UNIQUENESS OF WEIGHTED SOBOLEV SPACES WITH WEAKLY DIFFERENTIABLE WEIGHTS

JONAS M. TÖLLE

Abstract. We prove that weakly differentiable weights w which, together with their reciprocals, satisfy certain local integrability conditions, admit a unique associated first order p-Sobolev space, that is

$$H_0^{1,p}(\mathbb{R}^d, w\,\mathrm{d}x) = H^{1,p}(\mathbb{R}^d, w\,\mathrm{d}x) = W^{1,p}(\mathbb{R}^d, w\,\mathrm{d}x).$$

If w admits a (weak) logarithmic derivative $\nabla w/w$ which is in $L^q_{\mathrm{loc}}(w\,\mathrm{d}x;\mathbb{R}^d)$, we propose an alternative definition of the weighted p-Sobolev space based on an integration by parts formula involving $\nabla w/w$.

We prove that weights of the form $\exp(-\beta|\cdot|^q - W - V)$ are p-admissible, in particular, satisfy a Poincaré inequality, where $\beta \in (0,\infty)$, W, V are convex and bounded below such that $|\nabla W|$ satisfies a growth condition (depending on β and q) and V is bounded. We apply the uniqueness result to weights of this type.

1. Introduction

Consider the following quasi-linear PDE in $\mathbb{R}^d$ (in the weak sense)

$$-\operatorname{div}\left[w|\nabla u|^{p-2}\nabla u\right] = fw, \tag{1.1}$$

(here $1 < p < \infty$) where $w \geq 0$ is a locally integrable function, the *weight* and f is sufficiently regular (e.g $f \in L^q(w\,\mathrm{d}x)$, see below). Let $\mu(\mathrm{d}x) := w\,\mathrm{d}x$, $q := p/(p-1)$. The nonlinear weighted p-Laplace operator involved in (1.1) can be identified with the Gâteaux derivative of the convex functional

$$E_0 : u \mapsto \frac{1}{p}\int |\nabla u|^p\, w\,\mathrm{d}x. \tag{1.2}$$

By methods well-known in calculus of variations, solutions to (1.1) are characterized by minimizers of the convex functional

$$E_f : u \mapsto E_0(u) - \int fuw\,\mathrm{d}x. \tag{1.3}$$

Of course, the minimizer obtained depends on the energy space chosen for the functional (1.2). It is natural to demand that C_0^∞ is included in this energy space. Therefore, let $H_0^{1,p}(\mu)$ be the completion of C_0^∞ w.r.t. the Sobolev norm

$$\|\cdot\|_{1,p,\mu} := \left(\|\nabla\cdot\|^p_{L^p(\mu;\mathbb{R}^d)} + \|\cdot\|^p_{L^p(\mu)}\right)^{1/p}.$$

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$H^{1,p}_0(\mu)$ is referred to as the so-called *strong weighted Sobolev space*. Of course, in order to guarantee that $H^{1,p}_0(\mu)$ will be a space of functions we need a "closability condition", see equation (2.1) below.

Let V be a weighted Sobolev space such that

- $V \subset L^p(\mu)$,
- V admits a linear gradient-operator $\nabla^V : V \to L^p(\mu;\mathbb{R}^d)$ that respects μ-classes,
- V is complete w.r.t. the Sobolev norm,
- $C_0^\infty \subset V$ and $\nabla u = \nabla^V u$ μ-a.e. for $u \in C_0^\infty$ and hence $H^{1,p}_0(\mu) \subset V$.

In the case that

$$H^{1,p}_0(\mu) \subsetneqq V,$$

the so-called *Lavrent'ev phenomenon*, first described in [25], occurs if

$$\min_{u\in V} E_f(u) < \min_{u\in H^{1,p}_0(\mu)} E_f(u).$$

This leads to different variational solutions to equation (1.1), as discussed in detail in [30]. In order to prevent this possibility, we are concerned with the problem

$$H^{1,p}_0(\mu) = V,$$

which is equivalent to the density of C_0^∞ in V and therefore is called "smooth approximation". Classically, if $w \equiv 1$, the solution to this problem is known as the Meyers-Serrin Theorem [27] and briefly denoted by $H = W$. If $p = 2$, the problem is also known as "Markov uniqueness", see [5, 6, 9, 32, 33].

$H = W$ for weighted Sobolev spaces ($p \neq 2$) has been studied e.g. in [8, 20, 37]. $H = W$ is in particular useful for identifying a Mosco limit [21, 35]

We are going to investigate two types of weighted Sobolev space substituting V.

Let $\varphi := w^{1/p}$. Consider following condition:

$$\textbf{(Diff)} \qquad \varphi \in W^{1,p}_{\mathrm{loc}}(\mathrm{d}x), \quad \beta := p\frac{\nabla\varphi}{\varphi} \in L^q_{\mathrm{loc}}(\mu;\mathbb{R}^d).$$

If we assume (**Diff**), we can define the Sobolev space $H^{1,p}(\mu)$ (which extends $H^{1,p}_0(\mu)$) by saying that $f \in H^{1,p}(\mu)$ if $f \in L^p(\mu)$ and there is a gradient $\nabla^\mu f := (\partial_1^\mu f, \dots, \partial_d^\mu f) \in L^p(\mu;\mathbb{R}^d)$ such that the integration by parts formula

$$\int \partial_i^\mu f\eta \,\mathrm{d}\mu = -\int f\partial_i\eta\,\mathrm{d}\mu - \int f\eta\beta_i\,\mathrm{d}\mu \tag{1.4}$$

holds for all $\eta \in C_0^\infty(\mathbb{R}^d)$ and all $i \in \{1,\dots,d\}$. For $p = 2$, this framework has been carried out by Albeverio et. al. in [2, 3, 4, 6].

Assuming (**Diff**), equation (1.1) has the following heuristic reformulation

$$-\operatorname{div}\left[|\nabla u|^{p-2}\nabla u\right] - \left\langle |\nabla u|^{p-2}\nabla u, \beta\right\rangle = f,$$

which suggests that (1.1) can be regarded as a first-order perturbation of the unweighted p-Laplace equation.

Let us state our main result.

Theorem 1.1. *Assume* (**Diff**). *Then $C_0^\infty(\mathbb{R}^d)$ is dense in $H^{1,p}(\mu)$, and, in particular,*

$$H^{1,p}_0(\mu) = H^{1,p}(\mu).$$

For $p = 2$, Theorem 1.1 was proved by Röckner and Zhang [32, 33] using methods from the theory of Dirichlet forms depending strongly on the L^2-framework. Our proof is carried out in Section 3 and inspired by the work of Patrick Cattiaux and Myriam Fradon [7]. In contrary to their proof, in which they use Fourier transforms (depending on the L^2-framework), we shall use maximal functions in order to obtain the fundamental uniform estimate. Of course, formula (1.4) provides highly useful for the proof.

Consider the following well-known condition:

$$\text{(}\mathbf{Loc}\text{)} \qquad \varphi^{-q} \in L^1_{\mathrm{loc}}(\mathbb{R}^d).$$

Let D be the gradient in the sense of Schwartz distributions. Assuming (**Loc**), we define

$$W^{1,p}(\mu) := \left\{ u \in L^p(\mu) \mid \mathrm{D}u \in L^p(\mu; \mathbb{R}^d) \right\},$$

see e.g. [23]. It is well-known that $H^{1,p}_0(\mu) = W^{1,p}(\mu)$ is implied by the famous *p-Muckenhoupt condition*, i.e. there is a global constant $K > 0$ such that

$$\left(\fint_B \varphi^p \,\mathrm{d}x \right) \cdot \left(\fint_B \varphi^{-q} \,\mathrm{d}x \right)^{p-1} \le K, \tag{1.5}$$

for all balls $B \subset \mathbb{R}^d$. We refer to the lecture notes by Bengt Ove Turesson [36] for a detailed discussion of this class. See also [18, Ch. 15].

As a consequence of Theorem 1.1, we obtain the following result:

Corollary 1.2. *Assume* (**Loc**), (**Diff**). *Then*

$$H^{1,p}_0(\mu) = H^{1,p}(\mu) = W^{1,p}(\mu).$$

We shall give a precise proof in Section 4.

p-admissible weights. We shall give an example. For the notion of p-admissibility, see [18] or Definition 5.1 below. We say that a function $F : \mathbb{R}^d \to \mathbb{R}$ *has property* (D), if there are constants $c_1 \ge 1$, $c_2 \in \mathbb{R}$ such that $F(2x) \le c_1 F(x) + c_2$. If F is concave, it has property (D) with $c_1 = 2$ and $c_2 = F(0)$. With the help of the ideas of Hebisch and Zegarliński [16] we are able to prove:

Theorem 1.3. *Let $1 < p < \infty$, $q := p/(p-1)$. Let $\beta \in (0, \infty)$, let $W \in C^1(\mathbb{R}^d)$ be bounded below and suppose that*

$$|\nabla W(x)| \le \delta |x|^{q-1} + \gamma$$

for some $\delta < \beta q$ and $\gamma \in (0, \infty)$. Suppose also that $-W$ has property (D). *Let $V : \mathbb{R}^d \to \mathbb{R}$ be a measurable function such that* $\operatorname{osc} V := \sup V - \inf V < \infty$ *and $-V$ has property* (D).

Then

$$x \mapsto \exp(-\beta |x|^q - W(x) - V(x))$$

is a p-admissible weight. If, additionally, $V \in W^{1,\infty}_{\mathrm{loc}}(\mathrm{d}x)$, this weight satisfies the conditions of Corollary 1.2.

Remark 1.4. *If V is convex, then V is locally Lipschitz by* [31, Theorem 10.4] *and hence $V \in W^{1,\infty}_{\mathrm{loc}}(\mathrm{d}x)$ by* [10, §4.2.3, Theorem 5].

Remark 1.5. *If* $\operatorname{osc} V < \infty$, *then the weight* $\exp(-V)$ *obviously satisfies Muckenhoupt's condition* (1.5) *for all $1 < p < \infty$.*

As an application of the main result 1.1, the weighted Poincaré inequality

$$\int \left| f - \frac{\int f \, w \,\mathrm{d}x}{\int w \,\mathrm{d}x} \right|^p w \,\mathrm{d}x \le c \int |\nabla f|^p \, w \,\mathrm{d}x,$$

for the weight $w := \exp(-\beta |\cdot|^q - W - V)$ also holds for $f \in H^{1,p}(w \,\mathrm{d}x)$ and for $f \in W^{1,p}(w \,\mathrm{d}x)$.

Notation. Equip $\mathbb{R}^d$ with the Euclidean norm $|\cdot|$ and the Euclidean scalar product $\langle\cdot,\cdot\rangle$. For $i\in\{1,\dots,d\}$, denote by e_i the i-th unit vector in $\mathbb{R}^d$. For $\mathbb{R}^d$-valued functions v we indicate the projection on the i-th coordinate by v_i. We denote the (weak or strong) partial derivative $\frac{\partial}{\partial_{e_i}}$ by ∂_i. Also $\nabla := (\partial_1,\dots,\partial_d)$. We denote the standard Sobolev spaces on $\mathbb{R}^d$ by $W^{1,p}(\mathrm{d}x)$, $W^{1,p}_0(\mathrm{d}x)$ and $W^{1,p}_{\mathrm{loc}}(\mathrm{d}x)$, with $1\le p\le\infty$.

For $x\in\mathbb{R}^d$, let

$$\operatorname{sign}(x) := \begin{cases} \dfrac{x}{|x|}, & \text{if } x\neq 0,\\ 0, & \text{if } x=0.\end{cases}$$

Denote by D the gradient in the sense of Schwartz distributions. For $x\in\mathbb{R}^d$ and $\rho>0$, set $B(x,\rho) := \left\{y\in\mathbb{R}^d \;\middle|\; |x-y|<\rho\right\}$ and $\overline{B}(x,\rho) := \left\{y\in\mathbb{R}^d \;\middle|\; |x-y|\le\rho\right\}$. With a *standard mollifier* we mean a family of functions $\{\eta_\varepsilon\}_{\varepsilon>0}$ such that

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^d}\eta\left(\frac{x}{\varepsilon}\right),$$

where $\eta\in C_0^\infty(\mathbb{R}^d)$ with $\eta\ge 0$, $\eta(x)=\eta(|x|)$, $\operatorname{supp}\eta\subset\overline{B}(0,1)$ and $\int\eta\,\mathrm{d}x = 1$.

2. Weighted Sobolev spaces

For all what follows, fix $1<p<\infty$ and $d\in\{1,2,\dots\}$. Set $q:=p/(p-1)$.

Definition 2.1. *For an a.e.-nonnegative measurable function f on $\mathbb{R}^d$, we define the* regular set

$$R(f) := \left\{ y\in\mathbb{R}^d \;\middle|\; \int_{B(y,\varepsilon)} \frac{1}{f}\,\mathrm{d}x < \infty \;\text{ for some } \varepsilon>0\right\},$$

where we adopt the convention that $1/0 := +\infty$ and $1/+\infty := 0$.

Obviously, $R(f)$ is the largest open set $O\subset\mathbb{R}^d$, such that $1/f\in L^1_{\mathrm{loc}}(O)$. Also, it always holds that $f>0$ $\mathrm{d}x$-a.e. on $R(f)$.

Fix a *weight* w, that is a measurable function $w\in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $w\ge 0$ a.e. Set $\mu(\mathrm{d}x) := w\,\mathrm{d}x$. Following the notation of [32], we set $\varphi := w^{1/p}$.

Definition 2.2. *Consider the following conditions:*

(**Ham1**) *For each $i\in\{1,\dots,d\}$ and for ($d-1$-dim.) Lebesgue a.a. $y\in\{e_i\}^\perp$ it holds that $\varphi^p(y+\cdot e_i)=0$ $\mathrm{d}t$-a.e. on $\mathbb{R}\setminus R(\varphi^q(y+\cdot e_i))$.*

(**Ham2**) *$\varphi^p = 0$ $\mathrm{d}x$-a.e. on $\mathbb{R}^d\setminus R(\varphi^q)$.*

Both (**Ham1**), (**Ham2**) are called *Hamza's condition* ("on rays" resp. "on $\mathbb{R}^d$"), due to [15].

It is straightforward that the following implications hold

$$(\mathbf{Loc}) \quad\Longrightarrow\quad (\mathbf{Ham2}) \quad\Longrightarrow\quad (\mathbf{Ham1}).$$

Also, if (**Loc**) holds, μ and $\mathrm{d}x$ are equivalent measures.

Remark 2.3. *Suppose that for $\mathrm{d}x$-a.a. $x\in\{\varphi^p>0\}$,*

$$\operatorname*{ess\,inf}_{y\in B(x,\delta)} \varphi^p(y) > 0$$

*for some $\delta=\delta(x)>0$. Then (**Ham2**) holds. In particular, (**Ham2**) holds whenever $\varphi^p\ge 0$ is lower semi-continuous.*

The following Lemma is analogous to [3, Lemma 2.1].

Lemma 2.4. *Assume that* (**Ham2**) *holds. Then*

$$L^p(\mathbb{R}^d,\mu)\subset L^1_{\mathrm{loc}}(R(\varphi^q),\mathrm{d}x)$$

continuously.

Proof. Let $u\in L^p(\mathbb{R}^d,\mu)$ and let $B\subset\subset R(\varphi^q)$ be a ball. By Hölder's inequality,

$$\int_B |u|\,\mathrm{d}x\le\left(\int_{R(\varphi^q)}|u|^p\,\varphi^p\,\mathrm{d}x\right)^{1/p}\cdot\left(\int_B\varphi^{-q}\,\mathrm{d}x\right)^{1/q}.$$

$\int_B\varphi^{-q}\,\mathrm{d}x$ is finite by (**Ham2**). □

Definition 2.5. *Let*

$$X:=\left\{u\in C^\infty(\mathbb{R}^d)\;\middle|\;\|u\|_{1,p,\mu}:=\left(\|\nabla u\|^p_{L^p(\mu;\mathbb{R}^d)}+\|u\|^p_{L^p(\mu)}\right)^{1/p}<\infty\right\}.$$

Let $H^{1,p}_0(\mu):=\widetilde{X}$ *be the abstract completion of* X *w.r.t. the pre-norm* $\|\cdot\|_{1,p,\mu}$.

Lemma 2.6. *Suppose that* (**Ham1**) *holds. Then for all sequences* $\{u_n\}\subset C^\infty$ *the following condition holds:*

$$\lim_n\|u_n\|_{L^p(\mu)}=0 \text{ and } \{u_n\} \text{ is } \|\nabla\cdot\|_{L^p(\mu;\mathbb{R}^d)}\text{-Cauchy} \quad\text{always imply}\quad \lim_n\|\nabla u_n\|_{L^p(\mu;\mathbb{R}^d)}=0. \tag{2.1}$$

Condition (2.1) *is referred to as* closability.

Proof. We shall consider partial derivatives first. Fix $i\in\{1,\dots,d\}$.

Let $\{u_n\}\in C^\infty$ such that $\|u_n\|_{L^p(\mu)}\to 0$ and such that $\{u_n\}$ is $\|\partial_i\cdot\|_{L^p(\mu)}$-Cauchy. By the Riesz-Fischer theorem, $\{\partial_i u_n\}$ converges to some $v\in L^p(\mu)$. Fix $y\in\{e_i\}^\perp$. By (**Ham1**) and Lemma 2.4 for $d=1$, setting $I_y:=R(\varphi^q(y+\cdot e_i))$, we conclude that $\{\partial_i u_n(y+\cdot e_i)\}$ converges to $v(y+\cdot e_i)$ in $L^1_{\mathrm{loc}}(I_y)$. Let $\eta\in C_0^\infty(I_y)$,

$$\begin{aligned}0&=\lim_n\int_{I_y}u_n(y+te_i)\frac{\mathrm{d}}{\mathrm{d}s}\eta(s)\Big|_{s=t}\,\mathrm{d}t=-\lim_n\int_{\operatorname{supp}\eta\cap I_y}(\partial_i u_n)(y+te_i)\eta(t)\,\mathrm{d}t\\&=-\int_{\operatorname{supp}\eta\cap I_y}v(y+te_i)\eta(t)\,\mathrm{d}t.\end{aligned}$$

We conclude that $v(y+te_i)=0$ for $\mathrm{d}y$-a.e. $y\in\{e_i\}^\perp$ and $\mathrm{d}t$-a.e $t\in I_y$. By (**Ham1**) it follows that $v=0$ μ-a.e. on $\mathbb{R}^d$.

Assume now that $\{u_n\}\in C^\infty$ such that $\|u_n\|_{L^p(\mu)}\to 0$ and such that $\{u_n\}$ is $\|\nabla\cdot\|_{L^p(\mu;\mathbb{R}^d)}$-Cauchy. Since

$$c\sum_{i=1}^d|\partial_i|^p\le|\nabla|^p\le C\sum_{i=1}^d|\partial_i|^p,$$

(where $c>0$ and $C>0$ are constants depending only on d and p), clearly each $\{\partial_i u_n\}$ is a Cauchy-sequence in $L^p(\mu)$. Therefore,

$$\int_{\mathbb{R}^d}|\nabla u_n|^p\,\mathrm{d}\mu\le C\sum_{i=1}^d\int_{\mathbb{R}^d}|\partial_i u_n|^p\,\mathrm{d}\mu\to 0,$$

as $n\to\infty$ by the arguments above. □

Proposition 2.7. *Assume* (**Ham1**). *Then* $H^{1,p}_0(\mu)$ *is a space of* μ*-classes of functions and is continuously embedded into* $L^p(\mu)$. *Also,* $H^{1,p}_0(\mu)$ *is separable and reflexive.*

Proof. By (2.1), the gradient in $H_0^{1,p}(\mu)$ is unique and each element in $H_0^{1,p}(\mu)$ is uniquely characterized by its limit in $L^p(\mu)$. By our choice of norms, $H_0^{1,p}(\mu) \subset L^p(\mu)$ continuously. $H_0^{1,p}(\mu)$ can be identified with a closed subspace of $L^p(\mu;\mathbb{R}^{d+1})$ and is therefore separable and reflexive. □

Denote the (class of the) gradient of an element $u \in H_0^{1,p}(\mu)$ by $\nabla^\mu u$.

Proposition 2.8. *Assume* (**Ham1**). *The μ-classes of $C_0^\infty(\mathbb{R}^d)$ functions are dense in $H_0^{1,p}(\mu)$.*

Proof. The proof is a standard localization argument using partition of unity, see e.g. [18, Theorem 1.27]. □

2.1. **Integration by parts.** We follow the approach of Albeverio, Kusuoka and Röckner [2], which is to define a weighted Sobolev space via an integration by parts formula. Recall that $w = \varphi^p$. A function $f \in L^p(\mu)$ might fail to be a Schwartz distribution. Instead, consider $f\varphi^p$, which is in L^1_{loc} by Hölder's inequality and therefore $\mathrm{D}(f\varphi^p)$ is well-defined. For $f \in C_0^\infty$, the Leibniz formula yields

$$(\nabla f)\varphi^p = \mathrm{D}(f\varphi^p) - pf\frac{\mathrm{D}\varphi}{\varphi}\varphi^p, \tag{2.2}$$

which motivates the definition of the *logarithmic derivative* of μ:

$$\beta := p\frac{\mathrm{D}\varphi}{\varphi},$$

where we set $\beta \equiv 0$ on $\{\varphi = 0\}$. The name arises from the (solely formal) identity $\beta = \nabla(\log(\varphi^p))$.

Lemma 2.9. *Condition* (**Diff**) *implies* $\varphi^p \in W^{1,1}_{\text{loc}}(\mathrm{d}x)$ *and*

$$\beta = p\frac{\nabla\varphi}{\varphi} = \frac{\nabla(\varphi^p)}{\varphi^p}, \tag{2.3}$$

where ∇ denotes the usual weak gradient.

Moreover, $\beta \in L^p_{\text{loc}}(\mu;\mathbb{R}^d)$ and $|\nabla\varphi|\varphi^{p-2} \in L^q_{\text{loc}}$.

Proof. Assume (**Diff**). $\varphi^p \in L^1_{\text{loc}}$ is clear. We claim that

$$\nabla(\varphi^p) = p\varphi^{p-1}\nabla\varphi. \tag{2.4}$$

Let $\varphi_\varepsilon := \eta_\varepsilon * \varphi$, where $\{\eta_\varepsilon\}$ is a standard mollifier. It follows from the classical chain rule that for all $\varepsilon > 0$

$$\nabla((\varphi_\varepsilon)^p) = p\varphi_\varepsilon^{p-1}\nabla\varphi_\varepsilon.$$

Since $\varphi^{p-1} \in L^q_{\text{loc}}$ and $\nabla\varphi \in L^p_{\text{loc}}$, we can pass to the limit in L^1_{loc} and get that $\varphi^p \in W^{1,1}_{\text{loc}}(\mathrm{d}x)$. (2.4) follows now from the uniqueness of the gradient in $W^{1,1}_{\text{loc}}(\mathrm{d}x)$. The first equality in (2.3) is clear. The second follows from (2.4). $\beta \in L^p_{\text{loc}}(\mu;\mathbb{R}^d)$ is clear. The last equality follows from (**Diff**) by

$$\left|\frac{\nabla\varphi}{\varphi}\right|^q \varphi^p = \left(|\nabla\varphi|\varphi^{p-2}\right)^q.$$

□

Lemma 2.10. *Assume* (**Diff**). *Then $\varphi^{p-1} \in W^{1,q}_{\text{loc}}(\mathrm{d}x)$. Also,*

$$\nabla(\varphi^{p-1}) = (p-1)\varphi^{p-2}\nabla\varphi.$$

Proof. Fix $1 \le i \le d$. For $N \in \mathbb{N}$, define $\psi_N : \mathbb{R} \to \mathbb{R}$ by $\psi_N(t) := (|t| \vee N^{-1} \wedge N)^{p-1}$. Clearly, ψ_N is a Lipschitz function. By the chain rule for Sobolev functions [38, Theorem 2.1.11],

$$\partial_i \psi_N(\varphi) = (p-1) 1_{\{N^{-1} \le \varphi \le N\}} \frac{\varphi^{p-1}}{\varphi} \partial_i \varphi.$$

We have that $\psi_N(\varphi) \to \varphi^{p-1}$ $\mathrm{d}x$-a.s. as $N \to \infty$. Also,

$$|\psi_N(\varphi)|^q \le |(\varphi \vee N^{-1})^p| \le C|\varphi|^p + C \in L^1_{\mathrm{loc}}.$$

Furthermore, by Lemma 2.9,

$$\left| 1_{\{N^{-1} \le \varphi \le N\}} \frac{\varphi^{p-1}}{\varphi} \partial_i \varphi \right| \le |\varphi^{p-2} \partial_i \varphi| \in L^q_{\mathrm{loc}}.$$

Hence by Lebesgue's dominated convergence theorem, $\psi_N(\varphi) \to \varphi^{p-1}$ in L^q_{loc} and $\partial_i \psi_N(\varphi) \to (p-1)\varphi^{p-2}\partial_i\varphi$ in L^q_{loc}. The claim is proved. □

Lemma 2.11. *Fix $1 \le i \le d$. Suppose that* (**Diff**) *holds. Then there is a version $\widetilde{\varphi^p}$ of φ^p, such that for $y \in \{e_i\}^\perp$ the map $t \mapsto \widetilde{\varphi^p}(y + te_i)$ is absolutely continuous for almost all $y \in \{e_i\}^\perp$. Furthermore, for almost all $y \in \{e_i\}^\perp$,*

$$\mathbb{R} \setminus R(\varphi^q(y + \cdot e_i)) \supset \{t \,|\, \widetilde{\varphi^p}(y + te_i) = 0\}.$$

Recall that the dt*-almost sure inclusion "$\subset$" holds automatically.*

Proof. Note that $\varphi^p \in W^{1,1}_{\mathrm{loc}}(\mathrm{d}x)$ by Lemma 2.9. Then the first part follows from a well-known theorem due to Nikodým, cf. [28, Theorem 2.7]. The second part follows from absolute continuity and Remark 2.3 for $d = 1$. □

We immediately get that:

Corollary 2.12. *It holds that*

$$(\mathbf{Diff}) \quad \Longrightarrow \quad (\mathbf{Ham1}).$$

Motivated by (2.2), we shall define the weighted Sobolev space $H^{1,p}(\mu)$.

Definition 2.13. *If* (**Diff**) *holds, we define the space $H^{1,p}(\mu)$ to be the set of all μ-classes of functions $f \in L^p(\mu)$ such that there exists a gradient*

$$\nabla^\mu f = (\partial_1^\mu f, \ldots, \partial_d^\mu f) \in L^p(\mu; \mathbb{R}^d)$$

which satisfies

$$\int \partial_i^\mu f \eta \varphi^p \,\mathrm{d}x = -\int f \partial_i \eta \varphi^p \,\mathrm{d}x - \int f \eta \beta_i \varphi^p \,\mathrm{d}x \tag{2.5}$$

for all $i \in \{1, \ldots, d\}$ and all $\eta \in C_0^\infty(\mathbb{R}^d)$.

Define also $H^{1,p}_{\mathrm{loc}}(\mu)$ by replacing $L^p(\mu)$ and $L^p(\mu; \mathbb{R}^d)$ above by $L^p_{\mathrm{loc}}(\mu)$ and $L^p_{\mathrm{loc}}(\mu; \mathbb{R}^d)$ resp.

The first two integrals in (2.5) are obviously well-defined. The third integral is finite by (**Diff**). It follows immediately that the gradient ∇^μ is unique. Also, if $f \in C^1(\mathbb{R}^d)$, then $f \in H^{1,p}_{\mathrm{loc}}(\mu)$ and $\nabla f = \nabla^\mu f$ μ-a.s.

Proposition 2.14. *Assume* (**Diff**). *Then $H^{1,p}(\mu)$ is a Banach space with the obvious choice of a norm*

$$\|\cdot\|_{1,p,\mu} := \left(\|\nabla^\mu \cdot\|^p_{L^p(\mu;\mathbb{R}^d)} + \|\cdot\|^p_{L^p(\mu)} \right)^{1/p}.$$

Moreover, $H^{1,p}_0(\mu) \subset H^{1,p}(\mu)$ and their gradients coincide μ-a.e.

Proof. Let $\{f_n\} \subset H^{1,p}(\mu)$ be a $\|\cdot\|_{1,p,\mu}$-Cauchy sequence. By the Riesz-Fischer theorem, $\{f_n\}$ converges to some $f \in L^p(\mu)$ and $\{\nabla^\mu f_n\}$ converges to some $g \in L^p(\mu;\mathbb{R}^d)$. Let $i \in \{1,\ldots,d\}$ and $\eta \in C_0^\infty(\mathbb{R}^d)$. Passing on to the limit in

$$\int \partial_i^\mu f_n \eta\varphi^p \,\mathrm{d}x = -\int f_n \partial_i \eta \varphi^p \,\mathrm{d}x - \int f_n \eta\beta_i\varphi^p \,\mathrm{d}x$$

yields that

$$\int g_i \eta\varphi^p \,\mathrm{d}x = -\int f \partial_i \eta \varphi^p \,\mathrm{d}x - \int f \eta\beta_i\varphi^p \,\mathrm{d}x.$$

Therefore $g = \nabla^\mu f$ and $\|f_n - f\|_{1,p,\mu} \to 0$.

Let us prove the second part. Note that by Corollary 2.12 and the discussion above, $H_0^{1,p}(\mu)$ is a well-defined set of elements in $L^p(\mu)$.

Let $f \in C_0^\infty(\mathbb{R}^d) \subset H_0^{1,p}(\mu)$. By (**Diff**) and the Leibniz formula for unweighted Sobolev spaces, (2.2) is satisfied. By classical integration by parts, f satisfies (2.5) with $\nabla^\mu f = \nabla f$. We extend to all of $H_0^{1,p}(\mu)$ by Proposition 2.8 using that $H^{1,p}(\mu)$ is complete. □

For our main result further below, we need to be able to truncate $H^{1,p}(\mu)$-functions. In order to prove the necessary chain-rule for Lipschitz functions, we need another representation of functions in $H^{1,p}(\mu)$, broadly known as *absolute continuity on lines parallel to the coordinate axes.*

Proposition 2.15. *Suppose that* (**Diff**) *holds. Fix* $1 \le i \le d$. *Then* $f \in H^{1,p}(\mu)$ *has a representative* $\widetilde{f}^i$ *such that* $t \mapsto \widetilde{f}^i(y+te_i)$ *is absolutely continuous for ($d-1$-dim.) Lebesgue almost all* $y \in \{e_i\}^\perp$ *on any compact subinterval of* $R(\varphi^q(y+\cdot e_i))$. *In that case, for* $\mathrm{d}y$*-a.a.* $y \in \{e_i\}^\perp$, $\mathrm{d}t$*-a.a.* $t \in R(\varphi^q(y+\cdot e_i))$, *setting* $x := y + te_i$, $\partial_i^\mu f(x) = \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{f}^i(y+te_i)$.

Proof. We argue similar to [6, Proof of Lemma 2.2].

Fix $1 \le i \le d$. By Lemma 2.11, fix a version of φ^p (denoted also by φ^p) such that the map $t \mapsto w(y+te_i)$ is absolutely continuous on $\mathbb{R}$ for $\mathrm{d}y$-a.a. $y \in \{e_i\}^\perp$. By (2.5), for any $\eta \in C_0^\infty(\mathbb{R}^d)$,

$$\int \partial_i^\mu f \eta\varphi^p \,\mathrm{d}x = -\int f \partial_i \eta \varphi^p \,\mathrm{d}x - \int f \eta\beta_i\varphi^p \,\mathrm{d}x.$$

By Fubini's theorem for $\mathrm{d}y$-a.a. $y \in \{e_i\}^\perp$ and for all $\eta \in C_0^\infty(\mathbb{R})$

$$\begin{aligned} &-\int \left[\partial_i^\mu f(y+te_i) + f(y+te_i)\beta_i(y+te_i)\right] \varphi^p(y+te_i)\eta(t)\,\mathrm{d}t \\ &= \int \frac{\mathrm{d}}{\mathrm{d}t}\eta(t) f(y+te_i)\varphi^p(y+te_i)\,\mathrm{d}t, \end{aligned} \tag{2.6}$$

and hence for $\mathrm{d}y$-a.a. $y \in \{e_i\}^\perp$ the map

$$t \mapsto f(y+te_i)\varphi^p(y+te_i)$$

has a distributional derivative which lies in $L^1_{\mathrm{loc}}(\mathbb{R})$. Hence by a well-known theorem of Nikodým [28, Theorem 2.7] it has an absolutely continuous $\mathrm{d}t$-version on any compact interval in $\mathbb{R}$. By Lemma 2.11, $R(\varphi^q(y+\cdot e_i)) \supset \{\varphi^p(y+\cdot e_i) > 0\}$ $\mathrm{d}y$-a.s. and hence $R(\varphi^q(y+\cdot e_i)) = \{\varphi^p(y+\cdot e_i) > 0\}$ $\mathrm{d}y$-a.s. We conclude that $t \mapsto f(y+te_i)$ has a version $\widetilde{f}^i$ which is absolutely continuous on any compact subinterval of $R(\varphi^q(y+\cdot e_i))$ for almost all $y \in \{e_i\}^\perp$. By the Leibniz formula for absolutely continuous functions and integration by parts, (2.6) proves that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{f}^i(y+te_i) = \partial_i^\mu f(y+te_i)$$

where the equality holds in the sense of μ-classes. □

Picking appropriate absolutely continuous versions, one immediately obtains the following Leibniz formula:

Corollary 2.16. *Suppose that* (**Diff**) *holds. If* $f, g \in H^{1,p}(\mu)$ *and if* fg, $f\partial_i^\mu g$ *and* $g\partial_i^\mu f$ *are in* $L^p(\mu)$ *for all* $1 \le i \le d$, *then* $fg \in H^{1,p}(\mu)$ *and* $\partial_i^\mu(fg) = f\partial_i^\mu g + g\partial_i^\mu f$ *for all* $1 \le i \le d$. *Then also,* $\nabla^\mu(fg) = f\nabla^\mu g + g\nabla^\mu f$.

The following lemma guarantees that we can truncate Sobolev functions. This property is also known as the "sub-Markov property", "Dirichlet property" or "lattice property" of the Sobolev space.

Lemma 2.17. *Suppose that* (**Diff**) *holds. Suppose that* $f \in H^{1,p}(\mu)$ *and that* $F : \mathbb{R} \to \mathbb{R}$ *is Lipschitz. Then* $F \circ f \in H^{1,p}(\mu)$ *with*

$$\nabla^\mu(F \circ f) = (F' \circ f) \cdot \nabla^\mu f \quad \mu\text{-a.s.}$$

In particular, when $F(t) := N \wedge t \vee -N$, $N \in \mathbb{N}$ *is a cut-off function,*

$$|\nabla^\mu(F \circ f)| \le |\nabla^\mu f| \quad \mu\text{-a.s.} \tag{2.7}$$

Proof. The claim can be proved arguing similar to [38, Theorem 2.1.11]. □

We remark that, indeed, we are able to prove the lattice property now. The procedure is standard and can be excellently seen in [18, Theorem 1.18 et sqq.]. As another consequence, bounded and compactly supported functions are dense, which is crucial for our main result below.

Lemma 2.18. *Suppose that* (**Diff**) *holds. The set of bounded and compactly supported functions in* $H^{1,p}(\mu)$ *is dense in* $H^{1,p}(\mu)$.

Proof. The claim follows by a truncation argument from Corollary 2.16 and Lemma 2.17. We shall omit the proof. □

Note that the last two statements also hold for $H_0^{1,p}(\mu)$. Anyhow, the proof of Lemma 2.17 for $H_0^{1,p}(\mu)$ needs some caution, because the Lipschitz function has to be approximated by smooth functions. The method is well-known, we refer to [26, Proposition I.4.7, Example II.2.c)].

3. Proof of Theorem 1.1

We arrive at our main result. Our proof is inspired by that of Patrick Cattiaux and Myriam Fradon in [7]. See also [11]. However, our method in estimating (3.8) is different from theirs, as we use maximal function-estimates instead of Fourier transforms.

For all of this section, assume (**Diff**). By Lemma 2.18, bounded and compactly supported functions in $H^{1,p}(\mu)$ are dense. We will show that a subsequence of a standard mollifier of such a function f converges in $\|\cdot\|_{1,p,\mu}$-norm to f. The claim will then follow from Lemma 2.8.

For the approximation, we shall prove the following key-lemma. Compare with [7, Lemma 2.9].

Lemma 3.1. *Suppose that* (**Diff**) *holds. Let* $f \in H^{1,p}(\mu)$ *such that* f *is bounded. Then for every* $\zeta \in C_0^\infty(\mathbb{R}^d)$ *and every* $1 \le i \le d$

$$\int \partial_i^\mu f \zeta\varphi \,\mathrm{d}x + \int f\partial_i\zeta\varphi \,\mathrm{d}x + \int f\zeta\partial_i\varphi \,\mathrm{d}x = 0. \tag{3.1}$$

In particular, $f\varphi \in W^{1,1}_{\mathrm{loc}}(\mathrm{d}x)$ *and* $\partial_i(f\varphi) = \varphi\partial_i^\mu f + f\partial_i\varphi$.

Proof. For all of the proof fix $1 \le i \le d$. Let us first assure ourselves that all three integrals in (3.1) are well-defined. Clearly,

$$|\partial_i^\mu f\zeta\varphi|^p \le \|\zeta\|_\infty^p |\partial_i^\mu f|^p \varphi^p 1_{\operatorname{supp}\zeta} \in L^1(\mathrm{d}x),$$

and hence,

$$|\partial_i^\mu f\zeta\varphi| \in L^1(\mathrm{d}x).$$

A similar argument works for the second integral. The third integral is well-defined because by $\varphi \in W^{1,p}_{\mathrm{loc}}(\mathrm{d}x)$ we have that

$$|f\zeta\partial_i\varphi|^p \le \|f\zeta\|_\infty^p |\partial_i\varphi|^p 1_{\operatorname{supp}\zeta} \in L^1(\mathrm{d}x)$$

and hence,

$$|f\zeta\partial_i\varphi| \in L^1(\mathrm{d}x).$$

Let $M \in \mathbb{N}$ and $\vartheta_M \in C_0^\infty(\mathbb{R})$ with

$$\vartheta_M(t) = t \text{ for } t \in [-M, M],\ |\vartheta_M| \le M+1,\ |\vartheta_M'| \le 1$$

and

$$\operatorname{supp}(\vartheta_M) \subset [-3M, 3M].$$

Define

$$\varphi_M := \vartheta_M\left(\frac{1}{\varphi^{p-1}}\right) 1_{\{\varphi>0\}}.$$

Clearly, $\varphi_M \in L^p_{\mathrm{loc}}$. Furthermore, define

$$\Phi_M := (1-p)\vartheta_M'\left(\frac{1}{\varphi^{p-1}}\right)\frac{\partial_i\varphi}{\varphi^p} 1_{\{\varphi>0\}}.$$

Since $\vartheta_M'(1/\varphi^{p-1}) \equiv 0$ on $\{\varphi^{p-1} \le 1/(3M)\}$ and

$$|\Phi_M| \le (p-1)\frac{|\partial_i\varphi|}{\varphi^p} 1_{\{\varphi^{p-1}>1/(3M)\}} = (p-1)\frac{|\partial_i\varphi|}{\varphi^p} 1_{\{\varphi^p>(1/(3M))^q\}},$$

hence $\Phi_M \in L^p_{\mathrm{loc}}$. We claim that $\varphi_M \in W^{1,p}_{\mathrm{loc}}(\mathrm{d}x)$ and that $\partial_i\varphi_M = \Phi_M$. Let $\varepsilon > 0$ and define

$$\varphi_M^\varepsilon := \vartheta_M\left(\frac{1}{(\varphi+\varepsilon)^{p-1}}\right).$$

Clearly, $\varphi_M^\varepsilon \to \varphi_M$ in L^p_{loc} as $\varepsilon \searrow 0$. Also, by the chain rule for Sobolev functions (see e.g. [38, Theorem 2.1.11]),

$$\partial_i\varphi_M^\varepsilon = (1-p)\vartheta_M'\left(\frac{1}{(\varphi+\varepsilon)^{p-1}}\right)\frac{\partial_i\varphi}{(\varphi+\varepsilon)^p} 1_{\{\varphi+\varepsilon>(3M)^{-1/(p-1)}\}}$$

and

$$|\partial_i\varphi_M^\varepsilon| \le (p-1)\frac{|\partial_i\varphi|}{(\varphi+\varepsilon)^p} 1_{\{(\varphi+\varepsilon)^p>(1/(3M))^q\}} \in L^p_{\mathrm{loc}}.$$

Hence $\varphi_M^\varepsilon \in W^{1,p}_{\mathrm{loc}}(\mathrm{d}x)$ and $\partial_i\varphi_M^\varepsilon \to \Phi_M$ in L^p_{loc} as $\varepsilon \searrow 0$.

Since $\varphi \in W^{1,p}_{\mathrm{loc}}(\mathrm{d}x)$ and since φ_M is bounded, we have that $\varphi_M\partial_i\varphi \in L^p_{\mathrm{loc}}$. Also, $\varphi\partial_i\varphi_M \in L^p_{\mathrm{loc}}$, since

$$|\varphi\partial_i\varphi_M| \le (p-1)\frac{|\partial_i\varphi|}{\varphi^{p-1}} 1_{\{\varphi^{p-1}>1/(3M)\}} \le (p-1)3M|\partial_i\varphi|. \tag{3.2}$$

Now by the usual Leibniz rule for weak derivatives

$$\varphi\varphi_M \in W^{1,p}_{\mathrm{loc}}(\mathrm{d}x) \quad \text{and} \quad \partial_i(\varphi\varphi_M) = \varphi_M\partial_i\varphi + (1-p)\vartheta_M'\left(\frac{1}{\varphi^{p-1}}\right)\frac{\partial_i\varphi}{\varphi^{p-1}}$$

where by definition $\partial_i\varphi/\varphi^{p-1} \equiv 0$ on $\{\varphi = 0\}$. Consider the term $\varphi_M\varphi^p$. Recall that $\varphi^p \in W^{1,1}_{\mathrm{loc}}(\mathrm{d}x)$ by Lemma 2.9. As already seen, $\varphi\varphi_M \in W^{1,p}_{\mathrm{loc}}(\mathrm{d}x)$. By Lemma 2.10, $\varphi^{p-1} \in W^{1,q}_{\mathrm{loc}}(\mathrm{d}x)$ and $\partial_i(\varphi^{p-1}) = (p-1)\varphi^{p-2}\partial_i\varphi \in L^q_{\mathrm{loc}}$. Hence

$\varphi\varphi_M(\partial_i(\varphi^{p-1}))\in L^1_{\mathrm{loc}}$ and $\partial_i(\varphi\varphi_M)\varphi^{p-1}\in L^1_{\mathrm{loc}}$. It follows that $\varphi_M\varphi^p\in W^{1,1}_{\mathrm{loc}}(\mathrm{d}x)$ and by the Leibniz rule for weak derivatives

$$\partial_i(\varphi_M\varphi^p)=p\varphi_M\varphi^{p-1}\partial_i\varphi+(1-p)\vartheta_M'\left(\frac{1}{\varphi^{p-1}}\right)\partial_i\varphi\in L^1_{\mathrm{loc}}.$$

Let $\zeta\in C_0^\infty(\mathbb{R}^d)$. Applying integration by parts, we see that

$$\int\partial_i\zeta\varphi_M\varphi^p\,\mathrm{d}x=-p\int\zeta\varphi_M\frac{\partial_i\varphi}{\varphi}\varphi^p\,\mathrm{d}x+(p-1)\int\zeta\frac{\partial_i\varphi}{\varphi^p}\vartheta_M'\left(\frac{1}{\varphi^{p-1}}\right)\varphi^p\,\mathrm{d}x. \tag{3.3}$$

Moreover, by (3.2), $\partial_i\varphi_M\in L^p_{\mathrm{loc}}(\varphi^p\,\mathrm{d}x)$. $\varphi_M\in L^p_{\mathrm{loc}}(\varphi^p\,\mathrm{d}x)$ is clear. Therefore $\varphi_M\in H^{1,p}_{\mathrm{loc}}(\mu)$ and

$$\partial_i^\mu\varphi_M=(1-p)\frac{\partial_i\varphi}{\varphi^p}\vartheta_M'\left(\frac{1}{\varphi^{p-1}}\right).$$

The Leibniz rule in Corollary 2.16 also holds in $H^{1,p}_{\mathrm{loc}}(\mu)$, and so we would like to give sense to the expression $\partial_i^\mu(f\varphi_M)=\varphi_M\partial_i^\mu f+f\partial_i^\mu\varphi_M$. But $\varphi_M\in H^{1,p}_{\mathrm{loc}}(\mu)$, $f\in H^{1,p}(\mu)$ and f is bounded, $f\partial_i^\mu\varphi_M\in L^p_{\mathrm{loc}}(\mu)$ since f is bounded and finally $\varphi_M\partial_i^\mu f\in L^p_{\mathrm{loc}}(\mu)$ since φ_M is bounded. Hence $f\varphi_M\in H^{1,p}_{\mathrm{loc}}(\mu)$ and the Leibniz rule holds (locally). By definition of ∂_i^μ for $\zeta\in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned}\int\partial_i^\mu f\zeta\varphi_M\varphi^p\,\mathrm{d}x=&(p-1)\int f\zeta\frac{\partial_i\varphi}{\varphi^p}\vartheta_M'\left(\frac{1}{\varphi^{p-1}}\right)\varphi^p\,\mathrm{d}x\\&-\int f\partial_i\zeta\varphi_M\varphi^p\,\mathrm{d}x-p\int f\zeta\varphi_M\frac{\partial_i\varphi}{\varphi}\varphi^p\,\mathrm{d}x\end{aligned} \tag{3.4}$$

Now let $M\to\infty$ in (3.4). Note that

$$\varphi_M\to(1/\varphi^{p-1})1_{\{\varphi>0\}}$$

$\mathrm{d}x$-a.s. and

$$\vartheta_M'(1/\varphi^{p-1})\to1$$

$\mathrm{d}x$-a.s. In order to apply Lebesgue's dominated convergence theorem, we verify

$$|\partial_i^\mu f\zeta\varphi_M\varphi^p|\le2|\partial_i^\mu f\varphi|\,\|\zeta\|_\infty 1_{\operatorname{supp}\zeta}\in L^1(\mathrm{d}x),$$

where we have used that

$$|\varphi_M\varphi^{p-1}|\le1,$$

because ϑ_M is Lipschitz and $\vartheta_M(0)=0$, Furthermore,

$$|f\zeta\partial_i\varphi\vartheta_M'\left(1/\varphi^{p-1}\right)|\le|f\partial_i\varphi|\,\|\zeta\|_\infty 1_{\operatorname{supp}\zeta}\in L^1(\mathrm{d}x),$$

$$|f\partial_i\zeta\varphi_M\varphi^p|\le2|f\varphi|\,\|\partial_i\zeta\|_\infty 1_{\operatorname{supp}\zeta}\in L^1(\mathrm{d}x),$$

and

$$|f\zeta\varphi_M\partial_i\varphi\varphi^{p-1}|\le2|f\partial_i\varphi|\,\|\zeta\|_\infty 1_{\operatorname{supp}\zeta}\in L^1(\mathrm{d}x).$$

The formula obtained, when passing on to the limit $M\to\infty$ in (3.4), is exactly the desired statement. □

Let $f\in H^{1,p}(\mu)$ be (a class of) a function which is bounded and compactly supported. By Lemma 2.18, we are done if we can approximate f by C_0^∞-functions. Let $\{\eta_\varepsilon\}_{\varepsilon>0}$ be a standard mollifier. Since f is bounded and compactly supported, $\eta_\varepsilon*f\in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(\eta_\varepsilon*f)\subset\operatorname{supp}f+\varepsilon B(0,1)$ and $|\eta_\varepsilon*f|\le\|f\|_\infty$. We claim that there exists a sequence $\varepsilon_n\searrow0$ such that $\eta_{\varepsilon_n}*f$ converges to f in $H^{1,p}(\mu)$. The $L^p(\mu)$-part is easy. Since $\eta_\varepsilon*f,f\in L^1(\mathrm{d}x)$, $\lim_{\varepsilon\searrow0}\|\eta_\varepsilon*f-f\|_{L^1(\mathrm{d}x)}=0$. Therefore we can extract a subsequence $\{\varepsilon_n\}$ such that $\eta_{\varepsilon_n}*f\to f$ $\mathrm{d}x$-a.s. For $\varepsilon_n\le1$

$$|(\eta_{\varepsilon_n}*f)\varphi-f\varphi|^p\le2^p\|f\|_\infty^p|\varphi|^p1_{\operatorname{supp}f+B(0,1)}\in L^1(\mathrm{d}x).$$

By Lebesgue's dominated convergence theorem, $\lim_n\|\eta_{\varepsilon_n}*f-f\|_{L^p(\mu)}=0$.

Fix $1 \le i \le d$. We are left to prove $\partial_i(\eta_{\varepsilon_n} * f) \to \partial_i^\mu f$ in $L^p(\mu)$ for some sequence $\varepsilon_n \searrow 0$. Or equivalently,

$$\varphi\partial_i(\eta_{\varepsilon_n} * f) \to \varphi\partial_i^\mu f \quad \text{in } L^p(\mathrm{d}x).$$

Write

(3.5)
$$\begin{aligned}&\int |\varphi\partial_i(\eta_\varepsilon * f) - \varphi\partial_i^\mu f|^p \,\mathrm{d}x\\ \le& 2^{p-1}\left[\int |\varphi\partial_i^\mu f - (\eta_\varepsilon * (\varphi\partial_i^\mu f))|^p \,\mathrm{d}x + \int |(\eta_\varepsilon * (\varphi\partial_i^\mu f)) - \varphi\partial_i(\eta_\varepsilon * f)|^p \,\mathrm{d}x\right].\end{aligned}$$

The first term tends to zero as $\varepsilon \searrow 0$ by a well-known fact [34, Theorem III.2 (c), p. 62]. We continue with studying the second term. Recall that $\eta_\varepsilon(x) = \eta_\varepsilon(|x|)$.

$$\begin{aligned}&\int |\varphi\partial_i(\eta_\varepsilon * f) - (\eta_\varepsilon * (\varphi\partial_i^\mu f))|^p \,\mathrm{d}x\\
=&\int \left|\varphi(x)\int \partial_i\eta_\varepsilon(x-y)f(y)\,\mathrm{d}y - \int \eta_\varepsilon(x-y)\varphi(y)\partial_i^\mu f(y)\,\mathrm{d}y\right|^p \,\mathrm{d}x\\
=&\int \Bigg|\int \partial_i\eta_\varepsilon(x-y)f(y)[\varphi(x)-\varphi(y)]\,\mathrm{d}y\\
&+\int \partial_i\eta_\varepsilon(x-y)f(y)\varphi(y) - \eta_\varepsilon(x-y)\varphi(y)\partial_i^\mu f(y)\,\mathrm{d}y\Bigg|^p \,\mathrm{d}x\\
&\text{apply Lemma 3.1 with } \zeta(y) := \eta_\varepsilon(x-y)\\
&\text{and noting that } \partial_i\eta_\varepsilon(x-y) = \frac{\partial}{\partial x_i}\eta_\varepsilon(x-y) = -\frac{\partial}{\partial y_i}\eta_\varepsilon(x-y)\\
=&\int \left|\int \partial_i\eta_\varepsilon(x-y)f(y)[\varphi(x)-\varphi(y)]\,\mathrm{d}y + \int \eta_\varepsilon(x-y)f(y)\partial_i\varphi(y)\,\mathrm{d}y\right|^p \,\mathrm{d}x\\
\le& 2^{p-1}\left[\int \left|\int \partial_i\eta_\varepsilon(x-y)f(y)[\varphi(x)-\varphi(y)]\,\mathrm{d}y\right|^p \,\mathrm{d}x + \int |\eta_\varepsilon * (f\partial_i\varphi)|^p \,\mathrm{d}x\right]\\
\le& 2^{p-1}\int \left|\int \partial_i\eta_\varepsilon(x-y)f(y)[\varphi(x)-\varphi(y)]\,\mathrm{d}y\right|^p \,\mathrm{d}x + 2^{p-1}\|f\partial_i\varphi\|_{L^p(\mathrm{d}x)}^p.\end{aligned}$$

We would like to control the first term. Replace φ by $\widehat{\varphi} \in W_0^{1,p}(\mathrm{d}x)$ defined by:

$$\widehat{\varphi} = \varphi\xi \;\text{ with }\; \xi \in C_0^\infty(\mathbb{R}^d) \;\text{ and }\; 1_{\operatorname{supp} f + B(0,2)} \le \xi \le 1_{\operatorname{supp} f + B(0,3)}.$$

Let $h_\varepsilon : \mathbb{R}^d \to \mathbb{R}^d$, $h_\varepsilon(x) := -\varepsilon x$. Then upon substituting $y = x + \varepsilon z$ (which leads to $\mathrm{d}y = \varepsilon^d \,\mathrm{d}z$)

$$\begin{aligned}&\int \left|\int \partial_i\eta_\varepsilon(x-y)f(y)[\varphi(x)-\varphi(y)]\,\mathrm{d}y\right|^p \,\mathrm{d}x\\
=&\int \left|\int_{B(0,1)} \partial_i\eta_\varepsilon(-\varepsilon z)f(x+\varepsilon z)[\widehat{\varphi}(x)-\widehat{\varphi}(x+\varepsilon z)]\varepsilon^d \,\mathrm{d}z\right|^p \,\mathrm{d}x\end{aligned}$$

By the chain rule $-\varepsilon(\partial_i\eta_\varepsilon)(-\varepsilon z) = \partial_i(\eta_\varepsilon \circ h_\varepsilon)(z) = (1/\varepsilon^d)\partial_i(\eta)(z)$ and hence the latter is equal to

$$\int \left| \int_{B(0,1)} \partial_i\eta(z) f(x+\varepsilon z) \frac{\widehat{\varphi}(x) - \widehat{\varphi}(x+\varepsilon z)}{\varepsilon} \, \mathrm{d}z \right|^p \mathrm{d}x$$

$$\leq 2^{p-1} \int \left| \int_{B(0,1)} \partial_i\eta(z) f(x+\varepsilon z) \left\langle -\nabla\widehat{\varphi}(x+\varepsilon z), z \right\rangle \mathrm{d}z \right|^p \mathrm{d}x$$

$$+ 2^{p-1} \int \left| \int_{B(0,1)} \partial_i\eta(z) f(x+\varepsilon z) \left[\frac{\widehat{\varphi}(x) - \widehat{\varphi}(x+\varepsilon z)}{\varepsilon} + \langle \nabla\widehat{\varphi}(x+\varepsilon z), z\rangle \right] \mathrm{d}z \right|^p \mathrm{d}x$$

By Jensen's inequality and Fubini's theorem, the first term is bounded by

$$C(p,d) \left\| \partial_i\eta \right\|_\infty^p \sum_{j=1}^d \left\| f\partial_j\varphi \right\|_{L^p(\mathrm{d}x)}^p,$$

where $C(p,d)$ is a positive constant depending only on p and d.

Concerning the second term, we use again Jensen's inequality and Fubini's theorem to see that it is bounded by

$$C'(p,d) \left\| \partial_i\eta \right\|_\infty^p \left\| f \right\|_\infty^p \int_{B(0,1)} \int \left| \frac{\widehat{\varphi}(x) - \widehat{\varphi}(x+\varepsilon z)}{\varepsilon} + \langle \nabla\widehat{\varphi}(x+\varepsilon z), z\rangle \right|^p \mathrm{d}x \, \mathrm{d}z, \tag{3.6}$$

where $C'(p,d)$ is a positive constant depending only on p and d. Let us investigate the inner integral. We need a lemma on difference quotients. Compare with [13, Proof of Lemma 7.23] and [38, Theorem 2.1.6].

Lemma 3.2. *Let $z \in B(0,1) \subset \mathbb{R}^d$ and $u \in W^{1,p}(\mathrm{d}x)$. Set for $\varepsilon > 0$*

$$\Delta_\varepsilon u(x) := \frac{u(x-\varepsilon z) - u(x)}{\varepsilon}$$

for some representative of u. Then

$$\left\| \Delta_\varepsilon u + \langle \nabla u, z\rangle \right\|_{L^p(\mathrm{d}x)} \to 0$$

as $\varepsilon \searrow 0$.

Proof. Start with $u \in C^1 \cap W^{1,p}(\mathrm{d}x)$. By the fundamental theorem of calculus

$$\Delta_\varepsilon u(x) = -\frac{1}{\varepsilon} \int_0^\varepsilon \langle \nabla u(x-sz), z\rangle \, \mathrm{d}s.$$

Use Fubini's Theorem to get

$$\int \left| \Delta_\varepsilon u(x) + \langle \nabla u(x), z\rangle \right|^p \mathrm{d}x = \frac{1}{\varepsilon} \int_0^\varepsilon \int \left| \langle \nabla u(x-sz), z\rangle - \langle \nabla u(x), z\rangle \right|^p \mathrm{d}x \, \mathrm{d}s. \tag{3.7}$$

By a well-known property of L^p-norms [34, p. 63] the map

$$s \mapsto \int \left| \langle \nabla u(x-sz), z\rangle - \langle \nabla u(x), z\rangle \right|^p \mathrm{d}x$$

is continuous in zero. Hence $s = 0$ is a Lebesgue point of this map. Therefore the right hand side of (3.7) tends to zero as $\varepsilon \searrow 0$. The claim can be extended to functions in $W^{1,p}(\mathrm{d}x)$ by an approximation by smooth functions as e.g. in [38, Theorem 2.3.2]. □

By variable substitution, we get that the inner integral in (3.6) is equal to

$$\int \left| \frac{\widehat{\varphi}(x-\varepsilon z) - \widehat{\varphi}(x)}{\varepsilon} + \langle \nabla\widehat{\varphi}(x), z\rangle \right|^p \mathrm{d}x. \tag{3.8}$$

By the preceding lemma, the term converges to zero pointwise as $\varepsilon \searrow 0$ for each fixed $z \in B(0,1)$. Let for $g \in L^1_{\mathrm{loc}}$,

$$Mg(x) := \sup_{\rho>0} \fint_{B(x,\rho)} |g(y)| \,\mathrm{d}y,$$

be the *centered Hardy–Littlewood maximal function.* We shall need the useful inequality

$$|u(x) - u(y)| \leq c|x-y| \left[M|\nabla u|(x) + M|\nabla u|(y)\right] \tag{3.9}$$

for any $u \in W^{1,p}(\mathrm{d}x)$, for all $x, y \in \mathbb{R}^d \setminus N$, where N is a set of Lebesgue measure zero and c is a positive constant depending only on d and p. For a proof see e.g. [1, Corollary 4.3]. The inequality is credited to L. I. Hedberg [17].

Also for all $u \in L^p$

$$\|Mu\|_{L^p} \leq c' \|u\|_{L^p} \tag{3.10}$$

by the maximal function theorem [34, Theorem I.1 (c), p. 5] and $c' > 0$ depends only on d and p.

Hence for $\mathrm{d}z$-a.a. $z \in B(0,1)$

$$\int \left| \frac{\widehat{\varphi}(x-\varepsilon z) - \widehat{\varphi}(x)}{\varepsilon} + \langle \nabla\widehat{\varphi}(x), z\rangle \right|^p \mathrm{d}x \leq C(p,d) \|\nabla\widehat{\varphi}\|^p_{L^p(\mathrm{d}x)} |z|^p 1_{B(0,1)} \in L^1(\mathrm{d}z).$$

The desired convergence to zero as $\varepsilon \searrow 0$ follows now by the preceding discussion and Lebesgue's dominated convergence theorem.

We have proved that

$$\begin{aligned} &\int |\varphi\partial_i(\eta_\varepsilon * f) - (\eta_\varepsilon * (\varphi\partial_i^\mu f))|^p \,\mathrm{d}x \\ \leq &C(d,p,\operatorname{supp} f, \eta) \left[\sum_{j=1}^d \|f\partial_j\varphi\|^p_{L^p(\mathrm{d}x)} + \|f\|^p_\infty \theta(\varepsilon)\right] \end{aligned} \tag{3.11}$$

with $\theta(\varepsilon) \to 0$ as $\varepsilon \searrow 0$, and θ depends only on $\operatorname{supp} f$.

We shall go back to the right-hand side of (3.5). Let $f_\delta := \eta_\delta * f$ for $\delta > 0$. By Lebesgue's dominated convergence theorem again, we can prove that there is a subnet (also denoted by $\{f_\delta\}$), such that

$$\sum_{j=1}^d \|(f - f_\delta)\partial_j\varphi\|^p_{L^p(\mathrm{d}x)} \to 0 \tag{3.12}$$

as $\delta \searrow 0$. Taking (3.11) into account, (f replaced by $f - f_\delta$ therein), we get that

$$\begin{aligned} &\|\varphi\partial_i(\eta_\varepsilon * f) - (\eta_\varepsilon * (\varphi\partial_i^\mu f))\|^p_{L^p(\mathrm{d}x)} \\ \leq &2^{p-1} \|\varphi\partial_i(\eta_\varepsilon * (f - f_\delta)) - (\eta_\varepsilon * (\varphi\partial_i^\mu (f - f_\delta)))\|^p_{L^p(\mathrm{d}x)} \\ &+ 2^{p-1} \|\varphi\partial_i(\eta_\varepsilon * f_\delta) - (\eta_\varepsilon * (\varphi\partial_i^\mu f_\delta))\|^p_{L^p(\mathrm{d}x)} \\ \leq &C(d,p,\operatorname{supp} f) \left[\sum_{j=1}^d \|(f - f_\delta)\partial_j\varphi\|^p_{L^p(\mathrm{d}x)} + \|f - f_\delta\|^p_\infty \theta(\varepsilon)\right] \\ &+ 2^{p-1} \|\varphi\partial_i(\eta_\varepsilon * f_\delta) - (\eta_\varepsilon * (\varphi\partial_i^\mu f_\delta))\|^p_{L^p(\mathrm{d}x)}. \end{aligned}$$

The use of (3.11) is justified, since $\widehat{\varphi} = \varphi$ on $\operatorname{supp} f + B(0,2)$, thus on $\operatorname{supp}(f - f_\delta) + B(0,1)$. Taking (3.12) into account, by choosing first δ and then letting $\varepsilon \searrow 0$, the first term above can be controlled (since $\|f - f_\delta\|_\infty \leq 2\|f\|_\infty$). If we can prove for any $\zeta \in C_0^\infty$

$$\|\varphi\partial_i(\eta_\varepsilon * \zeta) - (\eta_\varepsilon * (\varphi\partial_i^\mu \zeta))\|^p_{L^p(\mathrm{d}x)} \to 0 \tag{3.13}$$

as $\varepsilon \searrow 0$, we can control the second term above and hence are done. But

$$\begin{aligned}&\|\varphi\partial_i(\eta_\varepsilon * \zeta) - (\eta_\varepsilon * (\varphi\partial_i\zeta))\|^p_{L^p(\mathrm{d}x)}\\ &\le \int \left| \int \eta_\varepsilon(x-y)\partial_i\zeta(y)\,[\varphi(x)-\varphi(y)]\,\mathrm{d}y\right|^p\,\mathrm{d}x.\end{aligned}$$

Substituting $y = x + \varepsilon z$ ($\mathrm{d}y = \varepsilon^d\,\mathrm{d}z$) and using Jensen's inequality and Fubini's theorem again, the latter is dominated by

$$C(d,p)\,\|\eta\|^p_\infty\,\|\partial_i\zeta\|^p_\infty \int_{B(0,1)} \|(\varphi\xi_\zeta)(\cdot) - (\varphi\xi_\zeta)(\cdot+\varepsilon z)\|^p_{L^p(\mathrm{d}x)}\,\mathrm{d}z,$$

where $\xi_\zeta \in C_0^\infty(\mathbb{R}^d)$ with $\xi_\zeta \equiv 1$ on $\operatorname{supp}\zeta + B(0,1)$.

$$\|(\varphi\xi_\zeta)(\cdot) - (\varphi\xi_\zeta)(\cdot+\varepsilon z)\|^p_{L^p(\mathrm{d}x)}$$

tends to zero as $\varepsilon \searrow 0$ again by [34, p. 63]. By inequalities (3.9) and (3.10) for $\mathrm{d}z$-a.a. $z \in B(0,1)$

$$\|(\varphi\xi_\zeta)(\cdot) - (\varphi\xi_\zeta)(\cdot+\varepsilon z)\|^p_{L^p(\mathrm{d}x)} \le c(d,p)\,\|\nabla(\varphi\xi_\zeta)\|^p_{L^p(\mathrm{d}x)}\,|\varepsilon z|^p 1_{B(0,1)} \in L^1(\mathrm{d}z),$$

thus we can apply Lebesgue's dominated convergence theorem.

The proof is complete.

4. The Kufner-Sobolev space $W^{1,p}(\mu)$

We shall briefly deal with the Kufner-Sobolev space $W^{1,p}(\mu)$ first introduced in [22] and studied e.g. in [23, 24, 29].

Definition 4.1. *Assume* (**Loc**). *Let*

$$W^{1,p}(\mu) := \left\{u \in L^p(\mu),\ \mid \mathrm{D}u \in L^p(\mu;\mathbb{R}^d)\right\}.$$

Note that in the above definition, by (**Loc**) and Lemma 2.4, $u \in L^1_{\mathrm{loc}}$ and hence $\mathrm{D}u$ is well-defined.

Proposition 4.2. *Assume* (**Loc**). *Then $W^{1,p}(\mu)$ is a Banach space with the obvious choice of a norm. Also, by definition $H_0^{1,p}(\mu) \subset W^{1,p}(\mu)$. Moreover, for all $u \in H_0^{1,p}(\mu)$, $\nabla^\mu u = \mathrm{D}u$ $\mathrm{d}x$-a.s.*

Proof. See [23, Theorem 1.11] and [18, §1.9]. □

Our contribution to the study of $W^{1,p}(\mu)$ is contained in the following Proposition. For $p = 2$ it was proved in [4].

Proposition 4.3. *Assume* (**Loc**), (**Diff**). *Then*

$$H_0^{1,p}(\mu) = H^{1,p}(\mu) = W^{1,p}(\mu).$$

Proof. The first equality follows from Theorem 1.1. Therefore by Proposition 4.2, $H^{1,p}(\mu) \subset W^{1,p}(\mu)$ and for $u \in H^{1,p}(\mu)$, $\nabla^\mu u = \mathrm{D}u$ both μ-a.e. and $\mathrm{d}x$-a.e. (recall that (**Loc**) implies that $\mathrm{d}x$ and μ are equivalent measures).

Conversely, let $f \in W^{1,p}(\mu) \cap L^\infty(\mu)$. Since by Lemma 2.9, $\varphi^p \in W^{1,1}_{\mathrm{loc}}(\mathrm{d}x)$, we have for each $i \in \{1,\dots,d\}$ and each $\eta \in C_0^\infty(\mathbb{R}^d)$ that

$$\int \mathrm{D}_i f \eta\varphi^p\,\mathrm{d}x = -\int f\partial_i(\eta\varphi^p)\,\mathrm{d}x,$$

where ∂_i is the usual weak derivative in $W^{1,1}_{\mathrm{loc}}(\mathrm{d}x)$. But, again by Lemma 2.9, the right hand side is equal to

$$-\int f\partial_i\eta\varphi^p\,\mathrm{d}x - \int f\eta\beta_i\varphi^p\,\mathrm{d}x.$$

Therefore $f \in H^{1,p}(\mu)$ and $\mathrm{D}f = \nabla^\mu f$ both μ-a.e. and $\mathrm{d}x$-a.e. It is well-known that, given (**Loc**), bounded functions in $W^{1,p}(\mu)$ are dense in $W^{1,p}(\mu)$ and hence $W^{1,p}(\mu) \subset H^{1,p}(\mu)$. □

5. A new class of p-admissible weights

Recall the definition of p-admissible weights as proposed by Heinonen, Kilpeläilen and Martio in [18]. Note the similarities between (5.2) and (2.1) above.

Definition 5.1. *A weight $w \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $w \geq 0$ is called p-admissible if the following four conditions are satisfied.*

- *$0 < w < \infty$ $\mathrm{d}x$-a.e. and the weight is doubling, i.e. there is a constant $C_1 > 0$ such that*

$$\int_{2B} w\,\mathrm{d}x \leq C_1 \int_B w\,\mathrm{d}x \quad \forall \text{ balls } B \subset \mathbb{R}^d. \tag{5.1}$$

- *If $\Omega \subset \mathbb{R}^d$ is open and $\{\eta_k\} \subset C^\infty(\Omega)$ is a sequence of functions such that*

$$\int_\Omega |\eta_k|^p w\,\mathrm{d}x \to 0 \quad \text{and} \quad \int_\Omega |\nabla \eta_k - v|^p w\,\mathrm{d}x \to 0 \tag{5.2}$$

for some $v \in L^p(\Omega, w\,\mathrm{d}x; \mathbb{R}^d)$, then $v \equiv 0 \in \mathbb{R}^d$.

- *There are constants $\kappa > 1$ and $C_3 > 0$ such that*

$$\left(\frac{1}{\int_B w\,\mathrm{d}x}\int_B |\eta|^{\kappa p} w\,\mathrm{d}x\right)^{1/(\kappa p)} \leq C_3 \operatorname{diam} B \left(\frac{1}{\int_B w\,\mathrm{d}x}\int_B |\nabla\eta|^p w\,\mathrm{d}x\right)^{1/p}, \tag{5.3}$$

whenever $B \subset \mathbb{R}^d$ is a ball and $\eta \in C_0^\infty(B)$.

- *There is a constant $C_4 > 0$ such that*

$$\int_B |\eta - \eta_B|^p w\,\mathrm{d}x \leq C_4 (\operatorname{diam} B)^p \int_B |\nabla \eta|^p w\,\mathrm{d}x, \tag{5.4}$$

whenever $B \subset \mathbb{R}^d$ is a ball and $\eta \in C_b^\infty(B)$. Here

$$\eta_B := \frac{1}{\int_B w\,\mathrm{d}x}\int_B \eta\, w\,\mathrm{d}x.$$

The next results were basically proved by Hebisch and Zegarliński in [16, Section 2]. We include the proofs in order to make this paper self-contained and obtain concrete bounds due to a more specific situation.

Lemma 5.2. *Let $1 < q < \infty$, $\beta \in (0,\infty)$. Let $\mu(\mathrm{d}x) := \exp(-\beta|x|^q)\,\mathrm{d}x$. Then for any $C \geq (\beta q)^{-1}$, any $\varepsilon > 0$ and any $D \geq (1+\varepsilon)^{q-1} + \varepsilon^{-1} C$, we have that*

$$\int |f||x|^{q-1}\,\mu(\mathrm{d}x) \leq C \int |\nabla f|\,\mu(\mathrm{d}x) + D \int |f|\,\mu(\mathrm{d}x), \tag{5.5}$$

for all $f \in C_0^1(\mathbb{R}^d)$.

Proof. Let $f \in C_0^1(\mathbb{R}^d)$ such that $f \geq 0$ and f is equal to zero on the unit ball. By the Leibniz rule we get that

$$(\nabla f) e^{-\beta|\cdot|^q} = \nabla\left(f e^{-\beta|\cdot|^q}\right) + \beta q f |\cdot|^{q-1} \operatorname{sign}(\cdot) e^{-\beta|\cdot|^q}.$$

Plugging into the functional $g \mapsto \int \langle g(x), \operatorname{sign}(x)\rangle\,\mathrm{d}x$ yields

$$\begin{aligned} &\int \langle \operatorname{sign}(x), \nabla f(x)\rangle\, e^{-\beta|x|^q}\,\mathrm{d}x \\ &= \int \left\langle \operatorname{sign}(x), \nabla\left(f e^{-\beta|x|^q}\right)\right\rangle\,\mathrm{d}x + \beta q \int f(x)|x|^{q-1} e^{-\beta|x|^q}\,\mathrm{d}x. \end{aligned} \tag{5.6}$$

Clearly, for the left-hand side,

$$\int \langle \operatorname{sign}(x), \nabla f(x)\rangle\, e^{-\beta|x|^q}\, \mathrm{d}x \le \int |\nabla f(x)| e^{-\beta|x|^q}\, \mathrm{d}x. \tag{5.7}$$

Denote by D the distributional gradient and by δ_x the Dirac measure in x. Recalling that $\mathrm{D}\operatorname{sign}(\cdot) = 2\delta_0$, after an approximation by mollifiers, we get the formula

$$\int \left\langle \operatorname{sign}(x), \nabla \left(f e^{-\beta|x|^q}\right)\right\rangle\, \mathrm{d}x = -2\int f e^{-\beta|x|^q}\, \mathrm{d}\delta_0 = -2f(0) = 0. \tag{5.8}$$

Gathering (5.6), (5.7) and (5.8) gives

$$\beta q \int f|x|^{q-1}\, \mu(\mathrm{d}x) \le \int |\nabla f|\, \mu(\mathrm{d}x). \tag{5.9}$$

Replacing f by $|f|$ and noting that $\nabla(|f|) = \operatorname{sign}(f)\nabla f$, we can extend to arbitrary $f \in C_0^1$ such that $f \equiv 0$ on $B(0,1)$.

Now, let $f \in C_0^1$ be arbitrary. Let $\varepsilon > 0$. Let $\varphi(x) := 1 \wedge (((1+\varepsilon) - |x|) \vee 0)$. Then $f = g + h$, where $g := \varphi f$ and $h := (1-\varphi)f$. Also, $h \equiv 0$ on $B(0,1)$. Now,

$$\begin{aligned}\int |f||x|^{q-1}\, \mu(\mathrm{d}x) &= \int_{|x|\le 1+\varepsilon} |f||x|^{q-1}\, \mu(\mathrm{d}x) + \int_{|x|>1+\varepsilon} |f||x|^{q-1}\, \mu(\mathrm{d}x) \\ &\le (1+\varepsilon)^{q-1} \int_{|x|\le 1+\varepsilon} |f|\, \mu(\mathrm{d}x) + \int_{|x|>1+\varepsilon} |h||x|^{q-1}\, \mu(\mathrm{d}x) \\ &\le (1+\varepsilon)^{q-1} \int |f|\, \mu(\mathrm{d}x) + \int |h||x|^{q-1}\, \mu(\mathrm{d}x).\end{aligned} \tag{5.10}$$

Note that $|\nabla h| \le |\nabla f| + \varepsilon^{-1}|f|$ $\mathrm{d}x$-a.s. Let $C \ge (\beta q)^{-1}$. By an approximation in $W^{1,\infty}$-norm, we see that (5.9) is also valid for h and hence

$$\int |h||x|^{q-1}\, \mu(\mathrm{d}x) \le C \int |\nabla h|\, \mu(\mathrm{d}x) \le C\int |\nabla f|\, \mu(\mathrm{d}x) + \varepsilon^{-1}C\int |f|\, \mu(\mathrm{d}x),$$

which, combined with (5.10), yields inequality (5.5) with $D \ge (1+\varepsilon)^{q-1} + \varepsilon^{-1}C$. $\square$

Lemma 5.3. *Let $1 < p < \infty$, $q := p/(p-1)$, $\beta \in (0,\infty)$. Let $\mu(\mathrm{d}x) := \exp(-\beta|x|^q)\,\mathrm{d}x$. Let $C \ge (\beta q)^{-1}$. Let $W \in C^1(\mathbb{R}^d)$ be a differentiable potential (in particular, is bounded below) such that*

$$|\nabla W(x)| \le \delta|x|^{q-1} + \gamma \tag{5.11}$$

with some constants $0 < \delta < C^{-1}$, $\gamma \in (0,\infty)$. Let V be measurable such that $\operatorname{osc} V := \sup V - \inf V < \infty$. Let $\mathrm{d}\nu := \exp(-W - V)\,\mathrm{d}\mu$. Then for any $\varepsilon_0 > 0$, any

$$C' \ge (1 - C\delta)^{-1}\varepsilon_0 p C e^{2\operatorname{osc} V},$$

any $\varepsilon_1 > 0$ and any

$$D' \ge (1-C\delta)^{-1} e^{2\operatorname{osc} V}\left((1+\varepsilon_1)^{q-1} + \varepsilon_1^{-1}C + (\varepsilon_0 p)^{-q/p}Cpq^{-1} + \gamma\right)$$

it holds that

$$\int |f|^p|x|^{q-1}\, \nu(\mathrm{d}x) \le C' \int |\nabla f|^p\, \nu(\mathrm{d}x) + D'\int |f|^p\, \nu(\mathrm{d}x), \tag{5.12}$$

for any $f \in C_0^1$.

Proof. Plug $|f|^p e^{-W}$ into (5.5). By Leibniz's rule we get that

$$\begin{aligned}&\int |f|^p|x|^{q-1}e^{-W}\, \mu(\mathrm{d}x) \\ &\le Cp\int |f|^{p-1}|\nabla f| e^{-W}\, \mu(\mathrm{d}x) + C\int |f|^p|\nabla W| e^{-W}\mu(\mathrm{d}x) + D\int |f|^p e^{-W}\mu(\mathrm{d}x).\end{aligned}$$

For the first term,

$$\begin{aligned}&Cp\int |f|^{p-1}|\nabla f|e^{-W}\,\mu(\mathrm{d}x)\\ \le &Cp\left(\int |\nabla f|^p e^{-W}\,\mu(\mathrm{d}x)\right)^{1/p}\cdot\left(\int |f|^p e^{-W}\,\mu(\mathrm{d}x)\right)^{1/q}\\ \le &\varepsilon_0 pC\int |\nabla f|^p e^{-W}\,\mu(\mathrm{d}x)+(\varepsilon_0 p)^{-q/p}Cpq^{-1}\int |f|^p e^{-W}\,\mu(\mathrm{d}x),\end{aligned}$$

by Hölder and Young inequalities resp. Since $\operatorname{osc} V<\infty$, the claim follows by an easy perturbation argument, see e.g. [12, preuve du théorème 3.4.1]. □

Usually, one would set $\varepsilon_0 := p^{-1}$ and $\varepsilon_1 := 1$.

Theorem 5.4. *Let $1<p<\infty$ and let w be a weight such that w satisfies a local p-Poincaré inequality* (5.4) *with constant $C_4>0$. Let β, W, V, $C'>0$, $D'>0$ be as in Lemma 5.3.*

Let $L>D'$. Let

$$a_L := \operatorname*{osc}_{B(0,L^{p-1})}\left[-\beta|\cdot|^q - W - V\right].$$

Let

$$c\ge 2^q\frac{e^{2a_L}C_4L^{p(p-1)}+\frac{C'}{L}}{1-\frac{D'}{L}}.$$

Suppose that $\mathrm{d}\nu_w := \exp(-\beta|\cdot|^q - W - V)\,w\,\mathrm{d}x$ is a finite measure. Then ν_w satisfies the Poincaré inequality

$$\int\left|f-\frac{\int f\,\mathrm{d}\nu_w}{\int \mathrm{d}\nu_w}\right|^p\mathrm{d}\nu_w \le c\int |\nabla f|^p\,\mathrm{d}\nu_w,$$

for all $f\in C_b^\infty(\mathbb{R}^d)$.

Proof. By the results of Lemma 5.3, we can apply [16, Theorem 3.1]. □

Before we prove Theorem 1.3, let us note that, under our assumptions, the results of Hebisch and Zegarliński (in this particular case) extend to $H^{1,p}(\mu)=W^{1,p}(\mu)$. Of course, other Poincaré and Sobolev type inequalities for smooth functions extend similarly to $H^{1,p}(\mu)$ if the weight satisfies (**Diff**).

Proof of Theorem 1.3. Let us prove that $\exp(-\beta|\cdot|^q - W - V)$ is doubling. Let $c_1^W, c_1^V\ge 1$, $c_2^W, c_2^V\in\mathbb{R}$ be the constants from property (D). Let $a:=\inf W$, $b:=\inf V$. Let $B\subset\mathbb{R}^d$ be any ball. Then

$$\begin{aligned}\int_{2B} e^{-\beta|x|^q - W(x)-V(x)}\,\mathrm{d}x &= 2\int_B e^{-2^q\beta|x|^q - W(2x)-V(2x)}\,\mathrm{d}x\\ &\le 2e^{-(c_1^W-1)a+c_2^W-(c_1^V-1)b+c_2^V}\int_B e^{-\beta|x|^q-W(x)-V(x)}\,\mathrm{d}x,\end{aligned}$$

which proves the doubling property.

By similar arguments as in the proof of Lemma 2.6, condition (5.2) is implied condition (**Loc**) which is obviously satisfied, since $\beta|\cdot|^q$, W and V are locally bounded. However, by a general result due to Semmes, (5.2) is implied by (5.1) and (5.4), see [19, Lemma 5.6].

The weighted Poincaré inequality (5.4) follows from Theorem 5.4 by noting that $\exp(-\beta|x|^q - W - V)\,\mathrm{d}x$ is a finite measure.

The weighted Sobolev inequality (5.3) follows from (5.1) and (5.4) by a general result of Hajłasz and Koskela [14].

Suppose now that $V \in W^{1,\infty}_{\mathrm{loc}}(\mathrm{d}x)$. Since $W \in C^1$, also $W \in W^{1,\infty}_{\mathrm{loc}}(\mathrm{d}x)$. A similar statement holds for $-\beta|\cdot|^q$. Therefore, it is an easy exercise to check that the conditions (**Loc**) and (**Diff**) are satisfied. $\square$

References

[1] D. Aalto and J. Kinnunen, *Maximal functions in Sobolev spaces*, Sobolev spaces in mathematics I, Sobolev type inequalities (V. Maz'ya, ed.), International Mathematical Series, vol. 8, Springer and Tamara Rozhkovskaya Publisher, 2009, pp. 25–67.

[2] S. Albeverio, S. Kusuoka, and M. Röckner, *On partial integration in infinite-dimensional space and applications to Dirichlet forms*, J. London Math. Soc. (2) **42** (1990), no. 1, 122–136.

[3] S. Albeverio and M. Röckner, *Classical Dirichlet forms on topological vector spaces - closability and a Cameron-Martin formula*, J. Funct. Anal. **88** (1990), no. 2, 395–436.

[4] ———, *New developments in the theory and application of Dirichlet forms*, Stochastic processes, physics and geometry (Ascona and Locarno, 1988), World Sci. Publ., Teaneck, NJ, 1990, pp. 27–76.

[5] S. Albeverio, M. Röckner, and T.-S. Zhang, *Markov uniqueness and its applications to martingale problems, stochastic differential equations and stochastic quantization*, C. R. Math. Rep. Acad. Sci. Canada **15** (1993), no. 1, 1–6.

[6] ———, *Markov uniqueness for a class of infinite dimensional Dirichlet operators*, Stochastic processes and optimal control, Stochastic Monographs 7, 1993, pp. 1–26.

[7] P. Cattiaux and M. Fradon, *Entropy, reversible diffusion processes, and Markov uniqueness*, J. Funct. Anal. **138** (1996), no. 1, 243–272.

[8] V. Chiadò Piat and F. Serra Cassano, *Some remarks about the density of smooth functions in weighted Sobolev spaces*, J. Convex Anal. **1** (1994), no. 2, 135–142.

[9] A. Eberle, *Uniqueness and non-uniqueness of semigroups generated by singular diffusion operators*, Lecture Notes in Mathematics, vol. 1718, Springer, Berlin–Heidelberg–New York, 1999.

[10] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, 1992.

[11] M. Fradon, *Diffusions réfléchies réversibles dégénérées*, Potential Anal. **6** (1997), no. 4, 369–414.

[12] I. Gentil, *Tensorisation et perturbation de l'inégalité de Sobolev logarithmique*, Sur les inégalités de Sobolev logarithmiques, Panoramas et Synthèses, no. 10, Société Mathématique de France, 2000.

[13] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der mathematischen Wissenschaften, vol. 224, Springer-Verlag, Berlin–Heidelberg–New York, 1977.

[14] P. Hajłasz and P. Koskela, *Sobolev meets Poincaré*, C. R. Acad. Sci. Paris, Série I Math. **320** (1995), no. 10, 1211–1215.

[15] M. M. Hamza, *Détermination des formes de Dirichlet sur* $\mathbb{R}^n$, Ph.D. thesis, Université Paris-Sud 11, Orsay, 1975.

[16] W. Hebisch and B. Zegarliński, *Coercive inequalities on metric measure spaces*, J. Funct. Anal. **258** (2010), no. 3, 814–851.

[17] L. I. Hedberg, *On certain convolution inequalities*, Proc. Amer. Math. Soc. **36** (1972), no. 2, 505–510.

[18] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Clarendon Press, Oxford University Press, 1993.

[19] J. Heinonen and P. Koskela, *Weighted Sobolev and Poincaré inequalities and quasiregular mappings of polynomial type*, Math. Scand. **77** (1995), no. 2, 251–271.

[20] T. Kilpeläinen, *Smooth approximation in weighted Sobolev spaces*, Comment. Math. Univ. Carolinæ **38** (1997), no. 1, 29–35.

[21] A. V. Kolesnikov, *Convergence of Dirichlet forms with changing speed measures on* $\mathbb{R}^d$, Forum Math. **17** (2005), no. 2, 225–259.

[22] A. Kufner, *Weighted Sobolev spaces*, Teubner–Texte zur Mathematik, vol. 31, Teubner, Stuttgart–Leipzig–Wiesbaden, 1980.

[23] A. Kufner and B. Opic, *How to define reasonably weighted Sobolev spaces*, Comment. Math. Univ. Carolinæ **25** (1984), no. 3, 537–554.

[24] A. Kufner and A.-M. Sändig, *Some applications of weighted Sobolev spaces*, Teubner–Texte zur Mathematik, vol. 100, Teubner, Stuttgart–Leipzig–Wiesbaden, 1987.

[25] M. Lavrent'ev, *Sur quelques problèmes du calcul des variations*, Ann. Math. Pura Appl. **4** (1926), 107–124.

[26] Z.-M. Ma and M. Röckner, *Introduction to the theory of (non-symmetric) Dirichlet forms*, Universitext, Springer-Verlag, Berlin–Heidelberg–New York, 1992.
[27] N. Meyers and J. Serrin, $H = W$, Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 1055–1056.
[28] S. Mizohata, *The theory of partial differential equations*, Cambridge University Press, London, New York, 1973.
[29] M. Ohtsuka, *Extremal length and precise functions*, GAKUTO international series mathematical sciences and applications, vol. 19, Gakkōtosho, 2003.
[30] S. E. Pastukhova, *Degenerate equations of monotone type: Lavrent'ev phenomenon and attainability problems*, Sb. Math. **198** (2007), no. 10, 1465–1494, translated from Math. Sb. **198** (2007), no. 10, 89–118.
[31] R. T. Rockafellar, *Convex analysis*, Princeton University Press, 1970.
[32] M. Röckner and T. S. Zhang, *Uniqueness of generalized Schrödinger operators and applications*, J. Funct. Anal. **105** (1992), 187–231.
[33] ———, *Uniqueness of generalized Schrödinger operators and applications II*, J. Funct. Anal. **119** (1994), 455–467.
[34] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton mathematical series, vol. 30, Princeton University Press, Princeton, New Jersey, 1970.
[35] J. M. Tölle, *Variational convergence of nonlinear partial differential operators on varying Banach spaces*, Ph.D. thesis, Universität Bielefeld, 2010, `http://www.math.uni-bielefeld.de/~bibos/preprints/E10-09-360.pdf`.
[36] B. O. Turesson, *Nonlinear potential theory and weighted Sobolev spaces*, Lecture Notes in Mathematics, vol. 1736, Springer, Berlin–Heidelberg–New York, 2000.
[37] V. V. Zhikov, *On weighted Sobolev spaces*, Sb. Math. **189** (1998), no. 8, 1139–1170, translated from Math. Sb. **189** (1998), no. 8, 27–58.
[38] W. P. Ziemer, *Weakly differentiable functions: Sobolev spaces and functions of bounded variation*, Graduate texts in mathematics, vol. 120, Springer-Verlag, Berlin–Heidelberg–New York, 1989.

Faculty of Mathematics, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany

E-mail address: `jtoelle1@math.uni-bielefeld.de`