Trace norm estimates at large coupling

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Abstract

Let $\mathcal{E}$ and $\mathcal{P}$ be nonnegative quadratic forms in a Hilbert space $\mathcal{H}$ such that $\mathcal{E}+\beta\mathcal{P}$ is densely defined and closed for all $\beta \geq 0$. Let H_β be the selfadjoint operator associated with $\mathcal{E}+\beta\mathcal{P}$. Let $0<r\leq 1$. We give both a sufficient condition and a necessary condition in order that

$$\limsup_{\beta\longrightarrow\infty} \beta^r \parallel (H_\beta+1)^{-1} - \lim_{\beta'\longrightarrow\infty} (H_{\beta'}+1)^{-1} \parallel_1 < \infty \tag{0.1}$$

where $\parallel \cdot \parallel_1$ denotes the trace norm. In the extremal case $r=1$ we derive even a condition that is necessary and sufficient in order that (0.1) holds true and present examples where this criterion is satisfied.

Key words: Schatten–von Neumann class, rate of convergence, Dirichlet form, equilibrium measure, polyharmonic oscillator, point interaction
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1 Introduction

In this paper we continue the discussion of problems related to various types of large coupling convergence in abstract Hilbert spaces that we initiated in previous works ([6], [4], [3]). Originally one has concentrated on large coupling convergence for Schrödinger operators (cf. [7] for recent results on this topic). Motivated by the work on point interactions ([1]) one has started to analyze also singular interactions ([8]). In addition, one has investigated large coupling convergence where the perturbation term is a differential operator ([9]).

The mentioned problems can be treated in a unified way. Let H be a nonnegative selfadjoint operator in a Hilbert space $\mathcal{H}$, $\mathcal{E}$ the closed quadratic form associated with H, $\mathcal{P}$ a nonnegative quadratic form in $\mathcal{H}$, such that $\mathcal{E}+\beta\mathcal{P}$ is densely defined and closed for one and therefore every $\beta>0$ and H_β the selfadjoint operator associated with $\mathcal{E}+\beta\mathcal{P}$. By Kato's monotone convergence theorem, the operators $(H_\beta+1)^{-1}$ converge strongly as β goes to infinity.

Under additional assumptions on H and $\mathcal{P}$ much stronger results have been achieved. Let

$$D_\beta := (H+1)^{-1} - (H_\beta+1)^{-1}, \quad D_\infty := \lim_{\beta\longrightarrow\infty} D_\beta.$$

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We have shown ([6, Proposition 1]), that

$$\liminf_{\beta \longrightarrow \infty} \beta \parallel D_\infty - D_\beta \parallel > 0,$$

i.e. convergence (w.r.t. the operator norm $\parallel \cdot \parallel$) faster than c/β is not possible in this context. We have derived ([4, Theorem 1]) a criterion in order that norm-convergence with maximal rate takes place, i.e. in order that

$$\limsup_{\beta \longrightarrow \infty} \beta \parallel D_\infty - D_\beta \parallel < \infty.$$

Let $0 < r < 1$. In the same article we have given a condition which is sufficient in order that $\parallel D_\infty - D_\beta \parallel = O(1/\beta^r)$. In a recent paper we have shown that this condition is 'almost' necessary ([3, Proposition 2.9]).

In scattering theory one is mainly interested in the stronger trace norm. For instance the wave operators $W^\pm(H, H_\beta)$ exist and are complete provided $(H+1)^{-k} - (H_\beta + 1)^{-k}$ belongs to the trace class for some $k \in \mathbb{N}$. There is also another motivation to study trace norms. If H has a purely discrete spectrum, then the same holds true for the perturbed operators H_β and estimates for the rate of convergence of $\parallel D_\infty - D_\beta \parallel_1$ can be used in order to get information on the eigenvalue distribution of H_β with the aid of the eigenvalue distribution of $\lim_{\beta \longrightarrow \infty}(H_\beta + 1)^{-1}$. At this point we would like to recall that it is often easier to study this limit operator than the operators H_β and that this is one of the main reasons why one investigates large coupling convergence.

For compact perturbations we have found a criterion for trace-class convergence of $D_\infty - D_\beta$ ([6, Theorem 3]). However, the problem of determining or estimating the speed of convergence of $\|D_\infty - D_\beta\|_1$ is still open. In this note, we shall give a criterion for convergence of $\|D_\infty - D_\beta\|_1$ with maximal rate. In addition, we shall derive a sufficient condition for convergence with rate $O(1/\beta^r)$ and show that this condition is 'almost' necessary.

In the last section we shall illustrate our results with the aid of several examples. For instance we shall discuss the case when H is a Schrödinger operator corresponding to a polyharmonic oscillator and $\mathcal{P}$ describes a point interaction.

Notation and hypothesis

- $\parallel \cdot \parallel_p$ denotes the norm in the Schatten-von Neumann class S_p of order p. By definition, $\parallel T \parallel_p = \infty$, if T does not belong to S_p.
- H is a nonnegative selfadjoint operator in a separable infinite-dimensional Hilbert space $(\mathcal{H}, (\cdot, \cdot))$ and $\mathcal{E}$ the associated closed quadratic form, i.e.

$$D(\mathcal{E}) = D(\sqrt{H}), \quad \mathcal{E}(u, u) = \parallel \sqrt{H} u \parallel^2 \quad \forall\, u \in D(\mathcal{E}).$$

- $\mathcal{E}_1(u, v) := \mathcal{E}(u, v) + (u, v)$.
- $K_1 = (H + 1)^{-1}$.

- $\mathcal{P}$ is a nonnegative quadratic form in $\mathcal{H}$ such that $\mathcal{E} + \beta\mathcal{P}$ is densely defined and closed for one and therefore all $\beta > 0$.
- H_β is the nonnegative selfadjoint operator in $\mathcal{H}$ associated with $\mathcal{E} + \beta\mathcal{P}$.
- $D_\beta := (H+1)^{-1} - (H_\beta + 1)^{-1}$ and $D_\infty := \lim_{\beta\longrightarrow\infty} D_\beta$.

By [3, Example 2.1 and Lemma 2.2], $\mathcal{P}$ is a nonnegative quadratic form in $\mathcal{H}$ such that $\mathcal{E}+\beta\mathcal{P}$ is densely defined and closed for one and therefore every $\beta > 0$ if and only if there exist an auxiliary Hilbert space $(\mathcal{H}_{aux}, (\cdot,\cdot)_{aux})$ and a closed operator J from $(D(\mathcal{E}), \mathcal{E}_1)$ to $\mathcal{H}_{aux}$ such that the range ran$J$ of J is dense in $\mathcal{H}_{aux}$ and

$$D(J) = D(\mathcal{E}) \cap D(\mathcal{P}), \qquad \mathcal{P}(u,u) = \|Ju\|^2_{aux} \quad \forall u \in D(J).$$

The nonnegative selfadjoint operator JJ^* in $\mathcal{H}_{aux}$ is invertible and we put

$$\check{H} := (JJ^*)^{-1}.$$

μ_g denotes the spectral measure of g w.r.t. $\check{H}$. We shall always assume that $\mathcal{H}_{aux} \neq \{0\}$ and

$$D(H) \subset D(J).$$

2 Convergence within the trace ideal

As a first step we shall give a sufficient condition ensuring convergence of $\|D_\infty - D_\beta\|$ in our general setting. We recall that

$$(JK_1)^* u = J^* u \quad \forall\, u \in D(J^*) \tag{2.1}$$

(since $(J^*u, v) = \mathcal{E}_1(J^*u, K_1 v) = (u, JK_1 v)_{aux} = ((JK_1)^*u, v)$ for all $v \in \mathcal{H}$) and the resolvent formula ([5, Lemma 3])

$$D_\beta = (JK_1)^*(1/\beta + JJ^*)^{-1} JK_1 = J^*(1/\beta + JJ^*)^{-1} JK_1. \tag{2.2}$$

Differentiating D_β w.r.t. β yields

$$\frac{d}{d\beta} D_\beta = \frac{1}{\beta^2} (JK_1)^*(1/\beta + JJ^*)^{-2} JK_1, \tag{2.3}$$

so that

$$D_{\beta'} - D_\beta = \int_\beta^{\beta'} \frac{1}{s^2} (JK_1)^*(1/s + JJ^*)^{-2} JK_1 \, ds, \tag{2.4}$$

and, by monotone convergence,

$$D_\infty - D_\beta = \int_\beta^{\infty} \frac{1}{s^2} (JK_1)^*(1/s + JJ^*)^{-2} JK_1 \, ds, \tag{2.5}$$

and

$$\begin{aligned}\|D_\infty - D_\beta\| &\le \int_\beta^\infty \frac{1}{s^2}\|(JK_1)^*(1/s+JJ^*)^{-2}JK_1\|\,ds\\ &= \int_\beta^\infty \frac{1}{s^2}\|(1/s+JJ^*)^{-1}JK_1\|_{aux}^2\,ds\\ &= \int_\beta^\infty \|J(H_s+1)^{-1}\|_{aux}^2\,ds. \end{aligned} \tag{2.6}$$

We thus derive the following

Proposition 2.1. *Assume that there is $\beta > 0$ such that*

$$\int_\beta^\infty \|J(H_s+1)^{-1}\|_{aux}^2\,ds < \infty. \tag{2.7}$$

Then $\lim_{\beta\to\infty}\|D_\beta - D_\infty\| = 0$.

If one is concerned with trace-class convergence of the difference $D_\infty - D_\beta$, then Proposition 2.1 can be improved so as to get a criterion which generalizes an old result of Baumgärtel–Demuth [2, Theorem 2], in the case where J is a projection (defined on $\mathcal{H}$) and the operator H is local w.r.t. the projection J.

Theorem 2.1. *Assume that $D(H) \subset D(J)$. Then the following equivalence holds true:*

$$\lim_{\beta\to\infty}\|D_\infty - D_\beta\|_1 = 0 \iff \int_\beta^\infty \|J(H_s+1)^{-1}\|_2^2\,ds < \infty, \text{ for some } \beta > 0. \tag{2.8}$$

Proof. Let $(f_k)_{k=1}^\infty$ be an O.N.B. of $\mathcal{H}$. From the integral representation (2.5), we get

$$\big((D_\infty - D_\beta)f_k, f_k\big) = \int_\beta^\infty \|J(H_\xi+1)^{-1}f_k\|_{aux}^2\,d\xi, \tag{2.9}$$

leading to

$$\begin{aligned}\|D_\beta - D_\infty\|_1 &= \sum_{k=1}^\infty \int_\beta^\infty \|J(H_\xi+1)^{-1}f_k\|_{aux}^2\,d\xi = \int_\beta^\infty \big(\sum_{k=1}^\infty \|J(H_\xi+1)^{-1}f_k\|_{aux}^2\big)\,d\xi\\ &= \int_\beta^\infty \|J(H_\xi+1)^{-1}\|_2^2\,d\xi, \end{aligned} \tag{2.10}$$

which yields the result.

□

We shall now give a criterion for trace norm convergence of the resolvent difference with maximal rate. We first quote that since the maximal rate of convergence of $\|D_\beta - D_\infty\|$ is proportional to $1/\beta$, then so is the maximal rate of convergence of $\|D_\beta - D_\infty\|_1$.

Theorem 2.2. *a) The mapping*

$$\beta \mapsto \beta \|D_\infty - D_\beta\|_1, \tag{2.11}$$

is nondecreasing and strictly positive.
b) The following holds true:

$$\lim_{\beta\to\infty} \beta \|D_\beta - D_\infty\|_1 < \infty, \tag{2.12}$$

if and only if $\check{H}JK_1$ is a Hilbert–Schmidt operator. If it is the case then

$$\lim_{\beta\to\infty} \beta \|D_\beta - D_\infty\|_1 = \|\check{H}JK_1\|_2^2. \tag{2.13}$$

Proof. Let $(f_k)_{k=1}^\infty$ be an O.N.B. of $\mathcal{H}$. It follows from the resolvent formula (2.2) and the spectral calculus (cf. [4, formula (20)]), that

$$((D_\infty - D_\beta)f_k, f_k) = \int \frac{\lambda^2}{\lambda+\beta}\, d\mu_{g_k}(\lambda)$$

where $g_k := JK_1 f_k$ and μ_g denotes the spectral measure of g w.r.t. $\check{H}$. Thus

$$\beta\|D_\beta - D_\infty\|_1 = \sum_{k=1}^\infty \beta\big((D_\infty - D_\beta)f_k, f_k\big) = \sum_{k=1}^\infty \int \frac{\beta\lambda^2}{\lambda+\beta}\, d\mu_{g_k}(\lambda). \tag{2.14}$$

Since $D_\infty - D_\beta$ is positive, each of the integrals $\int \frac{\beta\lambda^2}{\lambda+\beta}\, d\mu_{g_k}$ is nondecreasing w.r.t. β and assertion a) is proved.
Owing to the fact that the series are increasing w.r.t. to β and by monotone convergence, we get

$$\begin{aligned}
\limsup_{\beta\to\infty} \beta\|D_\beta - D_\infty\|_1 &= \lim_{\beta\to\infty} \beta\|D_\beta - D_\infty\|_1 = \lim_{\beta\to\infty} \sum_{k=1}^\infty \int \frac{\beta\lambda^2}{\lambda+\beta}\, d\mu_{g_k} \\
&= \sum_{k=1}^\infty \lim_{\beta\to\infty} \int \frac{\beta\lambda^2}{\lambda+\beta}\, d\mu_{g_k} = \sum_{k=1}^\infty \int \lambda^2\, d\mu_{g_k} \\
&= \sum_{k=1}^\infty \|\check{H}JK_1 f_k\|_{aux}^2 = \|\check{H}JK_1\|_2^2 \leq \infty, \qquad (2.15)
\end{aligned}$$

which achieves the proof. □

We turn our attention now to give conditions for $\|D_\beta - D_\infty\|_1$ to be $O(\beta^{-r})$ where $0 < r < 1$. In other words conditions for $\|D_\beta - D_\infty\|_1$ to converge at least as fast as $c\beta^{-r}$.

Proposition 2.2. *Let $0 < r < 1$ and $s_0 = \frac{1}{2} + \frac{r}{2}$. Suppose that $D(H) \subset D(J)$.*

a) *If* $\check{H}^{s_0} J K_1$*, is a Hilbert–Schmidt operator, then*

$$\|D_\beta - D_\infty\|_1 \le (1-r)^{1-r} r^r \|\check{H}^{s_0} J K_1\|_2^2 \frac{1}{\beta^r} \quad \forall\, \beta > 0. \tag{2.16}$$

b) *Assume that* $J : (D(\mathcal{E}), \mathcal{E}_1) \to \mathcal{H}_{aux}$ *is bounded. If*

$$\|D_\beta - D_\infty\|_1 \le \frac{c}{\beta^r} \quad \forall\, \beta > 0,$$

for some finite constant c, then $\check{H}^s J K_1$ *is a Hilbert-Schmidt operator for every* $\frac{1}{2} < s < s_0$.

Proof. a) Let $(f_k)_{k=1}^\infty$ be an O.N.B. of $\mathcal{H}$. Then, as in the last proof

$$\begin{aligned} \|D_\beta - D_\infty\|_1 &= \sum_{k=1}^\infty \big((D_\infty - D_\beta) f_k, f_k\big) = \sum_{k=1}^\infty \int \frac{\lambda^2}{\lambda+\beta}\, d\mu_{g_k}(\lambda) \\ &\le \max_{\lambda \in (0,\infty)} \frac{\lambda^{1-r}}{\lambda+\beta} \sum_{k=1}^\infty \int |\lambda^{1/2+r/2}|^2\, d\mu_{g_k}(\lambda). \end{aligned} \tag{2.17}$$

By elementary calculus,

$$\max_{\lambda \in (0,\infty)} \frac{\lambda^{1-r}}{\lambda+\beta} = \frac{(1-r)^{1-r}\, r^r}{\beta^r},$$

so that

$$\|D_\beta - D_\infty\|_1 \le \frac{(1-r)^{1-r}\, r^r}{\beta^r} \sum_{k=1}^\infty \|\check{H}^{s_0} J K_1 f_k\|_{aux}^2 = \frac{(1-r)^{1-r}\, r^r}{\beta^r} \|\check{H}^{s_0} J K_1\|_2^2. \tag{2.18}$$

b) Let $s \in (1/2, s_0)$. According to (2.10), $J(H_\beta + 1)^{-1}$ is a Hilbert-Schmidt operator for almost every $\beta > 0$.
On the other hand we have

$$J K_1 = \beta(1/\beta + J J^*) J (H_\beta + 1)^{-1}, \tag{2.19}$$

yielding that the operator

$$1_{[0,2)}(\check{H}) J K_1 = 1_{[0,2)}(\check{H}) \beta (1/\beta + J J^*) J (H_\beta + 1)^{-1}, \tag{2.20}$$

is a Hilbert–Schmidt operator.
Now set

$$C_k(\beta) := \beta^r \big((D_\infty - D_\beta) f_k, f_k\big) = \int \beta^r \frac{\lambda^2}{\lambda+\beta}\, d\mu_{g_k}(\lambda). \tag{2.21}$$

Then, by assumptions, $\sum_{k=1}^{\infty} C_k(\beta) \leq c < \infty$.
Choose $1 < t < \frac{r}{2s-1}$, $2 > \beta > 1$ and $\lambda \in [\beta, \beta^t]$. Then

$$\beta^r \frac{\lambda^{2-2s}}{\lambda + \beta} \geq \frac{\beta^r}{2}\lambda^{1-2s} \geq \frac{1}{2}\beta^{t(1-2s)+r}, \tag{2.22}$$

yielding

$$C_k(\beta) \geq \int_{[\beta,\beta^t)} \frac{1}{2}\beta^{t(1-2s)+r}\lambda^{2s}\, d\mu_{g_k}(\lambda). \tag{2.23}$$

Thus

$$\begin{aligned}\int_{[\beta,\infty)} \frac{1}{2}\lambda^{2s}\, d\mu_{g_k}(\lambda) &= \sum_{n=0}^{\infty}\int_{[\beta^{t^n},\, \beta^{t^{n+1}})} \frac{1}{2}\lambda^{2s}\, d\mu_{g_k}(\lambda) \\ &\leq \sum_{n=0}^{\infty} \beta^{-t^n(t(1-2s)+r)} C_k(\beta^{t^n}),\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}\|1_{[\beta,\infty)}(\check{H}^s)JK_1\|_2^2 &= \sum_{k=1}^{\infty}\|1_{[\beta,\infty)}(\check{H}^s)JK_1 f_k\|^2 = \sum_{k=1}^{\infty}\int_{[\beta,\infty)} \lambda^{2s}\, d\mu_{g_k}(\lambda) \\ &\leq 2\sum_{n=0}^{\infty}\beta^{-t^n(t(1-2s)+r)}\Big(\sum_{k=1}^{\infty} C_k(\beta^{t^n})\Big) \leq 2c\sum_{n=0}^{\infty}\beta^{-t^n(t(1-2s)+r)} \\ &< \infty \quad .\end{aligned} \tag{2.25}$$

Now putting (2.20) and (2.25) together yields that $\check{H}^s JK_1$ is a Hilbert-Schmidt operator, which was to be proved. □

Remark 2.1. a) If $r = 0$, then the following equivalence holds true:

$$\check{H}^{1/2}JK_1 \text{ is Hilbert} - \text{Schmidt} \iff \|D_\infty - D_\beta\|_1 \to 0, \tag{2.26}$$

so that we still get trace-class convergence but without information on the rate. Indeed, from the known fact that

$$D_\infty = (\check{H}^{1/2}JK_1)^* \check{H}^{1/2}JK_1, \tag{2.27}$$

we derive that $\check{H}^{1/2}JK_1$ is Hilbert-Schmidt if and only if D_∞ is trace-class, which, in turn, by [6, Theorem 3], is equivalent to $\|D_\infty - D_\beta\|_1 \to 0$.

b) We shall give an example (Example 3.3) where $\lim_{\beta\longrightarrow\infty}\beta^r \parallel D_\infty - D_\beta \parallel_1 < \infty$ although the operator $\check{H}^{s_0}JK_1$ is not even bounded.

The next result improves a bit [4], Corollary 6.

Proposition 2.3. *Assume that $\check{H}JK_1$ is bounded and that $\check{H}^{-1}$ is trace class. Then*

$$\lim_{\beta\to\infty}\|D_\infty - D_\beta\|_1 = 0. \tag{2.28}$$

Proof. We shall use Theorem 2.1 and prove that under the given assumptions,

$$\int_\beta^\infty \|J(H_s+1)^{-1}\|_2^2\,ds < \infty, \text{ for some } \beta. \tag{2.29}$$

We rewrite the operator $J(H_s+1)^{-1}$ as

$$\begin{aligned} J(H_s+1)^{-1} &= \frac{1}{s}(\frac{1}{s}+JJ^*)^{-1}JK_1 = \frac{1}{s}(\frac{1}{s}+\check{H}^{-1})^{-1}JK_1 \\ &= (s+\check{H})^{-1}\check{H}JK_1. \end{aligned} \tag{2.30}$$

We thereby derive

$$\|J(H_s+1)^{-1}\|_2^2 \le \|\check{H}JK_1\|^2\|(s+\check{H})^{-1}\|_2^2. \tag{2.31}$$

Let $(e_k)_{k\in I}$ be an O.N.B. of $\mathcal{H}_{aux}$ consisting of eigenvectors of $\check{H}^{-1}$ and for each $k\in I$ let $\mu_k>0$ be the corresponding eigenvalue of $\check{H}^{-1}$. We get

$$\begin{aligned} \int_\beta^\infty \|J(H_s+1)^{-1}\|_2^2\,ds &\le \|\check{H}JK_1\|^2\int_\beta^\infty \sum_{k\in I}\frac{1}{(s+\mu_k^{-1})^2}\,ds \\ &= \sum_{k\in I}\int_{\beta\mu_k}^\infty \frac{\mu_k}{(s+1)^2}\,ds = \sum_{k\in I}\frac{\mu_k}{\beta\mu_k+1} \le \sum_{k\in I}\mu_k \\ &= \|\check{H}^{-1}\|_1 < \infty, \end{aligned} \tag{2.32}$$

which was to be proved.

□

3 Examples

Originally one has concentrated on the important case when $H_\beta = -\Delta + V + \beta W$ for nonnegative locally integrable functions V and W. In this case the Schrödinger operator $H = -\Delta + V$ in $L^2(\mathbb{R}^d) := L^2(\mathbb{R}^d, dx)$ is associated with a regular Dirichlet form and $\mathcal{P}$ is a so called killing term. It turned out that this information on the unperturbed operator H and the perturbation form $\mathcal{P}$ is sufficient for many purposes. To work within this more general framework has helped to point out a variety of main ideas used for the investigation of large coupling convergence, has led to new results, e.g. that one gets norm convergence with maximal rate, provided the killing measure is an equilibrium measure ([6, Theorem 4]), and may be helpful for applications in other areas, e.g. in quantum graph theory.

In this section we shall illustrate our general results by several examples. In the first three examples the quadratic form $\mathcal{E}$ associated with the unperturbed operator H will be a regular Dirichlet form in $L^2(\mathbb{R})$ and we shall combine above operator-theoretical results with our results on regular Dirichlet forms in [4] and [6] in order to study large coupling convergence for certain Schrödinger operators. We shall use the following basic definitions and results: Let $\mathcal{E}$ be a regular Dirichlet form in $L^2(X,m)$ and μ a positive

Radon measure on X charging no set with capacity zero. We shall abuse notation and denote by u both an element of $D(\mathcal{E})$ and any fixed quasi-continuous representative of u. The operator J_μ from $(D(\mathcal{E}), \mathcal{E}_1)$ is defined as follows:

$$\begin{aligned} D(J_\mu) &:= \{u \in D(\mathcal{E}) : \int |u|^2 d\mu < \infty\}, \\ J_\mu u &:= u \quad \forall u \in D(\mathcal{E}). \end{aligned}$$

The operator J_μ is closed. Let P_μ be the orthogonal complement in $(D(\mathcal{E}), \mathcal{E}_1)$ onto the orthogonal complement of $\ker J_\mu$ and

$$\begin{aligned} D(\check{\mathcal{E}}_1^\mu) &:= \operatorname{ran} J_\mu, \\ \check{\mathcal{E}}_1^\mu(J_\mu u, J_\mu v) &:= \mathcal{E}_1(P_\mu u, P_\mu v) \quad \forall u, v \in \operatorname{ran} J_\mu. \end{aligned}$$

$\check{\mathcal{E}}_1^\mu$ is a closed quadratic form in $L^2(X, \mu)$ and called the trace of the Dirichlet form $\mathcal{E}_1$ w.r.t. the measure μ. $\check{H}^\mu := (J_\mu J_\mu^*)^{-1}$ is the selfadjoint operator in $L^2(X, \mu)$ associated with $\check{\mathcal{E}}_1^\mu$.

Example 3.1. We choose

$$\begin{aligned} D(\mathcal{E}) &:= H^1(\mathbb{R}) = W^{1,2}(\mathbb{R}), \\ \mathcal{E}(f, f) &:= \int_{\mathbb{R}} |f'(x)|^2 \, dx \quad \forall f \in H^1(\mathbb{R}), \text{ so that } H = -\frac{d^2}{dx^2}. \end{aligned}$$

Let $F \subset \mathbb{R}$ be compact, μ_F the equilibrium measure of F (w.r.t. the Dirichlet form $\mathcal{E}$) and $J = J_{\mu_F}$. Thus H_β is the selfadjoint operator in $L^2(\mathbb{R})$ associated with the form $\mathcal{E}^\beta$ in $L^2(\mathbb{R})$, where

$$\begin{aligned} D(\mathcal{E}^\beta) &= D(\mathcal{E}), \\ \mathcal{E}^\beta(f, f) &= \mathcal{E}(f, f) + \beta \int |f|^2 \, d\mu_F \quad \forall f \in D(\mathcal{E}). \end{aligned}$$

In this situation, according to [6, Theorem 4], we have uniform convergence of the resolvent difference $D_\infty - D_\beta$ with maximal rate which is, by [4, Theorem 1], equivalent to the boundedness of $\check{H} J K_1$.
On the other hand, setting G the integral kernel of K_1, i.e.,

$$G(x, y) = \frac{1}{2} e^{-|x-y|},$$

then it is known ([6, Example 1]) that

$$\check{H}^{-1} f = \int G(\cdot, y) f(y) \, d\mu_F(y) \ \mu_F - a.e. \forall f \in L^2(\mathbb{R}, \mu_F). \tag{3.1}$$

Thus

$$\|\check{H}^{-1}\|_1 = \int G(x, x) \, d\mu_F(x) = \frac{1}{2} \mu_F(F) < \infty, \tag{3.2}$$

and $\check{H}^{-1}$ is trace-class. Thus according to Proposition 2.3 we conclude

$$\|D_\infty - D_\beta\|_1 \to 0, \text{ as } \beta \to \infty. \tag{3.3}$$

The example extends with obvious modifications if one replaces $\mathcal{E}$ by the Dirichlet form associated to the α-stable process provided $d < 2\alpha$.

In the next example we shall use certain Schatten- von Neumann class properties of Schrödinger operators corresponding to polyharmonic oscillators, cf. the next lemma. The lemma is well known for the harmonic oscillator (the special case $k = 1$).

Lemma 3.1. *Let $k \in \mathbb{N}$. The resolvent $(H+1)^{-1}$ of the operator $H := -\Delta + |\,x\,|^{2k}$ in $L^2(\mathbb{R}^d)$ belongs to the Schatten-von Neumann class S_p for every $p > \dfrac{k+d}{2k}$.*

Proof: Let $p > 0$. Let $n \in \mathbb{N}$. By [10, Theorem XIII.81], and the hint below the proof of this theorem,

$$\dim \operatorname{ran} 1_{(-\infty, n^{2k}]}(H) \le c \int_{\{x: |\,x\,|^{2k} \le n^{2k}\}} \sqrt{n^{2k} - |\,x\,|^{2k}}\, dx$$

for some finite constant c, depending on the dimension d only. Thus

$$\dim \operatorname{ran} 1_{(-\infty, n^{2k}]}(H) \le c_1 n^{d+k} \tag{3.4}$$

for some finite constant c_1, depending on the dimension d only. Thus the operator $(H+1)^{-1}$ is compact and for $j = 0, 1, 2, 3, \ldots$ the operator H has at most $c_1 2^{(j+1)\,(d+k)}$ eigenvalues in the interval $(2^{2kj}, 2^{2k(j+1)}]$. Therefore $(H+1)^{-1}$ has at most $c_2 2^{(k+d)j}$ eigenvalues in the interval $[\frac{1}{1+2^{2k(j+1)}}, \frac{1}{1+2^{2kj}})$, where $c_2 := c_1\, 2^{k+d}$. Thus

$$\sum_{E:E \text{ eigenvalue of } (H+1)^{-1}} E^p \le c_3 + c_2 \sum_{j=0}^{\infty} 2^{(k+d)j} 2^{-2kjp} \tag{3.5}$$

for some finite constants c_2 and c_3, depending on p and d only. The right hand side of (3.5) is finite and hence $(H+1)^{-1} \in S_p$, if $p > \dfrac{k+d}{2k}$. □

Example 3.2. Let $k \in \mathbb{N}$. We shall study certain singular perturbations of the operator $H = -\dfrac{d^2}{dx^2} + x^{2k}$ in $L^2(\mathbb{R})$.
Let $\mu := \sum_{n\in\mathbb{Z}} |\,n\,|^k \delta_n$. By Sobolev's lemma, there exists a finite constant c, such that for every family $(a_n)_{n\in\mathbb{Z}}$ in $(0, \infty)$ and every $u \in H^1(\mathbb{R})$

$$|\,u(n)\,|^2 \le c\,\Big(a_n \int_{n-1/2}^{n+1/2} |\,u'(x)\,|^2\, dx + \frac{1}{a_n} \int_{n-1/2}^{n+1/2} |\,u(x)\,|^2\, dx\Big) \quad \forall\, n \in \mathbb{Z}. \tag{3.6}$$

If $u \in D(\sqrt{H})$, then $u \in H^1(\mathbb{R})$ and $\int x^{2k} \mid u(x) \mid^2 dx < \infty$. Thus (3.6), with $a_n = \dfrac{1}{\mid n \mid^k}$ for $n \neq 0$, implies

$$\begin{aligned} \int \mid u \mid^2 d\mu &= \sum_{n \in \mathbb{Z}} \mid n \mid^k \mid u(n) \mid^2 \\ &\leq c \sum_{n \in \mathbb{Z}} (\int_{n-1/2}^{n+1/2} \mid u'(x) \mid^2 dx + n^{2k} \int_{n-1/2}^{n+1/2} \mid u(x) \mid^2 dx) < \infty \end{aligned} \tag{3.7}$$

for every $u \in D(\sqrt{H})$. Thus $D(J_\mu) = D(\sqrt{H})$. Since $K_1^{1/2}$ is a bounded everywhere defined operator from $L^2(\mathbb{R})$ to $D(\sqrt{H})$ and J_μ an everywhere defined closed operator from $D(\sqrt{H})$ to $L^2(\mathbb{R}, \mu)$, the operator $J_\mu K_1^{1/2}$ from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}, \mu)$ is closed and everywhere defined. By the closed graph theorem, this implies that

$$J_\mu K_1^{1/2} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}, \mu) \text{ is bounded.} \tag{3.8}$$

Let $f \in L^2(\mathbb{R}, \mu)$. We put

$$\varphi_n(x) := (1 - 2 \mid n \mid^k \mid x - n \mid)^+ \quad \forall x \in \mathbb{R}, n \in \mathbb{Z},$$

and

$$u(x) := \sum_{n \neq 0} f(n) \varphi_n(x) \quad \forall x \in \mathbb{R}.$$

Then there exists a finite constant c such that

$$\int \mid \varphi_n'(x) \mid^2 dx + \int x^{2k} \varphi_n(x) \mid^2 dx \leq c \mid n \mid^k$$

for $n \neq 0$ and hence

$$\begin{aligned} \int \mid u'(x) \mid^2 dx + \int x^{2k} \mid u(x) \mid^2 dx &= \sum_{n \neq 0} \mid f(n) \mid^2 (\int \mid \varphi_n'(x) \mid^2 dx + \int x^{2k} \mid \varphi_n(x) \mid^2 dx) \\ &\leq c \sum_{n \in \mathbb{Z}} \mid n \mid^k \mid f(n) \mid^2 < \infty. \end{aligned}$$

Thus $u \in D(\sqrt{H})$ and $J_\mu u = f$. Thus

$$D(\check{\mathcal{E}}_1{}^\mu) = L^2(\mathbb{R}, \mu).$$

Since $\check{\mathcal{E}}_1{}^\mu$ is an everywhere defined closed quadratic form, the associated self-adjoint operator $\check{H}^\mu$ is bounded. In conjunction with (3.8) this implies that $\check{H}^\mu J_\mu K_1^{1/2}$ is bounded. By Lemma 3.1, $K_1^{1/2} \in S_p$ for every $p > \dfrac{k+1}{k}$. Thus $K_1^{1/2} \in S_2$ and

$$\check{H}^\mu J_\mu K_1 \in S_2, \tag{3.9}$$

provided $k > 1$. By Theorem 2.2, this implies that for $k > 1$

$$\lim_{\beta \longrightarrow \infty} \beta \parallel D_\infty - D_\beta \parallel_1 = \parallel \check{H}^\mu J_\mu K_1 \parallel_2^2 < \infty,$$

and, in particular, we have trace class convergence with maximal rate.

Example 3.3. As in Example 3.1 let $H = -\dfrac{d^2}{dx^2}$. Let $\mathcal{H}_{aux} = L^2(0,1)$ and $Jf = f_{|\,[0,1]}$ for every $f \in D(\mathcal{E}) = H^1(\mathbb{R})$. Then ([6, Example 3])

$$\lim_{\beta \longrightarrow \infty} \beta^{1/2} \| D_\infty - D_\beta \|_1 = \frac{3}{2}, \tag{3.10}$$

and $\check{H}$ is the selfadjoint realization of $-\dfrac{d^2}{dx^2} + 1$ in $L^2(0,1)$ characterized by the boundary conditions

$$f'(0) = f(0) \text{ and } f'(1) = -f(1). \tag{3.11}$$

$L^2(0,1)$ has an O.N.B. $(g_k)_{k \in \mathbb{N}}$ of real-valued eigenfunctions of $\check{H}$ satisfying

$$\lim_{k \longrightarrow \infty} |\, g_k(1) \,|^2 = 2, \tag{3.12}$$

$$g_k'' = -\eta_k^2 g_k \quad \forall\, k \in \mathbb{N} \tag{3.13}$$

for suitably chosen real numbers η_k. Furthermore $\eta_k^2 \leq ck^2$ for some finite constant c, that does not depend on k (actually $\eta_k \sim k$).

Since $1 + \eta_k^2$ is the eigenvalue of $\check{H}$ corresponding to the eigenfunction g_k of $\check{H}$ and $(g_k)_{k \in \mathbb{N}}$ is an O.N.B. of $L^2(0,1)$, the spectral calculus yields that $u \in D(\check{H}^s)$ if and only if

$$\sum_{k=1}^{\infty} |\, 1 + \eta_k^2 \,|^{2s} \,|\, (g_k, u)_{aux} \,|^2 < \infty. \tag{3.14}$$

Let $u(x) := e^x$ for all $x \in [0,1]$. Integrating by parts twice and taking into account (3.11) and (3.13), we get

$$(1 + \eta_k^2)\, (g_k, u)_{aux} = 2 g_k(1) e \quad \forall\, k \in \mathbb{N} \tag{3.15}$$

and hence

$$\sum_{k=1}^{\infty} |\, 1 + \eta_k^2 \,|^{2s} \,|\, (g_k, u)_{aux} \,|^2 = 4e^2 \sum_{k=1}^{\infty} |\, g_k(1) \,|^2 |\, 1 + \eta_k^2 \,|^{2s-2}.$$

By (3.12), this implies that

$$\sum_{k=1}^{\infty} |\, 1 + \eta_k^2 \,|^{2s} \,|\, (g_k, u)_{aux} \,|^2 \geq c_1 \sum_{k=1}^{\infty} |\, 1 + \eta_k^2 \,|^{2s-2} \tag{3.16}$$

for some strictly positive constant c_1. Since $\eta_k^2 \sim k^2$, the expression on the right hand side of (3.16) is finite if and only if $s < 3/4$. Thus for $s = \frac{1+1/2}{2}$ the inequality (3.14) does not hold and hence the operator $\check{H}^{3/4} J K_1$ is not even everywhere defined, although (3.10) is true.

In a variety of models in elasticity one considers perturbations of the Bi-Laplacian by differential operators of lower order, cf. [9]. In the next example we discuss large coupling convergence for an explicitly solvable model of this kind.

Example 3.4. Let $-\Delta_\Omega^D$ be the Dirichlet-Laplacian in a bounded domain Ω in $\mathbb{R}^d$. $L^2(\Omega)$ has an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ consisting of eigenfunctions of $-\Delta_\Omega^D$. Let λ_n, $n \in \mathbb{N}$, be the corresponding eigenvalues. Then

$$\inf_{n\in\mathbb{N}} \lambda_n = \min_{n\in\mathbb{N}} \lambda_n > 0 \text{ (Poincare's inequality)}, \tag{3.17}$$

$$\lambda_n \sim n^{2/d} \text{ (Weyl's asymptotic)}. \tag{3.18}$$

We shall consider the special case when $H = (\Delta_\Omega^D)^2$ in $L^2(\Omega)$ and

$$\begin{aligned} \mathcal{H}_{aux} &= L^2(\Omega), \\ J : D(\mathcal{E}) &\longrightarrow L^2(\Omega), \\ Ju = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\gamma_n (e_n, u)e_n \quad &\forall\, u \in D(\mathcal{E}), \end{aligned} \tag{3.19}$$

for some family $(\gamma_n)_{n\in\mathbb{N}}$ satisfying

$$0 < \inf_{n\in\mathbb{N}} \gamma_n \le \sup_{n\in\mathbb{N}} \gamma_n < \infty. \tag{3.20}$$

We get for $K_1 := (H + 1)^{-1}$ that

$$K_1 u = \sum_{n=1}^{\infty} \frac{1}{1+\lambda_n^2}(e_n, u)e_n \quad \forall\, u \in L^2(\Omega).$$

Thus

$$JK_1 u = \sum_{n=1}^{\infty} \frac{\gamma_n\sqrt{\lambda_n}}{1+\lambda_n^2}(e_n, u)e_n \quad \forall\, u \in L^2(\Omega) \tag{3.21}$$

and hence

$$(JK_1)^* u = \sum_{n=1}^{\infty} \frac{\gamma_n\sqrt{\lambda_n}}{1+\lambda_n^2}(e_n, u)e_n \quad \forall\, u \in L^2(\Omega). \tag{3.22}$$

By (2.1), $JJ^* = J(JK_1)^*$. By (3.19) and (3.22), this implies that

$$JJ^* u = \sum_{n=1}^{\infty} \frac{\gamma_n^2\lambda_n}{1+\lambda_n^2}(e_n, u)e_n \quad \forall\, u \in L^2(\Omega). \tag{3.23}$$

Since

$$D_\beta = (JK_1)^*(1/\beta + JJ^*)^{-1}JK_1,$$

it follows from (3.21), (3.22) and (3.23) that

$$D_\beta u = \sum_{n=1}^{\infty} \frac{1}{1+\lambda_n^2} \frac{\gamma_n^2 \lambda_n}{\gamma_n^2 \lambda_n + \frac{1+\lambda_n^2}{\beta}} (e_n, u) e_n \quad \forall\, u \in L^2(\Omega). \tag{3.24}$$

Passing to the limit as β goes to infinity yields

$$D_\infty = \sum_{n=1}^{\infty} \frac{1}{1+\lambda_n^2} (e_n, \cdot) e_n = (H+1)^{-1}. \tag{3.25}$$

Thus D_∞ is a nonnegative compact selfadjoint operator and, by the Weyl's asymptotic (3.18), $D_\infty \in S_p$ if and only if $p > d/4$. By [3, Corollary 2.20], $\| D_\infty - D_\beta \|_p \longrightarrow 0$ if and only if $D_\infty \in S_p$. Thus the operators D_β converge within S_p as β goes to infinity if and only if $p > d/4$ and, in particular, the operators D_β converge w.r.t. the trace norm if and only if $d \leq 3$. This trace norm convergence takes place with maximal rate, i.e. as fast as $O(1/\beta)$, if and only if $d = 1$, as it can be seen from the following consequence of (3.24) and (3.25):

$$D_\infty - D_\beta = \sum_{n=1}^{\infty} \frac{1}{1+\lambda_n^2 + \beta \gamma_n^2 \lambda_n} (e_n, \cdot) e_n.$$

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