CONVERGENCE OF SOLUTIONS TO THE STOCHASTIC p-LAPLACE EQUATIONS AS p GOES TO 1

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Abstract. We prove that the solutions to the stochastic p-Laplace equation on a bounded domain $\mathscr{O}\subset\mathbb{R}^d$, $d=1,2$, with Dirichlet boundary conditions and additive Gaussian noise converge, as $p\to 1$, $\mathbb{P}$-a.s. uniformly in time in $L^2(\mathscr{O})$ to the solution to the multi-valued stochastic 1-Laplace equation with Dirichlet boundary conditions and additive Gaussian noise. Due to the highly singular limit case, solutions are defined via stochastic variational inequalities. Convergence of invariant measures of the associated stochastic flows is investigated.

1. Introduction

In this paper, we are investigating the following family, $p\in(1,2]$, of stochastic diffusion equations on $L^2(\mathscr{O})$,

$$(\mathbf{E}_p)\begin{cases} \mathrm{d}X_p\,(t)=\operatorname{div}\left[\left|\nabla X_p\right|_d^{p-1}\operatorname{sgn}(\nabla X_p)\right]\mathrm{d}t+\sqrt{Q}\,\mathrm{d}W\,(t) & \text{in } (0,T)\times\mathscr{O},\\ X_p\,(t)=0 & \text{on } (0,T)\times\partial\mathscr{O},\\ X_p\,(0)=x & \text{in } \mathscr{O}.\end{cases}$$

The purpose of this paper is to prove that, for $p\to 1$, the sequence $\{X_p\}_p$ of solutions to the equations $(\mathbf{E}_p)$ is convergent to the solution of the following (multi-valued) stochastic diffusion equation in $L^2\,(\mathscr{O})$

$$(\mathbf{E}_1)\begin{cases} \mathrm{d}X_1\,(t)\in\operatorname{div}\left[\operatorname{sgn}\left(\nabla X_1\,(t)\right)\right]\mathrm{d}t+\sqrt{Q}\,\mathrm{d}W\,(t) & \text{in } (0,T)\times\mathscr{O},\\ X_1\,(t)=0 & \text{on } (0,T)\times\partial\mathscr{O},\\ X_1\,(0)=x & \text{in } \mathscr{O}.\end{cases}$$

In both cases, $\mathscr{O}$ is a bounded open subset of $\mathbb{R}^d$, $d=1,2$, such that its boundary $\partial\mathscr{O}$ is sufficiently smooth. Here, $|\cdot|_d$ denotes the Euclidean norm and the multi-valued sign-function $\operatorname{sgn}:\mathbb{R}^d\to 2^{\mathbb{R}^d}$ is defined by

$$\operatorname{sgn}(z)=\begin{cases}\dfrac{z}{|z|_d}, & \text{if } z\neq 0,\\ \left\{y\in\mathbb{R}^d\mid |y|_d\leqslant 1\right\}, & \text{if } z=0.\end{cases}$$

Furthermore, $W(t)$ is a cylindrical Wiener process on $L^2(\mathscr{O})$ of the form

$$W(t)=\sum_{n=1}^{\infty}\gamma_n(t)e_n,\quad t\geqslant 0,$$

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where $\{\gamma_n\}$ is a sequence of mutually independent real Brownian motions on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbb{P})$ and $\{e_n\}$ is an orthonormal basis of $L^2(\mathscr{O})$. We shall make further specifications. Q is assumed to be a linear, continuous, non-negative, symmetric operator on $L^2(\mathscr{O})$ with eigenbasis $\{e_n\}$ and corresponding sequence of eigenvalues $\{\lambda_n\}$. Let $(-\Delta, \mathrm{dom}(-\Delta))$ be the Dirichlet Laplacian in $L^2(\mathscr{O})$, in particular, $\mathrm{dom}(-\Delta) = H^2(\mathscr{O}) \cap H^1_0(\mathscr{O})$. Assume for simplicity that $\{e_n\}$ is an eigenbasis of $-\Delta$ with corresponding sequence of eigenvalues $\{\mu_n\}$. We shall assume that

$$\sum_{n=1}^{\infty} \lambda_n^{1+\kappa}\mu_n < \infty \tag{1.1}$$

for some $\kappa > 0$. For the situation considered in this paper, it is enough to set $Q := (-\Delta)^{-1-\delta}$ with $\delta > \frac{1}{2} + \kappa$ for $d = 1$ and $\delta > 1 + \kappa$ for $d = 2$.

The singular diffusion operators in equations $(\mathbf{E}_p)$, $(\mathbf{E}_1)$ are called *p-Laplacian* and *1-Laplacian* respectively. In the deterministic case, i.e. if $Q \equiv 0$, both equations of evolution-type are covered by the theory of nonlinear semigroups in Hilbert space [9]. Both operators are used in image restoration, see [5, Ch. 3] for a comprehensive treatment.

The deterministic p-Laplace equation arises from geometry, quasi-regular mappings, fluid dynamics and plasma physics, see [12, 13]. In [15], $(\mathbf{E}_p)$ with $Q \equiv 0$ is suggested as a model of motion of non-Newtonian fluids.

Due to the singularities in the diffusivity terms $|\nabla X_p|_d^{p-1}\operatorname{sgn}(\nabla X_p)$, $\operatorname{sgn}(\nabla X_1)$ resp., we shall define the operators involved variationally. Equation $(\mathbf{E}_p)$ can informally be rewritten as follows

$$\begin{cases} \mathrm{d}X_p(t) + \nabla\Phi^p(X_p(t))\,\mathrm{d}t = \sqrt{Q}\,\mathrm{d}W(t) & \text{in } (0,T)\times\mathscr{O}, \\ X_p(t) = 0 & \text{on } (0,T)\times\partial\mathscr{O}, \\ X_p(0) = x & \text{in } \mathscr{O}, \end{cases} \tag{1.2}$$

where $\nabla\Phi^p$ denotes the Gâteaux differential of the convex functional

$$\Phi^p(u) := \frac{1}{p}\int_{\mathscr{O}} |\nabla u|_d^p\,\mathrm{d}\xi, \quad u \in W_0^{1,p}(\mathscr{O}),$$

and where $W_0^{1,p}(\mathscr{O})$ denotes the standard first-order Sobolev space of p-integrable functions with Dirichlet boundary conditions.

Remark 1.1. *By the classical Sobolev-embedding theorem(s), if $d = 1, 2$ and if $p \in (1, 2]$,*

$$W_0^{1,p}(\mathscr{O}) \subset L^2(\mathscr{O})$$

continuously. See e.g. [1, Theorem 5.4].

For $p = 1$, the situation is more complicated. We would like to find a convex functional Φ^1 such that equation $(\mathbf{E}_1)$ can be written as

$$\begin{cases} \mathrm{d}X_1(t) \in -\partial\Phi^1(X_1(t))\,\mathrm{d}t + \sqrt{Q}\,\mathrm{d}W(t) & \text{in } (0,T)\times\mathscr{O}, \\ X_1(t) = 0 & \text{on } (0,T)\times\partial\mathscr{O}, \\ X_1(0) = x & \text{in } \mathscr{O}, \end{cases} \tag{1.3}$$

where $\partial\Phi^1$ is the subdifferential of Φ^1.

We shall need the spaces $BV(\mathscr{O})$ and $BV_0(\mathscr{O})$. For $f \in L^1_{\mathrm{loc}}(\mathscr{O})$, define the *total variation*

$$\|\mathrm{D}f\|(\mathscr{O}) = \sup\left\{ \int_{\mathscr{O}} f \operatorname{div}\psi\,\mathrm{d}\xi \;\middle|\; \psi \in C_0^\infty(\mathscr{O};\mathbb{R}^d).\ |\psi|_d \leqslant 1 \right\}$$

$BV(\mathscr{O})$ is defined to be equal to $\{f \in L^1(\mathscr{O}) \mid \|\mathrm{D}f\|(\mathscr{O}) < \infty\}$. It is a Banach space with norm $\|f\|_{BV(\mathscr{O})} := \|\mathrm{D}f\|(\mathscr{O}) + \|f\|_{L^1(\mathscr{O})}$. Denote the $d-1$-dimensional Hausdorff measure on $\partial\mathscr{O}$ by $\mathscr{H}^{d-1}$. For $f \in BV(\mathscr{O})$ there is an element $f^{\mathscr{O}} \in L^1(\partial\mathscr{O}, \mathrm{d}\mathscr{H}^{d-1})$ called the *trace* such that

$$\int_{\mathscr{O}} f \operatorname{div}\psi \,\mathrm{d}\xi = -\int_{\mathscr{O}} \langle \psi, \mathrm{d}[\mathrm{D}f]\rangle_d + \int_{\partial\mathscr{O}} \langle \psi, \nu\rangle_d f^{\mathscr{O}} \,\mathrm{d}\mathscr{H}^{d-1} \quad \forall \psi \in C^1(\overline{\mathscr{O}}; \mathbb{R}^d),$$

where $[\mathrm{D}f]$ denotes the distributional gradient of f on $\mathscr{O}$ (which is a $\mathbb{R}^d$-valued Radon measure here) and ν denotes the outer unit normal on $\partial\mathscr{O}$. With $BV_0(\mathscr{O})$ we denote the subspace of $BV(\mathscr{O})$ of elements with zero trace.

Remark 1.2. *By* [2, Corollary 3.49]*, if $d = 1, 2$, then*

$$W_0^{1,1}(\mathscr{O}) \subset BV_0(\mathscr{O}) \subset L^2(\mathscr{O})$$

continuously.

For spaces of functions of bounded variation, see e.g. [2, Ch. 3].

We shall return to equation (1.3). Recall that the subdifferential $\partial\Phi^1$ in $L^2(\mathscr{O})$ is defined by $\eta \in \partial\Phi^1(x)$ iff

$$\Phi^1(x) - \Phi^1(y) \leqslant \int_{\mathscr{O}} \eta\,(x-y)\,\mathrm{d}\xi, \quad \forall y \in \operatorname{dom}\Phi^1. \tag{1.4}$$

One possible choice for Φ^1 is the (homogeneous) energy

$$\widetilde{\Phi}(u) := \begin{cases} \int_{\mathscr{O}} |\nabla u|_d \,\mathrm{d}\xi, & \text{if } u \in W_0^{1,1}(\mathscr{O}), \\ +\infty, & \text{if } u \in L^2(\mathscr{O}) \setminus W_0^{1,1}(\mathscr{O}). \end{cases}$$

In this case, if $u \in W_0^{1,1}(\mathscr{O})$, and if $U := -\operatorname{div}(\operatorname{sgn}(\nabla u)) \subset L^2(\mathscr{O})$, then we have that $u \in \operatorname{dom}\partial\widetilde{\Phi}$ and $U = \partial\widetilde{\Phi}(u)$.

However, $\widetilde{\Phi}$ fails to be lower semi-continuous in $L^2(\mathscr{O})$ which is a necessary ingredient for the theory. Therefore, it is convenient to consider its relaxed functional in $L^2(\mathscr{O})$, which is equal to

$$\Phi^1(u) := \begin{cases} \|\mathrm{D}u\|(\mathscr{O}), & \text{if } u \in BV_0(\mathscr{O}), \\ +\infty, & \text{if } u \in L^2(\mathscr{O}) \setminus BV_0(\mathscr{O}). \end{cases}$$

Φ^1 is proper, convex and lower semi-continuous in $L^2(\mathscr{O})$ and an extension of $\widetilde{\Phi}$ in the sense that $\operatorname{dom}\Phi^1 \supset \operatorname{dom}\widetilde{\Phi}$ and $\Phi^1 \leqslant \widetilde{\Phi}$.

Following the approach of Barbu, Da Prato and Röckner [8], we shall give the definition of a solution for equations $(\mathbf{E}_p)$, $p \in [1,2]$.

Definition 1.3. *Set $V_p := W_0^{1,p}(\mathscr{O})$, $p \in (1,2]$, $V_1 := BV_0(\mathscr{O})$. A stochastic process $X = X(t,x)$ with $\mathbb{P}$-a.s. continuous sample paths in $H := L^2(\mathscr{O})$ is said to be a solution to equation $(\mathbf{E}_p)$, $p \in [1,2]$ if*

$$X \in C_W([0,T]; H) \cap L^p((0,T)\times\Omega, V_p), \quad X(0) = x \in H$$

and

$$\begin{aligned} &\frac{1}{2}\|X(t) - Y(t)\|_2^2 + \int_0^t (\Phi^p(X(s)) - \Phi^p(Y(s)))\,\mathrm{d}s \\ \leqslant\; &\frac{1}{2}\|x - Y(0)\|_2^2 + \int_0^t (G(s), X(s) - Y(s))_2\,\mathrm{d}s, \quad t \in [0,T], \end{aligned}$$

for all $G(t) \in L^2_W(0,T;H)$ and $Y \in C_W([0,T]; H) \cap L^p((0,T)\times\Omega; V_p)$ satisfying the equation

$$\mathrm{d}Y(t) + G(t)\,\mathrm{d}t = \sqrt{Q}\,\mathrm{d}W(t), \quad t \in [0,T]. \tag{1.5}$$

Suppose for a while that $1 < p < 2$, $d = 1, 2$. Arguing as in [17, Example 4.1.9, Theorem 4.2.4], we can easily prove existence and uniqueness of the solution X_p for equation $(\mathbf{E}_p)$, in the usual (strong) variational sense, as in Prévôt, Röckner [17, Definition 4.2.1]. Now, by Lemma A.1 in the Appendix A, we see that X_p is also a solution in the sense of the definition above.

Regarding equation $(\mathbf{E}_1)$, well-posedness of the problem as well as existence and uniqueness of the solution were proved by Barbu, Da Prato and Röckner in [8].

We are now able to formulate the main result of this paper.

Theorem 1.4. *The sequence of solutions* $\{X_p\}_p$ *to equations* $(\mathbf{E}_p)$ *is convergent for* $p \to 1$ *to the solution* X_1 *of equation* $(\mathbf{E}_1)$*, strongly in* $L^2(\mathscr{O})$*, uniformly on* $[0,T]$*,* $\mathbb{P}-a.s.$*, i.e.,*

$$\lim_{p\to 1} \sup_{t\in[0,T]} \|X_p(t) - X_1(t)\|_2 = 0, \quad \mathbb{P}-a.s.$$

We remark that our proof comprehends the situation where the limit equation could possibly be chosen any $(\mathbf{E}_{p_0})$, $p_0 \in [1,2]$. This suggests that our convergence result can as well be considered a continuity result (in the parameter p).

Our proof uses methods from variational convergence of convex functionals, see Attouch's book [3] for an introduction. We shall prove the analytic facts used for the main result in Section 2. Section 3 contains the proof of Theorem 1.4. In Section 4, we prove the convergence of ergodic invariant measures in the strongly monotone situation. The proof of tightness relies on a compactness argument in BV. Ergodicity of the semigroup associated to the singular stochastic p-Laplace equations has been studied in [16]. For all $p \in (1,2)$ it remains an open question. In this paper, we prove the following.

Theorem 1.5. *Let* $X_p = X_p(t,x)$ *be the solution to equation* $(\mathbf{E}_p)$*,* $p \in [1,2]$*. Let* $p_0 \in [1,2]$*,* $\{p_n\} \subset (p_0, 2]$ *such that* $\lim_n p_n = p_0$*. Let*

$$P_t^p \varphi(x) := \mathbb{E}\left[\varphi\left(X_p(t,x)\right)\right] \quad \varphi \in C_b(L^2(\mathscr{O}))$$

be the semigroup associated to equation $(\mathbf{E}_p)$*. Suppose that* $P_t^{p_n}$*,* $n \in \mathbb{N}$*,* $P_t^{p_0}$ *are ergodic, i.e. admit unique invariant measures* ν_{p_n}*,* $n \in \mathbb{N}$*,* ν_{p_0} *on* $L^2(\mathscr{O})$*. Then*

$$\nu_{p_n} \to \nu_{p_0} \quad \textit{in the weak sense.}$$

Notation. Throughout this paper we denote by H the Hilbert space $L^2(\mathscr{O})$ with the scalar product $(f,g)_2 := \int_{\mathscr{O}} fg \,\mathrm{d}\xi$ and the norm $\|f\|_2 := (f,f)_2^{1/2}$. The spaces $L^p(\mathscr{O})$, $p \geqslant 1$, and $W_0^{1,p}(\mathscr{O})$ are the standard spaces of integrable functions and Sobolev spaces on $\mathscr{O}$ with Dirichlet boundary conditions. We set $H_0^1(\mathscr{O}) := W_0^{1,2}(\mathscr{O})$ and $H^2(\mathscr{O}) := W^{2,2}(\mathscr{O})$. Depending on the context, we shall use the notation $\|\cdot\|_\infty$ both to denote the supremum-norm and the essential supremum norm. $|\cdot|_d$ and $\langle\cdot,\cdot\rangle_d$ denote the Euclidean norm and Euclidean scalar product respectively. For $p \in [1,\infty]$, let $p' := p/(p-1)$ be the conjugate exponent. The letter C will be used to denote several positive constants.

By $L_W^2(0,T;H)$ and $C_W^2([0,T];H)$ we denote the space of all square integrable (respectively continuous) functions from $[0,T]$ to $L^2(\Omega;H)$ which are adapted to $\{\mathscr{F}_t\}_{t\geqslant 0}$.

For $p \in (1,\infty)$, define $a_p : \mathbb{R}^d \to \mathbb{R}^d$ by $a_p(x) := |x|_d^{p-1} \operatorname{sgn}(x)$. Furthermore, let $A_p : W_0^{1,p}(\mathscr{O}) \to W_0^{-1,p'}(\mathscr{O})$ be defined by $A_p(y) := -\operatorname{div}\left[a_p(\nabla y)\right]$, where $y \in W_0^{1,p}(\mathscr{O})$. To be more specific,

$$_{W^{-1,p'}}\langle A_p(y), z\rangle_{W^{1,p}} = \int_{\mathscr{O}} \langle a_p(\nabla y), \nabla z\rangle_d \,\mathrm{d}\xi, \quad \forall z \in W_0^{1,p}(\mathscr{O}).$$

Furthermore, for $p \in [1,\infty)$, we shall define $j^p : \mathbb{R}^d \to \mathbb{R}$ by $j^p(x) := \frac{1}{p}|x|_d^p$. Obviously, if $p > 1$, each j^p is a convex C^1-function such that $a_p = \nabla j^p$ and

$$\Phi^p : W_0^{1,p}(\mathscr{O}) \to [0,\infty),$$
$$\Phi^p(y) := \int_{\mathscr{O}} j^p(\nabla y)\,\mathrm{d}\xi, \quad \forall y \in W_0^{1,p}(\mathscr{O})$$

is Fréchet differentiable and satisfies $\partial\Phi^p = \nabla\Phi^p = A_p$.

2. Some results on variational convergence

Before we prove Theorem 1.4, we need some preparations. The results of this section from variational convergence of convex functionals are only partly represented in the literature. We also collect the necessary facts in Appendix B.

Let $p \in [1,2]$. Let $j^p : \mathbb{R}^d \to \mathbb{R}$ be as in the last part of the introduction. For $\varepsilon > 0$, let $j^p_\varepsilon(x) := \inf_{y\in\mathbb{R}^d}\left[j^p(y) + \frac{1}{2\varepsilon}|x-y|_d^2\right]$ be its regularization. For $u \in L^2(\mathscr{O};\mathbb{R}^d)$, set

$$\Psi^p(u) := \int_{\mathscr{O}} j^p(u)\,\mathrm{d}\xi.$$

Ψ^p is a continuous convex functional on $L^2(\mathscr{O};\mathbb{R}^d)$ for each $p \in [1,2]$.

Lemma 2.1. *For $\varepsilon > 0$, let Ψ^p_ε be the Moreau-Yosida regularization of Ψ^p in $L^2(\mathscr{O};\mathbb{R}^d)$. Then*

$$\Psi^p_\varepsilon(v) = \int_{\mathscr{O}} j^p_\varepsilon(v)\,\mathrm{d}\xi \quad \forall v \in L^2(\mathscr{O};\mathbb{R}^d).$$

Proof. Let $v \in L^2(\mathscr{O};\mathbb{R}^d)$. Fix a representative $\overline{v}$ of v. For $\xi \in \mathscr{O}$, $x \in \mathbb{R}^d$, let

$$f_v(\xi,x) := j^p(x) + \frac{1}{2\varepsilon}|\overline{v}(\xi) - x|_d^2.$$

Obviously, f_v satisfies Carathéodory's conditions and is hence a normal integrand. Therefore, we can apply [18, Theorem 14.60] for the space $L^2(\mathscr{O};\mathbb{R}^d)$ in order to obtain

$$\Psi^p_\varepsilon(v) = \inf_{u\in L^2(\mathscr{O};\mathbb{R}^d)} \int_{\mathscr{O}} f_v(\xi,u(\xi))\,\mathrm{d}\xi = \int_{\mathscr{O}} \inf_{x\in\mathbb{R}^d} f_v(\xi,x)\,\mathrm{d}\xi = \int_{\mathscr{O}} j^p_\varepsilon(v(\xi))\,\mathrm{d}\xi,$$

where both integrals are finite. □

We would like to prove a convergence result, which shall be useful later. See the appendix for the terminology. Compare also with [4].

Lemma 2.2. *Let $\{p_n\} \subset [1,2]$ such that $\lim_n p_n = 1$. Then*

$$\Psi^{p_n} \xrightarrow{M} \Psi^1 \quad \textit{in the Mosco sense in } L^2(\mathscr{O};\mathbb{R}^d).$$

Proof. Let us prove (M1) in Definition B.1 first. Let $u_n \in L^2(\mathscr{O};\mathbb{R}^d)$, $n \in \mathbb{N}$, $u \in L^2(\mathscr{O};\mathbb{R}^d)$ such that $u_n \rightharpoonup u$ weakly in $L^2(\mathscr{O};\mathbb{R}^d)$. W.l.o.g. $\liminf_n \Psi^{p_n}(u_n) < +\infty$. Extract a subsequence (also denoted by $\{u_n\}$) such that

$$\liminf_n \Psi^{p_n}(u_n) = \lim_n \Psi^{p_n}(u_n).$$

Let $v \in L^\infty(\mathscr{O};\mathbb{R}^d)$. Clearly,

$$\lim_n \int_{\mathscr{O}} \langle u_n, v\rangle_d \,\mathrm{d}\xi = \int_{\mathscr{O}} \langle u, v\rangle_d \,\mathrm{d}\xi.$$

Also, by Hölder's inequality,

$$\frac{1}{p_n}\left|\int_{\mathscr{O}} \langle u_n, v\rangle_d \,\mathrm{d}\xi\right|^{p_n} \leqslant \Psi^{p_n}(u_n)|\mathscr{O}|^{p_n-1}\,\|v\|^{p_n}_{L^\infty(\mathscr{O};\mathbb{R}^d)},$$

(here $|\mathscr{O}| = \int_{\mathscr{O}} \mathrm{d}\xi$). Upon taking the limit $n \to \infty$, we get that

$$\left| \int_{\mathscr{O}} \langle u, v\rangle_d \,\mathrm{d}\xi \right| \leqslant \liminf_n \Psi^{p_n}(u_n) \, \|v\|_{L^\infty(\mathscr{O};\mathbb{R}^d)} .$$

Taking the supremum over all $v \in L^\infty(\mathscr{O};\mathbb{R}^d)$ with $\|v\|_{L^\infty(\mathscr{O};\mathbb{R}^d)} \leqslant 1$ and using the l.s.c. property of the supremum, we get that

$$\Psi^1(u) = \int_{\mathscr{O}} |u|_d \,\mathrm{d}\xi \leqslant \liminf_n \Psi^{p_n}(u_n).$$

Since the same argument works for any subsequence of $\{u_n\}$, we have proved (M1).

We are left to prove (M2) in Definition B.1. Let $u \in L^2(\mathscr{O};\mathbb{R}^d)$. Clearly for a.e. $\xi \in \mathscr{O}$

$$\lim_n \frac{1}{p_n} |u(\xi)|_d^{p_n} = |u(\xi)|_d.$$

But for all $p \in [1,2]$,

$$\frac{1}{p}|u|_d^p \leqslant 1_{\mathscr{O}} + |u|_d^2 \in L^1(\mathscr{O}).$$

Hence an application of Lebesgue's dominated convergence theorem yields

$$\lim_n \Psi^{p_n}(u) = \Psi^1(u).$$

(M2) is proved. □

Theorem B.2, Corollary B.3 and Lemmas 2.1, 2.2 together give:

Corollary 2.3. *Let $\{p_n\} \subset [1,2]$ such that $\lim_n p_n = 1$. Let $\varepsilon > 0$. Then for $u \in L^2(\mathscr{O};\mathbb{R}^d)$, we have that*

$$\lim_n \int_{\mathscr{O}} j_\varepsilon^{p_n}(u) \,\mathrm{d}\xi = \int_{\mathscr{O}} j_\varepsilon^1(u) \,\mathrm{d}\xi. \tag{2.1}$$

Furthermore, if $u_n \rightharpoonup u$ converges weakly in $L^2(\mathscr{O};\mathbb{R}^d)$, we have that

$$\liminf_n \int_{\mathscr{O}} j_\varepsilon^{p_n}(u_n) \,\mathrm{d}\xi \geqslant \int_{\mathscr{O}} j_\varepsilon^1(u) \,\mathrm{d}\xi. \tag{2.2}$$

For each $\varepsilon > 0$, let $R_\varepsilon := (1 - \varepsilon\Delta)^{-1}$ be the resolvent of the (Dirichlet) Laplace operator $(-\Delta, \mathrm{dom}(-\Delta))$, where $\mathrm{dom}(-\Delta) = H_0^{1,2}(\mathscr{O}) \cap H^{2,2}(\mathscr{O})$. For $p \in [1,2]$, $\varepsilon > 0$, let

$$\Phi_\varepsilon^p(u) := \int_{\mathscr{O}} j_\varepsilon^p(\nabla R_\varepsilon u) \,\mathrm{d}\xi, \quad u \in L^2(\mathscr{O}).$$

Lemma 2.4. *Let $\{p_n\} \subset [1,2]$ such that $\lim_n p_n = 1$. Let $\varepsilon > 0$. Then for $u \in L^2(\mathscr{O})$, we have that*

$$\lim_n \Phi_\varepsilon^{p_n}(u) = \Phi_\varepsilon^1(u). \tag{2.3}$$

Furthermore, if $u_n \rightharpoonup u$ converges weakly in $L^2(\mathscr{O})$, we have that

$$\liminf_n \Phi_\varepsilon^{p_n}(u_n) \geqslant \Phi_\varepsilon^1(u). \tag{2.4}$$

Also, each Φ_ε^p, $p \in [1,2]$, $\varepsilon > 0$, is continuous w.r.t. the weak topology of $L^2(\mathscr{O})$.

Proof. Since R_ε maps to $\mathrm{dom}(-\Delta) \subset H_0^1(\mathscr{O})$, it is clear that $\nabla R_\varepsilon u \in L^2(\mathscr{O};\mathbb{R}^d)$ and hence (2.3) follows from (2.1).

Let $u_n \in L^2(\mathscr{O})$, $n \in \mathbb{N}$, $u \in L^2(\mathscr{O})$, such that $u_n \rightharpoonup u$ weakly in $L^2(\mathscr{O})$. If we can proof that $\nabla R_\varepsilon u_n \rightharpoonup \nabla R_\varepsilon u$ weakly in $L^2(\mathscr{O};\mathbb{R}^d)$, we can apply (2.2) and (2.4) follows. Indeed, we even have that $\nabla R_\varepsilon u_n \to \nabla R_\varepsilon u$ strongly in $L^2(\mathscr{O};\mathbb{R}^d)$. To see this, one possibility is to proceed as follows: Equip $\mathrm{dom}(-\Delta)$ with the graph norm

$$\|\cdot\|_{\mathrm{dom}(-\Delta)} := \left(\|\cdot\|_{L^2(\mathscr{O})}^2 + \|\Delta\cdot\|_{L^2(\mathscr{O})}^2 \right)^{1/2} .$$

Now, by definition of the resolvent, and the fact that it is an $L^2(\mathscr{O})$-contraction, we see that $R_\varepsilon : L^2(\mathscr{O}) \to \mathrm{dom}(-\Delta)$ is strongly (and hence weakly) continuous. By [19, §4.2.4, Theorem], $\|\cdot\|_{\mathrm{dom}(-\Delta)}$ is an equivalent norm to the Sobolev-norm of $H^2(\mathscr{O})$. By the Rellich-Kondrachov Theorem (see e.g. [1, Theorem 6.2]), the embedding $H^2(\mathscr{O}) \subset H^1(\mathscr{O})$ is compact. Hence $R_\varepsilon u_n \to R_\varepsilon u$ strongly in $H^1(\mathscr{O})$ and so the corresponding gradients converge strongly in $L^2(\mathscr{O};\mathbb{R}^d)$ as claimed.

The last part follows by repeating the compactness argument above and the strong $L^2(\mathscr{O};\mathbb{R}^d)$-continuity of the Ψ_ε^p's. □

3. Proof of Theorem 1.4

We first consider the following approximating equations for $(\mathbf{E}_p)$

$$\begin{cases} \mathrm{d}X_p^\varepsilon(t) + A_p^\varepsilon\left(X_p^\varepsilon\right)\mathrm{d}t = \sqrt{Q}\,\mathrm{d}W(t) \\ X_p^\varepsilon(0) = x \end{cases} \tag{3.1}$$

where for any $u \in L^2(\mathscr{O})$,

$$A_p^\varepsilon(u) = -\left(1-\varepsilon\Delta\right)^{-1}\mathrm{div}\left[a_p^\varepsilon\left(\nabla\left(1-\varepsilon\Delta\right)^{-1}u\right)\right]$$

and a_p^ε is the Yosida approximation of a_p i.e., for any $r \in \mathbb{R}^d$,

$$a_p^\varepsilon(r) = \frac{1}{\varepsilon}\left(1 - \left(1+\varepsilon a_p\right)^{-1}(r)\right).$$

In particular, for $u, v \in L^2(\mathscr{O})$,

$$\left(A_p^\varepsilon(u), v\right)_2 = \int_{\mathscr{O}} \left\langle a_p^\varepsilon(\nabla R_\varepsilon u), \nabla R_\varepsilon(v)\right\rangle_d \mathrm{d}\xi,$$

where $R_\varepsilon := (1-\varepsilon\Delta)^{-1}$.

We shall consider a similar approximation for equation $(\mathbf{E}_1)$

$$\begin{cases} \mathrm{d}X_1^\varepsilon(t) + A^\varepsilon\left(X_1^\varepsilon\right)\mathrm{d}t = \sqrt{Q}\,\mathrm{d}W(t) \\ X_1^\varepsilon(0) = x \end{cases} \tag{3.2}$$

where for any $u \in L^2(\mathscr{O})$,

$$A^\varepsilon(u) = -\left(1-\varepsilon\Delta\right)^{-1}\mathrm{div}\left[\beta^\varepsilon\left(\nabla\left(1-\varepsilon\Delta\right)^{-1}u\right)\right].$$

with

$$\beta^\varepsilon(r) = \begin{cases} \dfrac{r}{\varepsilon}, \text{ if } |r|_d \leqslant \varepsilon, \\ \dfrac{r}{|r|_d}, \text{ if } |r|_d > \varepsilon. \end{cases}$$

In particular, for $u, v \in L^2(\mathscr{O})$,

$$\left(A^\varepsilon(u), v\right)_2 = \int_{\mathscr{O}} \left\langle \beta^\varepsilon(\nabla R_\varepsilon u), \nabla R_\varepsilon(v)\right\rangle_d \mathrm{d}\xi.$$

Note that β^ε is the Yosida approximation of the sign function, i.e., for any $r \in \mathbb{R}^d$,

$$\beta^\varepsilon(r) = \frac{1}{\varepsilon}\left(1 - \left(1+\varepsilon\ \mathrm{sgn}\right)^{-1}(r)\right).$$

In particular, $\beta^\varepsilon = \nabla j^\varepsilon$, where j_ε is the convex function defined by

$$j^\varepsilon(r) = \begin{cases} \dfrac{|r|_d^2}{2\varepsilon}, \text{ if } |r|_d \leqslant \varepsilon, \\ |r|_d - \dfrac{\varepsilon}{2}, \text{ if } |r|_d > \varepsilon. \end{cases}$$

We shall use the following strategy to prove the main result

$$\|X_p(t) - X_1(t)\|_2 \leqslant \left\|X_p(t) - X_p^\varepsilon(t)\right\|_2 + \left\|X_p^\varepsilon(t) - X_1^\varepsilon(t)\right\|_2 + \|X_1^\varepsilon(t) - X_1(t)\|_2$$

$\mathbb{P}$-a.s. and uniformly in $t \in [0,T]$.

At this point we need to prove the following lemma. We introduce the notation $r_\varepsilon^p(r) := (1+\varepsilon a_p)^{-1}(r)$.

Lemma 3.1. *Under our assumptions, if we let X_p^ε be the solution to* (3.1) *and $\widetilde{X}_p^\varepsilon := (1-\varepsilon\Delta)^{-1}X_p^\varepsilon$, we have that*

$$\mathbb{E}\int_0^t \int_{\mathscr{O}} \left| r_\varepsilon^p\left(\nabla \widetilde{X}_p^\varepsilon(s)\right)\right|_d^p \,\mathrm{d}\xi\,\mathrm{d}s \leqslant C_t\left(\|x\|_2^2 + \operatorname{Tr} Q\right), \tag{3.3}$$

for all $t \in [0,T]$.

Proof. We know by the definiton of a_p that

$$\langle a_p(r), r\rangle_d \geqslant |r|_d^p.$$

On the other hand we have by Itō's formula, applied to the function $u \mapsto \|u\|_2^2$, that

$$\mathbb{E}\left\|X_p^\varepsilon(t)\right\|_2^2 + 2\mathbb{E}\int_0^t\int_{\mathscr{O}} \left\langle a_p^\varepsilon\left(\nabla\widetilde{X}_p^\varepsilon(s)\right), \nabla\widetilde{X}_p^\varepsilon(s)\right\rangle_d \mathrm{d}\xi\,\mathrm{d}s \leqslant C_t\left(\|x\|_2^2 + \operatorname{Tr} Q\right). \tag{3.4}$$

By the definition of the Yosida approximation we have that

$$a_p^\varepsilon(r) = a_p(r_\varepsilon^p(r))$$

and

$$\left\langle a_p^\varepsilon(r), r\right\rangle_d = \left\langle a_p^\varepsilon(r_\varepsilon^p(r)), r_\varepsilon^p(r)\right\rangle_d + \frac{1}{\varepsilon}\left|r - r_\varepsilon^p(r)\right|_d^2.$$

We rewrite as follows

$$\begin{aligned} &\mathbb{E}\int_0^t\int_{\mathscr{O}} \left\langle a_p^\varepsilon\left(\nabla\widetilde{X}_p^\varepsilon(s)\right), \nabla\widetilde{X}_p^\varepsilon(s)\right\rangle_d \mathrm{d}\xi\,\mathrm{d}s \\ \geqslant\ &\mathbb{E}\int_0^t\int_{\mathscr{O}} \left\langle a_p\left(r_\varepsilon^p\left(\nabla\widetilde{X}_p^\varepsilon(s)\right)\right), r_\varepsilon^p\left(\nabla\widetilde{X}_p^\varepsilon(s)\right)\right\rangle_d \mathrm{d}\xi\,\mathrm{d}s \\ \geqslant\ &\mathbb{E}\int_0^t\int_{\mathscr{O}} \left| r_\varepsilon^p\left(\nabla\widetilde{X}_p^\varepsilon(s)\right)\right|_d^p \mathrm{d}\xi\,\mathrm{d}s. \end{aligned}$$

Plugging into (3.4) proves (3.3). ∎

Step I. We have from [8, equation (4.8)] that

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]} \|X_1^\varepsilon(t) - X_1(t)\|_2 = 0, \quad \mathbb{P}\text{-a.s.}$$

Step II. We shall prove now that

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\left\|X_p(t)-X_p^\varepsilon(t)\right\|_2=0,\quad \mathbb{P}\text{-a.s. uniformly in } p\in(1,2).$$

We set $\widetilde{X}_p^\varepsilon=(1-\varepsilon\Delta)^{-1}X_p^\varepsilon$ and $\widetilde{X}_p^\lambda=(1-\lambda\Delta)^{-1}X_p^\lambda$. Then by (3.1), we have that

$$\begin{aligned}&\frac{1}{2}\left\|X_p^\varepsilon(t)-X_p^\lambda(t)\right\|_2^2\\&\quad+\int_0^t\int_{\mathscr{O}}\left\langle a_p^\varepsilon\left(\nabla\widetilde{X}_p^\varepsilon(s)\right)-a_p^\lambda\left(\nabla\widetilde{X}_p^\lambda(s)\right),\nabla\widetilde{X}_p^\varepsilon(s)-\nabla\widetilde{X}_p^\lambda(s)\right\rangle_d \mathrm{d}\xi\,\mathrm{d}s=0.\end{aligned}$$

Setting $\nabla\widetilde{X}_p^\varepsilon(s)=u^\varepsilon$ and $\nabla\widetilde{X}_p^\lambda(s)=u^\lambda$ and using

$$a_p^\varepsilon(u)\in a_p\left((1+\varepsilon a_p)^{-1}(u)\right),$$

we get by the monotonicity of a_p that

$$\begin{aligned}&\left\langle a_p^\varepsilon(u^\varepsilon)-a_p^\lambda(u^\lambda),u^\varepsilon-u^\lambda\right\rangle_d\\\geqslant\ &\left\langle a_p^\varepsilon(u^\varepsilon)-a_p^\lambda(u^\lambda),\varepsilon a_p^\varepsilon(u^\varepsilon)-\lambda a_p^\lambda(u^\lambda)\right\rangle_d.\end{aligned}$$

This leads to

$$\begin{aligned}\frac{1}{2}&\left\|X_p^\varepsilon(t)-X_p^\lambda(t)\right\|_2^2\\&\leqslant\int_0^t\int_{\mathscr{O}}\left(\varepsilon\left|a_p^\varepsilon\left(\nabla\widetilde{X}_p^\varepsilon(s)\right)\right|_d^2+\lambda\left|a_p^\lambda\left(\nabla\widetilde{X}_p^\lambda(s)\right)\right|_d^2\right)\mathrm{d}\xi\,\mathrm{d}s.\end{aligned}\tag{3.5}$$

We can now prove that

$$\int_0^t\int_{\mathscr{O}}\left|a_p^\varepsilon\left(\nabla\widetilde{X}_p^\varepsilon(s)\right)\right|_d^2\mathrm{d}\xi\,\mathrm{d}s\leqslant C_t\quad\mathbb{P}\text{-a.s.}\tag{3.6}$$

for some C_t independent of p and ε.

Using Jensen's inequality (for $t\mapsto t^{p'/2}$) and taking into account that $|a_p(r)|_d\leqslant|r|_d^{p-1}$, we obtain

$$\begin{aligned}&\int_0^t\int_{\mathscr{O}}\left|a_p^\varepsilon\left(\nabla\widetilde{X}_p^\varepsilon(s)\right)\right|_d^2\mathrm{d}\xi\,\mathrm{d}s\\&\leqslant(t\,|\mathscr{O}|)^{1-2/p'}\left(\int_0^t\int_{\mathscr{O}}\left|a_p\left(r_\varepsilon^p\left(\nabla\widetilde{X}_p^\varepsilon(s)\right)\right)\right|_d^{p'}\mathrm{d}\xi\,\mathrm{d}s\right)^{2/p'}\\&\leqslant(1+t\,|\mathscr{O}|)\left(\int_0^t\int_{\mathscr{O}}\left|r_\varepsilon^p\left(\nabla\widetilde{X}_p^\varepsilon(s)\right)\right|_d^{p}\mathrm{d}\xi\,\mathrm{d}s\right)^{2/p'}\\&\leqslant C_t+C_t\left(\int_0^t\int_{\mathscr{O}}\left|r_\varepsilon^p\left(\nabla\widetilde{X}_p^\varepsilon(s)\right)\right|_d^{p}\mathrm{d}\xi\,\mathrm{d}s\right),\end{aligned}\tag{3.7}$$

where $|\mathscr{O}|=\int_{\mathscr{O}}\mathrm{d}\xi$.

Now by Lemma 3.1 we have (3.6) for a constant C_t independent of p and ε, and passing to the limit for $\varepsilon,\lambda\to 0$ in (3.5) we get that

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\left\|X_p(t)-X_p^\varepsilon(t)\right\|_2=0,\quad \mathbb{P}\text{-a.s. uniformly in } p\in(1,2).$$

Step III. In order to complete the proof we still need to show that for all $\varepsilon > 0$ fixed we have

$$\lim_{p\to 1}\sup_{t\in[0,T]}\left\|X_p^\varepsilon(t)-X_1^\varepsilon(t)\right\|_2=0,\quad \mathbb{P}\text{-a.s.}$$

To this aim, we consider the definiton of the solution for equations

$$\begin{cases} \mathrm{d}X_p^\varepsilon(t)+A_p^\varepsilon\left(X_p^\varepsilon\right)\mathrm{d}t=\sqrt{Q}\,\mathrm{d}W(t)\\ X_p^\varepsilon(0)=x\end{cases}$$

as

$$\begin{aligned}&\frac{1}{2}\left\|X_p^\varepsilon(t)-Y(t)\right\|_2^2+\int_0^t\left(\Phi_\varepsilon^p\left(X_p^\varepsilon(s)\right)-\Phi_\varepsilon^p(Y(s))\right)\mathrm{d}s\\ \leqslant\;&\frac{1}{2}\left\|x-Y(0)\right\|_2^2+\int_0^t\left(G(s),X_p^\varepsilon(s)-Y(s)\right)_2\mathrm{d}s,\\ &\text{for all } t\in[0,T],\ \mathbb{P}\text{-a.s.}\end{aligned}$$

We take $Y=X_1^\varepsilon$, the solution of equation

$$\begin{cases} \mathrm{d}X_1^\varepsilon(t)+A^\varepsilon\left(X_1^\varepsilon\right)\mathrm{d}t=\sqrt{Q}\,\mathrm{d}W(t)\\ X_1^\varepsilon(0)=x.\end{cases}$$

and using the definiton of the subdifferential we get that

(3.8)

$$\begin{aligned}&\frac{1}{2}\left\|X_p^\varepsilon(t)-X_1^\varepsilon(t)\right\|_2^2\\ &\quad+\int_0^t\left(\Phi_\varepsilon^p\left(X_p^\varepsilon(s)\right)-\Phi_\varepsilon^p\left(X_1^\varepsilon(s)\right)+\Phi_\varepsilon^1\left(X_1^\varepsilon(s)\right)-\Phi_\varepsilon^1\left(X_p^\varepsilon(s)\right)\right)\mathrm{d}s\\ \leqslant&\frac{1}{2}\left\|x-X_1^\varepsilon(0)\right\|_2^2=0,\end{aligned}$$

for $t\in[0,T]$ and $\mathbb{P}$-a.s. By estimate (3.4), we can extract a subsequence $\{p_n\}$ with $\lim_n p_n=1$ such that for $X_n^\varepsilon:=X_{p_n}^\varepsilon$ we have that for $\mathrm{d}t$-a.a. $t\in[0,T]$, $X_n^\varepsilon(t)\rightharpoonup Z^\varepsilon(t)$ weakly in $L^2(\mathscr{O})$ $\mathbb{P}$-a.s. for some $\mathrm{d}t\otimes\mathbb{P}$-measurable Z^ε that satisfies

$$\sup_{t\in[0,T]}\left\|Z^\varepsilon(t)\right\|_2\leqslant\liminf_n\sup_{t\in[0,T]}\left\|X_n(t)\right\|_2\quad\mathbb{P}\text{-a.s.}$$

We shall need following lemma. Set $\Phi_\varepsilon^n:=\Phi_\varepsilon^{p_n}$.

Lemma 3.2.

$$\Phi_\varepsilon^n(X_1^\varepsilon(\cdot))-\Phi_\varepsilon^n(X_n^\varepsilon(\cdot))+\Phi_\varepsilon^1(X_n^\varepsilon(\cdot))-\Phi_\varepsilon^1(X_1^\varepsilon(\cdot))$$

is $\mathbb{P}$-a.s. bounded above by a function in $L^\infty(0,T)$.

Proof. Set $u:=X_n^\varepsilon(\cdot)$, $v:=X_1^\varepsilon(\cdot)$. Recall that in our notation, $R_\varepsilon:=(1-\varepsilon\Delta)^{-1}$.

Let us treat the term $\Phi_\varepsilon^1(u)-\Phi_\varepsilon^1(v)$ first. By the definition of the subgradient it is bounded by $(\nabla\Phi_\varepsilon^1(u),u-v)_2$. But this term is equal to

$$\int_{\mathscr{O}}\left\langle\beta^\varepsilon(\nabla R_\varepsilon(u)),\nabla R_\varepsilon(u-v)\right\rangle_d\,\mathrm{d}\xi.$$

Since $|\beta^\varepsilon|_d\leqslant 1$, we get that the latter is bounded by $\|\nabla R_\varepsilon(u-v)\|_{L^2(\mathscr{O};\mathbb{R}^d)}$. By the proof of Lemma 2.4, ∇R_ε is a bounded operator from $L^2(\mathscr{O})$ to $L^2(\mathscr{O};\mathbb{R}^d)$.

We get that

$$\Phi_\varepsilon^1(X_n^\varepsilon(\cdot))-\Phi_\varepsilon^1(X_1^\varepsilon(\cdot))\leqslant C\sup_n\left\|X_n^\varepsilon(\cdot)\right\|_2+C\left\|X_1^\varepsilon(\cdot)\right\|_2$$

which is $\mathbb{P}$-a.s. in $L^\infty(0,T)$ again by estimate (3.4).

We continue with the term $\Phi^n_\varepsilon(v) - \Phi^n_\varepsilon(u)$. By the definition of the subgradient it is bounded by $(\nabla\Phi^n_\varepsilon(v), v-u)_2$, which is equal to

$$\int_{\mathscr{O}} \left\langle a^\varepsilon_p(\nabla R_\varepsilon(v)), \nabla R_\varepsilon(v-u)\right\rangle_d \,\mathrm{d}\xi.$$

Noticing that r^p_ε is a contraction on $\mathbb{R}^d$, we can use a similar estimate as in (3.7) to get that the latter is bounded by

$$C + C\left\|\nabla R_\varepsilon(v)\right\|_{L^2(\mathscr{O};\mathbb{R}^d)}\left\|\nabla R_\varepsilon(v-u)\right\|_{L^2(\mathscr{O};\mathbb{R}^d)}.$$

Arguing as above, we see that this term is bounded by

$$C + C\sup_n \left\|X^\varepsilon_n(\cdot)\right\|_2 \left\|X^\varepsilon_1(\cdot)\right\|_2 + C\left\|X^\varepsilon_1(\cdot)\right\|_2^2,$$

which is $\mathbb{P}$-a.s. in $L^\infty(0,T)$ by estimate (3.4). □

We take the limit superior in (3.8) and continue investigating

$$\limsup_n \int_0^t \left[\Phi^n_\varepsilon(X^\varepsilon_1(s)) - \Phi^n_\varepsilon(X^\varepsilon_n(s)) + \Phi^1_\varepsilon(X^\varepsilon_n(s)) - \Phi^1_\varepsilon(X^\varepsilon_1(s))\right] \mathrm{d}s.$$

By Lemma 3.2, we can apply (reverse) Fatou's lemma such that it is sufficient to prove that

$$\limsup_n \left[\Phi^n_\varepsilon(X^\varepsilon_1(s)) - \Phi^n_\varepsilon(X^\varepsilon_n(s)) + \Phi^1_\varepsilon(X^\varepsilon_n(s)) - \Phi^1_\varepsilon(X^\varepsilon_1(s))\right] \leqslant 0$$

$\mathbb{P}$-a.s. and for $\mathrm{d}s$-a.e. $s\in[0,T]$. At this point, we apply Lemma 2.4 and get that

$$\begin{aligned}
&\limsup_n \left[\Phi^n_\varepsilon(X^\varepsilon_1(s)) - \Phi^n_\varepsilon(X^\varepsilon_n(s)) + \Phi^1_\varepsilon(X^\varepsilon_n(s)) - \Phi^1_\varepsilon(X^\varepsilon_1(s))\right]\\
\leqslant&\limsup_n \Phi^n_\varepsilon(X^\varepsilon_1(s)) - \liminf_n \Phi^n_\varepsilon(X^\varepsilon_n(s)) + \limsup_n \Phi^1_\varepsilon(X^\varepsilon_n(s)) - \Phi^1_\varepsilon(X^\varepsilon_1(s))\\
\leqslant&\Phi^1_\varepsilon(X^\varepsilon_1(s)) - \Phi^1_\varepsilon(Z^\varepsilon(s)) + \Phi^1_\varepsilon(Z^\varepsilon(s)) - \Phi^1_\varepsilon(X^\varepsilon_1(s))\\
=&0,
\end{aligned}$$

$\mathbb{P}$-a.s. and for $\mathrm{d}s$-a.e. $s\in[0,T]$.

Final step. Going back to

$$\begin{aligned}
&\left\|X_p(t) - X_1(t)\right\|_2\\
&\qquad\leqslant \left\|X_p(t) - X^\varepsilon_p(t)\right\|_2 + \left\|X^\varepsilon_p(t) - X^\varepsilon_1(t)\right\|_2 + \left\|X^\varepsilon_1(t) - X_1(t)\right\|_2
\end{aligned}$$

$\mathbb{P}$-a.s. and uniformly in $t\in[0,T]$, we can complete the proof using Steps I–III as follows. Let $\delta>0$. Pick $\varepsilon_0>0$, independent of p, such that the first and the third term are less than $\delta/3$. Having fixed ε_0 in such a way, we can pick p such that the second term is less than $\delta/3$.

The proof of Theorem 1.4 is complete.

4. Convergence of invariant measures

For each $p\in[1,2]$, let $\gamma_p\geqslant 0$. Under similar assumptions, replace $(\mathbf{E}_p)$ ($p\in[1,2]$) by the equation

$$(\mathbf{EE}_p)\begin{cases}\mathrm{d}X_p(t) = \left[\operatorname{div}\left[|\nabla X_p|_d^{p-1}\operatorname{sgn}(\nabla X_p)\right] - \gamma_p X_p\right]\mathrm{d}t + \sqrt{Q}\,\mathrm{d}W(t)\\ X_p(0) = x,\end{cases}$$

with Dirichlet boundary conditions in $\mathscr{O}$.

By [17, Theorem 4.3.9], if $p>1$ and $\gamma_p>0$, then equation $(\mathbf{EE}_p)$ has exactly one ergodic invariant measure ν_p with second moments.

The following statement can be easily verified by replacing Φ^p in the previous section by $\Phi^p + \frac{\gamma_p}{2}\left\|\cdot\right\|_2^2$.

Corollary 4.1. *Let $X_p = X_p(t,x)$ be the variational solution to equation* ($\mathbf{EE}_p$)*. If $\lim_{p\to 1}\gamma_p = \gamma_1$, then*

$$\lim_{p\to 1}\sup_{t\in[0,T]}\|X_p(t,x)-X_1(t,x)\|_2 = 0 \quad \mathbb{P}\text{-}a.s.$$

In the strongly monotone situation, we have the following:

Theorem 4.2. *Assume that for each $p\in[1,2]$, $\gamma_p \geqslant k$ for some $k>0$, and that $\gamma_p\to\gamma_1$ as $p\to 1$. Then* ($\mathbf{EE}_1$) *has a unique ergodic invariant measure ν_1, the family of measures $\{\nu_p\}_{p\in(1,2)}$ is tight and $\nu_p\to\nu_1$ in the weak sense as $p\to 1$.*

Proof. Let us prove tightness of $\{\nu_p\}_{p\in(1,2)}$ first. Denote the norm of $V_p := W_0^{1,p}(\mathscr{O})$ by $\|\cdot\|_{1,p}$.

Let $X_p = X_p(t,x)$ be the variational solution to equation ($\mathbf{EE}_p$). By Itō's formula, (see [17]),

$$\mathbb{E}\,\|X_p(t,x)\|_2^2 + 2\mathbb{E}\int_0^t {}_{V_p^*}\langle A_p(\overline{X}_p(s,x)),\overline{X}_p(s,x)\rangle_{V_p}\,\mathrm{d}s = \|x\|_2^2 + t\operatorname{Tr}Q, \tag{4.1}$$

and hence,

$$\frac{1}{t}\mathbb{E}\int_0^t \left\|\overline{X}_p(s,x)\right\|_{1,p}^p\,\mathrm{d}s \leqslant \frac{1}{2t}\|x\|_2^2 + \frac{1}{2}\operatorname{Tr}Q. \tag{4.2}$$

Here $\overline{X}_p$ is some suitable progressively measurable V_p-valued version of X_p. By the Krylov–Bogoliubov theorem (and a truncation argument), passing to the limit $t\to+\infty$ yields the estimate

$$\int_H \|x\|_{1,p}^p\,\nu_p(\mathrm{d}x) \leqslant \frac{\operatorname{Tr}Q}{2} \quad \forall p\in(1,2). \tag{4.3}$$

By [14, Ch. 5.1], $V_p\subset BV_0(\mathscr{O})$ for every p. We have that

$$\|x\|_{1,p}^p = \int_{\mathscr{O}}|\nabla x|^p\,\mathrm{d}x \geqslant \int_{\mathscr{O}}|\nabla x|\,\mathrm{d}x - |\mathscr{O}| = \|\mathrm{D}x\|(\mathscr{O}) - |\mathscr{O}|.$$

Let $\theta>0$. Set

$$B_\theta := \left\{x\in H \;\middle|\; \|\mathrm{D}x\|(\mathscr{O}) \leqslant \theta^{-1} + |\mathscr{O}|\right\}.$$

By [2, Corollary 3.49], B_θ is compact in H. Now, for any $p\in(1,2)$, using (4.3),

$$\nu_p(B_\theta^c) = \nu_p\left(\{\|\mathrm{D}\cdot\|(\mathscr{O}) - |\mathscr{O}| > \theta^{-1}\}\right) \leqslant \theta\int_H \|x\|_{1,p}^p\,\nu_p(\mathrm{d}x) \leqslant \theta\frac{\operatorname{Tr}Q}{2}.$$

Let $p_n\to 1$, $p_n\in(1,2)$. Let μ be a weak accumulation point of $\{\nu_{p_n}\}_{n\in\mathbb{N}}$, such that $\nu_{p_{n_k}}\to\mu$ weakly in H for some subsequence $\{p_{n_k}\}$. Denote by

$$P_t^p\varphi(x) := \mathbb{E}[\varphi(X_p(t,x))] \quad \varphi\in C_b(H)$$

the semigroup associated to equation ($\mathbf{EE}_p$).

Let $\{T_l\}$ be exactly the sequence of positive numbers such that $T_l\uparrow+\infty$ which is used in the proof of [8, Theorem 5.1]. By the Krylov–Bogoliubov theorem, we have for all $\varphi\in C_b(H)$ that

$$\begin{aligned}\int_H \varphi(x)\,\nu_{p_{n_k}}(\mathrm{d}x) &= \lim_l \frac{1}{T_l}\int_0^{T_l} P_t^{p_{n_k}}\varphi(x)\,\mathrm{d}t\\ &= \lim_l \frac{1}{T_l}\int_0^{T_l}\left(P_t^{p_{n_k}}\varphi(x) - P_t^1\varphi(x)\right)\mathrm{d}t\\ &\quad + \lim_l \frac{1}{T_l}\int_0^{T_l} P_t^1\varphi(x)\,\mathrm{d}t.\end{aligned}$$

Since φ is bounded, we can apply Lebesgue's dominated convergence theorem (in $L^2(\Omega)$) to Corollary 4.1 in order to obtain the convergence

$$P_t^{p_{n_k}}\varphi(x) \to P_t^1\varphi(x).$$

By [8, Theorem 5.1], we get the existence of an invariant measure for equation ($\mathbf{EE}_1$), which we denote by ν. By the above,

$$\int_H \varphi(x)\,\mu(\mathrm{d}x) = \int_H \varphi(x)\,\nu(\mathrm{d}x) \quad \forall \varphi \in C_b(H).$$

We are left to prove ergodicity of ν. Let $x, y \in H$. Let $X_p(t,x)$, $X_p(t,y)$ be solutions to ($\mathbf{EE}_p$) starting in x and y resp. By Itō's formula and strong monotonicity we have that $\mathbb{P}$-a.s.

$$\frac{1}{2}\left\|X_p(t,x) - X_p(t,y)\right\|_2^2 \leqslant \frac{1}{2}\left\|x-y\right\|_2^2 - \gamma_p \int_0^t \left\|X_p(s,x) - X_p(s,y)\right\|_2^2 \,\mathrm{d}s.$$

By Corollary 4.1, we can pass to the limit $p \to 1$ and get that $\mathbb{P}$-a.s.

$$\frac{1}{2}\left\|X_1(t,x) - X_1(t,y)\right\|_2^2 \leqslant \frac{1}{2}\left\|x-y\right\|_2^2 - \gamma_1 \int_0^t \left\|X_1(s,x) - X_1(s,y)\right\|_2^2 \,\mathrm{d}s.$$

Hence by Grönwall's lemma, $\mathbb{P}$-a.s.

$$\left\|X_1(t,x) - X_1(t,y)\right\|_2^2 \leqslant \left\|x-y\right\|_2^2 \exp(-2\gamma_1 t).$$

We know that by [8, Theorem 5.1], ν is an invariant measure for the semigroup

$$P_t^1\varphi := \mathbb{E}\left[\varphi(X_1(t,x))\right], \quad \varphi \in C_b(H),$$

If $\varphi \in C_b^1(H)$, we have that

$$\left|P_t\varphi(x) - \int_H \varphi(y)\,\nu(\mathrm{d}y)\right| \leqslant \int_H \left|P_t\varphi(x) - P_t\varphi(y)\right|\nu(\mathrm{d}y).$$

Furthermore,

$$\begin{aligned}\left|P_t\varphi(x) - \int_H \varphi(y)\,\nu(\mathrm{d}y)\right| &\leqslant \|\varphi\|_{1,\infty} \int_H \mathbb{E}\left\|X_1(t,x) - X_1(t,y)\right\|_2^2 \,\nu(\mathrm{d}y)\\ &\leqslant \|\varphi\|_{1,\infty} \int_H \left\|x-y\right\|_2^2 \,\nu(\mathrm{d}y)\exp(-2\gamma_1 t).\end{aligned}$$

By weak convergence, ν has second moments. Now, since $C_b^1(H)$ is dense in $L^2(H,\nu)$, it follows for any $f \in L^2(H,\nu)$ that

$$\lim_{t\to+\infty} P_t f(x) = \int_H f\,\mathrm{d}\nu, \quad \forall x \in H.$$

Therefore, ν is ergodic and strongly mixing by [11, Theorem 3.4.2]. □

With uniqueness of invariant measures assumed a priori, Theorem 1.5 can be proved exactly by the steps above, where $\gamma_p = 0$, $p \in [1,2]$.

Appendix A. Two notions of solutions

Lemma A.1. *A solution to equation* ($\mathbf{E}_p$), $p \in (1,2]$, *in the sense of Prévôt, Röckner* [17, Definition 4.2.1] *is also a solution in the sense of Definition 1.3.*

Proof. Let $X_p(t,x)$ be a solution to ($\mathbf{E}_p$) in the sense of Prévôt and Röckner. Let $\widehat{X}_p$ be its $\mathrm{d}t\otimes\mathbb{P}$-equivalence class. Then $\widehat{X}_p \in L^p([0,T]\times\Omega, \mathrm{d}t\otimes\mathbb{P}; V_p) \cap L^2([0,T]\times\Omega, \mathrm{d}t\otimes\mathbb{P}; H)$ and $\mathbb{P}$-a.s.

$$X_p(t,x) = x - \int_0^t \partial\Phi^p(\overline{X}_p(s,x))\,\mathrm{d}s + \int_0^t \sqrt{Q}\,\mathrm{d}W(s)$$

for any V_p-valued progressively measurable $\mathrm{d}t\otimes\mathbb{P}$-version $\overline{X}_p$ of $\widehat{X}_p$.

Now, let $Y \in C_W([0,T];H) \cap L^p((0,T)\times\Omega;V_p)$ and $G \in L^2_W(0,T;H)$ such that $\mathbb{P}$-a.s.

$$Y(t) = Y(0) - \int_0^t G(s)\,\mathrm{d}s + \int_0^t \sqrt{Q}\,\mathrm{d}W(s).$$

Take the difference of the two equations to obtain the V_p^*-valued process

$$X_p(t,x) - Y(t) = x - Y(0) - \int_0^t \partial\Phi^p(\overline{X}_p(s,x))\,\mathrm{d}s + \int_0^t G(s)\,\mathrm{d}s.$$

By the Itō-formula [17, Theorem 4.2.5], we get that

$$\begin{aligned}&\frac{1}{2}\left\|\overline{X}_p(t,x) - Y(t)\right\|_H^2 + \int_0^t {}_{V_p^*}\big\langle \partial\Phi^p(\overline{X}_p(s,x)), \overline{X}_p(s,x) - Y(s)\big\rangle_{V_p}\,\mathrm{d}s\\ =&\frac{1}{2}\left\|x - Y(0)\right\|_H^2 + \int_0^t {}_{V_p^*}\big\langle G(s), \overline{X}_p(s,x) - Y(s)\big\rangle_{V_p}\,\mathrm{d}s\end{aligned}$$

for any V_p-valued progressively measurable $\mathrm{d}t\otimes\mathbb{P}$-version $\overline{X}_p$ of $\widehat{X}_p$. Using the definition of the subgradient and the fact that G is H-valued yields

$$\begin{aligned}&\frac{1}{2}\left\|\overline{X}_p(t,x) - Y(t)\right\|_H^2 + \int_0^t (\Phi^p(\overline{X}_p(s,x)) - \Phi^p(Y(s)))\,\mathrm{d}s\\ \leqslant&\frac{1}{2}\left\|x - Y(0)\right\|_H^2 + \int_0^t (G(s), \overline{X}_p(s,x) - Y(s))_H\,\mathrm{d}s,\end{aligned}$$

which completes the proof. □

Appendix B. Mosco convergence

Let H be a separable Hilbert space. For a proper, convex functional $\Phi : H \to (-\infty,+\infty]$, the *Legendre transform* Φ^* is defined by

$$\Phi^*(y) := \sup_{x\in H}\left[(x,y)_H - \Phi(x)\right], \quad y \in H.$$

For two functionals $F, G : H \to (-\infty,+\infty]$ the *infimal convolution* $F\#G$ is defined by

$$(F\#G)(y) := \inf_{x\in H}\left[F(x) + G(y-x)\right], \quad y \in H.$$

For a proper, convex, l.s.c. functional $\Phi : H \to (-\infty,+\infty]$, for each $\varepsilon > 0$, define the *Moreau-Yosida regularization*

$$\Phi_\varepsilon := \Phi\#\frac{1}{2\varepsilon}\left\|\cdot\right\|_H^2.$$

Φ_ε is a continuous convex function. Also, $\lim_{\varepsilon\searrow 0}\Phi_\varepsilon = \Phi$ pointwise.

It holds that

$$(\Phi_\varepsilon)^* = \Phi^* + \frac{\varepsilon}{2}\left\|\cdot\right\|_H^2. \tag{B.1}$$

see e.g. [6, §2.2] and [3, Ch. 3].

Recall following definition.

Definition B.1 (Mosco convergence)**.** *Let* $\Phi^n : H \to (-\infty,+\infty]$, $n \in \mathbb{N}$, $\Phi : H \to (-\infty,+\infty]$ *be proper, convex, l.s.c. functionals. We say that* $\Phi^n \xrightarrow{M} \Phi$ *in the* Mosco sense *if*

(M1) $\forall x \in H\ \forall x_n \in H,\ n \in \mathbb{N},\ x_n \rightharpoonup x$ *weakly in* H: $\quad \liminf_n \Phi^n(x_n) \geqslant \Phi(x)$.

(M2) $\forall y \in H\ \exists y_n \in H,\ n \in \mathbb{N},\ y_n \to y$ *strongly in* H: $\quad \limsup_n \Phi^n(y_n) \leqslant \Phi(y)$.

We shall need following theorem.

Theorem B.2. *Let* $\Phi^n : H \to (-\infty, +\infty]$, $n \in \mathbb{N}$, $\Phi : H \to (-\infty, +\infty]$ *be proper, convex, l.s.c. functionals. Then the following conditions are equivalent.*

(i) $\Phi^n \xrightarrow{M} \Phi$.
(ii) $(\Phi^n)^* \xrightarrow{M} \Phi^*$.
(iii) $\forall \varepsilon > 0$, $\forall x \in H$: $\lim_n \Phi^n_\varepsilon(x) = \Phi_\varepsilon(x)$.

Proof. See [3, Theorems 3.18 and 3.26]. □

Corollary B.3. *Suppose that* $\Phi^n \xrightarrow{M} \Phi$. *Then for each* $\varepsilon > 0$, $\Phi^n_\varepsilon \xrightarrow{M} \Phi_\varepsilon$, *too.*

Proof. Suppose that $\Phi^n \xrightarrow{M} \Phi$. By Theorem B.2, $(\Phi^n)^* \xrightarrow{M} \Phi^*$, too.

If we can prove for each $\varepsilon > 0$ that $(\Phi^n_\varepsilon)^* \xrightarrow{M} (\Phi_\varepsilon)^*$, we are done by Theorem B.2. (M2) in Definition B.1 follows easily, using equation (B.1) and (M2) for $\{(\Phi_n)^*\}$ and Φ^*.

Let $x_n \in H$, $n \in \mathbb{N}$, $x \in H$ such that $x_n \rightharpoonup x$ weakly in H. By (B.1), weak lower semi-continuity of the norm and (M1) in Definition B.1 for $\{(\Phi_n)^*\}$ and Φ^* we get that

$$\begin{aligned}\liminf_n (\Phi^n_\varepsilon)^*(x_n) &= \liminf_n \left[(\Phi^n)^*(x_n) + \frac{\varepsilon}{2}\|x_n\|_H^2\right] \\ &\geqslant \liminf_n (\Phi^n)^*(x_n) + \liminf_n \frac{\varepsilon}{2}\|x_n\|_H^2 \geqslant \Phi^*(x) + \frac{\varepsilon}{2}\|x\|_H^2 = (\Phi_\varepsilon)^*(x).\end{aligned}$$

□

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