi_\lambda(e^{-\mu} Y_\lambda) + \lambda e^{-\mu} Y_\lambda) - \frac{1}{2}\, \widetilde{\mu} Y_\lambda,\\
&Y_\lambda(0) = X, \quad Y_\lambda \in L^2(0, T; H_0^1(\mathcal{O})),
\end{aligned} \tag{4.5}$$

which corresponds to (3.1), i.e. $Y_\lambda = e^{\mu} X_\lambda$. Multiplying (4.5) by Y_λ and integrating over $\mathcal{O}$, we obtain

$$\begin{aligned}
\frac{1}{2}\, \frac{d}{dt}\, |Y_\lambda(t)|_2^2 &+ \int_{\mathcal{O}} \nabla(\psi_\lambda(e^{-\mu} Y_\lambda) + \lambda e^{-\mu} Y_\lambda) \nabla(e^{\mu} Y_\lambda) d\xi\\
&= -\frac{1}{2} \int_{\mathcal{O}} Y_\lambda^2 \widetilde{\mu} d\xi \leq 0, \ \text{for a.e. } t > 0,
\end{aligned} \tag{4.6}$$

because $\widetilde{\mu} \geq 0$, a.e. on $\mathcal{O} \times \Omega$. On the other hand, recalling that

$$\nabla(\psi_\lambda)(z) = \begin{cases} \dfrac{1}{\lambda}\nabla z & \text{if } |z| < \lambda, \\ 0 & \text{if } |z| \geq \lambda, \end{cases} \tag{4.7}$$

we get, by (4.6),

$$\begin{aligned} \frac{1}{2}\,\frac{d}{dt}\,|Y_\lambda(t)|_2^2 \;&+\; \frac{1}{\lambda}\int_{\mathcal{O}} \mathbb{1}_\lambda^{**} e^{2\mu}[|\nabla X_\lambda|^2 + 2X_\lambda \nabla X_\lambda \cdot \nabla\mu]d\xi \\ &+\; \lambda\int_{\mathcal{O}} e^{2\mu}[|\nabla X_\lambda|^2 + 2X_\lambda \nabla X_\lambda \cdot \nabla\mu]d\xi \leq 0 \quad \text{a.e. } t \geq 0, \end{aligned}$$

where $\mathbb{1}_\lambda^{**}$ is the characteristic function of $\{(t,\xi);\ 0 \leq (e^{-\mu}Y_\lambda)(t,\xi) \leq \lambda\}$. This yields

$$\begin{aligned} \frac{d}{dt}\,|Y_\lambda(t)|_2^2 \;&\leq\; 2\int_{\mathcal{O}} \left(\frac{1}{\lambda}\,\mathbb{1}_\lambda^{**} + \lambda\right)|X_\lambda|^2 e^{2\mu}|\nabla\mu|^2 d\xi \\ &\leq\; 2\lambda\int_{\mathcal{O}} (1 + |X_\lambda|^2)e^{2\mu}|\nabla\mu|^2 d\xi, \quad \text{a.e. } t > 0. \end{aligned}$$

Integrating, we obtain

$$|Y_\lambda(t)|_2^2 \leq |x|_2^2 + 2\lambda\int_0^t |(1 + X_\lambda^2)^{1/2} e^\mu |\nabla\mu||_2^2 ds, \quad \forall t \geq 0,\ \mathbb{P}\text{-a.s.} \tag{4.8}$$

Defining $Y_\lambda^{(N)} := X_\lambda(e^\mu \wedge N)$ and $Y^{(N)} := X(e^\mu \wedge N)$, $N \in \mathbb{N}$, we deduce from (3.2) that for all $\rho \in L^\infty(\Omega)$, $\rho \geq 0$, as $\lambda \to \infty$,

$$Y_\lambda^{(N)} \to Y^{(N)} \text{ weak}^* \text{ in } L^\infty(0,T;L^2(\Omega,\rho\mathbb{P};L^2(\mathcal{O})).$$

Hence

$$\operatorname*{ess\,sup}_{t\in[0,T]} \mathbb{E}[|Y^{(N)}(t)|_2^2\rho] \leq \liminf_{\lambda\to\infty}\ \operatorname*{ess\,sup}_{t\in[0,T]} \mathbb{E}[|Y_\lambda^{(N)}(t)|_2^2\rho]. \tag{4.9}$$

But, by (4.8), for all $N \in \mathbb{N}$,

$$\mathbb{E}[|Y_\lambda^{(N)}(t)|_2^2\rho] \leq |x|_2^2\mathbb{E}[\rho] + 2\lambda\|\rho\|_\infty C, \quad \forall t \geq 0,\ \mathbb{P}\text{-a.s.},$$

where

$$C := \int_0^T (\mathbb{E}|e^{\mu}|\nabla\mu||_4^4)^{1/2} dt \cdot \sup_{\lambda\in(0,1)} \operatorname*{ess\,sup}_{t\in[0,T]} (\mathbb{E}|1+X_\lambda|_4^4)^{1/2}$$

is finite by [5, Lemma 3.1]. Hence, letting first $\lambda \to 0$ and then $N \to \infty$ in (4.9), since $\rho \in L^\infty(\Omega)$, $\rho \geq 0$, was arbitrary, we obtain that

$$|Y(t)|_2^2 \leq |x|_2^2 \quad \text{for a.e. } t > 0, \ \mathbb{P}\text{-a.s.}$$

Now (4.4) follows, since $\mathbb{P}$-a.s. $t \to |Y(t)|_2^2$ is lower-semicontinuous.

Now, let us turn to the proof of Theorem 2.3.

To prove (i), let us assume that for some sequence $t_n \to \infty$ we have that

$$|Y(t_n)|_1 \geq \delta > 0, \quad \forall n \in \mathbb{N}. \tag{4.10}$$

Here and below $Y(t) = Y(t,\omega)$ for a fixed $\omega \in \Omega$ such that (4.4) holds. By (4.4), selecting a subsequence if necessary, we have $Y(t_n) \to g$ weakly in $L^2(\mathcal{O})$ as $n \to \infty$. We have that $g \geq 0$ and by (4.10)

$$g \not\equiv 0. \tag{4.11}$$

We recall from the proof of Theorem 2.2 that $t \mapsto \int_{\mathcal{O}} X(t)d\xi$ is continuous, hence so is $t \to \int_{\mathcal{O}} Y(t)d\xi$. So, for every $n \in \mathbb{N}$, there exists $\varepsilon_n > 0$ such that

$$\left| \int_{\mathcal{O}} Y(t)d\xi - \int_{\mathcal{O}} Y(t_n)d\xi \right| \leq \frac{1}{n}, \quad \forall t \in (t_n - \varepsilon_n, t_n + \varepsilon_n). \tag{4.12}$$

It follows by (2.7) that for some subsequence $t_{n_k} \to \infty$ there exist $s_k \in (t_{n_k} - \varepsilon_{n_k}, t_{n_k} + \varepsilon_{n_k})$, $k \in \mathbb{N}$, such that

$$\int \mathbb{1}_{\{X(s_k)\neq 0\}} d\xi = m(\mathcal{O} \setminus \mathcal{O}_0^{s_k}) \to 0 \ \text{ as } k \to \infty.$$

Hence, selecting another subsequence if necessary, we have

$$\mathbb{1}_{\{X(s_k)\neq 0\}} \to 0 \ \text{ a.e. as } k \to \infty$$

and by (4.4) that $X(s_k) \to \widetilde{g}$ weakly in $L^2(\mathcal{O})$.

As a consequence of the first, we obtain

$$Y(s_k) = Y(s_k)\mathbb{1}_{\{X(s_k)\neq 0\}} \to 0 \ \text{ a.e. as } k \to \infty,$$

which, in turn, implies that $\widetilde{g} = 0$. Hence, by (4.12)

$$\int g\, d\xi = \lim_{k\to\infty} \int Y(t_{n_k}) d\xi = \lim_{k\to\infty} \int Y(s_k) d\xi = \int \widetilde{g}\, d\xi = 0.$$

Hence, $g = 0$ a.e., since $g \geq 0$. This contradiction to (4.11) proves that a sequence $t_n \to 0$ with (4.10) does not exist and assertion (i) follows.

Clearly, to prove (ii), it suffices to prove the exponential decay part of Theorem 2.3 (ii). So, additionally, assume that (2.10) holds and let $K \subset \mathcal{O}$, K compact, and $K' \subset \mathcal{O}$ a compact neighborhood of K, i.e., $K \subset \overset{\circ}{K'}$. Let $\mu^* \in C_0^\infty(\mathcal{O})$ such that $0 \leq \mu^* \leq 1$, $\mu^* = 1$ on K and $\mu^* = 0$ on $\mathcal{O} \setminus K'$. Furthermore, let $C_K := \inf_{K'} \widetilde{\mu}$. We multiply equation (4.5) by $\mu^* Y_\lambda$ and integrate over $\mathcal{O}$ to obtain

$$\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}\,|(\mu^*)^{\frac{1}{2}} Y_\lambda(t)|_2^2 + \frac{C_K}{2}\,|(\mu^*)^{\frac{1}{2}} Y_\lambda(t)|_2^2 \\
&\quad \leq \frac{1}{2}\,\frac{d}{dt}\,|(\mu^*)^{\frac{1}{2}} Y_\lambda(t)|_2^2 + \frac{1}{2}\int_{\mathcal{O}} \widetilde{\mu}\mu^* Y_\lambda^2 d\xi \\
&\quad = -\int_{\mathcal{O}} \nabla(\psi_\lambda(e^{-\mu} Y_\lambda)) \cdot \nabla(e^{\mu}\mu^* Y_\lambda) d\xi \\
&\quad -\lambda \int_{\mathcal{O}} \nabla(e^{-\mu} Y_\lambda) \cdot \nabla(e^{\mu}\mu^* Y_\lambda) d\xi \\
&\quad = -\frac{1}{\lambda}\int_{\mathcal{O}} \mathbb{1}_\lambda^{**} \left[|\nabla X_\lambda|^2 + 2X_\lambda \nabla X_\lambda \cdot \left(\nabla\mu + \frac{1}{2}\,\frac{\nabla\mu^*}{\mu^*}\right)\right] e^{2\mu}\mu^* d\xi \qquad (4.13)\\
&\quad -\lambda \int_{\mathcal{O}} \left[|\nabla X_\lambda|^2 + 2X_\lambda \nabla X_\lambda \cdot \left(\nabla\mu + \frac{1}{2}\,\frac{\nabla\mu^*}{\mu^*}\right)\right] e^{2\mu}\mu^* d\xi \\
&\quad \leq \int_{\mathcal{O}} \left(\frac{1}{\lambda}\,\mathbb{1}_\lambda^{*} + \lambda\right) X_\lambda^2 \left[2|\nabla\mu|^2\mu^* + \frac{1}{2}\,\frac{|\nabla\mu^*|^2}{\mu^*}\right] e^{2\mu} d\xi \\
&\quad \leq 2\lambda \int_{\mathcal{O}} (1 + X_\lambda^2)[|\nabla\mu|^2\mu^* + |\nabla(\mu^*)^{\frac{1}{2}}|^2] e^{2\mu} d\xi.
\end{aligned}$$

Denoting the latter by $\frac{\lambda}{2}\,\eta_\lambda(t)$, we deduce that

$$\frac{d}{dt}\,(|(\mu^*)^{\frac{1}{2}} Y_\lambda(t)|_2^2 e^{C_K t}) \leq \eta_\lambda(t) e^{C_K t}, \ \text{ for a.e. } t > 0,\ \mathbb{P}\text{-a.s.}$$

Integrating from 0 to t, we obtain

$$(4.14)\quad |(\mu^*)^{\frac{1}{2}}Y_\lambda(t)|_2^2 \le |(\mu^*)^{\frac{1}{2}}x|_2^2 e^{-C_K t} + \lambda \int_0^t e^{C_K(s-t)}\eta_\lambda(s)ds,\ t\ge 0,\ \mathbb{P}\text{-a.s.}$$

Now analogous arguments as in the proof of Lemma 4.1 imply that after letting $\lambda \to 0$, inequality (4.14) turns into

$$(4.15)\qquad \begin{aligned} |(\mu^*)^{\frac{1}{2}}Y(t)|_2^2 &\le e^{-C_K t}|(\mu^*)^{\frac{1}{2}}x|_2^2, \quad t\ge 0,\ \mathbb{P}\text{-a.s.},\\ &\le e^{-C_K t}|x|_2^2. \end{aligned}$$

Hence

$$\begin{aligned} \int_K X(t)d\xi &= \int_K Y(t)e^{\mu(t)}d\xi \\ &\le |(\mu^*)^{\frac{1}{2}}Y(t)|_2 \left(\int_K \exp\left(2(\widetilde{\mu})^{\frac{1}{2}}\left(\sum_{k=1}^N \beta_k(t)^2\right)^{\frac{1}{2}}\right)d\xi\right)^{\frac{1}{2}} \\ &\le e^{-\frac{C_K}{2}t}|x|_2 \exp\left(\sup_K(\widetilde{\mu})^{\frac{1}{2}}\left(\sum_{k=1}^N \beta_k(t)^2\right)^{\frac{1}{2}}\right) m(K)^{\frac{1}{2}}, \end{aligned}$$

i.e. (2.11) is proved.

Remark 4.2 For existence of solutions to equation (2.1) in the special case (2.9), it is not absolutely necessary to assume that $\{e_k\} \subset H_0^1(\mathcal{O})$ is a basis of eigenfunctions for A. It suffices to assume that $e_k \in C^2(\overline{\mathcal{O}})$ and the proof of Theorem 2.3 is essentially the same. Then one might choose e_k, $1 \le k \le N$, such that

$$\inf\{\widetilde{\mu}(\xi);\ \xi \in \overline{\mathcal{O}}\} = \rho > 0$$

and, in this case, the exponential decay in Theorem 2.3 is global in $\mathcal{O}$. More precisely, in (2.3) the compact sets K and K' can be replaced by $\mathcal{O}$, and, in this case, (2.11) strengthens to

$$(4.16)\qquad \lim_{t\to\infty} X(t) = 0 \quad \text{in } L^1(\mathcal{O}),\ \mathbb{P}\text{-a.s.},$$

and, therefore, $\ell = 0$, $\mathbb{P}$-a.s. The details are omitted.

Remark 4.3 If condition (2.10) does not hold, the following slightly weaker statements still hold. Since $\widetilde{\mu}$ is analytic on $\mathcal{O}$, the set $\{\xi_j \in \mathcal{O};\ \widetilde{\mu}(\xi_j)\}$ is countable and, therefore, $\widetilde{\mu}(\xi) \geq \rho_K > 0$, $\forall \xi \in K$, for any compact $K \subset \mathcal{O} \setminus \{\xi_j\}$. Then the proof of Theorem 2.3 applies word by word and we have (2.11) and (2.12) in this case, too.

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