Existence and Uniqueness of Solutions to Nonlinear Evolution Equations with Locally Monotone Operators *

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Abstract

In this paper we establish the existence and uniqueness of solutions for nonlinear evolution equations on Banach space with locally monotone operators, which is a generalization of the classical result by J.L. Lions for monotone operators. In particular, we show that local monotonicity implies the pseudo-monotonicity. The main result is applied to various types of PDE such as reaction-diffusion equations, generalized Burgers equation, Navier-Stokes equation, 3D Leray-α model and p-Laplace equation with non-monotone perturbations.

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1 Main results

Consider the following Gelfand triple

$$V \subseteq H \equiv H^* \subseteq V^*,$$

i.e. $(H, \langle \cdot, \cdot \rangle_H)$ is a real separable Hilbert space and identified with its dual space H^* by the Riesz isomorphism, V is a real reflexive Banach space such that it is continuously and

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densely embedded into H. If $\langle\cdot,\cdot\rangle_V$ denotes the dualization between V and its dual space V^*, then it follows that

$$\langle u,v\rangle_V = \langle u,v\rangle_H, \quad u\in H, v\in V.$$

The main aim of this paper is to establish the existence and uniqueness of solutions for general nonlinear evolution equations

$$u'(t) = A(t,u(t)) + b(t),\ 0<t<T,\ u(0)=u_0\in H, \tag{1.1}$$

where $T>0$, u' is the generalized derivative of u on $(0,T)$ and $A:[0,T]\times V\to V^*, b:[0,T]\to V^*$ is measurable, *i.e.* for each $u\in L^1([0,T];V)$, $A(t,u(t))$ is V^*-measurable on $[0,T]$.

It's well known that (1.1) has a unique solution if A satisfies the monotone and coercivity conditions (cf. [23, 19, 40]). The proof is mainly based on the Galerkin approximation and the monotonicity tricks. The theory of monotone operators started from the substantial work of Minty [30, 31], then it was studied systematically by Browder [7, 8] in order to obtain existence theorems for quasi-linear elliptic and parabolic partial differential equations. The existence results of Browder were generalized to more general classes of quasi-linear elliptic differential equations by Leray and Lions [22], and Hartman and Stampacchia [17]. We refer to [6, 23, 38, 40] for more detailed exposition and references.

One of most important extensions of monotone operator is the pseudo-monotone operator, which was first introduced by Brézis in [5]. The prototype of a pseudo-monotone operator is the sum of a monotone operator and a strongly continuous operator (*i.e.* a operator maps a weakly convergent sequence into a strongly convergent sequence). Hence the theory of pseudo-monotone operator unifies both the monotonicity arguments and the compactness arguments. For example, it can be applied to show the existence of solutions for general quasi-linear elliptic equations with lower order terms which satisfy no monotonicity condition (cf. [9, 38, 40]).

This variational approach has also been adapted for analyzing stochastic partial differential equations (SPDE). The existence and uniqueness of solutions to SPDE was first developed by Pardoux [32], Krylov and Rozovskii [19], we refer to [15, 34] for further generalizations. Within this framework many different types of properties have been established recently, *e.g.* see [26, 35] for the small noise large deviation principle, [16] for discretization approximation schemes to the solution of SPDE, [24, 25, 39] for the Harnack inequality and consequent ergodicity, compactness and contractivity for the associated transition semigroups, and [27, 4, 14] for the invariance of subspaces and existence of random attractors for corresponding random dynamical systems.

In this work we establish the existence, uniqueness and continuous dependence on initial conditions of solutions to (1.1) by using the local monotonicity condition instead of the classical monotonicity condition. The analogous result for stochastic PDE has been established in [28]. The standard growth condition on A (cf. [23, 19, 40]) is also replaced by a much weaker condition such that the main result can be applied to larger class of examples. One of the key observations is that we show the local monotonicity implies the pseudo-monotonicity (see Lemma 2.2), which may have some independent interests. The main result is applied to establish the existence and uniqueness of solutions for a large class of nonlinear evolution equations such as reaction diffusion equations, generalized Burgers type equations, generalized p-Laplace equations, 2-D Navier-Stokes equation and 3D Leray-α model of turbulence.

Suppose for fixed $\alpha > 1, \beta \geq 0$ there exist constants $\delta > 0$, C and a positive function $f \in L^1([0,T];\mathbb{R})$ such that the following conditions hold for all $t \in [0,T]$ and $v, v_1, v_2 \in V$.

$(H1)$ (Hemicontinuity) The map $s \mapsto \langle A(t, v_1 + sv_2), v\rangle_V$ is continuous on $\mathbb{R}$.

$(H2)$ (Local monotonicity)

$$\langle A(t,v_1) - A(t,v_2), v_1 - v_2\rangle_V \leq (C + \rho(v_1) + \eta(v_2))\, \|v_1 - v_2\|_H^2,$$

where $\rho, \eta : V \to [0, +\infty)$ are measurable functions and locally bounded in V.

$(H3)$ (Coercivity)

$$2\langle A(t,v), v\rangle_V \leq -\delta \|v\|_V^\alpha + C\|v\|_H^2 + f(t).$$

$(H4)$ (Growth)

$$\|A(t,v)\|_{V^*} \leq \left(f(t)^{\frac{\alpha-1}{\alpha}} + C\|v\|_V^{\alpha-1}\right)\left(1 + \|v\|_H^\beta\right).$$

Remark 1.1. (1) If $\beta = 0$ and $\rho = \eta \equiv 0$, then $(H1) - (H4)$ are the classical monotone and coercive conditions in [40, Theorem 30.A] (see also [23, 19, 33]). It can be applied to many quasi-linear PDE such as porous medium equation and p-Laplace equation (cf. [40, 33]).

(2) One typical form of $(H2)$ in applications is

$$\rho(v) = \eta(v) = C\|v\|^\gamma,$$

where $\|\cdot\|$ is some norm on V and C, γ are some constants. The typical examples are 2-D Navier-Stokes equation on bounded or unbounded domain and Burgers equation, which satisfy $(H2)$ but do not satisfy the classical monotonicity condition (*i.e.* $\rho = \eta \equiv 0$). We refer to section 3 for more examples.

(3) If $\rho \equiv 0$ in $(H2)$, then the existence and uniqueness of solutions to (1.1) with general random noise has been established in [28] by using some different techniques.

(4) $(H4)$ is also weaker than the following standard growth condition assumed in the literature (cf. [19, 40, 33]:

$$\|A(t,v)\|_{V^*} \leq f(t)^{\frac{\alpha-1}{\alpha}} + C\|v\|_V^{\alpha-1}. \tag{1.2}$$

The advantage of $(H4)$ is, *e.g.*, to include many semilinear type equations with nonlinear perturbation terms. For example, if we consider the reaction-diffusion type equation, *i.e.* $A(u) = \Delta u + F(u)$, then for verifying the coercivity $(H3)$ we have $\alpha = 2$. Hence (1.2) implies that F has at most linear growth. However, we can allow F to have some polynomial growth by using $(H4)$ here. We refer to section 3 for more details.

Now we can state the main result, which gives a unified framework to analyze various classes of nonlinear evolution equations.

Theorem 1.1. *Suppose that $V \subseteq H$ is compact and $(H1)$-$(H4)$ hold, then for any $u_0 \in H,\ b \in L^{\frac{\alpha}{\alpha-1}}([0,T];V^*)$* (1.1) *has a solution*

$$u \in L^\alpha([0,T];V) \cap C([0,T];H),\ u' \in L^{\frac{\alpha}{\alpha-1}}([0,T];V^*)$$

such that

$$\langle u(t), v\rangle_H = \langle u_0, v\rangle_H + \int_0^t \langle A(s,u(s)) + b(s), v\rangle_V ds,\ t \in [0,T], v \in V.$$

Moreover, if there exist constants C and γ such that

$$\rho(v) + \eta(v) \le C(1 + \|v\|_V^\alpha)(1 + \|v\|_H^\gamma),\ v \in V, \tag{1.3}$$

then the solution of (1.1) *is unique.*

Remark 1.2. (1) The proof is based on Galerkin approximation. Moreover, by the Lions-Aubin theorem (cf. [38, Chapter III, Proposition 1.3]), the compact embedding of $V \subseteq H$ implies the following embedding

$$W_\alpha^1(0,T;V,H) := \{u \in L^\alpha([0,T];V) : u' \in L^{\frac{\alpha}{\alpha-1}}([0,T];V^*)\} \subseteq L^\alpha(0,T;H)$$

is also compact. Hence there exists a subsequence of the solutions of the Galerkin approximated equations (see (2.5) in Section 2) strongly converges to the solution of (1.1) in $L^\alpha(0,T;H)$.

(2) One can easily see from the proof that the solution of (1.1) is unique if all solutions of (1.1) satisfy

$$\int_0^T (\rho(u(s)) + \eta(u(s)))\, ds < \infty.$$

(3) The compact embedding $V \subseteq H$ is required in the main result. For (global) monotonicity one can easily drop this assumption. In fact, the classical monotonicity tricks only works in general for the operator satisfies $(H2)$ with $C = \rho = \eta = 0$. For $C > 0$ (but $\rho = \eta = 0$) one can apply a standard exponential transformation to (1.1) to reduce the case $C > 0$ to the case $C = 0$. However, this kind of techniques does not work for the locally monotone case. In order to verify the pseudo-monotonicity of $A(t,\cdot)$, we have to split it into the sum of $A(t,\cdot) - cI$ and cI. And I is pseudo-monotone if and only if the embedding $V \subset H$ is compact.

(4) We can also establish a similar result for stochastic evolution equations in Hilbert space with additive noise:

$$dX(t) = A(t, X(t))dt + BdN(t),\ t \ge 0,\ X(0) = x. \tag{1.4}$$

Here $A : [0,T] \times V \to V^*$ and $B \in L(U;H)$, where U is another Hilbert space and $N(t)$ is a U-valued adapted stochastic process definded on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ (cf. [14, 33]). By a standard transformation (substitution), (1.4) can be reduced to deterministic evolution equations with a random parameter which Thoerem 1.1 can be applied to. This result and some further applications will be investigated in a separated paper.

Next result is the continuous dependence of solution of (1.1) on u_0 and b.

Theorem 1.2. *Suppose that $V \subseteq H$ is compact and $(H1)$-$(H4)$ hold, u_i are the solution of (1.1) with $u_{i,0} \in H$ and $b_i \in L^{\frac{\alpha}{\alpha-1}}([0,T];V^*) \cap L^2([0,T];H)$, $i=1,2$ respectively and satisfy*

$$\int_0^T (\rho(u_1(s)) + \eta(u_2(s)))\, ds < \infty.$$

Then there exists a constant C such that

$$\begin{aligned} \|u_1(t)-u_2(t)\|_H^2 \le & \exp\left[\int_0^t (C+\rho(u_1(s))+\eta(u_2(s)))\,ds\right] \\ & \cdot \left(\|u_{1,0}-u_{2,0}\|_H^2 + \int_0^t \|b_1(s)-b_2(s)\|_H^2 ds\right),\ t\in[0,T]. \end{aligned} \tag{1.5}$$

The paper is organized as follows. The proofs of the main results are given in the next section. In Section 3 we apply the main results to several concrete semilinear and quasi-linear evolution equations on Banach space.

2 Proofs of Main Theorems

2.1 Proof of Theorem 1.1

In order to make the proof easier to follow, we first give the outline of the proof for the reader's convenience.

Step 1: Galerkin approximation; local monotonicity and coercivity implies the existence (and uniqueness) of solutions to the approximated equations;

Step 2: A priori estimates was obtained from coercivity;

Step 3: Verify the weak limits by using modified monotonicity tricks;

Step 4: Uniqueness follows from local monotonicity.

The main difficulty is in the third step. The classical monotonicity tricks does not work for locally monotone operators. The crucial part for overcoming this difficulty is the following result: every locally monotone operator is pseudo-monotone. Then by using some techniques related with pseudo-monotonicity one can establish the existence of solutions.

We first recall the definition of pseudo-monotone operator introduced first by Brézis in [5]. We use the standard notation "$\rightharpoonup$" for weak convergence in Banach space.

Definition 2.1. The operator $A: V \to V^*$ is called pseudo-monotone if $v_n \rightharpoonup v$ in V and

$$\liminf_{n\to\infty} \langle A(v_n), v_n - v\rangle_V \ge 0$$

implies for all $u \in V$

$$\langle A(v), v-u\rangle_V \ge \limsup_{n\to\infty} \langle A(v_n), v_n - u\rangle_V.$$

Remark 2.1. (1) We remark that the definition of pseudo-monotone operator here coincides with the definition in [40] (one should replace A here by $-A$ in [40] due to different form of the formulation for evolution equations).

(2) The class of pseudo-monotone operators is stable under summation (*i.e.* the sum of two pseudo-monotone operators is still pseudo-monotone) and strictly smaller than the class of operator of type (M) (cf. [40, 38]). And note that the class of operator of type (M) is not stable under summation. A counterexample can be found in [38].

Proposition 2.1. *If A is pseudo-monotone, then $v_n \rightharpoonup v$ in V implies that*

$$\liminf_{n\to\infty}\langle A(v_n), v_n - v\rangle_V \leq 0.$$

Proof. If the conclusion is not true, then there exists $v_n \rightharpoonup v$ in V such that

$$\liminf_{n\to\infty}\langle A(v_n), v_n - v\rangle_V > 0.$$

Then we can extract a subsequence such that $v_{n_k} \rightharpoonup v$ and

$$\lim_{k\to\infty}\langle A(v_{n_k}), v_{n_k} - v\rangle_V > 0. \tag{2.1}$$

Then by the pseudo-monotonicity of A we have for all $u \in V$

$$\langle A(v), v-u\rangle_V \geq \limsup_{k\to\infty}\langle A(v_{n_k}), v_{n_k} - u\rangle_V.$$

By taking $u = v$ we obtain

$$\limsup_{k\to\infty}\langle A(v_{n_k}), v_{n_k} - v\rangle_V \leq 0,$$

which is a contradiction to (2.1).

Hence the proof is completed. □

Remark 2.2. We also recall a slightly modified definition of pseudo-monotone operator by Browder (cf. [13]): The operator $A : V \to V^*$ is called pseudo-monotone if $v_n \rightharpoonup v$ in V and

$$\liminf_{n\to\infty}\langle A(v_n), v_n - v\rangle_V \geq 0$$

implies

$$A(v_n) \rightharpoonup A(v) \quad \text{and} \quad \lim_{n\to\infty}\langle A(v_n), v_n\rangle_V = \langle A(v), v\rangle_V.$$

From this definition one clearly see the role of pseudo-monotone operator for verifying the limit of weakly convergent sequence under nonlinear operator.

If A is bounded (*i.e.* A maps bounded set into bounded set), then it's easy to show that these two definitions are equivalent by Proposition 2.1. In particular, under the assumption of ($H4$), these two definitions are equivalent.

Lemma 2.2. *If the embedding $V \subseteq H$ is compact, then $(H1)$ and $(H2)$ implies that $A(t,\cdot)$ is pseudo-monotone for any $t \in [0,T]$.*

Proof. For simplicity, we denote $A(t,\cdot)$ by $A(\cdot)$ for any fixed $t \in [0,T]$.

Suppose $v_n \rightharpoonup v$ in V and

$$\liminf_{n\to\infty} \langle A(v_n), v_n - v\rangle_V \geq 0, \tag{2.2}$$

then for any $u \in V$ we need to show

$$\langle A(v), v-u\rangle_V \geq \limsup_{n\to\infty} \langle A(v_n), v_n - u\rangle_V. \tag{2.3}$$

Given u and the constant C in $(H2)$, we take

$$K = \|u\|_V + \|v\|_V + \sup_n \|v_n\|_V; \quad C_1 = \sup_{v:\|v\|_V \leq 2K} (C + \rho(v) + \eta(v)) < \infty.$$

Since the embedding $V \subseteq H$ is compact, we have $v_n \to v$ in V^* and

$$\langle C_1 v, v-u\rangle_V = \lim_{n\to\infty} \langle C_1 v_n, v_n - u\rangle_V.$$

Hence for proving (2.3) it's sufficient to show that

$$\langle A_0(v), v-u\rangle_V \geq \limsup_{n\to\infty} \langle A_0(v_n), v_n - u\rangle_V,$$

where $A_0 = A - C_1 I$ (I is the identity operator).

Then $(H2)$ implies that

$$\limsup_{n\to\infty} \langle A_0(v_n), v_n - v\rangle_V \leq \limsup_{n\to\infty} \langle A_0(v), v_n - v\rangle_V = 0.$$

By (2.2) we obtain

$$\lim_{n\to\infty} \langle A_0(v_n), v_n - v\rangle_V = 0. \tag{2.4}$$

Let $z = v + t(u-v)$ with $t \in (0, \frac{1}{2})$, then the local monotonicity $(H2)$ implies that

$$\langle A_0(v_n) - A_0(z), v_n - z\rangle_V \leq 0,$$

i.e.

$$t\langle A_0(z), v-u\rangle_V - (1-t)\langle A_0(v_n), v_n - v\rangle_V \geq t\langle A_0(v_n), v_n - u\rangle_V - \langle A_0(z), v_n - v\rangle_V.$$

By taking $\limsup$ on both sides and using (2.4) we have

$$\langle A_0(z), v-u\rangle_V \geq \limsup_{n\to\infty} \langle A_0(v_n), v_n - u\rangle_V.$$

Then letting $t \to 0$, by the hemicontinuity $(H1)$ we obtain

$$\langle A_0(v), v-u\rangle_V \geq \limsup_{n\to\infty} \langle A_0(v_n), v_n - u\rangle_V.$$

Therefore, A is pseudo-monotone. □

Remark 2.3. For some concrete operators, the local monotonicity $(H2)$ might be easier to check by explicit calculations than the definition of pseudo-monotonicity. Hence the above result can be also seen as a computable sufficient condition for the pseudo-monotonicity in applications.

The proof of Theorem 1.1 is split into a few lemmas. Let $X := L^\alpha([0,T];V)$, then $X^* = L^{\frac{\alpha}{\alpha-1}}([0,T];V^*)$. We denote by $W^1_\alpha(0,T;V,H)$ the Banach space

$$W^1_\alpha(0,T;V,H) = \{u \in X : u' \in X^*\},$$

where the norm is defined by

$$\|u\|_W := \|u\|_X + \|u'\|_{X^*} = \left(\int_0^T \|u(t)\|_V^\alpha dt\right)^{\frac{1}{\alpha}} + \left(\int_0^T \|u'(t)\|_{V^*}^{\frac{\alpha}{\alpha-1}} dt\right)^{\frac{\alpha-1}{\alpha}}.$$

It's well known that $W^1_\alpha(0,T;V,H)$ is a reflexive Banach space and it is continuously imbedded into $C([0,T];H)$ (cf. [40]). Moreover, we also have the following integration by parts formula

$$\langle u(t), v(t)\rangle_H - \langle u(0), v(0)\rangle_H = \int_0^t \langle u'(s), v(s)\rangle_V ds + \int_0^t \langle v'(s), u(s)\rangle_V ds,$$
$$t \in [0,T],\ u, v \in W^1_\alpha(0,T;V,H).$$

We first consider the Galerkin approximation to (1.1).

Let $\{e_1, e_2, \cdots\} \subset V$ be an orthonormal basis in H and let $H_n := span\{e_1, \cdots, e_n\}$ such that $span\{e_1, e_2, \cdots\}$ is dense in V. Let $P_n : V^* \to H_n$ be defined by

$$P_n y := \sum_{i=1}^n \langle y, e_i\rangle_V e_i,\ y \in V^*.$$

Obviously, $P_n|_H$ is just the orthogonal projection onto H_n in H and we have

$$\langle P_n A(t,u), v\rangle_V = \langle P_n A(t,u), v\rangle_H = \langle A(t,u), v\rangle_V,\ u \in V, v \in H_n.$$

For each finite $n \in \mathbb{N}$ we consider the following evolution equation on H_n:

$$u_n'(t) = P_n A(t, u_n(t)) + P_n b(t),\ 0 < t < T,\ u_n(0) = P_n u_0 \in H_n. \tag{2.5}$$

It is easy to show that $P_n A$ is locally monotone and coercive on H_n (finite dimensional space). According to the classical result of Krylov (cf. [20] or [33, Theorem 3.1.1]), there exists a unique solution u_n to (2.5) such that

$$u_n \in L^\alpha([0,T];H_n) \cap C([0,T];H_n),\ u_n' \in L^{\frac{\alpha}{\alpha-1}}([0,T];H_n).$$

Lemma 2.3. *Under the assumptions of Theorem 1.1, there exists a constant $K > 0$ such that*

$$\|u_n\|_X + \sup_{t\in[0,T]} \|u_n\|_H + \|A(\cdot, u_n)\|_{X^*} \le K,\ n \ge 1. \tag{2.6}$$

Proof. By the integration by parts formula and $(H3)$ we have

$$\begin{aligned}
&\|u_n(t)\|_H^2 - \|u_n(0)\|_H^2 \\
&= 2\int_0^t \langle u_n'(s), u_n(s)\rangle_V ds \\
&= 2\int_0^t \langle P_n A(s, u_n(s)) + P_n b(s), u_n(s)\rangle_V ds \\
&= 2\int_0^t \langle A(s, u_n(s)) + b(s), u_n(s)\rangle_V ds \qquad (2.7)\\
&\le \int_0^t \left(-\delta\|u_n(s)\|_V^\alpha + C\|u_n(s)\|_H^2 + f(s) + \|b(s)\|_{V^*}\|u_n(s)\|_V\right) ds \\
&\le \int_0^t \left(-\frac{\delta}{2}\|u_n(s)\|_V^\alpha + C\|u_n(s)\|_H^2 + f(s) + C_1\|b(s)\|_{V^*}^{\frac{\alpha}{\alpha-1}}\right) ds,
\end{aligned}$$

where C_1 is a constant induced from Young's inequality.

Hence we have for $t \in [0, T]$,

$$\|u_n(t)\|_H^2 + \frac{\delta}{2}\int_0^t \|u_n(s)\|_V^\alpha ds \le \|u(0)\|_H^2 + C\int_0^t \|u_n(s)\|_H^2 ds + \int_0^t \left(f(s) + C_1\|b(s)\|_{V^*}^{\frac{\alpha}{\alpha-1}}\right) ds.$$

Then by Gronwall's lemma we have

$$\|u_n(t)\|_H^2 \le e^{Ct}\left(\|u(0)\|_H^2 + \int_0^t e^{-Cs}\left(f(s) + C_1\|b(s)\|_{V^*}^{\frac{\alpha}{\alpha-1}}\right) ds\right), \ t \in [0, T].$$

$$\frac{\delta}{2}\int_0^t \|u_n(s)\|_V^\alpha ds \le e^{Ct}\left(\|u(0)\|_H^2 + \int_0^t e^{-Cs}\left(f(s) + C_1\|b(s)\|_{V^*}^{\frac{\alpha}{\alpha-1}}\right) ds\right), \ t \in [0, T].$$

Therefore, there exists a constant C_2 such that

$$\|u_n\|_X + \sup_{t\in[0,T]} \|u_n(t)\|_H \le C_2, \ n \ge 1.$$

Then by $(H4)$ there exists a constant C_3 such that

$$\|A(\cdot, u_n)\|_{X^*} \le C_3, \ n \ge 1.$$

Hence the proof is complete. □

Note that X, X^* and H are reflexive spaces, by the estimates in Lemma 2.3, there exists a subsequence, again denote by u_n, such that as $n \to \infty$

$$\begin{aligned}
u_n &\rightharpoonup u \ \text{ in } X \text{ (also in } W_\alpha^1(0, T; V, H));\\
A(\cdot, u_n) &\rightharpoonup w \ \text{ in } X^*;\\
u_n(T) &\rightharpoonup z \ \text{ in } H.
\end{aligned}$$

Recall that $u_n(0) = P_n u_0 \to u_0$ in H as $n \to \infty$.

Lemma 2.4. *Under the assumptions of Theorem 1.1, the limit elements u, w and z satisfy $u \in W^1_\alpha(0,T;V,H)$ and*

$$u'(t) = w(t) + b(t),\ 0 < t < T,\ u(0) = u_0,\ u(T) = z.$$

Proof. The proof is standard (cf. [40, Lemma 30.5]), we include it here for the completeness. Recall the following integration by parts formula

$$\langle u(T), v(T)\rangle_H - \langle u(0), v(0)\rangle_H = \int_0^T \langle u'(t), v(t)\rangle_V dt + \int_0^T \langle v'(t), u(t)\rangle_V dt,\ u, v \in W^1_\alpha(0,T;V,H).$$

Then, for $\psi \in C^\infty([0,T])$ and $v \in H_n$, by (2.5) we have

$$\begin{aligned} &\langle u_n(T), \psi(T)v\rangle_H - \langle u_n(0), \psi(0)v\rangle_H \\ &= \int_0^T \langle u_n'(t), \psi(t)v\rangle_V dt + \int_0^T \langle \psi'(t)v, u_n(t)\rangle_V dt \\ &= \int_0^T \langle A(t, u_n(t)) + b(t), \psi(t)v\rangle_V dt + \int_0^T \langle \psi'(t)v, u_n(t)\rangle_V dt. \end{aligned} \tag{2.8}$$

Letting $n \to \infty$ we obtain for all $v \in \bigcup_n H_n$,

$$\langle z, \psi(T)v\rangle_H - \langle u_0, \psi(0)v\rangle_H = \int_0^T \langle w(t) + b(t), \psi(t)v\rangle_V dt + \int_0^T \langle \psi'(t)v, u(t)\rangle_V dt. \tag{2.9}$$

Since $\bigcup_n H_n$ is dense in V, it's easy to show that (2.9) hold for all $v \in V, \psi \in C^\infty([0,T])$.

If $\psi(T) = \psi(0) = 0$, then we have

$$\int_0^T \langle w(t) + b(t), v\rangle_V \psi(t) dt = -\int_0^T \langle u(t), v\rangle_V \psi'(t) dt.$$

This implies that $u' = w + b, t \in (0,T)$. In particular, we have $u \in W^1_\alpha(0,T;V,H)$.

Then by the integration by parts formula we have

$$\begin{aligned} &\langle u(T), \psi(T)v\rangle_H - \langle u(0), \psi(0)v\rangle_H \\ &= \int_0^T \langle u'(t), \psi(t)v\rangle_V dt + \int_0^T \langle \psi'(t)v, u(t)\rangle_V dt \\ &= \int_0^T \langle w(t) + b(t), \psi(t)v\rangle_V dt + \int_0^T \langle \psi'(t)v, u(t)\rangle_V dt. \end{aligned}$$

Hence by (2.9) we obtain

$$\langle u(T), \psi(T)v\rangle_H - \langle u(0), \psi(0)v\rangle_H = \langle z, \psi(T)v\rangle_H - \langle u_0, \psi(0)v\rangle_H.$$

Then by choosing $\psi(T) = 1, \psi(0) = 0$ and $\psi(T) = 0, \psi(0) = 1$ respectively we obtain that

$$u(T) = z,\ u(0) = u_0.$$

Hence the proof is complete. □

Next lemma is very crucial for the proof of Theorem 1.1. The result basically says that A is also a pseudo-monotone operator from X to X^*, hence one can still use a modified monotonicity tricks to verify the limit of the Galerkin approximation as a solution to (1.1). The techniques used in the proof is inspired by the works of Hirano and Shioji [18, 37].

Lemma 2.5. *Under the assumptions of Theorem 1.1, suppose that*

$$\liminf_{n\to\infty} \int_0^T \langle A(t,u_n(t)), u_n(t)\rangle_V dt \geq \int_0^T \langle w(t), u(t)\rangle_V dt, \tag{2.10}$$

then for any $v \in X$ we have

$$\int_0^T \langle A(t,u(t)), u(t)-v(t)\rangle_V dt \geq \limsup_{n\to\infty} \int_0^T \langle A(t,u_n(t)), u_n(t)-v(t)\rangle_V dt. \tag{2.11}$$

Proof. Since $W^1_\alpha(0,T;V,H) \subset C([0,T];H)$ is a continuous embedding, we have that $u_n(t)$ converges to $u(t)$ weakly in H for all $t \in [0,T]$. Hence $u_n(t)$ also converges to $u(t)$ weakly in V for all $t \in [0,T]$.

Claim 1: For all $t \in [0,T]$ we have

$$\limsup_{n\to\infty} \langle A(t,u_n(t)), u_n(t)-u(t)\rangle_V \leq 0. \tag{2.12}$$

Suppose there exists a t_0 such that

$$\limsup_{n\to\infty} \langle A(t_0,u_n(t_0)), u_n(t_0)-u(t_0)\rangle_V > 0.$$

Then we can take a subsequence such that

$$\lim_{i\to\infty} \langle A(t_0,u_{n_i}(t_0)), u_{n_i}(t_0)-u(t_0)\rangle_V > 0.$$

Note that $u_{n_i}(t_0)$ converges to $u(t_0)$ weakly in V and $A(t_0,\cdot)$ is pseudo-monotone, we have

$$\langle A(t_0,u(t_0)), u(t_0)-v\rangle_V \geq \limsup_{i\to\infty} \langle A(t_0,u_{n_i}(t_0)), u_{n_i}(t_0)-v\rangle_V,\ \ v\in V.$$

In particular, we have

$$\limsup_{i\to\infty} \langle A(t_0,u_{n_i}(t_0)), u_{n_i}(t_0)-u(t_0)\rangle_V \leq 0,$$

which is a contradiction with the definition of this subsequence.

Hence (2.12) holds.

By $(H3)$ and $(H4)$ there exists a constant K such that

$$\begin{aligned}\langle A(t,u_n(t)), u_n(t)-v(t)\rangle_V \leq & -\frac{\delta}{2}\|u_n(t)\|_V^\alpha + K\left(f(t)+\|u_n(t)\|_H^2\right)\\ & + K\left(1+\|u_n(t)\|_H^{\alpha\beta}\right)\|v(t)\|_V^\alpha,\ \ v\in X.\end{aligned}$$

Then by Lemma 2.3, Fatou's lemma, (2.10) and (2.12) we have

$$\begin{aligned} 0 &\le \liminf_{n\to\infty} \int_0^T \langle A(t,u_n(t)), u_n(t)-u(t)\rangle_V dt \\ &\le \limsup_{n\to\infty} \int_0^T \langle A(t,u_n(t)), u_n(t)-u(t)\rangle_V dt \\ &\le \int_0^T \limsup_{n\to\infty} \langle A(t,u_n(t)), u_n(t)-u(t)\rangle_V dt \le 0. \end{aligned} \tag{2.13}$$

Hence

$$\lim_{n\to\infty} \int_0^T \langle A(t,u_n(t)), u_n(t)-u(t)\rangle_V dt = 0.$$

Claim 2: There exists a subsequence $\{u_{n_i}\}$ such that

$$\lim_{i\to\infty} \langle A(t,u_{n_i}(t)), u_{n_i}(t)-u(t)\rangle_V = 0 \ \text{ for } a.e.\ t \in [0,T]. \tag{2.14}$$

Define $g_n(t) = \langle A(t,u_n(t)), u_n(t)-u(t)\rangle_V,\ t \in [0,T]$, then

$$\lim_{n\to\infty} \int_0^T g_n(t)dt = 0, \quad \limsup_{n\to\infty} g_n(t) \le 0, \quad t \in [0,T].$$

Then by Lebesgue's dominated convergence theorem we have

$$\lim_{n\to\infty} \int_0^T g_n^+(t)dt = 0,$$

where $g_n^+(t) := max\{g_n(t), 0\}$.

Note that $|g_n(t)| = 2g_n^+(t) - g_n(t)$, hence we have

$$\lim_{n\to\infty} \int_0^T |g_n(t)|dt = 0.$$

Therefore, we can take a subsequence $\{g_{n_i}(t)\}$ such that

$$\lim_{i\to\infty} g_{n_i}(t) = 0 \ \text{ for } a.e.\ t \in [0,T],$$

i.e. (2.14) holds.

Therefore, for any $v \in X$, we can choose a subsequence $\{u_{n_i}\}$ such that

$$\lim_{i\to\infty} \int_0^T \langle A(t,u_{n_i}(t)), u_{n_i}(t)-v(t)\rangle_V dt = \limsup_{n\to\infty} \int_0^T \langle A(t,u_n(t)), u_n(t)-v(t)\rangle_V dt;$$

$$\lim_{i\to\infty} \langle A(t,u_{n_i}(t)), u_{n_i}(t)-u(t)\rangle_V = 0 \ \text{ for } a.e.\ t \in [0,T].$$

Since A is pseudo-monotone, we have

$$\langle A(t,u(t)), u(t)-v(t)\rangle_V \ge \limsup_{i\to\infty} \langle A(t,u_{n_i}(t)), u_{n_i}(t)-v(t)\rangle_V, \quad t \in [0,T].$$

By Fatou's lemma we obtain

$$
\begin{aligned}
\int_0^T \langle A(t,u(t)), u(t)-v(t)\rangle_V dt &\geq \int_0^T \limsup_{i\to\infty}\langle A(t,u_{n_i}(t)), u_{n_i}(t)-v(t)\rangle_V dt \\
&\geq \limsup_{i\to\infty} \int_0^T \langle A(t,u_{n_i}(t)), u_{n_i}(t)-v(t)\rangle_V dt \\
&= \limsup_{n\to\infty} \int_0^T \langle A(t,u_n(t)), u_n(t)-v(t)\rangle_V dt.
\end{aligned} \tag{2.15}
$$

Hence the proof is complete. □

Proof of Theorem 1.1 (i) Existence: The integration by parts formula implies that

$$
\|u_n(T)\|_H^2 - \|u_n(0)\|_H^2 = 2\int_0^T \langle A(t,u_n(t)) + b(t), u_n(t)\rangle_V dt;
$$

$$
\|u(T)\|_H^2 - \|u(0)\|_H^2 = 2\int_0^T \langle w(t) + b(t), u(t)\rangle_V dt.
$$

Since $u_n(T) \rightharpoonup z$ in H, by the lower semicontinuity of $\|\cdot\|_H$ we have

$$
\liminf_{n\to\infty} \|u_n(T)\|_H^2 \geq \|z\|_H^2 = \|u(T)\|_H^2.
$$

Hence we have

$$
\begin{aligned}
\liminf_{n\to\infty} \int_0^T \langle A(t,u_n(t)), u_n(t)\rangle_V dt &\geq \frac{1}{2}\left(\|u(T)\|_H^2 - \|u(0)\|_H^2\right) - \int_0^T \langle b(t), u(t)\rangle_V dt \\
&= \int_0^T \langle w(t), u(t)\rangle_V dt.
\end{aligned}
$$

By Lemma 2.5 we have for any $v \in X$,

$$
\begin{aligned}
\int_0^T \langle A(t,u(t)), u(t)-v(t)\rangle_V dt &\geq \limsup_{n\to\infty} \int_0^T \langle A(t,u_n(t)), u_n(t)-v(t)\rangle_V dt \\
&\geq \liminf_{n\to\infty} \int_0^T \langle A(t,u_n(t)), u_n(t)-v(t)\rangle_V dt \\
&\geq \int_0^T \langle w(t), u(t)\rangle_V dt - \int_0^T \langle w(t), v(t)\rangle_V dt \\
&= \int_0^T \langle w(t), u(t)-v(t)\rangle_V dt.
\end{aligned}
$$

Since $v \in X$ is arbitrary, we have $A(\cdot, u) = w$ as the element in X^*.

Hence u is a solution to (1.1).

(ii) Uniqueness: Suppose $u(\cdot, u_0), v(\cdot, v_0)$ are the solutions to (1.1) with starting points u_0, v_0 respectively, then by the integration by parts formula we have for $t \in [0, T]$,

$$\begin{aligned}\|u(t)-v(t)\|_H^2 &= \|u_0 - v_0\|_H^2 + 2\int_0^t \langle A(s,u(s)) - A(s,v(s)), u(s)-v(s)\rangle_V ds \\ &\le \|u_0 - v_0\|_H^2 + 2\int_0^t \left(C + \rho(u(s)) + \eta(v(s))\right) \|u(s)-v(s)\|_H^2 ds.\end{aligned}$$

By (1.3) we know that

$$\int_0^T \left(C + \rho(u(s)) + \eta(v(s))\right) ds < \infty.$$

Then by Gronwall's lemma we obtain

$$\|u(t)-v(t)\|_H^2 \le \|u_0 - v_0\|_H^2 \exp\left[2\int_0^t \left(C + \rho(u(s)) + \eta(v(s))\right) ds\right], \ t \in [0,T]. \tag{2.16}$$

In particular, if $u_0 = v_0$, this implies the uniqueness of the solution of (1.1). □

2.2 Proof of Theorem 1.2

By $(H2)$ we have

$$\begin{aligned}\|u_1(t)-u_2(t)\|_H^2 &= \|u_{1,0} - u_{2,0}\|_H^2 + 2\int_0^t \langle A(s,u_1(s)) - A(s,u_2(s)), u_1(s)-u_2(s)\rangle_V ds \\ &\quad + 2\int_0^t \langle b_1(s) - b_2(s), u_1(s)-u_2(s)\rangle_V ds \\ &\le \|u_{1,0} - u_{2,0}\|_H^2 + \int_0^t \|b_1(s)-b_2(s)\|_H^2 ds \\ &\quad + \int_0^t \left(C + \rho(u_1(s)) + \eta(u_2(s))\right) \|u_1(s)-u_2(s)\|_H^2 ds, \ t \in [0,T],\end{aligned}$$

where C is a constant.

Then by Gronwall's lemma we have

$$\begin{aligned}\|u_1(t)-u_2(t)\|_H^2 \le &\exp\left[\int_0^t \left(C + \rho(u_1(s)) + \eta(u_2(s))\right) ds\right] \\ &\cdot \left(\|u_{1,0} - u_{2,0}\|_H^2 + \int_0^t \|b_1(s)-b_2(s)\|_H^2 ds\right), \ t \in [0,T].\end{aligned}$$

□

3 Application to examples

It's obvious that the main results can be applied to nonlinear evolution equations with monotone operators (*e.g.* porous medium equation, p-Laplace equation) perturbated by some non-monotone terms (*e.g.* some locally Lipschitz perturbation). Moreover, we also formulate some examples where the coefficients are only locally monotone. For simplicity here we only formulate the examples where the coefficients are time independent, but one can easily adapt all these examples to the time dependent case.

Here we use the notation D_i to denote the spatial derivative $\frac{\partial}{\partial x_i}$, $\Lambda \subseteq \mathbb{R}^d$ is an open bounded domain with smooth boundary. For standard Sobolev space $W_0^{1,p}(\Lambda)$ $(p \geq 2)$ we always use the following (equivalent) Sobolev norm:

$$\|u\|_{1,p} := \left(\int_\Lambda |\nabla u(x)|^p dx \right)^{\frac{1}{p}}.$$

As preparation we first give a lemma for verifying $(H2)$.

Lemma 3.1. *Consider the Gelfand triple*

$$V := W_0^{1,2}(\Lambda) \subseteq H := L^2(\Lambda) \subseteq W^{-1,2}(\Lambda)$$

and the operator

$$A(u) = \Delta u + \sum_{i=1}^d f_i(u) D_i u,$$

where f_i $(i = 1, \cdots, d)$ are Lipschitz functions on $\mathbb{R}$.

(1) If $d < 3$, then there exists a constant K such that

$$2\langle A(u) - A(v), u - v\rangle_V \leq -\|u - v\|_V^2 + \left(K + K\|u\|_{L^4}^4 + K\|v\|_V^2\right) \|u - v\|_H^2, \ u, v \in V.$$

In particular, if f_i are bounded functions for $i = 1, \cdots, d$, then we have

$$2\langle A(u) - A(v), u - v\rangle_V \leq -\|u - v\|_V^2 + \left(K + K\|v\|_V^2\right) \|u - v\|_H^2, \ u, v \in V.$$

(2) For $d = 3$ we have

$$2\langle A(u) - A(v), u - v\rangle_V \leq -\|u - v\|_V^2 + \left(K + K\|u\|_{L^4}^8 + K\|v\|_V^4\right) \|u - v\|_H^2, \ u, v \in V.$$

In particular, if f_i are bounded functions for $i = 1, \cdots, d$, we have

$$2\langle A(u) - A(v), u - v\rangle_V \leq -\|u - v\|_V^2 + \left(K + K\|v\|_V^4\right) \|u - v\|_H^2, \ u, v \in V.$$

(3) If f_i are bounded measurable functions on Λ and independent of u for $i = 1, \cdots, d$, i.e.

$$A(u) = \Delta u + \sum_{i=1}^d f_i \cdot D_i u,$$

then for any $d \geq 1$ we have

$$2\langle A(u) - A(v), u - v\rangle_V \leq -\|u - v\|_V^2 + K\|u - v\|_H^2, \ u, v \in V.$$

Proof. (1) Since all f_i are Lipschitz and linear growth, we have

$$
\begin{aligned}
&\langle A(u)-A(v),u-v\rangle_V \\
&= -\|u-v\|_V^2 + \sum_{i=1}^d \int_\Lambda \left(f_i(u)D_iu - f_i(v)D_iv\right)(u-v)\,dx \\
&= -\|u-v\|_V^2 + \sum_{i=1}^d \int_\Lambda \left(f_i(u)(D_iu - D_iv) + D_iv(f_i(u)-f_i(v))\right)(u-v)\,dx \\
&\le -\|u-v\|_V^2 + \sum_{i=1}^d \Bigg[\left(\int_\Lambda (D_iu-D_iv)^2dx\right)^{1/2} \left(\int_\Lambda f_i^2(u)(u-v)^2dx\right)^{1/2} \\
&\quad + \left(\int_\Lambda (D_iv)^2dx\right)^{1/2} \left(\int_\Lambda \left(f_i(u)-f_i(v)\right)^2(u-v)^2dx\right)^{1/2} \Bigg] \\
&\le -\|u-v\|_V^2 + K\|u-v\|_V \left(\int_\Lambda \left(1+u^4\right)dx\right)^{1/4} \left(\int_\Lambda (u-v)^4dx\right)^{1/4} + K\|v\|_V \left(\int_\Lambda (u-v)^4dx\right)^{1/2} \\
&\le -\|u-v\|_V^2 + K\|u-v\|_V^{3/2}\|u-v\|_H^{1/2}\left(1+\|u\|_{L^4}\right) + 2K\|v\|_V\|u-v\|_V\|u-v\|_H \\
&\le -\frac{1}{2}\|u-v\|_V^2 + \left(K + K\|v\|_V^2 + K\|u\|_{L^4}^4\right)\|u-v\|_H^2,\ \ u,v\in V,
\end{aligned}
$$

where K is a constant that may change from line to line, and we also used the following well known estimate on $\mathbb{R}^2$ (see [29, Lemma 2.1])

$$\|u\|_{L^4}^4 \le 2\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2,\ \ u\in W_0^{1,2}(\Lambda). \tag{3.1}$$

(2) For $d=3$ we use the following estimate (cf. [29])

$$\|u\|_{L^4}^4 \le 4\|u\|_{L^2}\|\nabla u\|_{L^2}^3,\ \ u\in W_0^{1,2}(\Lambda), \tag{3.2}$$

then the second assertion can be derived similarly by using Young's inequality.

(3) This assertion obviously follows from the estimates in (i). □

Remark 3.1. (1) If all f_i are bounded, then the local monotonicity $(H2)$ also implies the coercivity $(H3)$.

(2) If we write the operator in the following vector form

$$A(u) = \Delta u + \nabla\cdot\vec{F}(u),$$

where $\vec{F}(x) = (F_1(x),\cdots,F_d(x)) : \mathbb{R}\to\mathbb{R}^d$ satisfies

$$|\vec{F}(x)-\vec{F}(y)| \le C(1+|x|+|y|)|x-y|,\ \ x,y\in\mathbb{R}.$$

Then by using the divergence theorem (or Stokes' theorem) one can show that for $d<3$,

$$2\langle A(u)-A(v),u-v\rangle_V \le -\|u-v\|_V^2 + \left(K + K\|u\|_{L^4}^4 + K\|v\|_{L^4}^4\right)\|u-v\|_H^2,\ \ u,v\in V,$$

for $d=3$ we have

$$2\langle A(u)-A(v),u-v\rangle_V\le -\|u-v\|_V^2+\left(K+K\|u\|_{L^4}^8+K\|v\|_{L^4}^8\right)\|u-v\|_H^2,\ u,v\in V.$$

And it's also easy to show the coercivity $(H3)$ holds since we have

$$\langle\nabla\cdot\vec{F}(u),u\rangle_V=-\int_\Lambda \vec{F}(u)\cdot\nabla u dx=0,\ u\in W_0^{1,2}(\Lambda).$$

The first example is a general semilinear equation on $\Lambda\subseteq\mathbb{R}$, which unifies the classical reaction-diffusion equation and Burgers equation.

Example 3.2. *Consider the following equation*

$$u'=\frac{\partial^2 u}{\partial x^2}+\frac{\partial F}{\partial x}(u)+g(u)+h,\ u(0)=u_0\in L^2(\Lambda). \tag{3.3}$$

Suppose the following conditions hold for some constant $C>0$:

(i) F is a function on $\mathbb{R}$ satisfies

$$|F(x)-F(y)|\le C(1+|x|+|y|)|x-y|,\ x,y\in\mathbb{R}.$$

(ii) g is a continuous function on $\mathbb{R}$ such that

$$\begin{aligned}g(x)x&\le C(x^2+1),\ x\in\mathbb{R};\\ |g(x)|&\le C(|x|^3+1),\ x\in\mathbb{R};\\ (g(x)-g(y))(x-y)&\le C(1+|x|^t+|y|^t)(x-y)^2,\ x,y\in\mathbb{R},\end{aligned}\tag{3.4}$$

where $t\ge 1$ is a constant.

(iii) $h\in W^{-1,2}(\Lambda)$.

Then (3.3) *has a solution $u\in W_2^1(0,T;W_0^{1,2}(\Lambda),L^2(\Lambda))$. Moreover, if $t\le 2$, then the solution of* (3.3) *is also unique.*

Proof. We define the Gelfand triple

$$V:=W_0^{1,2}(\Lambda)\subseteq H:=L^2(\Lambda)\subseteq W^{-1,2}(\Lambda)$$

and the operator

$$A(u)=\frac{\partial^2 u}{\partial x^2}+\frac{\partial F}{\partial x}(u)+g(u),\ u\in V.$$

It is easy to show that $(H1)$ holds by the continuity of F and g.

Similar to Lemma 3.1, one can easily show that

$$\begin{aligned}&\langle\frac{\partial F}{\partial x}(u)-\frac{\partial F}{\partial x}(v),u-v\rangle_V\\ =&-\int_\Lambda\Big(F(u)-F(v)\Big)\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}\right)dx\\ \le&\frac{1}{4}\|u-v\|_V^2+C\left(1+\|u\|_{L^4}^4+\|v\|_{L^4}^4\right)\|u-v\|_H^2,\ u,v\in V.\end{aligned}\tag{3.5}$$

By integration by parts formula we have

$$\langle \frac{\partial F}{\partial x}(u), u\rangle_V = 0,\ \ u \in V.$$

(3.4) and (3.1) implies that

$$\begin{aligned}
&\langle g(u) - g(v), u - v\rangle_V \\
\le& C\left(1 + \|u\|_{L^{2t}}^t + \|v\|_{L^{2t}}^t\right)\|u - v\|_{L^4}^2 \\
\le& \frac{1}{4}\|u - v\|_V^2 + C\left(1 + \|u\|_{L^{2t}}^{2t} + \|v\|_{L^{2t}}^{2t}\right)\|u - v\|_H^2,\ \ u, v \in V.
\end{aligned} \tag{3.6}$$

Therefore, we have

$$2\langle A(u) - A(v), u - v\rangle_V \le -\|u - v\|_V^2 + C\left(1 + \|u\|_{L^4}^4 + \|u\|_{L^{2t}}^{2t} + \|v\|_{L^4}^4 + \|v\|_{L^{2t}}^{2t}\right)\|u - v\|_H^2,\ u, v \in V,$$

$i.e.$ $(H2)$ holds.

Note that by (3.4) we have

$$\langle g(u), u\rangle_V \le C\left(1 + \|u\|_H^2\right),\ \ u \in V,$$

hence $(H3)$ holds with $\alpha = 2$.

By the Sobolev embedding theorem we have

$$\|g(u)\|_{V^*} \le C\left(1 + \|u\|_{L^3}^3\right) \le C\left(1 + \|u\|_V\|u\|_H^2\right),\ \ u \in V,$$

$$\|\frac{\partial F}{\partial x}(u)\|_{V^*} \le \|F(u)\|_H \le C\left(1 + \|u\|_{L^4}^2\right) \le C\left(1 + \|u\|_V\|u\|_H\right),\ \ u \in V.$$

Hence $(H4)$ also holds (with $\beta = 2$).

Therefore, the assertions follow from Theorem 1.1. □

Remark 3.2. If we take $F(x) = x^2$ and $g = h = 0$, then (3.3) is the classical Burgers equation in fluid mechanics. If we take $g(x) = x - x^3$ and $F = h = 0$, then (3.3) is the well known reaction-diffusion equation.

Example 3.3. *Consider the following equation*

$$u' = \Delta u + \sum_{i=1}^{d} f_i(u)D_i u + g(u) + h,\ \ u(0) = u_0 \in L^2(\Lambda). \tag{3.7}$$

Suppose the following conditions hold for some constant $C > 0$*:*

(i) f_i *are bounded Lipschitz functions on* $\mathbb{R}$ *for* $i = 1, \cdots, d$*;*

(ii) g *is a continuous function on* $\mathbb{R}$ *such that*

$$\begin{aligned}
g(x)x &\le C(x^2 + 1),\ \ x \in \mathbb{R}; \\
|g(x)| &\le C(|x|^r + 1),\ \ x \in \mathbb{R}; \\
(g(x) - g(y))(x - y) &\le C(1 + |x|^t + |y|^t)(x - y)^2,\ \ x, y \in \mathbb{R},
\end{aligned} \tag{3.8}$$

where $r, t \geq 1$ are some constants.

(iii) $h \in W^{-1,2}(\Lambda)$.

Then we have

(1) if $d = 2$, $r = \frac{7}{3}$ and $t = 2$, (3.7) has a unique solution $u \in W_2^1(0, T; W_0^{1,2}(\Lambda), L^2(\Lambda))$.

(2) if $d = 3$, $r = \frac{7}{3}$ and $t \leq 3$, (3.7) has a solution $u \in W_2^1(0, T; W_0^{1,2}(\Lambda), L^2(\Lambda))$. Moreover, if $t = \frac{4}{3}$, $f_i, i = 1, 2, 3$ are bounded measurable functions on Λ and independent of u, then the solution of (3.7) is also unique.

Proof. We define the Gelfand triple

$$V := W_0^{1,2}(\Lambda) \subseteq H := L^2(\Lambda) \subseteq W^{-1,2}(\Lambda)$$

and the operator

$$A(u) = \Delta u + \sum_{i=1}^{d} f_i(u) D_i u + g(u), \ u \in V.$$

By assumption (3.8) we have

$$\langle g(u) - g(v), u - v \rangle_V \leq C \left(1 + \|u\|_{L^{2t}}^t + \|v\|_{L^{2t}}^t\right) \|u - v\|_{L^4}^2.$$

Then from (3.1) or (3.2) and Lemma 3.1 we have for $d = 2$

$$2\langle A(u) - A(v), u - v \rangle_V \leq -\|u - v\|_V^2 + K \left(1 + \|v\|_V^2 + \|u\|_{L^{2t}}^{2t} + \|v\|_{L^{2t}}^{2t}\right) \|u - v\|_H^2, \ u, v \in V,$$

and for $d = 3$,

$$2\langle A(u) - A(v), u - v \rangle_V \leq -\|u - v\|_V^2 + K \left(1 + \|v\|_V^4 + \|u\|_{L^{2t}}^{4t} + \|v\|_{L^{2t}}^{4t}\right) \|u - v\|_H^2, \ u, v \in V,$$

i.e. $(H2)$ holds.

Note that

$$\langle g(u), u \rangle_V \leq C \left(1 + \|u\|_H^2\right), \ u \in V.$$

Then by Lemma 3.1 and Remark 3.1 we know that $(H3)$ holds with $\alpha = 2$.

For $d = 2, 3$ we have

$$\|g(u)\|_{V^*} \leq C \left(1 + \|u\|_{L^{6r/5}}^r\right), \ u \in V.$$

For $r = \frac{7}{3}$, by the interpolation theorem we have

$$\|u\|_{L^{6r/5}} \leq \|u\|_{L^2}^{4/7} \|u\|_{L^6}^{3/7}, \ u \in W_0^{1,2}(\Lambda) \subseteq L^6(\Lambda).$$

Then

$$\|g(u)\|_{V^*} \leq C \left(1 + \|u\|_{L^{6r/5}}^r\right) \leq C \left(1 + \|u\|_H^{4/3} \|u\|_V\right), \ u \in V.$$

Hence $(H4)$ holds.

The hemicontinuity $(H1)$ follows easily from the continuity of f and g.

Therefore, all assertions follow from Theorem 1.1.

In particular, if $d = 3$ and $f_i, i = 1, 2, 3$ are bounded measurable functions on Λ and independent of u, then we have

$$2\langle A(u) - A(v), u - v\rangle_V \le -\|u - v\|_V^2 + K\left(1 + \|u\|_{L^{2t}}^{4t} + \|v\|_{L^{2t}}^{4t}\right)\|u - v\|_H^2,\ u, v \in V.$$

Since $t = \frac{4}{3}$, by the interpolation inequality we have

$$\|u\|_{L^{2t}} \le \|u\|_{L^2}^{5/8}\|u\|_{L^6}^{3/8},\ u \in V.$$

Therefore

$$\|u\|_{L^{2t}}^{4t} \le C\|u\|_H^{10/3}\|u\|_V^2,\ u \in V.$$

Hence the solution of of (3.7) is unique. □

Remark 3.3. (1) As we mentioned in Remark 1.1, the classical result for monotone operators can not be applied to the above example. The typical example of monotone perturbation is to assume all f_i are independent of unknown solution u, g is monotone (*e.g.* Lipschitz) and has linear growth ($r = 1$). However, here we allow g is locally monotone (*e.g.* locally Lipschitz) and has certain polynomial growth ($r > 1$).

(2) The boundedness of f_i is only assumed in order to verify the coercivity ($H3$). This assumption can be removed if we formulate (3.7) in vector form as explained in Remark 3.1.

We may also consider the following quasi-linear evolution equations on $\mathbb{R}^d$ ($d \ge 3$).

Example 3.4. *Consider the Gelfand triple*

$$V := W_0^{1,p}(\Lambda) \subseteq H := L^2(\Lambda) \subseteq W^{-1,q}(\Lambda)$$

and the following equation on $\mathbb{R}^d$ *for* $p > 2$

$$u' = \sum_{i=1}^{d} D_i\left(|D_i u|^{p-2} D_i u\right) + g(u) + h,\ u(0) = u_0 \in L^2(\Lambda). \tag{3.9}$$

Suppose the following conditions hold:

(i) g is a continuous function on $\mathbb{R}$ such that

$$\begin{aligned} g(x)x &\le C(|x|^{\frac{p}{2}+1} + 1),\ x \in \mathbb{R}; \\ |g(x)| &\le C(|x|^r + 1),\ x \in \mathbb{R}; \\ (g(x) - g(y))(x - y) &\le C(1 + |x|^t + |y|^t)|x - y|^s,\ x, y \in \mathbb{R}, \end{aligned} \tag{3.10}$$

where $C > 0$ and $r, s, t \ge 1$ are some constants.

(ii) $h \in W^{-1,q}(\Lambda)$, $p^{-1} + q^{-1} = 1$.

Then we have

(1) if $d < p$, $s = 2$ and $r = p + 1$, (3.9) *has a solution. Moreover, if $t \le p$ also holds, then the solution is unique.*

(2) if $d > p$, $2 < s < p$, $r = \frac{2p}{d} + p - 1$ and $t \le \frac{p^2(s-2)}{(d-p)(p-2)}$, (3.9) *has a solution. The solution is unique if $t \le \frac{p(p-s)}{p-2}$ also holds.*

Proof. (1) It's well known that $\sum_{i=1}^d D_i\left(|D_iu|^{p-2}D_iu\right)$ satisfy $(H1)$-$(H4)$ (cf. [24, 26]). In particular, there exists a constant $\delta>0$ such that

$$\sum_{i=1}^d \langle D_i\left(|D_iu|^{p-2}D_iu\right)-D_i\left(|D_iv|^{p-2}D_iv\right), u-v\rangle_V \le -\delta\|u-v\|_V^p,\ u,v\in W_0^{1,p}(\Lambda). \tag{3.11}$$

Recall that for $d<p$ we have the following Sobolev embedding

$$W_0^{1,p}(\Lambda)\subseteq L^\infty(\Lambda).$$

Hence we have

$$\begin{aligned}\langle g(u)-g(v),u-v\rangle_V &\le C\int_\Lambda \left(1+|u|^t+|v|^t\right)|u-v|^2dx\\ &\le C\left(1+\|u\|_{L^\infty}^t+\|v\|_{L^\infty}^t\right)\|u-v\|_{L^2}^2\\ &\le C\left(1+\|u\|_V^t+\|v\|_V^t\right)\|u-v\|_H^2,\ u,v\in V,\end{aligned} \tag{3.12}$$

where C is a constant may change from line to line.

Hence $(H2)$ holds.

Note that from (3.10) we have

$$\begin{aligned}\langle g(u),u\rangle_V &\le C\int_\Lambda (1+|u|^{\frac{p}{2}+1})dx\\ &\le C\left(1+\|u\|_{L^\infty}^{p/2}\|u\|_H\right)\\ &\le \frac{\delta}{2}\|u\|_V^p+C\left(1+\|u\|_H^2\right),\ u\in V.\end{aligned} \tag{3.13}$$

Therefore, $(H3)$ holds with $\alpha=p$ by (3.11).

$(H4)$ follows from the following estimate:

$$\|g(u)\|_{V^*}\le C\left(1+\|u\|_{L^{p+1}}^{p+1}\right)\le C\left(1+\|u\|_{L^\infty}^{p-1}\|u\|_H^2\right),\ u\in V.$$

Hence the assertions follow from Theorem 1.1.

(2) Note that for $d>p$ we have the following Sobolev embedding

$$W_0^{1,p}(\Lambda)\subseteq L^{p_0}(\Lambda),\ p_0=\frac{dp}{d-p}.$$

Let $t_0=\frac{p(s-2)}{s(p-2)}\in(0,1)$ and $p_1\in(2,p_0)$ such that

$$\frac{1}{p_1}=\frac{1-t_0}{2}+\frac{t_0}{p_0}.$$

Then we have the following interpolation inequality

$$\|u\|_{L^{p_1}}\le\|u\|_{L^2}^{1-t_0}\|u\|_{L^{p_0}}^{t_0},\ u\in W_0^{1,p}(\Lambda).$$

Since $2 < s < p$, it is easy to show that $s < p_1$.

Let $p_2 = \frac{p_1}{p_1 - s}$, then by assumption (3.10) we have

$$\begin{aligned}\langle g(u) - g(v), u - v\rangle_V &\le C \int_\Lambda \left(1 + |u|^t + |v|^t\right) |u - v|^s dx \\ &\le C\left(1 + \|u\|_{L^{tp_2}}^t + \|v\|_{L^{tp_2}}^t\right) \|u - v\|_{L^{p_1}}^s \\ &\le C\left(1 + \|u\|_{L^{tp_2}}^t + \|v\|_{L^{tp_2}}^t\right) \|u - v\|_{L^2}^{s(1-t_0)} \|u - v\|_{L^{p_0}}^{st_0} \\ &\le \varepsilon \|u - v\|_{L^{p_0}}^p + C_\varepsilon \left(1 + \|u\|_{L^{tp_2}}^{tb} + \|v\|_{L^{tp_2}}^{tb}\right) \|u - v\|_{L^2}^2,\end{aligned} \tag{3.14}$$

where $\varepsilon, C_\varepsilon$ are some constants and the last step follows from the following Young inequality

$$xy \le \varepsilon x^a + C_\varepsilon y^b, \ x, y \in \mathbb{R}, \ a = \frac{p-2}{s-2}, \ b = \frac{p-2}{p-s}.$$

By calculation we have

$$\frac{s}{p_1} = \frac{p-s}{p-2} + \frac{p(s-2)}{p_0(p-2)}, \ p_2 = \frac{p_0(p-2)}{(p_0-p)(s-2)}.$$

Hence if $t \le \frac{(p_0-p)(s-2)}{p-2}$, then

$$\|u\|_{L^{tp_2}} \le C\|u\|_{L^{p_0}} \le C\|u\|_V, \ v \in V.$$

Therefore, $(H2)$ follows from (3.11) and (3.14).

$(H3)$ can be verified for $\alpha = p$ in a similar way.

For $r = \frac{2p}{d} + p - 1$, by the interpolation inequality we have

$$\|g(u)\|_{V^*} \le C\left(1 + \|u\|_{L^{rp_0'}}^r\right) \le C\left(1 + \|u\|_{p_0}^{p-1} \|u\|_H^\beta\right), \ u \in V,$$

where

$$\frac{1}{p_0} + \frac{1}{p_0'}, \quad \beta = \frac{2p}{d}.$$

Therefore, $(H4)$ also holds.

Then all assertions follow from Theorem 1.1. □

Remark 3.4. One further generalization is to replace $\sum_{i=1}^d D_i\left(|D_i u|^{p-2} D_i u\right)$ by more general quasi-linear differential operator

$$\sum_{|\alpha| \le m} (-1)^{|\alpha|} D_\alpha A_\alpha(x, Du(x,t); t),$$

where $Du = (D_\beta u)_{|\beta| \le m}$. Under certain assumptions (cf. [40, Proposition 30.10]) this operator satisfies the monotonicity and coercivity condition.

According to Theorem 1.1, we can obtain the existence and uniqueness of solutions to this type of quasi-linear PDE with some non-monotone perturbations (*e.g.* some locally Lipschitz lower order terms).

Now we apply Theorem 1.1 to the Navier-Stokes equation.

Let Λ be a bounded domain in $\mathbb{R}^2$ with smooth boundary. It's well known that by means of divergence free Hilbert spaces V, H and the Helmhotz-Leray orthogonal projection P_H, the classical form of the Navier-Stokes equation can be formulated in the following form:

$$u' = Au + B(u) + f, \; u(0) = u_0 \in H, \tag{3.15}$$

where

$$V = \left\{ v \in W_0^{1,2}(\Lambda, \mathbb{R}^2) : \nabla \cdot v = 0 \; a.e. \text{ in } \Lambda \right\}, \; \|v\|_V := \left(\int_\Lambda |\nabla v|^2 dx \right)^{1/2},$$

and H is the closure of V in the following norm

$$\|v\|_H := \left(\int_\Lambda |v|^2 dx \right)^{1/2}.$$

The linear operator P_H (Helmhotz-Leray projection) and A (Stokes operator with viscosity constant ν) are defined by

$$P_H : L^2(\Lambda, \mathbb{R}^2) \to H, \quad \text{orthogonal projection;}$$

$$A := W^{2,2}(\Lambda, \mathbb{R}^2) \cap V \to H, \; Au = \nu P_H \Delta u,$$

and the nonlinear operator

$$B : \mathcal{D}_B \subset H \times V \to H, \; B(u, v) = -P_H \left[(u \cdot \nabla) v \right], B(u) = B(u, u).$$

It's well known that by using the Gelfand triple

$$V \subseteq H \equiv H^* \subseteq V^*$$

the following mappings

$$A : V \to V^*, \; B : V \times V \to V^*$$

are well defined. In particular, we have

$$\langle B(u, v), w \rangle_V = -\langle B(u, w), v \rangle_V, \; \langle B(u, v), v \rangle_V = 0, \; u, v, w \in V.$$

Example 3.5. *(2D Navier-Stokes equation) For $f \in L^2(0, T; V^*)$ and $u_0 \in H$,* (3.15) *has a unique solution.*

Proof. The hemicontinuity $(H1)$ is easy to show since B is a bilinear map.

Note that $\langle B(v), v \rangle_V = 0$, it's also easy to get the coercivity $(H3)$

$$\langle Av + B(v) + f, v \rangle_V \le -\nu \|v\|_V^2 + \|f\|_{V^*} \|v\|_V \le -\frac{\nu}{2} \|v\|_V^2 + C \|f\|_{V^*}^2, \; v \in V.$$

Recall the following estimate (cf. [29, Lemma 2.1, 2.2])

$$
\begin{aligned}
&|\langle B(w), v\rangle_V| \le 2\|w\|_{L^4(\Lambda;\mathbb{R}^2)}\|v\|_V;\\
&|\langle B(w), v\rangle_V| \le 2\|w\|_V^{3/2}\|w\|_H^{1/2}\|v\|_{L^4(\Lambda;\mathbb{R}^2)}, v, w\in V.
\end{aligned}
\tag{3.16}
$$

Then we have

$$
\begin{aligned}
\langle B(u)-B(v), u-v\rangle_V &= -\langle B(u,u-v), v\rangle_V + \langle B(v,u-v),v\rangle_V\\
&= -\langle B(u-v), v\rangle_V\\
&\le 2\|u-v\|_V^{3/2}\|u-v\|_H^{1/2}\|v\|_{L^4(\Lambda;\mathbb{R}^2)}\\
&\le \frac{\nu}{2}\|u-v\|_V^2 + \frac{32}{\nu^3}\|v\|_{L^4(\Lambda;\mathbb{R}^2)}^4\|u-v\|_H^2,\ u,v\in V.
\end{aligned}
\tag{3.17}
$$

Hence we have the local monotonicity $(H2)$

$$\langle Au + B(u) - Av - B(v), u-v\rangle_V \le -\frac{\nu}{2}\|u-v\|_V^2 + \frac{32}{\nu^3}\|v\|_{L^4(\Lambda;\mathbb{R}^2)}^4\|u-v\|_H^2.$$

The growth $(H4)$ follows from (3.16) and (3.1).

Hence the existence of solution to (3.15) follows from Theorem 1.1.

By definition any solution u of (3.15) is a element in $L^2([0,T];V)$ and $C([0,T];H)$, then (3.1) implies that

$$\int_0^T \|u(t)\|_{L^4}^4 dt \le 2\sup_{t\in[0,T]}\|u(t)\|_H^2\int_0^T\|u(t)\|_V^2 dt < \infty.$$

Hence the solution of (3.15) is also unique. □

Remark 3.5. (1) The main result can be also applied to some other classes of two dimensional hydrodynamical models such as magneto-hydrodynamic equations, the Boussinesq model for the Bénard convection and 2D magnetic Bénard problem. We refer to [12] (and the references therein) for the details of these models. Note that the assumption $(C1)$ in [12] implies a special type of local monotonicity (e.g. see (2.8) in [12]).

(2) For the 3D Navier-Stokes equation, we recall the following estimate (cf. [29, (2.5)])

$$\|\psi\|_{L^4}^4 \le 4\|\psi\|_{L^2}\|\nabla\psi\|_{L^2}^3,\ \psi\in W_0^{1,2}(\Lambda;\mathbb{R}^3).$$

Then one can show that

$$
\begin{aligned}
\langle B(u)-B(v), u-v\rangle_V &= -\langle B(u-v), v\rangle_V\\
&\le 2\|u-v\|_V^{7/4}\|u-v\|_H^{1/4}\|v\|_{L^4(\Lambda;\mathbb{R}^3)}\\
&\le \frac{\nu}{2}\|u-v\|_V^2 + \frac{2^{12}}{\nu^7}\|v\|_{L^4(\Lambda;\mathbb{R}^3)}^8\|u-v\|_H^2,\ u,v\in V.
\end{aligned}
\tag{3.18}
$$

Hence we have the following local monotonicity $(H2)$

$$\langle Au + B(u) - Av - B(v), u-v\rangle_V \le -\frac{\nu}{2}\|u-v\|_V^2 + \frac{2^{12}}{\nu^7}\|v\|_{L^4(\Lambda;\mathbb{R}^3)}^8\|u-v\|_H^2.$$

However, we only have the following growth condition in the 3D case:

$$\|B(u)\|_{V^*} \leq 2\|u\|^2_{L^4(\Lambda;\mathbb{R}^3)} \leq 4\|u\|_H^{1/2}\|u\|_V^{3/2},\ u \in V.$$

Unfortunately, this is not enough to verify $(H4)$ in Theorem 1.1.

Now we apply the main result to the 3D Leray-α model of turbulence, which is a regularization of the 3D Navier-Stokes equation. It was first considered by Leray [21] in order to prove the existence of a solution to the Navier-Stokes equation in $\mathbb{R}^3$. Here we use a special smoothing kernel in the 3D Leray-α model, which was first considered in [11] (cf. [10] for more references). It has been shown in [11] that the 3D Leray-α model compares successfully with empirical data from turbulent channel and pipe flows for a wide range of Reynolds numbers. This model has a great potential to become a good sub-grid-scale large-eddy simulation model of turbulence. The Leray-α model can be formulated as follows:

$$\begin{aligned} &u' = \nu\Delta u - (v\cdot\nabla)u - \nabla p + f, \\ &\nabla\cdot u = 0,\ u = v - \alpha^2\Delta v \end{aligned} \tag{3.19}$$

where $\nu > 0$ is the viscosity, u is the velocity, p is the pressure and f is a given body-forcing term.

By using the same divergence free Hilbert spaces V, H (but in 3D) one can rewrite the Leray-α model into the following abstract form:

$$u' = Au + B(u,u) + f,\ u(0) = u_0 \in H, \tag{3.20}$$

where

$$Au = \nu P_H\Delta u, B(u,v) = -P_H\left[\left(\left(I-\alpha^2\Delta\right)^{-1}u\cdot\nabla\right)v\right].$$

Example 3.6. *(3D Leray-α model) For $f \in L^2(0,T;V^*)$ and $u_0 \in H$,* (3.20) *has a unique solution.*

Proof. $(H1)$ holds obviously since B is a bilinear map.

Note that $\langle B(u,v),v\rangle_V = 0$, it's also easy to get the coercivity $(H3)$:

$$\langle Av + B(v,v) + f, v\rangle_V \leq -\nu\|v\|_V^2 + \|f\|_{V^*}\|v\|_V \leq -\frac{\nu}{2}\|v\|_V^2 + C\|f\|_{V^*}^2,\ v\in V.$$

Recall the following well-known estimate (cf. [29, Lemma 2.1, 2.2])

$$\begin{aligned} &|\langle B(u,v),w\rangle_V| \\ \leq& c\|(I-\alpha^2\Delta)^{-1}u\|_H^{1/4}\|(I-\alpha^2\Delta)^{-1}u\|_V^{3/4}\|v\|_H^{1/4}\|v\|_V^{3/4}\|w\|_V \\ \leq& C\|u\|_H\|v\|_H^{1/4}\|v\|_V^{3/4}\|w\|_V,\ u,v,w\in V, \end{aligned} \tag{3.21}$$

where c, C are some constants.

Then we have

$$
\begin{aligned}
&\langle B(u,u)-B(v,v),u-v\rangle_V\\
=&-\langle B(u,u-v),v\rangle_V+\langle B(v,u-v),v\rangle_V\\
=&-\langle B(u-v,u-v),v\rangle_V\\
\le& C\|u-v\|_H^{5/4}\|u-v\|_V^{3/4}\|v\|_V\\
\le&\frac{\nu}{2}\|u-v\|_V^2+C_\nu\|v\|_V^{8/5}\|u-v\|_H^2,\ \ u,v\in V.
\end{aligned}
\tag{3.22}
$$

Hence we have the local monotonicity $(H2)$:

$$
\langle Au+B(u,u)-Av-B(v,v),u-v\rangle_V\le-\frac{\nu}{2}\|u-v\|_V^2+C_\nu\|v\|_V^{8/5}\|u-v\|_H^2.
$$

Note that (3.21) also implies that $(H4)$ holds.

Hence the existence and uniqueness of solutions to (3.20) follows from Theorem 1.1. □

Remark 3.6. The main result can also be applied to some other equations such as 3D Tamed Navier-Stokes equation, which is also a modified version of 3D Navier-Stokes equation. One may refer to [28, 36] for more details.

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