

Additive derivations on generalized Arens algebras

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Abstract

Given a von Neumann algebra M with a faithful normal finite trace τ denote by $L^\Lambda(M, \tau)$ the generalized Arens algebra with respect to M . We give a complete description of all additive derivations on the algebra $L^\Lambda(M, \tau)$. In particular each additive derivation on the algebra $L^\Lambda(M, \tau)$, where M is a type II von Neumann algebra, is inner.

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1. Introduction

The present paper continues the series of papers [2]-[9] devoted to the study and a description of derivations on the algebra $LS(M)$ of locally measurable operators affiliated with a von Neumann algebra M and on its various subalgebras.

Let \mathcal{A} be an algebra over the field complex number \mathbb{C} . A linear (additive) operator $D : \mathcal{A} \rightarrow \mathcal{A}$ is called a linear (additive) *derivation* if it satisfies the identity $D(xy) = D(x)y + xD(y)$ for all $x, y \in \mathcal{A}$ (Leibniz rule). Each element $a \in \mathcal{A}$ defines a linear derivation D_a on \mathcal{A} given as $D_a(x) = ax - xa$, $x \in \mathcal{A}$. Such derivations D_a are said to be *inner derivations*. If the element a implementing the derivation D_a on \mathcal{A} , belongs to a larger algebra \mathcal{B} , containing \mathcal{A} (as a proper ideal as usual) then D_a is called a *spatial derivation*.

One of the main problems in the theory of derivations is to prove the automatic continuity, innerness or spatialness of derivations or to show the existence of non inner and discontinuous derivations on various topological algebras.

On this way A. F. Ber, F. A. Sukochev, V. I. Chilin [10] obtained necessary and sufficient conditions for the existence of non trivial derivations on commutative regular algebras. In particular they have proved that the algebra $L^0(0, 1)$ of all (classes of equivalence of) complex measurable functions on the interval $(0, 1)$ admits non trivial derivations. Independently A. G. Kusraev [16] by means of Boolean-valued analysis has also proved the existence of non trivial derivations and automorphisms on $L^0(0, 1)$. It is clear that these derivations are discontinuous in the measure topology, and therefore they are neither inner nor spatial. It was conjectured that the existence of such exotic examples of derivations deeply depends on the commutativity of the underlying von Neumann algebra M . In this connection we have initiated the study of the above problems in the non commutative case [2]-[6], by considering derivations on the algebra $LS(M)$ of all locally measurable operators affiliated with a von Neumann algebra M and on various subalgebras of $LS(M)$. In [2] so-called noncommutative Arens algebras $L^\omega(M, \tau) = \bigcap_{p \geq 1} L^p(M, \tau)$ and related algebras associated with a von Neumann algebra M and a faithful normal semi-finite trace τ have been considered. It has been proved that every derivation on this algebra is spatial, and, if the trace τ is finite, then all derivations are inner. In [5] and [6] the mentioned conjecture concerning derivations on

on the algebra $LS(M)$ has been confirmed for type I von Neumann algebras.

Recently this conjecture was also independently confirmed for the type I case in the paper of A.F. Ber, B. de Pagter and A.F. Sukochev [11] by means of a representation of measurable operators as operator valued functions. Another approach to similar problems in the framework of type I AW^* -algebras has been outlined in the paper of A.F. Gutman, A.G.Kusraev and S.S. Kutateladze [13].

In [5] we considered derivations on the algebra $LS(M)$ of all locally measurable operators affiliated with a type I von Neumann algebra M , and also on its subalgebras $S(M)$ – of measurable operators and $S(M, \tau)$ of τ -measurable operators, where τ is a faithful normal semi-finite trace on M . It was proved that an arbitrary derivation D on each of these algebras can be uniquely decomposed into the sum $D = D_a + D_\delta$ where the derivation D_a is inner (for $LS(M)$, $S(M)$ and $S(M, \tau)$) while the derivation D_δ is an extension of a derivation δ (possibly non trivial) on the center of the corresponding algebra.

In the present paper we consider additive derivations on generalized Arens algebras in the sense of Kunze [15] with respect to a von Neumann algebra with a faithful normal finite trace.

In section 1 we give some necessary properties of the generalized Arens algebra $L^\Lambda(M, \tau)$.

Section 2 is devoted to study of additive derivations on generalized Arens algebras. We prove that an arbitrary additive derivation D on the algebra $L^\Lambda(M, \tau)$ can be uniquely decomposed into the sum $D = D_a + D_\delta$, where the derivation D_a is inner while the derivation D_δ is an extension of some additive derivation δ on the center of the algebra $L^\Lambda(M, \tau)$. In particular, if M is a type II von Neumann algebra then every additive derivation on the algebra $L^\Lambda(M, \tau)$ is inner.

2. Generalized Arens algebras

Let H be a complex Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on H . Consider a von Neumann algebra M in $B(H)$ with the operator norm $\|\cdot\|_M$. Denote by $P(M)$ the lattice of projections in M .

A linear subspace \mathcal{D} in H is said to be *affiliated* with M (denoted as $\mathcal{D}\eta M$), if

$u(\mathcal{D}) \subset \mathcal{D}$ for every unitary u from the commutant

$$M' = \{y \in B(H) : xy = yx, \forall x \in M\}$$

of the von Neumann algebra M .

A linear operator x on H with the domain $\mathcal{D}(x)$ is said to be *affiliated* with M (denoted as $x\eta M$) if $\mathcal{D}(x)\eta M$ and $u(x(\xi)) = x(u(\xi))$ for all $\xi \in \mathcal{D}(x)$.

Let τ be a faithful normal semi-finite trace on M . We recall that a closed linear operator x is said to be τ -*measurable* with respect to the von Neumann algebra M , if $x\eta M$ and $\mathcal{D}(x)$ is τ -dense in H , i.e. $\mathcal{D}(x)\eta M$ and given $\varepsilon > 0$ there exists a projection $p \in M$ such that $p(H) \subset \mathcal{D}(x)$ and $\tau(p^\perp) < \varepsilon$. The set $S(M, \tau)$ of all τ -measurable operators with respect to M is a unital *-algebra when equipped with the algebraic operations of strong addition and multiplication and taking the adjoint of an operator (see [18]).

Consider the topology t_τ of convergence in measure or *measure topology* on $S(M, \tau)$, which is defined by the following neighborhoods of zero:

$$V(\varepsilon, \delta) = \{x \in S(M, \tau) : \exists e \in P(M), \tau(e^\perp) \leq \delta, xe \in M, \|xe\|_M \leq \varepsilon\},$$

where ε, δ are positive numbers, and $\|\cdot\|_M$ denotes the operator norm on M .

It is well-known [18] that $S(M, \tau)$ equipped with the measure topology is a complete metrizable topological *-algebra.

Recall [14] that ϕ is a Young function, if

$$\phi(t) = \int_0^t \varphi(s) ds, \quad t \geq 0,$$

where the real-valued function φ defined on $[0, \infty)$ has the following properties:

- (i) $\varphi(0) = 0$, $\varphi(s) > 0$ for $s > 0$ and $\lim_{s \rightarrow \infty} \varphi(s) = \infty$,
- (ii) φ is right continuous,
- (iii) φ is nondecreasing on $(0, \infty)$.

Every Young function is a continuous, convex and strictly increasing function. For every Young function ϕ there is a complementary Young function ψ given by the density

$$\tilde{\psi}(t) = \sup\{s : \phi(s) \leq t\}.$$

The complement of ψ is ϕ again. Further a Young function ϕ is said to satisfy the Δ_2 -condition, shortly $\phi \in \Delta_2$, if there exists a $k > 0$ and $T \geq 0$ such that:

$$\phi(2t) \leq k\phi(t)$$

for all $t \geq T$.

Put

$$K_\phi = \{x \in S(M, \tau) : \tau(\phi(|x|)) \leq 1\}$$

and

$$L_\phi(M, \tau) = \bigcup_{n=1}^{\infty} nK_\phi.$$

It is known [17] (see also [15]) that $L_\phi(M, \tau)$ is a Banach space with respect to the norm

$$\|x\|_\phi = \inf \left\{ \lambda > 0 : \frac{1}{\lambda}x \in K_\phi \right\}, \quad x \in L_\phi(M, \tau).$$

Recall [15] that $\phi_1 \prec \phi_2$, if there exist two nonnegative constants c and T such that $\phi_1(t) \leq \phi_2(ct)$ for all $t \geq T$. Let Λ be a generating family of Young functions, i.e. for $\phi_1, \phi_2 \in \Lambda$ there is a $\psi \in \Lambda$ with $\phi_1, \phi_2 \prec \psi$. A generating family Λ of Young functions is said to be quadratic, if for any $\phi \in \Lambda$ there is a $\psi \in \Lambda$ such that the composition of ϕ and the squaring function as a Young function is smaller than ψ regarding the partial order \prec , i.e. there are $c > 0$ and $T \geq 0$ with $\phi(t^2) \leq \psi(ct)$ for all $t \geq T$. For a quadratic family Λ of Young functions we define

$$L^\Lambda(M, \tau) = \bigcap_{\phi \in \Lambda} L_\phi(M, \tau).$$

On the space $L^\Lambda(M, \tau)$ one can consider the topology t_Λ , generated by the system of norms $\{\|\cdot\|_\phi : \phi \in \Lambda\}$.

It is known [15, Proposition 4.1] that if Λ is a quadratic family of Young functions, then $(L^\Lambda(M, \tau), t_\Lambda)$ is a complete locally convex *-algebra with jointly continuous multiplications.

Note that if $\Lambda = \{t^p : p \geq 1\}$ we have that

$$L^\Lambda(M, \tau) = L^\omega(M, \tau) = \bigcap_{p \geq 1} L^p(M, \tau).$$

Non-commutative Arens algebras $L^\omega(M, \tau)$ were introduced by Inoue [12] and their properties were investigated in [1]. Generalized Arens algebras were introduced by Kunze [15].

Let $\varphi \in \Lambda$ be a Young function. Then there exists a Young function $\phi \in \Lambda$ and $k > 0$ such that

$$\|xy\|_\varphi \leq k\|x\|_\phi\|y\|_\phi \quad (1)$$

for all $x, y \in L^\Lambda(M, \tau)$ (see [15]).

Let us remark that, if τ is a finite trace, then $t \prec \phi(t)$ for every Young function and for any quadratic family Λ of Young functions we obtain that

$$L^\Lambda(M, \tau) \subset L^\omega(M, \tau). \quad (2)$$

Further, if every $\phi \in \Lambda$ satisfies the Δ_2 -condition then

$$L^\omega(M, \tau) \subset L^\Lambda(M, \tau).$$

It is known [15] that if N is a von Neumann subalgebra of M then

$$L_\phi(N, \tau_N) = S(N, \tau_N) \cap L_\phi(M, \tau),$$

where τ_N is the restriction of the trace τ onto N .

It should be noted, that if M is a finite von Neumann algebra with a faithful normal semi-finite trace τ , then the restriction τ_Z of the trace τ onto the center $Z(M)$ of M is also semi-finite.

Further we shall need the description of the center of the algebra $L^\Lambda(M, \tau)$ for finite von Neumann algebras.

Proposition 2.1. *Let M be a von Neumann algebra with a faithful normal finite trace τ and with the center $Z(M)$. Then*

$$Z(L^\Lambda(M, \tau)) = L^\Lambda(Z(M), \tau_Z).$$

Proof. Using the equality

$$L_\phi(N, \tau_N) = S(N, \tau_N) \cap L_\phi(M, \tau).$$

we obtain that

$$L^\Lambda(N, \tau_N) = S(N, \tau_N) \cap L^\Lambda(M, \tau).$$

Hence

$$\begin{aligned} L^\Lambda(Z(M), \tau_Z) &= S(Z(M), \tau_Z) \cap L^\Lambda(M, \tau) = \\ &= Z(S(M, \tau)) \cap L^\Lambda(M, \tau) = Z(L^\Lambda(M, \tau)), \end{aligned}$$

i.e.

$$Z(L^\Lambda(M, \tau)) = L^\Lambda(Z(M), \tau_Z).$$

The proof is complete. ■

3. Derivations on the generalized Arens algebras

In this section we give a complete description of all additive derivations on the algebra $L^\Lambda(M, \tau)$.

Let \mathcal{A} be an algebra with the center $Z(\mathcal{A})$ and let $D : \mathcal{A} \rightarrow \mathcal{A}$ be an additive derivation. Given any $x \in \mathcal{A}$ and a central element $a \in Z(\mathcal{A})$ we have

$$D(ax) = D(a)x + aD(x)$$

and

$$D(xa) = D(x)a + xD(a).$$

Since $ax = xa$ and $aD(x) = D(x)a$, it follows that $D(a)x = xD(a)$ for any $a \in \mathcal{A}$. This means that $D(a) \in Z(\mathcal{A})$, i.e. $D(Z(\mathcal{A})) \subseteq Z(\mathcal{A})$. Therefore given any additive derivation D on the algebra \mathcal{A} we can consider its restriction $\delta : Z(\mathcal{A}) \rightarrow Z(\mathcal{A})$.

We shall need some facts about additive derivations $\delta : \mathbb{C} \rightarrow \mathbb{C}$. Every such derivation vanishes at every algebraic number. On the other hand, if $\lambda \in \mathbb{C}$ is transcendental then there is a additive derivation $\delta : \mathbb{C} \rightarrow \mathbb{C}$ which does not vanish at λ (see [20]).

Let $M_n(\mathbb{C})$ be the algebra of $n \times n$ matrices over \mathbb{C} . If $e_{i,j}$, $i, j = \overline{1, n}$, are the matrix units in $M_n(\mathbb{C})$, then each element $x \in M_n(\mathbb{C})$ has the form

$$x = \sum_{i,j=1}^n \lambda_{ij} e_{ij}, \quad \lambda_{i,j} \in \mathbb{C}, \quad i, j = \overline{1, n}.$$

Let $\delta : \mathbb{C} \rightarrow \mathbb{C}$ be an additive derivation. Setting

$$D_\delta \left(\sum_{i,j=1}^n \lambda_{ij} e_{ij} \right) = \sum_{i,j=1}^n \delta(\lambda_{ij}) e_{ij} \quad (3)$$

we obtain a well-defined additive operator D_δ on the algebra $M_n(\mathbb{C})$. Moreover D_δ is an additive derivation on the algebra $M_n(\mathbb{C})$ and its restriction onto the center of the algebra $M_n(\mathbb{C})$ coincides with the given δ .

It is known [21, Theorem 2.2] that if M be a von Neumann factor of type I_n , $n \in \mathbb{N}$ then every additive derivation D on the algebra M can be uniquely represented as a sum

$$D = D_a + D_\delta,$$

where D_a is an inner derivation implemented by an element $a \in M$ while D_δ is the additive derivation of the form (3) generated by an additive derivation δ on the center of M identified with \mathbb{C} .

Note that if M is a finite-dimensional von Neumann algebra then $L^\Lambda(M, \tau) = M$ for any faithful normal finite trace τ .

Now let M be an arbitrary finite-dimensional von Neumann algebra with the center $Z(M)$. There exist a family of mutually orthogonal central projections $\{z_1, z_2, \dots, z_k\}$ from M with $\bigvee_{i=1}^k z_i = \mathbf{1}$ and $n_1, n_2, \dots, n_k \in \mathbb{N}$ such that the algebra M is *-isomorphic with the C^* -product of von Neumann factors $z_i M$ of type I_{n_i} respectively, i.e.

$$M \cong M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \dots \oplus M_{n_k}(\mathbb{C}).$$

Suppose that D is an additive derivation on M , and δ is its restriction onto its center $Z(M)$. Since $\delta(zx) = z\delta(x)$ for all central projection $z \in Z(M)$ and $x \in M$ then δ maps each $z_i Z(M) \cong \mathbb{C}$ into itself, δ generates an additive derivation δ_i on \mathbb{C} for each $i = \overline{1, k}$.

Let D_{δ_i} be the additive derivation on the matrix algebra $M_{n_i}(\mathbb{C})$, $i = \overline{1, k}$, defined as in (3). Put

$$D_\delta((x_i)_{i=1}^k) = (D_{\delta_i}(x_i)), (x_i)_{i=1}^k \in M. \quad (4)$$

Then the map D_δ is an additive derivation on M .

Lemma 3.1. *Let M be a finite-dimensional von Neumann algebra. Each additive derivation D on the algebra M can be uniquely represented in the form*

$$D = D_a + D_\delta,$$

where D_a is an inner derivation implemented by an element $a \in M$, and D_δ is an additive derivation given as (4).

Proof. Let D be an additive derivation on M , and let δ be its restriction onto $Z(M)$. Consider an additive derivation D_δ on $Z(M)$ of the form (4), generated by an additive derivation δ . Since an additive derivations D and D_δ coincides on $Z(M)$ then an additive derivation $D - D_\delta$ is a linear derivation. Hence by Sakai's theorem [19, Theorem 4.1.6] $D - D_\delta$ is an inner derivation. This means that there exists an element $a \in M$ such that $D_a = D - D_\delta$ and therefore $D = D_a + D_\delta$. The proof is complete. ■

Now let M be a commutative von Neumann algebra with a faithful normal finite trace τ . Given an arbitrary additive derivation δ on $L^\Lambda(M, \tau)$ the element

$$z_\delta = \inf\{z \in P(M) : z\delta = \delta\}$$

is called the support of the derivation δ .

Suppose that M is a commutative von Neumann algebra with a faithful normal finite trace τ and q_1, q_2, \dots, q_k are atoms in M . Then

$$L^\Lambda(M, \tau) \cong q_1\mathbb{C} \oplus q_2\mathbb{C} \oplus \dots \oplus q_k\mathbb{C} \oplus pL^\Lambda(M, \tau),$$

where $p = \mathbf{1} - \bigvee_{i=1}^k q_i$.

Now if $\delta_i : \mathbb{C} \rightarrow \mathbb{C}$ is an additive derivation then

$$\delta(x) = (\delta_1(q_1x), \dots, \delta_k(q_kx), 0), \quad x \in L^\Lambda(M, \tau) \tag{5}$$

is also an additive derivation. Note that $z_\delta = \bigvee\{q_i : \delta_i \neq 0, 1 \leq i \leq k\}$.

Lemma 3.2. *Let M be a commutative von Neumann algebra with a faithful normal finite trace τ . For any non trivial additive derivation $\delta : L^\Lambda(M, \tau) \rightarrow L^\Lambda(M, \tau)$ there exists a sequence $\{a_n\}_{n=1}^\infty$ in M with $|a_n| \leq \mathbf{1}$, $n \in \mathbb{N}$, such that*

$$|\delta(a_n)| \geq nz_\delta$$

for all $n \in \mathbb{N}$.

In [5, Lemma 2.6] (see also [11, Lemma 4.6]) this assertion was proved for linear derivations on the algebra $S(M)$, but the proof is the same for additive derivations on $L^\Lambda(M, \tau)$.

The following result shows that the above construction (5) is the general form of additive derivations on the generalized Arens algebras in the commutative case.

Lemma 3.3. *Let M be a commutative von Neumann algebra with a faithful normal finite trace τ and let δ be an additive derivation on the algebra $L^\Lambda(M, \tau)$. Then $z_\delta M$ is a finite-dimensional algebra.*

Proof. Suppose that $z_\delta M$ is infinite-dimensional. Then there exists a sequence of mutually orthogonal projections $\{z_n\}_{n=1}^\infty$ in M such that $\bigvee_{n=1}^\infty z_n = z_\delta$. By Lemma 3.2 there exists a sequence $\{a_n\}_{n=1}^\infty$ in M with $|a_n| \leq \mathbf{1}$, $n \in \mathbb{N}$, such that

$$|\delta(a_n)| \geq 2^n \tau(z_n)^{-1} z_\delta \quad (6)$$

for all $n \in \mathbb{N}$. Put

$$a = \sum_{n=1}^{\infty} \frac{a_n z_n}{2^n}.$$

Then $a \in M \subset L^\Lambda(M, \tau)$ and

$$\delta(a) = \delta \left(\sum_{n=1}^{\infty} \frac{a_n z_n}{2^n} \right) = \sum_{n=1}^{\infty} \frac{z_n}{2^n} \delta(a_n).$$

From (6) we obtain that

$$|\delta(a)| = \sum_{n=1}^{\infty} \frac{z_n}{2^n} |\delta(a_n)| \geq \sum_{n=1}^{\infty} \frac{z_n}{2^n} 2^n \tau(z_n)^{-1} z_\delta,$$

i.e.

$$|\delta(a)| \geq \sum_{n=1}^{\infty} \tau(z_n)^{-1} z_n.$$

Thus

$$\tau(|\delta(a)|) \geq \sum_{n=1}^{\infty} \tau(z_n)^{-1} \tau(z_n) = \sum_{n=1}^{\infty} 1 = \infty.$$

This means that $\delta(a) \notin L^1(M, \tau)$. Then by (2) we have that $\delta(a) \notin L^\Lambda(M, \tau)$. This contradiction implies that $z_\delta M$ is a finite-dimensional algebra. The proof is complete. \blacksquare

Lemma 3.3 implies the following

Corollary 3.1. *Let M be a commutative von Neumann algebra with a faithful normal finite trace τ such that a Boolean algebra $P(M)$ of all projections of M is continuous. Then every additive derivation on the algebra $L^\Lambda(M, \tau)$ is zero.*

Note that properties of additive derivations on the algebras $S(M, \tau)$ and $L^\Lambda(M, \tau)$, where M be a commutative von Neumann algebra with a faithful normal finite trace τ , are quite opposite. Indeed, if the Boolean algebra $P(M)$ is continuous then the algebra

$S(M, \tau)$ admits a non-zero linear, in particular additive, derivation, (see [10, Theorem 3.3]), whereas the algebra $L^\Lambda(M, \tau)$ in this case does not admit a non-zero additive derivation (see Corollary 3.1).

Now we consider the noncommutative case.

We shall need following result ([7, Theorem 4.1], see also [9, Theorem 6.8]).

Theorem 3.1. *Let M be a von Neumann algebra with a faithful normal finite trace τ . If $A \subseteq L^\omega(M, \tau)$ is a solid $*$ -subalgebra such that $M \subseteq A$, then every linear derivation on A is inner.*

The following theorem is one of the main results of this paper.

Theorem 3.2. *Let M be a type II von Neumann algebra with a faithful normal finite trace τ . Then every additive derivation on the algebra $L^\Lambda(M, \tau)$ is inner.*

The proof of the theorem 3.2 follows from Theorem 3.1 and the following assertion.

Lemma 3.4. *Let M be a type II von Neumann algebra with a faithful normal finite trace τ , and suppose that $D : L^\Lambda(M, \tau) \rightarrow L^\Lambda(M, \tau)$ is an additive derivation. Then $D|_{Z(L^\Lambda(M, \tau))} \equiv 0$, in particular, D is a linear.*

Proof. Let D be an additive derivation on $L^\Lambda(M, \tau)$, and let δ be its restriction onto $Z(L^\Lambda(M, \tau))$.

Since M is of type II there exists a sequence of mutually orthogonal projections $\{p_n\}_{n=1}^\infty$ in M with central covers $\mathbf{1}$ (i.e. faithful projections). For any bounded sequence $B = \{b_n\}_{n \in \mathbb{N}}$ in $Z(M)$ define an operator x_B by

$$x_B = \sum_{n=1}^{\infty} b_n p_n.$$

Then

$$x_B p_n = p_n x_B = b_n p_n \tag{7}$$

for all $n \in \mathbb{N}$.

Take $b \in Z(M)$ and $n \in \mathbb{N}$. From the identity

$$D(bp_n) = D(b)p_n + bD(p_n)$$

multiplying it by p_n from the both sides we obtain

$$p_n D(bp_n) p_n = p_n D(b) p_n + b p_n D(p_n) p_n.$$

Since p_n is a projection, one has that $p_n D(p_n) p_n = 0$, and since $D(b) = \delta(b) \in Z(L^\Lambda(M, \tau))$, we have

$$p_n D(b p_n) p_n = \delta(b) p_n. \quad (8)$$

Now from the identity

$$D(x_B p_n) = D(x_B) p_n + x_B D(p_n),$$

in view of (7) one has similarly

$$p_n D(b_n p_n) p_n = p_n D(x_B) p_n + b_n p_n D(p_n) p_n,$$

i.e.

$$p_n D(b_n p_n) p_n = p_n D(x_B) p_n. \quad (9)$$

Now (8) and (9) imply

$$p_n D(x_B) p_n = \delta(b_n) p_n. \quad (10)$$

Let $\varphi \in \Lambda$. By (1) there are $\phi, \psi \in \Lambda$ and $k > 0$ such that

$$\|x_1 x_2 x_3\|_\varphi \leq k \|x_1\|_\phi \|x_2\|_\phi \|x_3\|_\psi$$

for all $x_1, x_2, x_3 \in L^\Lambda(M, \tau)$. If we suppose that $\delta \neq 0$ then $z_\delta \neq 0$. By Lemma 3.2 there exists a bounded sequence $B = \{b_n\}_{n \in \mathbb{N}}$ in $Z(M)$ such that

$$|\delta(b_n)| \geq n c_n z_\delta$$

for all $n \in \mathbb{N}$, where $c_n = k \|p_n\|_\phi^2 \|p_n z_\delta\|_\varphi^{-1}$. Then in view of (10) we obtain

$$\begin{aligned} k \|p_n\|_\phi \|D(x)\|_\psi \|p_n\|_\phi &\geq \|p_n D(x) p_n\|_\varphi = \\ &= \|\delta(b_n) p_n\|_\varphi \geq \|n c_n p_n z_\delta\|_\varphi = n c_n \|p_n z_\delta\|_\varphi, \end{aligned}$$

i.e.

$$\|D(x)\|_\psi \geq n c_n k^{-1} \|p_n\|_\phi^{-2} \|p_n z_\delta\|_\varphi.$$

Hence

$$\|D(x)\|_\psi \geq n$$

for all $n \in \mathbb{N}$. This contradiction implies that $\delta \equiv 0$, i.e. D is identically zero on the center of $L^\Lambda(M, \tau)$, and therefore it is linear. The proof is complete. ■

Now consider an additive derivation D on $L^\Lambda(M, \tau)$ and let δ be its restriction onto its center $Z(L^\Lambda(M, \tau))$. By Lemma 3.3 $z_\delta M$ is a finite-dimensional and $z_\delta^\perp \delta \equiv 0$, i.e. $\delta = z_\delta \delta$.

Let D_δ be the derivation on $z_\delta L^\Lambda(M, \tau) = z_\delta M$ defined as in (4) and consider its extension D_δ on $L^\Lambda(M, \tau) = z_\delta L^\Lambda(M, \tau) \oplus z_\delta^\perp L^\Lambda(M, \tau)$ which is defined as

$$D_\delta(x_1 + x_2) := D_\delta(x_1), \quad x_1 \in z_\delta L^\Lambda(M, \tau), x_2 \in z_\delta^\perp L^\Lambda(M, \tau). \quad (11)$$

The following theorem is the main result of this paper, and gives the general form of derivations on the algebra $L^\Lambda(M, \tau)$.

Theorem 3.3. *Let M be a von Neumann algebra with faithful normal finite trace τ . Each additive derivation D on $L^\Lambda(M, \tau)$ can be uniquely represented in the form*

$$D = D_a + D_\delta$$

where D_a is an inner derivation implemented by an element $a \in L^\Lambda(M, \tau)$, and D_δ is an additive derivation of the form (11), generated by an additive derivation δ on the center of $L^\Lambda(M, \tau)$.

Proof. Let D be an additive derivation on $L^\Lambda(M, \tau)$, and let δ be its restriction onto $Z(L^\Lambda(M, \tau)) = L^\Lambda(Z(M), \tau_Z)$. By Lemma 3.3 $z_\delta Z(M)$ is a finite-dimensional. Thus $z_\delta M$ is a C^* -product of finite number of von Neumann factors of type I_n or II. Since by Lemma 3.4 any additive derivation on $L^\Lambda(M, \tau)$, where M is a type II algebra, is linear then by Theorem 3.2 it is inner. Therefore $z_\delta M$ is a C^* -product of finite number of von Neumann factors of type I_n .

Now consider an additive derivation D_δ on $L^\Lambda(M, \tau)$ of the form (11), generated by a derivation δ . Since the derivations D and D_δ coincide on $L^\Lambda(Z(M), \tau)$ then $D - D_\delta$ is a linear derivation. Hence Theorem 3.2 implies that the derivation $D - D_\delta$ is inner. This means that there exists an element $a \in L^\Lambda(M, \tau)$ such that $D_a = D - D_\delta$ and therefore $D = D_a + D_\delta$. The proof is complete. ■

Corollary 3.2. *Let M be a von Neumann algebra without type I_n direct summands and with a faithful normal finite trace τ . Then each additive derivation on $L^\Lambda(M, \tau)$ is inner.*

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