Elliptic and parabolic equations for measures

V.I. Bogachev, N.V. Krylov, M. Röckner

The article gives a detailed account of recent investigations of weak elliptic and parabolic equations for measures with unbounded and possibly singular coefficients. The existence and differentiability of densities are studied, and lower and upper bounds for them are discussed. Semigroups associated with second order elliptic operators in L^p-spaces with respect to infinitesimally invariant measures are investigated.

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Introduction

The goal of this survey is to give a systematic presentation of the results obtained over the last 10–15 years for elliptic and parabolic equations for measures, typical examples of which are the Fokker–Planck and Kolmogorov equations. This direction goes back to Kolmogorov's works [**104**], [**105**]. One of our principal objects is a second order elliptic operator

$$L_{A,b}f = \mathrm{trace}(Af'') + (b, \nabla f), \quad f \in C_0^\infty(\Omega),$$

where $A = (a^{ij})$ is a mapping on a domain $\Omega \subset \mathbb{R}^d$ with values in the space of nonnegative symmetric linear operators on $\mathbb{R}^d$ and $b = (b^i)$ is a vector field on Ω. In coordinate form, $L_{A,b}$ is given by

$$L_{A,b}f = a^{ij}\partial_{x_i}\partial_{x_j}f + b^i\partial_{x_i}f,$$

where we always assume that summation is taken over all repeated indices.

With this operator $L_{A,b}$, we associate the weak elliptic equation

$$L_{A,b}^*\mu = 0 \tag{1}$$

for Borel measures on Ω, which is understood in the following weak sense:

$$\int_\Omega L_{A,b} f\, d\mu = 0 \quad \forall\, f \in C_0^\infty(\Omega), \tag{2}$$

where we assume that $|b|$, $a^{ij} \in L^1_{\mathrm{loc}}(\mu)$. If μ has a density ϱ, then ϱ is sometimes called "an adjoint solution" and the equation is called "an equation in double divergence form".

Similarly, one can consider parabolic operators and parabolic equations for measures on $\mathbb{R}^d \times (0,T)$.

A typical feature of the direction discussed in this survey is that equation (1) is meaningful under very broad assumptions on A and b: only their local integrability with respect to a solution μ is needed. These coefficients may be quite singular with respect to Lebesgue measure even if the solution admits a smooth density. For example, for an arbitrary infinitely differentiable probability density ϱ on $\mathbb{R}^d$ the measure $\mu = \varrho\, dx$ satisfies the above equation with $A = \mathrm{I}$ and $b = \nabla\varrho/\varrho$, where we set $\nabla\varrho(x)/\varrho(x) = 0$ whenever $\varrho(x) = 0$. This is obvious from the integration by parts formula

$$\int_{\mathbb{R}^d} [\Delta f + \langle \nabla\varrho/\varrho, \nabla f\rangle]\varrho\, dx = \int_{\mathbb{R}^d} \varrho\Delta f + \int_{\mathbb{R}^d} \langle \nabla\varrho, \nabla f\rangle\, dx = 0.$$

Since ϱ may vanish on an arbitrary proper closed subset of $\mathbb{R}^d$, the vector field b can fail to be locally integrable with respect to Lebesgue measure, but it is locally integrable with respect to μ. Note also that in general our solutions cannot be more regular than the coefficients (unlike the case of usual elliptic equations). For example, if $d = 1$ and $b = 0$, then for an arbitrary positive probability density ϱ the measure $\mu = \varrho\, dx$ satisfies the equation $L^*_{A,0}\mu = 0$ with $A = \varrho^{-1}$; in particular, for Hölder continuous A, our solution may be just Hölder continuous and nothing more (and for discontinuous A it may be discontinuous).

In this general setting, a study of weak elliptic equations for measures on finite- and infinite-dimensional spaces was initiated in [**36**], [**37**], [**31**], [**32**], [**4**], [**38**], [**33**], [**39**], [**46**], [**47**]. Actually, the infinite-dimensional case was even a starting point, which was motivated by investigations of infinite-dimensional diffusion processes and other applications in infinite-dimensional stochastic analysis; for example, related problems arose in stochastic quantization in the approach developed by A.I. Kirillov [**99**], [**100**], [**101**], [**102**] and in the theory of Dirichlet forms (see [**125**]). It was realized in the course of these investigations that even infinite-dimensional equations with very nice coefficients often require results on finite-dimensional equations with quite general coefficients. For example, we shall see in §3.2 that the finite-dimensional projections μ_n of a measure μ satisfying an elliptic equation on an infinite-dimensional space satisfy elliptic equations whose coefficients are the conditional expectations of the original coefficients with respect to the σ-algebras generated by the considered projection operators. As a result, even for smooth infinite-dimensional coefficients the only information about their conditional expectations is related to their integrability with respect to μ_n, not with respect to Lebesgue measure; in particular, no local boundedness is given. The theory of elliptic and parabolic equations for measures is now a rapidly growing area with deep and interesting connections to many directions in real analysis, partial differential equations, and stochastic analysis.

Suppose that ξ_t is a diffusion process in $\mathbb{R}^d$ governed by the following stochastic differential equation:

$$d\xi_t = \sqrt{2A(\xi_t)}dW_t + b(\xi_t)dt.$$

Then any invariant measure μ of ξ_t satisfies (1) and the transition probabilities of ξ_t satisfy the corresponding parabolic equation. We draw the reader's attention to the fact that for the diffusion governed by the stochastic equation

$$d\xi_t = \sigma(\xi_t)dW_t + b(\xi_t)dt,$$

the generator of the transition semigroup has the form $L_{A,b}$, where $A = \sigma\sigma^*/2$. The vector field b is called the *drift coefficient* or just the *drift*. The matrix $A = (a^{ij})$ in the operator $L_{A,b}$ will be called the *diffusion matrix* or *diffusion coefficient*; this differs from the standard form of the diffusion generator by the absence of the factor $1/2$ at the second order derivatives.

Measures satisfying (1) are called *infinitesimally invariant* because this equation has deep connections with invariance with respect to corresponding operator semigroups.

Let $(X, \mathcal{B})$ be a measurable space and let $B(X)$ be the space of all bounded $\mathcal{B}$-measurable functions on X equipped with the sup-norm. We recall that if $(T_t)_{t\geq 0}$ is a semigroup of bounded linear operators on the space $B(X)$, then a bounded measure μ on $\mathcal{B}$ is called invariant for $(T_t)_{t\geq 0}$ (or $(T_t)_{t\geq 0}$-invariant) if

$$\int_X T_t f\, d\mu = \int_X f\, d\mu \quad \forall\, f \in B(X). \tag{3}$$

Such semigroups extend naturally to $L^1(\mu)$ and are strongly continuous there in many cases, e.g., when they are given by transition probabilities of solutions to stochastic equations. Let L be the corresponding generator with domain $D(L)$. Then (3) is equivalent to the equality

$$\int_X Lf\, d\mu = 0 \quad \forall\, f \in D(L).$$

Under reasonable assumptions on A and b, the generator of the semigroup associated with the diffusion governed by the indicated stochastic equation coincides with $L_{A,b}$ on $C_0^\infty(\mathbb{R}^d)$. As we shall see, invariance of the measure in the sense of (3) is not the same as (2). The point is that the class $C_0^\infty(\mathbb{R}^d)$ may be much smaller than $D(L)$. What is important is that the equation is meaningful and can have solutions under assumptions that are much weaker than those needed for the existence of a diffusion, so that this equation can be investigated without any assumptions about the existence of semigroups. On the other hand, there exist very interesting and fruitful relations between the two equations; such relations will be one of the central topics in this work.

In the first two chapters we shall consider the following problems on $\mathbb{R}^d$ (or, more generally, on finite-dimensional Riemannian manifolds).

1) Regularity of solutions of equation (2), for example, the existence of densities with respect to Lebesgue measure, the continuity and smoothness of these densities, and certain related estimates (such as L^2-estimates for logarithmic gradients of solutions). In particular, we shall see in §§1.1 and 1.2 that the measure μ is always absolutely continuous on the set $\{\text{detA} > 0\}$ and has a continuous density from the Sobolev class $W^{p,1}_{\text{loc}}$ with $p > d$ provided that the coefficients a^{ij} are in this class, $|b| \in L^p_{\text{loc}}(dx)$ or $|b| \in L^p_{\text{loc}}(\mu)$, and the matrix A is positive. The assumptions $|b| \in L^p_{\text{loc}}(dx)$ or $|b| \in L^p_{\text{loc}}(\mu)$ on the drift are not at all always equivalent. For example, the first yields the positivity of densities of nonnegative solutions, and the second can hold even when the density has zeros (the case of a singular drift); it is interesting, however, that the local μ-integrability of $\exp(c|b|)$ already ensures the positivity of the density. Global properties of solutions of equations with unbounded coefficients are studied in §§1.8 and 1.9, where certain global upper and

lower estimates for densities are obtained. We shall also obtain analogous results for parabolic equations; with respect to applications to diffusion processes, this means regularity of transition probabilities. The results about a priori regularity of solutions play an important role in many other problems related, for instance, to uniqueness and existence of solutions. Since we are interested in singular coefficients or coefficients of low regularity, we do not discuss at all the regularity problems for degenerate operators with smooth coefficients.

2) Existence of solutions to elliptic equation (2) and existence of invariant measures in the sense of (3) as well as the relations between the two concepts. In particular, we shall see in §§1.4 and 1.5 that under rather general assumptions, for a given probability measure μ satisfying our elliptic equation (2) one can construct a strongly continuous Markov semigroup $(T_t^\mu)_{t\geq 0}$ on $L^1(\mu)$ such that μ is $(T_t^\mu)_{t\geq 0}$-invariant and the generator of $(T_t^\mu)_{t\geq 0}$ coincides with $L_{A,b}$ on $C_0^\infty(\mathbb{R}^n)$. For this, a condition simple for verification is the existence of a Lyapunov function for $L_{A,b}$. In the general case (without such additional assumptions), a bit less is true, namely, μ is only sub-invariant for $(T_t)_{t\geq 0}$. We shall see examples where this really occurs, i.e., where μ is not invariant.

3) Various uniqueness problems; in particular, uniqueness of invariant measures in the sense of (3) and uniqueness of solutions to (2) in the class of all probability measures. Related interesting problems concern uniqueness of associated semigroups $(T_t^\mu)_{t\geq 0}$ and the essential self-adjointness of the operator $L_{A,b}$ on $C_0^\infty(\mathbb{R}^d)$ in the case when it is symmetric. These topics are considered in §§1.5–1.7.

First we concentrate on the elliptic case, to which we devote the whole first chapter. In the second chapter similar problems are studied for parabolic equations. The last chapter is devoted to a brief discussion of infinite-dimensional analogs of problems 1)–3). Unlike the finite-dimensional case, where most of the results presented are obtained in great generality (sometimes in maximal generality), there does not (and maybe cannot) exist any general infinite-dimensional theory. The results obtained so far in the infinite-dimensional setting apply to various particular situations, but it is very important that they cover many concrete examples arising in applications such as stochastic partial differential equations, infinite particle systems, Gibbs measures, and so on. Some of the finite-dimensional results are not valid in infinite dimensions, and the validity of some others (when suitably formulated) is still unknown. The infinite-dimensional case will be the subject of a separate survey; the purpose of the last chapter is mainly just to comment on some developments of infinite-dimensional methods, results, and techniques.

In this survey, we have used our own work (including our joint papers with a number of other authors, in particular, [**24**], [**26**], [**30**], [**41**], [**42**], [**43**], [**48**], [**49**]), as well as works of many other authors. Due to limitations on the size of this article, the list of references contains less than a quarter of the bibliography we collected. A number of new results are given with proofs. We are grateful for useful discussions to S. Albeverio, G. Da Prato, A. Eberle, D. Elworthy, B. Goldys, A.I. Kirillov, V.A. Kondratiev, Yu.G. Kondratiev, G. Leha, V.A. Liskevich, P. Malliavin, G. Metafune, D. Pallara, E. Pardoux, A. Rhandi, G. Ritter, S.V. Shaposhnikov, I. Shigekawa, W. Stannat, N.S. Trudinger, A.Yu. Veretennikov, F.Y. Wang, J. Zabczyk, M. Zakai, T.S. Zhang, and V.V. Zhikov. This work was supported by the RFBR projects 07-01-00536, 08-01-91205-JF, 08-01-90431-Ukr, 09-01-12180-ofi-m, and the SFB 701 at Bielefeld University.

CHAPTER 1

Invariant and infinitesimally invariant measures

1.1. Elliptic equations for measures and existence of densities

Throughout we shall use the following standard notation. The class of all smooth functions with compact support in an open set $U \subset \mathbb{R}^d$ is denoted by $C_0^\infty(U)$. The class of all bounded functions on U with bounded derivatives of all orders is denoted by $C_b^\infty(U)$; classes like $C_b^2(U)$, $C_0^2(U)$, and so on are defined in an analogous way.

The term "a Borel measure μ" will normally mean a finite (possibly signed) countably additive measure on the σ-algebra of Borel sets; the cases where infinite measures (say, locally finite measures) are considered will always be specified, except for Lebesgue measure. The integrability of a function with respect to such a measure is understood as its integrability with respect to the total variation $|\mu|$ of the measure μ; the corresponding classes will be denoted by $L^p(\mu)$ or by $L^p(\Omega,\mu)$ in the case where μ is considered on a fixed set Ω. The notation $L^p(U)$ always refers to Lebesgue measure; sometimes we write $L^p(U,dx)$ in order to stress this. As usual, for $p \in [1,+\infty)$ we set $p' := p/(p-1)$.

The class of all Borel probability measures on Ω will be denoted by $\mathcal{P}(\Omega)$.

Given an open set $U \subset \mathbb{R}^d$ and $p \in [1,+\infty)$, we denote by $W^{p,1}(U)$ or $H^{p,1}(U)$ the Sobolev class of functions $f \in L^p(U)$ whose generalized partial derivatives $\partial_{x_i} f$ belong to $L^p(U)$. This space is equipped with the Sobolev norm

$$\|f\|_{p,1} := \|f\|_{L^p} + \sum_{i=1}^{d} \|\partial_{x_i} f\|_{L^p}.$$

In some places we also use higher-order Sobolev classes $W^{p,k}(U) = H^{p,k}$ with $k \in \mathbb{N}$, consisting of functions whose partial derivatives up to order k belong to $L^p(U)$, and fractional Sobolev spaces $H^{p,r}(U)$; the notation with H will normally be used in the case of fractional or parabolic Sobolev classes. In a few places we use similarly defined Sobolev spaces $W^{p,1}(\mu)$ with respect to a measure μ on $\mathbb{R}^d$ (in such cases the measure μ has some additional properties, e.g., has a continuous positive density or a weakly differentiable density, so that the weighted Sobolev classes are well-defined, see, e.g., [**22**]).

The symbols like $W^{p,1}_{\mathrm{loc}}(\mathbb{R}^d)$, $W^{p,1}_{\mathrm{loc}}(U)$, $L^p_{\mathrm{loc}}(U,\mu)$ etc. denote the classes of functions f such that ζf belongs to the corresponding class without the index "loc" for every $\zeta \in C_0^\infty(U)$.

In expressions like $a^{ij} b_i$ the standard summation rule with respect to repeated indices is meant. The inner product and norm in $\mathbb{R}^d$ are denoted by $\langle \cdot,\cdot \rangle$ and $|\cdot|$, respectively. The identity matrix is denoted by I.

Suppose we are given a locally finite Borel measure μ (possibly signed) on an open set $\Omega \subset \mathbb{R}^d$, a Borel vector field b on Ω, and a matrix-valued mapping

$A = (a^{ij})_{i,j\le d}$ on Ω such that the functions a^{ij} are Borel measurable. Let us set

$$L_{A,b}\varphi := \sum_{i,j\le d} a^{ij}\partial_{x_i}\partial_{x_j}\varphi + \sum_{i\le d} b^i\partial_{x_i}\varphi, \quad \varphi\in C_0^\infty(\Omega).$$

Given a function c on Ω, we set

$$L_{A,b,c}\varphi = L_{A,b}\varphi + c\varphi.$$

We shall also consider the divergence form operators

$$\mathcal{L}_{A,b}\varphi := \sum_{i,j\le d} \partial_{x_i}(a^{ij}\partial_{x_j}\varphi) + \sum_{i\le d} b^i\partial_{x_i}\varphi, \quad \varphi\in C_0^\infty(\Omega)$$

and the correspondingly defined operators $\mathcal{L}_{A,b,c}$.

1.1.1. Definition. *We say that μ satisfies the equation*

$$L^*_{A,b}\mu = 0 \tag{1.1.1}$$

in Ω if $a^{ij}, b^i \in L^1_{\rm loc}(|\mu|)$ and one has

$$\int_\Omega L_{A,b}\varphi(x)\,\mu(dx) = 0 \quad \forall\varphi\in C_0^\infty(\Omega).$$

For a given measure ν on Ω the equation

$$L^*_{A,b}\mu = \nu$$

is defined similarly. Finally, for a given function $c\in L^1_{\rm loc}(|\mu|)$ the equation $L^*_{A,b,c}\mu = \nu$ is defined in the same sense.

The equation

$$\mathcal{L}^*_{A,b}\mu = 0$$

is defined similarly, but it requires additional assumptions either on a^{ij} or on μ (which will be made in appropriate places).

For a fixed domain Ω we set

$$\mathcal{M}^{A,b}_{\rm ell} := \Bigl\{\mu\in\mathcal{P}(\Omega):\ L^*_{A,b}\mu = 0\Bigr\}. \tag{1.1.2}$$

In what follows we shall deal with the case where the matrix A is symmetric and nonnegative, but this is not needed for the definition (unlike for most of the results).

In general, equation (1.1.1) can fail to have nonzero solutions in the class of bounded measures (take $\Omega=\mathbb{R}^1$, $A=1$, $b=0$), it can have many solutions even in the class of probability measures, and its solutions can be quite singular (e.g., if $A=0$ and $b=0$, then any measure is a solution). However, even in the generality under consideration some positive information is available.

The one-dimensional case is much simpler than the multidimensional case.

1.1.2. Proposition. *Let $d=1$ and let Ω be an interval. Suppose that $A>0$ on Ω. Then, any measure μ satisfying the equation $L^*_{A,b,c}\mu=\nu$ is absolutely continuous with respect to Lebesgue measure and has a density ϱ of the form $\varrho=\varrho_0/A$, where ϱ_0 is absolutely continuous on every compact subinterval in Ω.*

If $A=1$, $c=0$, $\nu=0$, $\Omega=(-1,1)$, and b is locally Lebesgue integrable on $(-1,1)$, then

$$\varrho(x) = \biggl(k_1 + k_2\int_0^x \exp\Bigl(-\int_0^s b(t)\,dt\Bigr)\,ds\biggr)\exp\int_0^x b(t)\,dt,$$

where k_1 and k_2 are constants.

PROOF. We have the identity

$$\int_\Omega (A\varphi'' + b\varphi' + c\varphi)\, d\mu = \int_\Omega \varphi\, d\nu \quad \forall \varphi \in C_0^\infty(\Omega),$$

which can be written as the equality

$$(A\mu)'' - (b\mu)' + c\mu = \nu$$

in the sense of distributions. Hence the distributional derivative of $(A\mu)' - b\mu$ is a locally bounded measure. This shows that the distributional derivative of $A\mu$ is a locally bounded measure as well. Hence $A\mu$ is absolutely continuous and has a density ϱ_0. Therefore, μ is absolutely continuous. Now it is seen from our reasoning that $(A\mu)' - b\mu$ is a function of locally bounded variation, hence the distributional derivative of $A\mu$ is a locally integrable function, so that ϱ_0 admits a locally absolutely continuous version. In the case $A = 1$, $c = 0$, $\nu = 0$, we arrive at the equation $\mu'' - (b\mu)' = 0$, whence $\mu' - b\mu = k_2$ for some constant k_2. If b is locally Lebesgue integrable, this equation can be explicitly solved. □

Even in this simplest one dimensional case we observe that a solution μ can fail to have a continuous density if A is positive but not continuous. We actually see that in the case of nondegenerate A (i.e., $\det A \neq 0$) the regularity of solution is essentially the regularity of A. We shall see below that in higher dimensions the picture is similar, although the proofs involve much deeper techniques. Another simple observation is that without any assumptions of nondegeneracy on A we obtain that the measure $A\cdot\mu$ is absolutely continuous. A highly nontrivial analogue of this is valid also in the multidimensional case.

Let us consider one more instructive example.

1.1.3. EXAMPLE. Let $\varrho \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ and let $\mu = \varrho\, dx$. Then μ satisfies the equation $L^*_{\mathrm{I},b}\mu = 0$ with

$$b := \frac{\nabla\varrho}{\varrho}, \quad \text{where } b(x) := 0 \text{ whenever } \varrho(x) = 0.$$

Indeed, $|b|$ is locally $|\mu|$-integrable. For any $\varphi \in C_0^\infty(\mathbb{R}^d)$, by the integration by parts formula we have

$$\int [\Delta\varphi + \langle b, \nabla\varphi\rangle]\, \varrho\, dx = \int [-\langle \nabla\varphi, \nabla\varrho\rangle + \langle b, \nabla\varphi\rangle\varrho]\, dx = 0$$

since $b\varrho = \nabla\varrho$ almost everywhere due to the fact that $\nabla\varrho$ vanishes almost everywhere on the set $\{\varrho = 0\}$.

The mapping $\nabla\varrho/\varrho$ is called the *logarithmic gradient* of the measure μ or the density ϱ; it is defined if $\varrho \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$.

In this example, we can even choose ϱ to be infinitely differentiable, but b can be quite singular with respect to Lebesgue measure. For instance, given a proper closed subset $Z \subset \mathbb{R}^d$, we can find a probability density $\varrho \in C^\infty(\mathbb{R}^d)$ with $Z = \{\varrho = 0\}$; in this way one can even obtain b that is not Lebesgue locally integrable on a closed set of positive Lebesgue measure. The simplest example of a singularity is this:

$$\varrho(x) = x^2 \exp(-x^2/2)/\sqrt{2\pi}, \quad b(x) = x + 2/x.$$

Now we formulate our main results on the existence of densities; see [**33**] for the proofs.

1.1.4. THEOREM. *Suppose that the matrices $A(x)$ are symmetric and nonnegative. Let μ be a locally finite Borel measure on Ω such that $a^{ij} \in L^1_{loc}(\Omega, \mu)$, and for some $C > 0$ one has*

$$\int_\Omega a^{ij}\partial_i\partial_j\varphi\, d\mu \le C(\sup_\Omega|\varphi| + \sup_\Omega|\nabla\varphi|) \tag{1.1.3}$$

for all nonnegative $\varphi \in C_0^\infty(\Omega)$. Then the following assertions are true.

(i) *If μ is nonnegative, then* $(\det A)^{1/d}\mu$ *has a density in $L^{d'}_{\rm loc}(\Omega, dx)$.*

(ii) *If A is locally Hölder continuous and* $\det A > 0$*, then μ has a density which belongs to $L^r_{\rm loc}(\Omega, dx)$ for every $r \in [1, d')$.*

We do not know whether assertion (i) remains true for signed measures.

1.1.5. COROLLARY. *Let μ be a locally finite signed Borel measure on Ω and let a^{ij}, b^i, $c \in L^1_{\rm loc}(\Omega, \mu)$. Assume that*

$$\int_\Omega (L_{A,b}\varphi + c\varphi)\, d\mu \le 0 \quad \textit{for all nonnegative} \quad \varphi \in C_0^\infty(\Omega). \tag{1.1.4}$$

Then the following assertions are true.

(i) *If μ is nonnegative, then* $(\det A)^{1/d}\mu$ *has a density in $L^{d'}_{\rm loc}(\Omega, dx)$.*

(ii) *If A is locally Hölder continuous and* $\det A > 0$*, then μ has a density which belongs to $L^r_{\rm loc}(\Omega, dx)$ for every $r \in [1, d')$.*

In particular, the above statements are true if (1.1.1) *holds.*

In assertion (ii) of this corollary one cannot expect the density of μ to be Hölder continuous since for $d = 1$ and $A = 1$ one can take $\mu(dx) = \exp(\int_0^x b(t)\, dt)\, dx$ with a suitable function b.

The previous corollary has the following important generalization.

1.1.6. COROLLARY. *Let μ and ν be two locally finite signed Borel measures on Ω and let a^{ij}, b^i, $c \in L^1_{\rm loc}(\Omega, \mu)$. Assume that*

$$\int_\Omega \Big[L_{A,b}\varphi + c\varphi\Big]\, d\mu = \int_\Omega \varphi\, d\nu \quad \textit{for all nonnegative} \quad \varphi \in C_0^\infty(\Omega). \tag{1.1.5}$$

Then the following assertions are true.

(i) *If μ is nonnegative, then* $(\det A)^{1/d}\mu$ *has a density in $L^{d'}_{\rm loc}(\Omega, dx)$.*

(ii) *If A is locally Hölder continuous and* $\det A > 0$*, then μ has a density which belongs to $L^r_{\rm loc}(\Omega, dx)$ for every $r \in [1, d')$.*

1.1.7. REMARK. (i) Assertions (i) of Theorem 1.1.4, Corollary 1.1.5, and Corollary 1.1.6 for nonnegative measures extend to the case when μ is a σ-finite nonnegative Borel measure on Ω (not necessarily locally bounded). Indeed, (1.1.3), (1.1.4), and (1.1.5) make sense also for σ-finite μ provided that $a^{ij}, b^i, c \in L^1_{\rm loc}(\Omega, \mu)$. One can find a probability measure μ_0 such that $\mu = f\,\mu_0$, where f is a positive Borel function. Let

$$a_0^{ij} := fa^{ij},\ b_0^i := fb^i,\ c_0 := fc,\ A_0 = (a_0^{ij})_{i,j\le d},\ b_0 = (b_0^i)_{i\le d}.$$

Clearly, $a_0^{ij}, b_0^i, c_0 \in L^1_{\rm loc}(\mu_0)$ and μ_0 satisfies the hypotheses of the above mentioned assertions with A_0, b_0, and c_0 in place of A, b, and c. Hence the measure $(\det A_0)^{1/d}\mu_0$ has a density $\varrho \in L^{d'}_{\rm loc}(\Omega, dx)$. Since $(\det A_0)^{1/d} = f(\det A)^{1/d}$, this means that $(\det A)^{1/d}\mu$ has the same density.

(ii) Assume that the hypotheses of Corollary 1.1.5(ii) are fulfilled. Let $B_{R_1}(x_0)$ be a ball in Ω of radius $R_1 > 0$ centered at x_0. Then, for every $R < R_1$ and $r < d'$, there exists N depending only on R_1, R, r, d, $\inf_{B_{R_1}} \det A$, $\sup_{i,j} \sup_{B_{R_1}} |a^{ij}|$, and the Hölder norm of A on B_{R_1} such that the density ϱ of μ satisfies

$$\|\varrho\|_{L^r(B_R)} \le N \big\| 1 + |b| + |c| \big\|_{L^1(B_{R_1}, \mu)}.$$

In addition, for fixed d, the number N is a locally bounded function of the indicated quantities. This follows from the proof of Theorem 1.1.4 in [**33**].

1.2. Local properties of solutions

We now proceed to the regularity results. Throughout the rest of this section we assume that $A(x)$ is symmetric and positive and $A(x)$ *is continuous in* x. By the Sobolev embedding theorem, the continuity assumption is automatically satisfied for some version of A if $a^{ij} \in W^{p,1}_{\mathrm{loc}}$, where $p > d$. We do not discuss here the case of smooth coefficients and possibly degenerate A under Hörmander's condition and its analogs in the Malliavin calculus (see references in [**22**]).

First of all we recall a classical result.

1.2.1. THEOREM. *Suppose that the functions a^{ij}, b^i, c are infinitely differentiable and* $\det A > 0$ *in* Ω. *If ν has a infinitely differentiable density, then any solution of the equation $L^*_{A,b,c}\mu = \nu$ possesses an infinitely differentiable density.*

Next we consider the case where the coefficients are only Hölder continuous. The following result was proved in [**165**].

1.2.2. THEOREM. *Suppose that the coefficients a^{ij}, b^i, c are locally Hölder continuous in Ω and* $\det A > 0$. *Then any solution μ of the equation $L^*_{A,b,c}\mu = 0$ has a locally Hölder continuous density.*

Note that the solutions in [**165**] were a priori locally integrable functions, but by the above results the theorem remains valid for measures. It would be interesting to study the case where only the coefficients a^{ij} are Hölder continuous. The continuity of all coefficients does not guarantee the Hölder continuity of a solution even if $d = 1$ and $A > 0$. However, it is not clear whether densities of solutions are continuous in the case where the coefficients are just continuous and A is nondegenerate.

We now proceed to the most difficult case where the diffusion coefficient is somewhat better than Hölder continuous, but is not smooth, and we want to have some Sobolev regularity of the density of solutions. One of the reasons why this is important is that, having established the Sobolev regularity of our solution, we can rewrite the equation $L^*_{A,b,c}\mu = 0$ for μ as a classical equation for its density ϱ in the sense of weak solutions: indeed, integrating by parts we find that

$$\int_\Omega [a^{ij}\partial_{x_i}\varrho\partial_{x_j}\varphi + \partial_{x_i}a^{ij}\partial_{x_j}\varphi\varrho + b^i\partial_{x_i}\varphi\varrho + c\varrho]\, dx = 0$$

for all $\varphi \in C_0^\infty(\Omega)$.

1.2.3. THEOREM. *Let $d \ge 2$, $p \ge d$, $q \in (1,\infty)$, $R_1 > 0$, let $a^{ij} \in W^{p,1}(B_{R_1})$ and let $A \ge \lambda \mathrm{I}$, where $\lambda > 0$. Then there exist numbers $R_0 > 0$ and $N_0 > 0$ with the following properties. Let $R < R_0$ and let μ be a measure of finite total variation on B_R such that for any $\varphi \in C_0^2(B_R) := C^2(\overline{B_R}) \cap \big\{u\colon\ u|_{\partial B_R} = 0\big\}$ we have*

$$\left| \int_{B_R} a^{ij}\partial_i\partial_j\varphi\, d\mu \right| \le N ||\nabla\varphi||_{L^q(B_R)} \qquad (1.2.1)$$

with N independent of φ. Furthermore, assume one of the following:

a) *$p > d$ or*

b) *$p = d > q'$ and $\mu \in \bigcup_{r>1} L^r(B_R)$.*

Then $\mu \in W_0^{q'\wedge p,1}(B_R)$ *(where we identify* μ *with its density) and*

$$||\mu||_{W_0^{q'\wedge p,1}(B_R)} \le N_0.$$

In addition, the radius R_0 *depends only on* p, q, d, λ, R_1, $\|a^{ij}\|_{W^{p,1}(B_{R_1})}$, *and the rate of decrease of* $\|\nabla a^{ij}\|_{L^d(B_R)}$ *as* $R \to 0$, *and* N_0 *depends on the same quantities and* N.

1.2.4. Remark. The proof of this theorem actually shows that if μ has compact support in B_{R_1} and (1.2.1) holds for all $\varphi \in C_0^\infty(B_{R_1})$, then $\mu \in W_0^{q'\wedge p,1}(B_R)$ for some $R < R_1$. Moreover, even without the assumption of compactness of support, one can show that $\mu \in W_{\rm loc}^{q'\wedge p,1}(B_R)$, but this requires some extra work.

This theorem at once yields a certain low regularity of solutions to our elliptic equations.

1.2.5. Corollary. *Suppose that* $p > d \ge 2$, $a^{ij} \in W_{\rm loc}^{p,1}(\Omega)$, $\det A > 0$, *and* μ *satisfies the equation* $L_{A,b}^*\mu = 0$, *where* $b \in L_{\rm loc}^r(\mu)$ *for some* $r > 1$. *Then* μ *has a density in the class* $W_{\rm loc}^{\alpha,1}(\Omega)$ *for each* $\alpha < dr/(dr - r + 1)$.

More can be obtained if b better integrable.

1.2.6. Theorem. *Let* $p > d$, $r \in (p', \infty)$, $\mu = \varrho\, dx$, $\varrho \in L_{\rm loc}^r(\Omega, dx)$, *and* $a^{ij} \in W_{\rm loc}^{p,1}(\Omega)$, *and let either* $\beta \in L_{\rm loc}^p(\Omega, dx)$ *or* $\beta \in L_{\rm loc}^p(\Omega, \mu)$. *Suppose that* A^{-1} *is locally bounded. Assume that for every* $\varphi \in C_0^\infty(\Omega)$ *we have*

$$\left|\int_\Omega a^{ij}\partial_i\partial_j\varphi\,\mu(dx)\right| \le \int_\Omega (|\varphi| + |\nabla\varphi|)|\beta|\,|\mu|(dx).$$

Then $\varrho \in W_{\rm loc}^{p,1}(\Omega)$.

We shall say that A is locally uniformly nondegenerate if A^{-1} is locally bounded.

1.2.7. Remark. It should be noted that the assertion of Theorem 1.2.6 is valid under the following alternative assumptions on a^{ij}, β, μ: μ is a locally bounded Borel measure on Ω (without assumptions on its density), A^{-1} is locally bounded, $\beta \in L_{\rm loc}^p(\Omega, \mu)$ or $\beta \in L_{\rm loc}^1(\Omega, \mu) \bigcap L_{\rm loc}^p(\Omega, dx)$. This follows by Theorem 1.1.4.

1.2.8. Corollary. *Let* μ *be a locally finite Borel measure on* Ω. *Let* A^{-1} *be locally bounded in* Ω *with* $a^{ij} \in W_{\rm loc}^{p,1}(\Omega)$, *where* $p > d$, *and let either* (i) b^i, $c \in L_{\rm loc}^p(\Omega, dx)$ *or* (ii) b^i, $c \in L_{\rm loc}^p(\Omega, \mu)$. *Assume that, for every* $\varphi \in C_0^\infty(\Omega)$, *one has*

$$\int_\Omega \Big[a^{ij}\partial_i\partial_j\varphi + b^i\partial_i\varphi + c\varphi\Big]\,d\mu = 0,$$

where b^i *and* c *are also assumed to be locally* μ*-integrable in case* (i). *Then* μ *has a density in* $W_{\rm loc}^{p,1}(\Omega)$ *that is locally Hölder continuous.*

The following result, proved in [**43**], is a useful modification of the previous theorem.

1.2.9. Theorem. *Let* $p > d$, $r \in (p', \infty)$, *and let* μ *be a measure on* B_R *with a density* $\varrho \in L_{\rm loc}^r(B_R)$. *Let* $a^{ij} \in W_{\rm loc}^{p,1}(B_R)$, $\beta_1 \in L_{\rm loc}^p(B_R)$, *and* $\beta_2 \in L_{\rm loc}^p(\mu)$, *where* A^{-1} *is locally bounded on* B_R. *Assume that, for every* $\varphi \in C_0^\infty(B_R)$, *we have*

$$\left|\int_{B_R} a^{ij}\partial_{x_i}\partial_{x_j}\varphi\,d\mu\right| \le \int_{B_R}\Big(|\varphi| + |\nabla_x\varphi|\Big)(|\beta_1\varrho| + |\beta_2\varrho|)\,dx.$$

Then $\varrho \in W_{\rm loc}^{p,1}(B_R)$, *hence* ϱ *has a locally Hölder continuous version.*

1.2.10. Corollary. *Let μ be a locally finite Borel measure on B_R. Let A^{-1} be locally bounded on B_R with $a^{ij} \in W^{p,1}_{\text{loc}}(B_R)$, where $p > d$, $\partial_{x_i} a^{ij} \in L^p_{\text{loc}}(\mu)$, and let b^i, $c \in L^p_{\text{loc}}(\mu)$. Suppose that*

$$\int_{B_R} \Bigl[a^{ij}\partial_{x_i}\partial_{x_j}\varphi + \partial_{x_i} a^{ij}\partial_{x_j}\varphi + b^i\partial_{x_i}\varphi + c\varphi\Bigr]\, d\mu = 0 \quad \forall\, \varphi \in C_0^\infty(B_R).$$

Then μ has a density in $W^{p,1}_{\text{loc}}(B_R)$ that is locally Hölder continuous.

Bensoussan [**16**, Ch. II, Theorem 5.5] proved $W^{p,1}_{\text{loc}}$-regularity for probability measures on $\mathbb{R}^d$ satisfying the equation $L^*_{\mathrm{I},b}\mu = 0$ with $b = b_1 + b_2$, where b_1 is Lipschitzian and b_2 is bounded, under the additional assumption of the existence of a certain Lyapunov function.

The classical Harnack inequality yields the following.

1.2.11. Corollary. *In the situation of Corollary 1.2.8, let $b^i \in L^p_{\text{loc}}(\Omega, dx)$, let μ be nonnegative, and let ϱ be its continuous density. Then, for every compact set K contained in a connected open set U with compact closure in Ω, one has*

$$\sup_K \varrho \le C \inf_K \varrho,$$

where the constant C depends only on $\|a^{ij}\|_{W^{p,1}(U)}$, $\|b\|_{L^p(U)}$, $\inf_U \det A$, and K (if K is a ball, then the dependence on K is through the distance from K to ∂U). In particular, ϱ does not vanish in U if it is not identically zero in U.

The dependence of C on the indicated quantities will be studied below in §1.9.

1.2.12. Corollary. *Suppose that in the situation of Corollary 1.2.8 one has $b^i \in L^p_{\text{loc}}(\Omega, dx)$ and that $c \le 0$. Assume also that the continuous density ϱ of μ is strictly positive on the boundary of a bounded open set $U \subset \overline{U} \subset \Omega$. Then ϱ is strictly positive on U.*

Clearly, the assumption that $b^i \in L^p_{\text{loc}}(\Omega, dx)$ in Corollary 1.2.11 and Corollary 1.2.12 cannot be replaced by the alternative assumption that $b^i \in L^p_{\text{loc}}(\Omega, \mu)$ from Corollary 1.2.8. Indeed, it suffices to take $b = \nabla\varrho/\varrho$ such that ϱ is a probability density which has zeros, but $|b| \in L^p(\mu)$ (for example, we can take ϱ which behaves like $\exp(-x^{-2})$ in a neighborhood of the origin).

Corollary 1.2.8 can be generalized as follows.

1.2.13. Corollary. *Let $p > d$, let $a^{ij} \in W^{p,1}_{\text{loc}}(\Omega)$, b^i, f^i, $c \in L^p_{\text{loc}}(\Omega)$, $i, j = 1, \ldots, d$, and let A^{-1} be locally bounded in Ω. Assume that μ is a locally finite Borel measure on Ω such that b^i, $c \in L^1_{\text{loc}}(\Omega, \mu)$ and, for every function $\varphi \in C_0^\infty(\Omega)$, one has*

$$\int_\Omega \Bigl[a^{ij}\partial_i\partial_j\varphi + b^i\partial_i\varphi + c\varphi\Bigr]\, d\mu = \int_\Omega f^i\partial_i\varphi\, dx.$$

Then μ has a density in $W^{p,1}_{\text{loc}}(\Omega)$.

1.2.14. Remark. It is easily seen that in Corollary 1.2.8 one cannot omit the hypotheses that A^{-1} is locally bounded and $a^{ij} \in W^{p,1}_{\text{loc}}$. Indeed, if A and b vanish at a point x_0, then Dirac's measure at x_0 satisfies our elliptic equation. In particular, if it is not given in advance that μ is absolutely continuous, then one cannot take an arbitrary Lebesgue version of A. In order to see that μ cannot be more regular than A (which has already been noted in the introduction), let us take a probability measure μ with a smooth density that satisfies $L^*_{\mathrm{I},b_0}\mu = 0$, e.g., let μ be the standard Gaussian measure and $b_0(x) = -x$. If now g is any Borel function with $1 \le g \le 2$, then the measure $g\cdot\mu$ satisfies the equation $L^*_{A,b}\mu = 0$ with $A = g^{-1}\mathrm{I}$ and $b = g^{-1}b_0$.

In particular, we obtain an example, where A and b are Hölder continuous and A is uniformly nondegenerate, but the density of μ is not weakly differentiable. Also, the condition $p > d$ is essential for the membership of μ in a Sobolev class even if $A = \mathrm{I}$ (see the example below). However, if μ is a probability measure on $\mathbb{R}^d$, then the condition $|b| \in L^2(\mathbb{R}^d, \mu)$ implies that $\mu = \varrho\, dx$ with $\varrho \in W^{1,1}(\mathbb{R}^d)$ and $|\nabla\varrho|^2/\varrho \in L^1(\mathbb{R}^d)$ (see §1.8).

1.2.15. Example. Let $d > 3$ and

$$L^*F = \Delta F + \alpha\partial_{x^i}(x^i|x|^{-2}F) - F,$$

where $\alpha = d - 3$. Then the function $F(x) = (e^r - e^{-r})r^{-(d-2)}$, $r = |x|$, is locally Lebesgue integrable and $L^*F = 0$ in the sense of distributions, but F is not in $W^{2,1}_{\mathrm{loc}}(\mathbb{R}^d)$. Here

$$b(x) = -\alpha x|x|^{-2} = \nabla(|x|^{-\alpha})/|x|^{-\alpha}$$

and $|b| \in L^{d-\varepsilon}_{\mathrm{loc}}(\mathbb{R}^d)$ for all $\varepsilon > 0$. In a similar way, if the term $-F$ is omitted in the equation above, then the function $F(x) = r^{-(d-3)}$ has the same properties.

Proof. Observe that $\partial_{x^i}F$, $\partial_{x^i}\partial_{x^j}F$ are locally Lebesgue integrable. Hence the equation $L^*F = 0$ follows easily from the equation

$$f'' + \frac{(d-1+\alpha)}{r}f' + \alpha\frac{d-2}{r^2}f - f = 0$$

on $(0, \infty)$, which is satisfied for the function $f(r) = (e^r - e^{-r})r^{-(d-2)}$. It remains to note that F, ∇F, and ΔF are locally Lebesgue integrable, since $f(r)r^{d-1}$, $f'(r)r^{d-1}$, and $f''(r)r^{d-1}$ are locally bounded, but ∇F is not Lebesgue square-integrable at the origin. If $d \geq 6$, then F is also not Lebesgue square-integrable at the origin. In the case without the term $-F$ in the equation similar calculations show that $F(x) = r^{-(d-3)}$ has the same properties. □

The following theorem is helpful in the study of local properties of densities, in particular, for control of various constants that appear in local estimates. It precises a particular case of a more general result which was formulated by Morrey in his book [**136**, p. 156] in a way that is not completely correct (with $\Omega' = \Omega$). The statement below with $\Omega' = \Omega$ would be false, for example, for the Laplace equation in a ball. A proof of Morrey's estimate with an investigation of the dependence of the constant on the coefficients was given in [**155**] with the same inaccuracy as in [**136**]. In fact, the reasoning from [**155**] yields the estimate given below as explained in [**157**], but an estimate with $\Omega = \Omega'$ is possible only for solutions with zero boundary condition on a domain with a sufficiently regular boundary. We note that in the existing applications of Morrey's theorem, only the correct assertion proven below is actually used, although in some papers it is formulated with the indicated inaccuracy (see, e.g., [**38**] and [**33**]). For the proof, see [**42**] or [**157**], where a more general fact is established.

1.2.16. Theorem. *Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded domain, $A \in C^{0,\delta}(\Omega)$, where $\delta > 0$, and there is a number $\alpha > 0$ such that $A(x) \geq \alpha \cdot \mathrm{I}$ for all $x \in \Omega$. Let $h^i \in L^q(\Omega)$, where $q > d$. If a function u from $W^{q,1}(\Omega)$ satisfies the equation $\partial_{x_i}(a^{ij}\partial_{x_j}u + h^i) = 0$ in the weak sense on Ω, then for every domain Ω' with closure in Ω the following estimate holds*:

$$\|u\|_{W^{q,1}(\Omega')} \leq C\big(\|u\|_{L^q(\Omega)} + \|h\|_{L^q(\Omega)}\big), \quad h := (h^1, \ldots, h^d),$$

where the number C depends only on $d, q, \alpha, \Omega, \Omega'$, and $\|A\|_{C^{0,\delta}}$.

Moreover, if $u \in W^{q,1}(\Omega)$ *and* $\partial_{x_i}(a^{ij}\partial_{x_j}u - b^i u + h^i) = 0$ *in the weak sense and* $b^i \in L^q(\Omega)$*, then a stronger estimate*

$$\|u\|_{W^{q,1}(\Omega')} \le C\bigl(\|u\|_{L^1(\Omega)} + \|h\|_{L^q(\Omega)}\bigr)$$

holds, where C *depends only on* $d, q, \alpha, \Omega, \Omega'$*,* $\|A\|_{C^{0,\delta}}$*, and* $\|b^i\|_{L^q(\Omega)}$.

1.2.17. Remark. Let a Borel measure μ on a ball B_{R_1} of radius $R_1 > 0$ in Ω satisfy the equation $L^*_{A,b}\mu + c\mu = f\,dx$, where $a^{ij}, b^i, c \in L^1(B_{R_1}, \mu)$, $f \in L^p(B_{R_1})$, and

$$A \ge \lambda_1 \mathrm{I}, \quad \|a^{ij}\|_{W^{p,1}(B_{R_1})} \le \lambda_2, \quad \|b^i\|_{L^p(B_{R_1})} + \|c\|_{L^p(B_{R_1})} \le \lambda_2, \ p > d.$$

Then it follows from the above results that, for any $R < R_1$, the measure μ has a continuous density $u \in W^{p,1}(B_R)$ that satisfies the equation

$$\partial_i(a^{ij}\partial_j u + \partial_j a^{ij} u - b^i u) + cu = f.$$

Therefore,

$$\begin{aligned}\|u\|_{W^{p,1}(B_R)} &\le \Lambda_{d,R_1}(\lambda_1, \lambda_2, R)\Bigl[\|u\|_{L^1(B_{R_1})} + \|f\|_{L^p(B_{R_1})}\Bigr] \\ &\le \Lambda_{d,R_1}(\lambda_1, \lambda_2, R)\Bigl[\mathrm{mes}(B_{R_1}) \sup_{B_{R_1}} |u| + \|f\|_{L^p(B_{R_1})}\Bigr], \qquad (1.2.2)\end{aligned}$$

where $\Lambda_{d,R_1}(\lambda_1, \lambda_2, R)$ is a locally bounded function on $(0, +\infty)^2 \times (0, R_1)$ and is independent of u.

1.2.18. Proposition. *Let* $A_k = (a^{ij}_k)$ *be a sequence of continuous mappings on* $\mathbb{R}^d$ *with values in the space of symmetric matrices and let* $b_k = (b^i_k)$ *be a sequence of Borel vector fields on* $\mathbb{R}^d$*. Suppose that for every ball* $B_r \subset \mathbb{R}^d$ *there exist numbers* $c_r > 0$*,* $\alpha_r > 0$*, and* $p = p_r > d$ *such that*

$$A_k \ge c_r \mathrm{I}, \quad \|a^{ij}_k\|_{W^{p,1}(B_r)} + \|b^i_k\|_{L^p(B_r)} \le \alpha_r \quad \text{for all } i, j, k.$$

Assume that there exist probability measures μ_k *on* $\mathbb{R}^d$ *such that* $L^*_{A_k,b_k}\mu_k = 0$*. Then the measures* μ_k *have continuous strictly positive densities that are uniformly Hölder continuous on every ball. If, in addition, the sequence* $\{\mu_k\}$ *is uniformly tight, then it has compact closure with respect to the variation norm, and every measure* μ *in its closure has a continuous strictly positive density that belongs to* $W^{p,1}(B_r)$ *for every* $r > 0$.

Proof. It follows from our hypotheses and Theorem 1.2.6 that the measures μ_k have continuous densities f_k. Since the functions f_k are probability densities, we obtain by (1.2.2) that, for every $r > 0$, the sequence $\{f_k\}$ is bounded in $W^{p,1}(B_r)$. By the Sobolev embedding theorem (see, e.g., [**17**]), $\{f_k\}$ is uniformly Hölder continuous on B_r, in particular, has compact closure with respect to the sup-norm. If $\{\mu_k\}$ is uniformly tight, then some subsequence $\{\mu_{k_i}\}$ converges weakly to some probability measure μ. Passing to a subsequence once again we may assume that the functions f_{k_i} converge uniformly on compacts and are uniformly bounded in $W^{p,1}(B_r)$ for each $r > 0$. Hence μ has a density $f \in W^{p,1}(B_r)$. Then we obtain a continuous and strictly positive version of f. Therefore, the probability measures μ_{k_i} converge to μ in the variation norm. This reasoning applies to any subsequence in $\{\mu_k\}$, whence we obtain the desired conclusion. □

1.2.19. Remark. The above proposition can be generalized as follows. Let Ω be an open set in $\mathbb{R}^d$ and let Ω be a union of increasing open sets Ω_k such that the closure of Ω_k is compact and contained in Ω_{k+1}. Let μ_k be probability measures on Ω_k satisfying the equations $L^*_{A_k,b_k}\mu_k = 0$ on Ω_k, where each A_k is a continuous mapping on Ω_k with values in the set of nonnegative symmetric matrices, the mappings A_k are uniformly bounded on compact sets in the $W^{p,1}$-norm with some $p > 1$, the mappings A_k^{-1} are uniformly bounded on compact sets, and Borel vector fields b_k on the sets Ω_k are uniformly bounded in the $L^p(\mathbb{R}^d)$-norm on compact sets. Then the analogue of the statement of the previous proposition is true. Moreover, the same is true for Riemannian manifolds of dimension d.

In the case where the diffusion matrix A is infinitely differentiable a somewhat more special result holds. In its proof we use the following well-known lemma.

1.2.20. Lemma. *Suppose that $a^{ij} \in C^\infty(\Omega)$ and $\det A > 0$.*

(i) *Let $r \in (-\infty, \infty)$ and $p > 1$. If u is a distribution such that $L_A u \in H^{p,r}_{\text{loc}}(\Omega)$, then $u \in H^{p,r+2}_{\text{loc}}(\Omega)$; also if $u \in H^{p,r}_{\text{loc}}(\Omega)$, then $\partial_{x_i} u \in H^{p,r-1}_{\text{loc}}(\Omega)$, $1 \le i \le d$.*

(ii) *We have $H^{p,1}_{\text{loc}}(\Omega) \subset L^{dp/(d-p)}_{\text{loc}}(\Omega)$ and $L^p_{\text{loc}}(\Omega) \subset H^{dp/(d-p),-1}_{\text{loc}}(\Omega)$ whenever $1 < p < d$, and $H^{p,1}_{\text{loc}}(\Omega) \subset C^{1-d/p}_{\text{loc}}(\Omega)$ if $p > d$, so that in the latter case elements of $H^{p,1}_{\text{loc}}(\Omega)$ are locally bounded. In addition, for $q > p > 1$ we have $L^p_{\text{loc}}(\Omega) \subset H^{q,d/q-d/p}_{\text{loc}}(\Omega)$.*

(iii) *If μ is a locally bounded Radon measure on Ω, then $\mu \in H^{p,-m}_{\text{loc}}(\Omega)$ whenever $p > 1$ and $m > d(1-1/p)$.*

Proof. Assertion (i) is well-known. Specifically, its first statement is a well-known elliptic regularity result and the second statement follows from the boundedness of Riesz's transforms. Assertion (ii) is just the Sobolev embedding theorem. Assertion (iii) follows from this embedding theorem, since for regular subdomains U of Ω one has $H^{q,m}(U) \subset C(\overline{U})$ if $qm > d$ whence by duality the space $H^{q/(q-1),-m}(U) = [H^{q,m}(U)]^*$ contains all finite measures on U. □

We formulate the following result for $d > 1$ just because the case $d = 1$ is elementary and has already been discussed. In addition, we include in the formulation some assertions which follow also from the already mentioned results (the proof is direct and does not use the results above).

1.2.21. Theorem. *Under the same assumptions on A as in the lemma, let $d \ge 2$ and let μ, ν be (possibly, signed) Radon measures on Ω. Let a mapping $b = (b^i)\colon \Omega \to \mathbb{R}^d$ and a function $c\colon \Omega \to \mathbb{R}$ be such that $|b|,\ c \in L^1_{\text{loc}}(\Omega, \mu)$. Suppose that $L^*_{A,b,c}\mu = \nu$. Then the following assertions are true.*

(i) *One has $\mu \in H^{p,1-d(p-1)/p-\varepsilon}_{\text{loc}}(\Omega)$ for any $p \ge 1$ and any $\varepsilon > 0$. Here $1 - d(p-1)/p > 0$ if $p \in [1, \frac{d}{d-1})$ and, in particular, μ admits a density $F \in L^p_{\text{loc}}(\Omega)$ for any $p \in [1, \frac{d}{d-1})$.*

(ii) *If $|b| \in L^\gamma_{\text{loc}}(\Omega, \mu)$, $c \in L^{\gamma/2}_{\text{loc}}(\Omega, \mu)$ and $\nu \in L^{d/(d-\gamma+2)}_{\text{loc}}(\Omega)$ where $d \ge \gamma > 1$, then $F := d\mu/dx \in H^{p,1}_{\text{loc}}(\Omega)$ for any $p \in [1, d/(d-\gamma+1))$. In particular, $F \in L^p_{\text{loc}}(\Omega)$ for any $p \in [1, d/(d-\gamma))$, where we set $\frac{d}{d-\gamma} := \infty$ if $\gamma = d$.*

(iii) *If $\gamma > d$ and either*

(a) *$|b| \in L^\gamma_{\text{loc}}(\Omega)$ and $c, \nu \in L^{\gamma d/(d+\gamma)}_{\text{loc}}(\Omega)$,*

or

(b) *$|b| \in L^\gamma_{\text{loc}}(\Omega, \mu)$, $c \in L^{\gamma d/(d+\gamma)}_{\text{loc}}(\Omega, \mu)$, and $\nu \in L^{\gamma d/(d+\gamma)}_{\text{loc}}(\Omega)$,*

then μ admits a density $F \in H^{\gamma,1}_{\text{loc}}(\Omega)$. In particular, $F \in C^{1-d/\gamma}_{\text{loc}}(\Omega)$.

Proof. (i) We have in the sense of distributions

$$\Delta\mu = \partial_{x^i}(b^i\mu) - c\mu + \nu \tag{1.2.3}$$

on Ω. Here by Lemma 1.2.20(iii) the right-hand side belongs to $H^{p,-m-1}_{\mathrm{loc}}(\Omega)$ if $m > d(1-1/p)$. By Lemma 1.2.20(i) we conclude $\mu \in H^{p,-m+1}_{\mathrm{loc}}(\Omega)$, which leads to the result after substituting $m = d(1-1/p)+\varepsilon$.

Before we prove (ii) and (iii) we need some preparations. Fix $p_1 > 1$ and assume that $F = d\mu/dx \in L^{p_1}_{\mathrm{loc}}(\Omega)$. Such a number p_1 exists by (i). Set

$$r := r(p_1) := \frac{\gamma p_1}{\gamma - 1 + p_1} \tag{1.2.4}$$

and observe that owing to the inequalities $1 < \gamma$ and $p_1 > 1$, we have $1 < r < \gamma$. Next, starting with the formula

$$|bF|^r = (|b||F|^{1/\gamma})^r |F|^{r-r/\gamma}$$

and using Hölder's inequality (with $s = \frac{\gamma}{r} > 1$ and $t := \frac{s}{s-1} = \frac{\gamma}{\gamma - r}$) and the relations $|b||F|^{1/\gamma} \in L^{\gamma}_{\mathrm{loc}}(\Omega)$ and $F \in L^{p_1}_{\mathrm{loc}}(\Omega)$, we obtain that $b^iF \in L^r_{\mathrm{loc}}(\Omega)$. By Lemma 1.2.20(i) one has

$$b^iF \in H^{r,0}_{\mathrm{loc}}(\Omega), \quad (b^iF)_{x^i} \in H^{r,-1}_{\mathrm{loc}}(\Omega). \tag{1.2.5}$$

(ii) Set

$$q := q(p_1) := \frac{\gamma p_1}{\gamma - 2 + 2p_1} \vee 1, \tag{1.2.6}$$

and note that $q > 1 \Leftrightarrow \gamma > 2 \Leftrightarrow q < \frac{\gamma}{2}$, in particular, $q < \gamma$ in any case. Hence repeating the above argument with the triple c, $\gamma/2$, q in place of $|b|$, γ, r, we obtain that

$$cF \in L^q_{\mathrm{loc}}(\Omega). \tag{1.2.7}$$

Fix $p_1 > 1$ such that $F := \frac{d\mu}{dx} \in L^{p_1}_{\mathrm{loc}}(\Omega)$ and let r, q be as in (1.2.4), (1.2.6), correspondingly. Since $\gamma \le d$, we have $q < d$, which by (1.2.7) and assertions (ii) and (iii) of Lemma 1.2.20 implies that $cF \in H^{dq/(d-q),-1}_{\mathrm{loc}}(\Omega)$ if $q > 1$ and that $cF \in H^{s,-1}_{\mathrm{loc}}(\Omega)$ for any $s \in (1, d/(d-1))$ if $q = 1$.

It turns out that if $p_1 < d/(d-\gamma)$, then

$$cF \in H^{r,-1}_{\mathrm{loc}}(\Omega). \tag{1.2.8}$$

Indeed, if $q > 1$, then (1.2.8) follows from the fact that if $p_1 \in (1, d/(d-\gamma))$, then the inequality $r \le dq/(d-q)$ holds. If $q = 1$, then $\gamma \le 2$ and (1.2.8) follows from the fact that $r < d/(d-\gamma+1) \le d/(d-1)$ for $p_1 < d/(d-\gamma)$.

Finally by Lemma 1.2.20 (ii) we have $\nu \in H^{d/(d-\gamma+1),-1}_{\mathrm{loc}}(\Omega)$ if $\gamma > 2$ and $\nu \in H^{s,-1}_{\mathrm{loc}}(\Omega)$ for any $s \in (1, d/(d-1))$ if $\gamma \le 2$. In the same way as above, $\nu \in H^{r,-1}_{\mathrm{loc}}(\Omega)$ whenever $1 < p_1 < d/(d-\gamma)$. This along with (1.2.5) and (1.2.8) shows that the right-hand side of (1.2.3) is now in $H^{r,-1}_{\mathrm{loc}}(\Omega)$. By Lemma 1.2.20(i) we have

$$\mu \in H^{r,1}_{\mathrm{loc}}(\Omega) \tag{1.2.9}$$

and by Lemma 1.2.20(ii) we have $F \in L^{p_2}_{\mathrm{loc}}(\Omega)$, where

$$p_2 := \frac{dr}{d-r} = \frac{d\gamma p_1}{d\gamma - d + (d-\gamma)p_1} =: f(p_1).$$

Thus, we obtain

$$p_1 \in \left(1, \frac{d}{d-\gamma}\right) \text{ and } F \in L^{p_1}_{\mathrm{loc}}(\Omega) \Longrightarrow F \in L^{f(p_1)}_{\mathrm{loc}}(\Omega).$$

One can easily check that $p_2 = f(p_1) > p_1$ if $p_1 < d/(d-\gamma)$, and that the only positive solution of the equation $q = f(q)$ is $q = d/(d-\gamma)$. Therefore, by taking p_1 in $(1, d/(d-1))$, which is possible by (i), and by defining $p_{k+1} = f(p_k)$ we obtain an increasing sequence of numbers $p_k \uparrow d/(d-\gamma)$, which implies that $F \in L^p_{\mathrm{loc}}(\Omega)$ for any $p < d/(d-\gamma)$.

But as $p_k \nearrow d/(d-\gamma)$, the sequence of numbers $r(p_k)$ defined according to (1.2.4) increases to the limit

$$\frac{\gamma d/(d-\gamma)}{\gamma - 1 + d/(d-\gamma)} = \frac{d}{d-\gamma+1} \ .$$

By (1.2.9) this proves (ii).

(iii) First we consider case (b). By the last assertion in (ii) we have $F \in L^{p_1}_{\mathrm{loc}}(\Omega)$ for any finite $p_1 > 1$. Let $r := r(p_1)$ be defined as in (1.2.4). Then $1 < r < \gamma$ and (1.2.5) holds. Set

$$q := q(p_1) := \frac{\frac{d\gamma}{d+\gamma}\, p_1}{\frac{d\gamma}{d+\gamma} - 1 + p_1} \ . \tag{1.2.10}$$

If $2 \le d < \gamma$, then $\frac{d\gamma}{d+\gamma} > 1$. Therefore, since $p_1 > 1$, it follows that $1 < q < \frac{d\gamma}{d+\gamma}$. Hence repeating the arguments that led to (1.2.5) with the triple c, $\frac{d\gamma}{d+\gamma}$, q in place of $|b|$, γ, r we obtain $cF \in L^q_{\mathrm{loc}}(\Omega)$, thus $cF \in H^{dq/(d-q),-1}_{\mathrm{loc}}(\Omega)$ by assertion (ii) of the lemma. Observe that as $p_1 \to \infty$ we have

$$r \uparrow \gamma, \quad q \uparrow d\gamma/(d+\gamma), \quad dq/(d-q) \uparrow \gamma.$$

Therefore, combining this with our assumption that ν is contained in the class $L^{d\gamma/(d+\gamma)}_{\mathrm{loc}}(\Omega)$, which by assertion (ii) of the lemma is contained in $H^{\gamma,-1}_{\mathrm{loc}}(\Omega)$, and by taking p_1 large enough, we see that the right-hand side of (1.2.3) is in $H^{\gamma-\varepsilon,-1}_{\mathrm{loc}}(\Omega)$ for any $\varepsilon \in (0, \gamma-1)$. By Lemma 1.2.20(ii) we conclude that $F \in H^{\gamma-\varepsilon,1}_{\mathrm{loc}}(\Omega)$. Since $\gamma > d$, the function F is locally bounded. Now we see that above we can take $p_1 = \infty$ and therefore the right-hand side of (1.2.3) is in $H^{\gamma,-1}_{\mathrm{loc}}(\Omega)$, which by assertion (i) of the lemma gives us the desired result.

In the remaining case (a) we take $p_1 > \gamma/(\gamma-1)$ and assume that $F \in L^{p_1}_{\mathrm{loc}}(\Omega)$. Then instead of (1.2.4) and (1.2.10) we define

$$r := r(p_1) := \frac{\gamma p_1}{\gamma + p_1}, \quad q := q(p_1) := \frac{\frac{d\gamma}{d+\gamma}\, p_1}{\frac{d\gamma}{d+\gamma} + p_1} \vee 1 \tag{1.2.11}$$

and observe that since $p_1 > \gamma/(\gamma-1)$ we have $r > 1$, which (because of the relation $p_1^{-1} + \gamma^{-1} = r^{-1}$) allows us to apply Hölder's inequality starting with $|bF|^r = |b|^r |F|^r$ to conclude that (1.2.5) holds. Since $c \in L^1_{\mathrm{loc}}(\Omega, \mu)$,

$$\frac{d\gamma}{d+\gamma} > 1 \quad \text{and} \quad \left(\frac{d\gamma}{d+\gamma}\right)^{-1} + p_1^{-1} = q^{-1},$$

we also have that $cF \in L^q_{\mathrm{loc}}(\Omega)$. Obviously, $q < d$. As in part (ii) this yields that $cF \in H^{dq/(d-q),-1}_{\mathrm{loc}}(\Omega)$ if $q > 1$ and $cF \in H^{s,-1}_{\mathrm{loc}}(\Omega)$ for any $s \in (1, d/(d-1))$ if $q = 1$. We assert that (1.2.8) holds (with $r = r(p_1)$ as in (1.2.11)) for all $p_1 > \gamma/(\gamma-1)$, $p_1 \ne d\gamma/(d\gamma - d - \gamma)$.

Indeed, if $q > 1$, then $dq/(d-q) = r$. If $q = 1$, then $p_1 \le d\gamma/(d\gamma - d - \gamma)$. But since $p_1 \ne d\gamma/(d\gamma - d - \gamma)$, we have $p_1 < d\gamma/(d\gamma - d - \gamma)$, which is equivalent to the inequality $r < d/(d-1)$.

Thus, since $\nu \in L^{d\gamma/(d+\gamma)}_{\mathrm{loc}}(\Omega) \subset H^{\gamma,-1}_{\mathrm{loc}}(\Omega) \subset H^{r,-1}_{\mathrm{loc}}(\Omega)$, because $r < \gamma$, assertion (i) in the lemma yields the following:

$$\Big(p_1 > \frac{\gamma}{\gamma-1},\ p_1 \ne \frac{d\gamma}{d\gamma - d - \gamma},\ F \in L^{p_1}_{\mathrm{loc}}(\Omega)\Big) \Longrightarrow F \in H^{r,1}_{\mathrm{loc}}(\Omega). \tag{1.2.12}$$

If $r < d$, then the latter in turn implies by assertion (ii) in Lemma 1.2.20 that $F \in L^{p_2}_{\rm loc}(\Omega)$. Let us summarize what has been shown:

$$\Bigl(p_1 > \tfrac{\gamma}{\gamma-1},\ p_1 \neq \tfrac{d\gamma}{d\gamma-d-\gamma},\ r := \tfrac{\gamma p_1}{\gamma+p_1} < d,\ F \in L^{p_1}_{\rm loc}(\Omega)\Bigr) \Longrightarrow F \in L^{p_2}_{\rm loc}(\Omega), \tag{1.2.13}$$

where

$$p_2 := \frac{dr}{d-r} = \frac{d\gamma p_1}{d\gamma-(\gamma-d)p_1} > \frac{d\gamma}{d\gamma-(\gamma-d)}p_1.$$

Also, note that $\gamma/(\gamma-1) < d/(d-1) < \frac{d\gamma}{\gamma d-d-\gamma}$, so that by (i) we can find a number p_1 to start with. Then starting with p_1 close enough to $d/(d-1)$, by iterating (1.2.13) we always increase p by some factor greater than $d\gamma/(d\gamma-(\gamma-d)) > 1$. While doing this, we can obviously choose the first p so that the iterated numbers p will never be equal to $d\gamma/(d\gamma-d-\gamma)$ and the corresponding numbers r will not coincide with d. After several steps we shall come to the situation where $r > d$, and then we conclude from (1.2.12) that F is locally bounded (one cannot keep iterating (1.2.13) infinitely because of the restriction $r < d$). As in case (b), we can now easily complete the proof. □

Example 1.2.15 shows that assertion (iii) of this theorem may fail if $\gamma > d$ is replaced by $\gamma = d-\varepsilon$. Then F does not even need to be in $H^{2,1}_{\rm loc}(\Omega)$.

1.2.22. Proposition. *Suppose that the hypotheses of Corollary 1.2.8 are fulfilled. Let μ be some positive measure on Ω satisfying the equation $L^*_{A,b,c}\mu = 0$. Then, any other solution μ_0 can be written as $\mu_0 = f\cdot\mu$, where $f \in W^{p,2}_{\rm loc}(\Omega)$.*

Proof. Suppose first that $d > 1$. Then $p > 2$. We know that μ and μ_0 have continuous densities ϱ and ϱ_0, respectively, in the class $W^{p,1}_{\rm loc}(\Omega)$ and that ϱ has no zeros in Ω. Set $f = \varrho/\varrho_0$. Then $\mu_0 = f\cdot\mu$ and $f \in W^{p,1}_{\rm loc}(\Omega)$. We have $a^i := \sum_{j=1}^d \partial_{x_j} a^{ij} \in L^p_{\rm loc}(\Omega)$. Set $a := (a^i)$. Let us verify that f satisfies the elliptic equation

$$a^{ij}\varrho\partial_{x_i}\partial_{x_j} f + \langle\nabla f, 2\varrho a + 2A\nabla\varrho - \varrho b\rangle = 0$$

in the sense of weak solutions in the class $W^{p,1}_{\rm loc}(\Omega)$, i.e., in the sense of the identity

$$\int [-\varphi\partial_{x_i}(a^{ij}\varrho)\partial_{x_j} f - \langle\varrho A\nabla f, \nabla\varphi\rangle + \langle\nabla f, 2\varrho a + 2A\nabla\varrho - \varrho b\rangle\varphi]\,dx = 0 \tag{1.2.14}$$

for all $\varphi \in C_0^\infty(\Omega)$. This will yield the desired inclusion $f \in W^{p,2}_{\rm loc}(\Omega)$, since we have $\varrho a^{ij} \in W^{p,1}_{\rm loc}(\Omega)$, ϱA is nondegenerate, $a\varrho$, $b\varrho$, $c\varrho \in L^p_{\rm loc}(\Omega)$. In order to establish (1.2.14) we observe that the equality $L^*_{A,b,c}\mu_0 = L^*_{A,b,c}\mu = 0$ and the integration by parts formula give the identities

$$\int [-\partial_{x_i}(a^{ij}\varrho f)\partial_{x_j}\varphi + \langle f\varrho b, \nabla\varphi\rangle + c\varrho f\varphi]\,dx = 0, \tag{1.2.15}$$

$$\int [-\partial_{x_i}(a^{ij}\varrho)\partial_{x_j}\varphi + \langle\varrho b, \nabla\varphi\rangle + c\varrho\varphi]\,dx = 0$$

for all $\varphi \in C_0^\infty(\Omega)$. Since $a^{ij}, \varrho, f \in W^{p,1}_{\rm loc}(\Omega)$ and $p > 2$, it follows that the second relation remains true for all functions φ of the form $\varphi = f\psi$ with $\psi \in C_0^\infty(\Omega)$. This yields the identity

$$\int [-\partial_{x_i}(a^{ij}\varrho)f\partial_{x_j}\varphi - \partial_{x_i}(a^{ij}\varrho)\varphi\partial_{x_j} f + \langle\varrho b, f\nabla\varphi\rangle + \langle\varrho b, \varphi\nabla f\rangle + c\varrho f\varphi]\,dx = 0$$

for all $\varphi \in C_0^\infty(\Omega)$. Subtracting the last equality from (1.2.15) and differentiating the products by the Leibnitz formula we arrive at (1.2.14). In the case $d = 1$ this reasoning does not apply if $p < 2$, but in this case a simple direct proof works: we have $(Af\varrho)' = f\varrho b + \psi$ and $(A\varrho)' = b\varrho + k$, where ψ is the indefinite integral of $cf\varrho$ and k is constant. Then $f' = (\psi - kf)(A\varrho)^{-1}$. □

We have already seen from Harnack's inequality that the continuous version of a density of a nonnegative measure μ satisfying the equation $L^*_{A,b}\mu = 0$ on a ball Ω is strictly positive provided that $a^{ij} \in W^{p,1}_{\text{loc}}(\Omega)$, A is strictly positive, and $b^i \in L^p_{\text{loc}}(\Omega)$ with some $p > d$. Unlike the Sobolev regularity result, the latter condition on b cannot be replaced by the alternative assumption $b^i \in L^p_{\text{loc}}(\mu)$. However, the next result of S.V. Shaposhnikov [**156**] gives a sufficient condition for the strict positivity of densities entirely in terms of integrability with respect to μ.

1.2.23. Theorem. *Let μ be a nonzero nonnegative Borel measure on a ball Ω satisfying the equation $L^*_{\mathrm{I},b}\mu = 0$, where* $\exp(\varepsilon|b|) \in L^1_{\text{loc}}(\mu)$ *for some $\varepsilon > 0$. Then μ has a continuous strictly positive density.*

In §1.9 below we discuss lower bounds for densities under similar global exponential integrability assumptions.

Let us mention several works containing various results related to weak elliptic equations for measures (or equations for functions that are satisfied by densities of solutions to equations for measures): [**120**], [**15**], [**76**], [**107**], [**108**], [**109**], [**98**]. Finally, let us note that the local regularity results can be used to strengthen the results on smoothness of invariant measures with respect to a parameter obtained in [**178**].

1.3. Some a priori estimates

In this section we establish certain general a priori estimates which will be useful in proving the existence of solutions. These estimates provide bounds on the integrals of certain given functions with respect to unknown solutions.

1.3.1. Theorem. *Let $V \geq 0$ be a continuous function of the class $W^{2,1}_{\text{loc}}(\mathbb{R}^d)$, let $U := \{V < c\}$ be bounded, and let μ be a nonnegative Borel measure on U satisfying the equation $L^*_{A,b}\mu = 0$, where $A = (a^{ij})$ is a mapping on U with values in the space of positive symmetric linear operators on $\mathbb{R}^d$, $a^{ij} \in W^{p,1}_{\text{loc}}(U)$ for some $p > d$, $b = (b^i)$ is a Borel mapping from U to $\mathbb{R}^d$ such that either $|b| \in L^p_{\text{loc}}(\mu)$ or $|b| \in L^p_{\text{loc}}(U)$. Suppose that there exist a Borel function $\Psi \in L^1(\mu)$ and a Borel function $\Phi \geq 0$ such that*

$$LV \leq \Psi - \Phi \quad \mu\text{-a.e. on } U.$$

Then one has

$$\int_U \Phi\, d\mu \leq \int_U \Psi\, d\mu.$$

If μ is a probability measure on $\mathbb{R}^d$ satisfying the equation $L^*_{A,b}\mu = 0$ on the whole space, where A and b satisfy the above assumptions locally, and if $LV \leq \Psi - \Phi$ on $\mathbb{R}^d$, where all the sets $\{V \leq c\}$ are compact, then

$$\int_{\mathbb{R}^d} \Phi\, d\mu \leq \int_{\mathbb{R}^d} \Psi\, d\mu.$$

Proof. It suffices to prove our assertion for bounded Φ. Indeed, once this is done, we consider the functions $\Phi_k = \min(\Phi, k)$, for which $LV \leq \Psi - \Phi_k$, and then apply Fatou's theorem. Assuming that Φ is bounded, given $\varepsilon > 0$, we can find a compact set $K \subset U$ such that $\mu(U) < \mu(K) + \varepsilon$ and

$$\int_U \Phi\, d\mu < \int_K \Phi\, d\mu + \varepsilon.$$

By the continuity of V there is a number $r < c$ such that

$$K \subset \{V < r\} \subset \{V \le r\} \subset U.$$

We know that μ has a continuous density ϱ. Hence there is a number $r_1 < r$ such that $K \subset \{V < r_1\}$ and

$$\int_{\{r_1 \le V \le r\}} |\langle \nabla V, b\rangle|\, d\mu < \varepsilon.$$

In the case $d > 1$ this follows by the Cauchy–Bunyakowskii inequality, and if $d = 1$, then V' is locally bounded. Let us take the function φ on the real line such that $\varphi(t) = t$ if $t \le r_1$, $\varphi(t) = (r_1 + r)/2$ if $t \ge r$, $\varphi''(t) = -1/|r - r_1|$ on $[r_1, r]$, and $\varphi''(t) = 0$ outside $[r_1, r]$. Then $0 \le \varphi' \le 1$, $\varphi'' \le 0$, and

$$L_{A,b}(\varphi \circ V) = \varphi''(V)\langle A\nabla V, \nabla V\rangle + \varphi'(V) L_{A,b} V,$$

which vanishes outside $\{V \le r\}$ since $\varphi \circ V$ is constant outside $\{V \le r\}$, and $L_{A,b}(\varphi \circ V) = L_{A,b} V$ on $\{V \le r_1\}$. Since $\varphi \circ V \in W^{2,1}_{\text{loc}}(U)$ and $\varphi \circ V$ is constant outside $\{V \le r\}$, we have

$$\int_U L_{A,b}(\varphi \circ V)\, d\mu = 0.$$

Taking into account that $\varphi''(V) \le 0$ and $0 \le \varphi'(V) \le 1$, we obtain that s

$$\begin{aligned}\int_{\{V \le r_1\}} [\Psi - \Phi]\, d\mu \ge \int_{\{V \le r_1\}} L(\varphi \circ V)\, d\mu &= -\int_{\{r_1 < V \le r\}} L(\varphi \circ V)\, d\mu \\ &\ge -\int_{\{r_1 < V \le r\}} |\langle \nabla V, b\rangle|\, d\mu \ge -\varepsilon,\end{aligned}$$

whence

$$\int_{\{V \le r_1\}} \Phi\, d\mu \le \int_{\{V \le r_1\}} \Psi\, d\mu + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary and the numbers r and r_1 can be chosen as close to c as we like, the desired estimate is proven. The assertion concerning the whole space follows at once since we can take increasing sets $\{V < j\}$ whose union is $\mathbb{R}^d$. □

A function V is called compact if all the sets $\{V \le c\}$ are compact. For a continuous function V on $\mathbb{R}^d$ this is equivalent to $\lim\limits_{|x| \to \infty} V(x) = +\infty$. We shall call a function V *quasi-compact* if the space can be represented as the union of increasing compact sets $\{V \le c_k\}$ for some increasing sequence of numbers c_k. For example, any even continuous function on the real line which is increasing on $[0, +\infty)$ to some number c but does not assume the value c is quasi-compact.

1.3.2. Theorem. *Suppose that we are given mappings $A_k = (a_k^{ij})$ on $\mathbb{R}^d$ with values in the space of positive symmetric linear operators on $\mathbb{R}^d$ and Borel vector fields $b_k = (b_k^i)$ on $\mathbb{R}^d$ such that, for every $R > 0$, there exist $\alpha_R > d$, $\beta_R > 0$, and $\gamma_R > 0$ for which*

$$\kappa_R := \sup_k \int_{|x| \le R} \Big[|b_k(x)|^{\alpha_R} + \sum_{i,j=1}^d \|\nabla a_k^{ij}(x)\|^{\alpha_R} \Big]\, dx < \infty, \tag{1.3.1}$$

$$\beta_R \mathrm{I} \le A_k(x) \le \gamma_R \mathrm{I} \quad \forall\, k,\ \forall\, x \in \{y\colon\ |y| \le R\}. \tag{1.3.2}$$

Suppose also that there exists a continuous quasi-compact (*e.g., compact*) *function* $V \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^d)$ *such that*

$$\lim_{|x|\to\infty} \sup_k L_{A_k,b_k} V(x) = -\infty. \tag{1.3.3}$$

Let $\{\mu_k\}$ *be a sequence of probability measures on* $\mathbb{R}^d$ *such that*

$$L^*_{A_k,b_k}\mu_k = 0.$$

Set $U_R := \{V < R\}$. *Then the following assertions are true.*

(i) *The measures* μ_k *admit continuous densities* ϱ_k *such that, for every fixed* $R > 1$, *the functions* $\varrho_k|_{U_R}$ *are uniformly bounded, uniformly Hölder continuous, and uniformly bounded in* $W^{\alpha_R,1}(U_{R-1})$.

(ii) *One has* $\inf\limits_k \inf\limits_{x\in U_R} \varrho_k(x) > 0$ *for every* $R > 0$.

(iii) *The sequence* $\{\mu_k\}$ *is relatively weakly compact on* $\mathbb{R}^d$.

Furthermore, suppose that for every k *a probability measure* μ_k *is defined on* $U_k = \{V < c_k\}$, *where* $c_k \uparrow \infty$, *and satisfies the equation* $L^*_{A_k,b_k}\mu_k = 0$ *only in* U_k. *Then assertions* (i)–(iii) *remain valid with the following changes*: *only those* k *with* $c_k > R$ *are considered in* (i) *and* (ii).

Proof. Assertions (i) and (ii) follow directly from the results in §1.2. In order to prove (iii) let us take a compact set E such that

$$L_{A_k,b_k}V(x) \le -1, \quad \forall k,\ \forall x \notin E.$$

Let $\Psi_k := |L_{A_k,b_k}V| I_E$ and $\Phi_k := -(L_{A_k,b_k}V) I_{\mathbb{R}^d\setminus E}$. Then $L_{A_k,b_k}V \le \Psi_k - \Phi_k$ and $\Phi_k \ge 0$ since $\Phi_k = -L_{A_k,b_k}V$ outside E. Theorem 1.3.1 yields

$$\int_{\mathbb{R}^d} \Phi_k\, d\mu_k \le \int_{\mathbb{R}^d} \Psi_k\, d\mu_k.$$

It follows from our assumptions that

$$S := \sup_k \int_E |L_{A_k,b_k}V|\, d\mu_k < \infty.$$

Therefore, the previous estimate gives

$$\int_{\mathbb{R}^d\setminus E} |L_{A_k,b_k}V|\, d\mu_k \le S,$$

whence we find that

$$\sup_k \int_{\mathbb{R}^d} |L_{A_k,b_k}V|\, d\mu_k \le 2S.$$

Therefore, given $C > 0$, we can find a number R such that $L_{A_k,b_k}V \le -C$ outside $\{V \le R\}$ for all k. Hence for all k we have

$$\mu(\mathbb{R}^d\setminus\{V \le R\}) \le 2S/C,$$

which means the uniform tightness of $\{\mu_k\}$ and, consequently, relative weak compactness of $\{\mu_k\}$.

Let us prove the last assertion in which each measure μ_k is defined only on the set $\{V < c_k\}$. There is no change in the proof of assertions (i) and (ii) except that now one has to consider only k with $c_k > R$. The same proof of (iii) works also in this case, we just consider sufficiently large k such that $E \subset \{V < c_k\}$. □

1.3.3. Corollary. *In the situation of Theorem* 1.3.2, *the sequence* $\{\mu_k\}$ *is relatively compact in the variation norm.*

Proof. It follows from Theorem 1.3.2 that every subsequence of this sequence has a subsequence $\{\nu_k\}$ such that $\{\nu_k\}$ converges weakly to some probability measure ν and the continuous densities ϱ_k of the measures ν_k converge locally uniformly to a continuous limit ϱ_0. Clearly, ϱ_0 serves as a density for ν. Therefore, we have $\|\varrho_k - \varrho_0\|_{L^1(\mathbb{R}^d)} \to 0$, which is our claim. □

1.3.4. Corollary. *Suppose that the hypotheses of Theorem* 1.3.2 *are satisfied except for* (1.3.3)*, which is replaced by*

$$\lim_{|x|\to\infty} \sup_k \langle b_k(x), x\rangle = -\infty. \tag{1.3.4}$$

Assume, in addition, that $\sup_R \gamma_R < \infty$*. Then, for every* $p \geq 1$*, one has*

$$M_p := \sup_k \int_{\mathbb{R}^d} \langle x, x\rangle^p \, \mu_k(dx) < \infty.$$

Proof. We may assume that $p \in \mathbb{N}$. By (1.3.4) the function $V(x) = \langle x, x\rangle^p$ satisfies condition (1.3.3) since

$$\begin{aligned} L_{A_k,b_k} V(x) &= 4p(p-1)\langle x, x\rangle^{p-2} \sum_{i,j=1}^d a_k^{ij}(x) x_i x_j \\ &+2p\langle x, x\rangle^{p-1} \text{trace}\, A_k(x) + 2p\langle x, x\rangle^{p-1}\langle x, b_k(x)\rangle \\ &\leq \langle x, x\rangle^{p-1}\Big[4p(p-1)\|A_k(x)\| + 2pd\|A_k(x)\| + 2p\langle x, b_k(x)\rangle\Big]. \end{aligned}$$

By (1.3.4) and the uniform boundedness of $\|A_k(x)\|$ there is $R > 0$ and two positive numbers c_1 and c_2 such that

$$L_{A_k,b_k} V(x) \leq c_1 - c_2\langle x, x\rangle^{p-1} \quad \forall\, k \geq 1$$

whenever $|x| > R$. It follows from our assumptions and the above theorem that

$$\sup_k \int_{\{|x|\leq R\}} |L_{A_k,b_k} V(x)| \, \mu_k(dx) < \infty.$$

Hence $M_{p-1} < \infty$ by Theorem 1.3.1. □

1.3.5. Remark. (i) We observe that the proof of tightness of $\{\mu_k\}$ only used (1.3.3). Hence assertion (ii) of Theorem 1.3.2 is valid under (1.3.3) for measures μ_k satisfying equations $L^*_{A_k,b_k}\mu_k = 0$ with arbitrary coefficients A_k and b_k that are locally μ_k-integrable.

(ii) Clearly, conditions (1.3.1) and (1.3.2) can be imposed locally, i.e., for every point x there exist a neighborhood W of x and numbers $\alpha_W > d$, $\beta_W > 0$, and $\gamma_W > 0$ such that (1.3.1) and (1.3.2) hold with W, α_W, β_W, and γ_W in place of U_R, α_R, β_R, and γ_R, respectively.

1.4. Existence of solutions

Here we present sufficient conditions for the existence of solutions expressed in terms of Lyapunov functions, hence verifiable explicitly in terms of the coefficients of our operators. This method goes back to R.Z. Hasminskii [**92**], [**93**], and the presented results are borrowed from [**38**] with some improvements.

1.4.1. Theorem. *Suppose that* $A = (a^{ij})_{i,j\leq d}$ *is a mapping on* $\mathbb{R}^d$ *with values in the space of nonnegative symmetric linear operators on* $\mathbb{R}^d$ *and let* $b\colon \mathbb{R}^d \to \mathbb{R}^d$

be a Borel mapping such that, for every ball U_R, there exist numbers $\alpha_R > d$ and $\beta_R > 0$ such that $\beta_R \mathrm{I} \le A(x)$ for all $x \in U_R$ and

$$a^{ij}|_{U_R} \in W^{\alpha_R,1}(U_R), \quad |b|\big|_{U_R} \in L^{\alpha_R}(U_R).$$

Assume, in addition, that there exists a quasi-compact (e.g., compact) function $V \in C^2(\mathbb{R}^d)$ such that

$$L_{A,b}V(x) \to -\infty \quad \text{as } |x| \to \infty. \tag{1.4.1}$$

Then there exists a Borel probability measure μ on $\mathbb{R}^d$ such that

$$|b| \in L^1_{\mathrm{loc}}(\mu) \quad \text{and} \quad L^*_{A,b}\mu = 0.$$

Moreover, the measure μ admits a continuous strictly positive density ϱ such that $\varrho|_{U_{R-1}} \in W^{\alpha_R,1}(U_{R-1})$ for every $R > 1$.

Proof. By our condition on V we have increasing domains $U_k = \{V < c_k\}$ with compact closures of U_k in U_{k+1}, whose union is the whole space. Let us first find a sequence of nonnegative functions p_k that are not identically zero and satisfy our equation in the domains U_k. To this end, take a ball B_k containing U_k. It is known (see [**172**, Theorem 3.2 and the remark at the end of §3]) that, for every k, there exists a nonnegative function $p_k \in W^{\alpha_k,1}(B_k)$ such that $p_k = 1$ on the boundary of B_k (in the sense explained in [**172**]) while in the interior of B_k one has

$$\partial_{x_i}\Big(a^{ij}\partial_{x_j}p_k - b^i p_k + (\partial_{x_j}a^{ij})p_k\Big) = 0 \tag{1.4.2}$$

in the weak sense. The fact that these solutions are nonnegative follows by the weak maximum principle [**173**, Theorem 7]. Our case corresponds to $\gamma = b^i = b = 0$ in the theorem cited (see also [**172**, Exercise 8.1]). It is worth mentioning that the existence of functions p_k with the aforementioned properties can be derived from the solvability of the boundary problem for smooth coefficients. Indeed, there exists a sequence of smooth mappings $A_m = (a^{ij}_m)^d_{i,j=1}$ on a larger ball $B'_k \supset B_k$ with values in the space of the positive symmetric linear operators on $\mathbb{R}^d$ such that the mappings A_m are uniformly nondegenerate and converge to A in the norm of the Sobolev class $W^{\alpha_k,1}(B'_k)$. In addition, there exists a sequence of smooth mappings $b_m = (b^i_m)^d_{i=1}$ convergent to b in $L^{\alpha_k}(B'_k, \mathbb{R}^d)$. Let us denote by f_m the solution of the boundary value problem

$$\partial_{x_i}\Big(a^{ij}_m(x)\partial_{x_j}f_m - b^i_m f_m + \partial_{x_j}a^{ij}_m f_m\Big) = 0$$

with condition $f_m = 1$ on the boundary of B'_k. Such solutions exist according to [**89**, Theorem 8.3 and the remark at the end of §8.2]. Let κ_m be such that the minimum of $\widetilde{f}_m := \kappa_m f_m$ on B'_k is 1. According to the results in §1.2, the sequence $\{\widetilde{f}_m\}$ is uniformly bounded on B_k. In addition, the restrictions of $\widetilde{f}_m$ have uniformly bounded norms in the space $W^{\alpha_k,1}(B_k)$. Choosing a subsequence from $\{\widetilde{f}_m\}$ that is uniformly convergent on B_k, we conclude that its limit after normalization can be taken for p_k since it satisfies equation (1.4.2) in B_k in the weak sense and is estimated by 1 from below. It follows from our reasoning that $p_k \in W^{\alpha_k,1}(U_k)$. Multiplying the solutions p_{n_k} by positive constants and letting $p_k = 0$ outside U_k, we obtain probability measures μ_k satisfying the elliptic equation $L^*_{A,b}\mu_k = 0$ in U_k. According to Theorem 1.3.2, the sequence $\{\mu_k\}$ contains a subsequence $\{\mu_{k_i}\}$ convergent in the variation norm to a probability measure μ with a strictly positive continuous density ϱ. In addition, for every ball B, there is k_0 such that the densities

ϱ_{k_i} of the measures μ_{k_i} with $k_i \geq k_0$ converge uniformly on B. Therefore, for any $\varphi \in C_0^\infty(\mathbb{R}^d)$, we obtain by Lebesgue's dominated convergence theorem that

$$0 = \lim_{i\to\infty} \int_{\mathbb{R}^d} L_{A,b}\varphi(x)\varrho_{k_i}(x)\, dx = \int_{\mathbb{R}^d} L_{A,b}\varphi(x)\varrho(x)\, dx.$$

Therefore, $\mu = \varrho\, dx$ is the desired solution. □

1.4.2. Corollary. *The assertion of the previous theorem is true if A is a locally Lipschitzian uniformly bounded mapping with values in the space of positive symmetric operators on $\mathbb{R}^d$ and b is a Borel mapping such that $|b| \in L^\alpha_{\mathrm{loc}}(\mathbb{R}^d)$, where $\alpha > d$, and $\lim\limits_{|x|\to\infty} \langle b(x), x\rangle = -\infty$.*

Proof. It suffices to take $V(x) = \langle x, x\rangle$. □

For example, the previous corollary applies to the situation where $A = \mathrm{I}$ and $b(x) = -k(x)x$, where $k \in L^\alpha_{\mathrm{loc}}(\mathbb{R}^d)$ is such that

$$k(x)|x|^2 \to +\infty \quad \text{as} \quad |x| \to \infty.$$

More generally, if $A = \mathrm{I}$, then it suffices to have a weaker relation

$$\limsup_{|x|\to\infty} \langle x, x\rangle^{\gamma-1}\Big[2(\gamma-1) + d + \langle b(x), x\rangle\Big] = -\infty$$

for some $\gamma \geq 1$ (then the function $V(x) = \langle x, x\rangle^\gamma$ can be used).

Another example covered by Theorem 1.4.1 is this: $A = \mathrm{I}$ and for some $\gamma \geq 1$ one has $\langle b(x), x\rangle \leq -r < -d + 2 - 2\gamma$ outside some ball (of course, we assume that $|b|$ is locally integrable to a power bigger than d). Then we take the function $V(x) = \langle x, x\rangle^\gamma$. The same coercivity condition was assumed in [**175**] for locally bounded b (in order to get the equation from [**175**] one should take $A = \mathrm{I}/2$ in our case). Note that, as shown in [**175**], one can reduce to this case the more general case where the diffusion term is continuous, nondegenerate, and satisfies certain uniform bounds, provided that the drift is locally bounded and coercive and that the corresponding stochastic differential equation has a unique weak solution.

The next result was proved in [**38**].

1.4.3. Corollary. *Let $A = (a^{ij})$ be a continuous mapping on $\mathbb{R}^d$ with values in the space of positive symmetric linear operators on $\mathbb{R}^d$ and let b be a Borel vector field on $\mathbb{R}^d$. Suppose that there exists a quasi-compact function $V \in C^2(\mathbb{R}^d)$ such that* (1.4.1) *is fulfilled. Then the following assertions are true.*

(i) *If b is continuous, then there exists a probability measure μ satisfying the equation $L^*_{A,b}\mu = 0$.*

(ii) *If $\det A > 0$ and b is locally bounded, then there exists a probability measure μ which has a density in the class $L^{d/(d-1)}_{\mathrm{loc}}(\mathbb{R}^d)$ and satisfies the equation $L^*_{A,b}\mu = 0$.*

1.4.4. Remark. It is obvious that in the case of nondegenerate A the Borel measurability of b can be replaced by its Lebesgue measurability and that our solution does not depend on a Lebesgue equivalent version of b because any solution has a density.

Let us observe that (1.4.1) holds if there are positive numbers c_1 and c_2 such that

$$LV(x) \leq c_1 - c_2 V(x) \quad \text{and} \quad \lim_{|x|\to\infty} V(x) = +\infty.$$

It should be noted that in general even for $A = \mathrm{I}$ the assumption that

$$\lim_{|x|\to\infty} L_{A,b}V(x) = -\infty$$

cannot be replaced by the weaker assumption that

$$L_{A,b}V(x) \le c < 0 \quad \text{outside some ball } U. \tag{1.4.3}$$

Indeed, let $d = 1$, $A = 1$, and $b(x) = -\operatorname{sign} x/(1 + |x|)$. We have $xb(x) \to -1$ as $|x| \to \infty$. Assume that μ is a probability measure satisfying the equation $L^*_{1,b}\mu = 0$. Then μ has a locally absolutely continuous density ϱ such that $\varrho'' + (b\varrho)' = 0$ in the sense of distributions, whence $\varrho' + b\varrho = 0$, since $b\varrho$ is integrable (it suffices to integrate over the intervals $[t_k, s_k]$, where $t_k \to -\infty$, $s_k \to +\infty$, $\varrho(t_k) \to 0$, $\varrho(s_k) \to 0$). In addition, $\varrho > 0$, so that $(\ln \varrho)' = -\ln(1 + |x|)'$, whence we obtain that $\varrho(x) = c/(1+|x|)$, which is a non-integrable function. In [**16**], the existence of an invariant probability is established under assumption (1.4.3) and the additional assumption that $|\nabla V|^2/V \le c$ for some number $c > 0$. However, in this case we can take a new Lyapunov function $W = (2V)^\alpha$, where $\alpha = 1 + c^{-1}$, which satisfies (1.4.1).

It is clear that the presented existence results can be modified for divergence form operators. For example, if the functions a^{ij} are Lipschitzian and $\langle b(x), x\rangle \le c_1 - c_2|x|\ln|x|$ outside some ball, then the extra term in the drift which appears when we write the equation in non-divergence form does not destroy condition (1.4.1).

For other sufficient existence conditions expressed in terms of A and b, see [**50**].

Note also that in the case where b is a gradient, more special sufficient conditions are available that guarantee the existence of a probability measure whose logarithmic gradient is b (see [**22**, §7.5]), but even in this case the general sufficient conditions presented above turn out to be very efficient.

1.5. Associated L^1-semigroups

Here we discuss semigroups associated with solutions to our elliptic equations for measures, along with relations between such equations and the proper invariance of measures with respect to their associated semigroups. We recall that in the Introduction we defined the concept of a measure invariant with respect to a semigroup $(T_t)_{t\ge 0}$ of bounded linear operators on the space $B_b(X)$ of bounded measurable functions. Similarly one defines invariance in the case where X is a topological space, μ is a Borel measure on X, and $(T_t)_{t\ge0}$ is a semigroup of bounded linear operators on the space $C_b(X)$ of bounded continuous functions. An obvious modification of the latter concept arises if the operators T_t are just operators from $C_b(X)$ to $L^\infty(\mu)$, not necessarily forming a semigroup. If for all bounded Borel functions f and g one has

$$\int_X T_t\, f(x)\, g(x)\, \mu(dx) = \int_X f(x)\, T_t\, g(x)\, \mu(dx),$$

then the semigroup is called *symmetric* and the measure μ is called *symmetric invariant*. It is clear that this case is characterized by the property that the generator L of the semigroup is symmetric in $L^2(\mu)$. If $(T_t)_{t\ge0}$ is the transition semigroup of a Markov process, then this process is called *μ-symmetric*. A process is called *symmetrizable* if there is a measure μ such that it is μ-symmetric.

We are interested in the case where the semigroup $(T_t)_{t\ge0}$ is in a certain sense generated by the elliptic operator $L_{A,b}$ for which there exists a probability measure μ satisfying the equation $L^*_{A,b}\mu = 0$. However, one should be very careful with a possible interpretation of the term "generated". We shall use instead the term "associated" and this will simply mean the following: the semigroup $(T_t)_{t\ge0}$ on the space of bounded Borel functions will be called associated with $L_{A,b}$ and μ if it extends to a strongly continuous semigroup on $L^1(\mu)$ whose generator coincides with $L_{A,b}$ on the class of smooth compactly supported functions. Typically, such a semigroup is not continuous on B_b or C_b. Moreover, in general it is not unique (we shall see such examples; however, a certain special associated semigroup will be singled out), and this is closely connected with the non-uniqueness of solutions to the equation $L^*_{A,b}\mu = 0$ in the class of probability measures. On the other hand, we shall find conditions that guarantee the uniqueness of an associated semigroup, and this uniqueness turns out to be equivalent to the proper invariance of μ with respect to its special associated semigroup in the case where the drift coefficient b is locally Lebesgue integrable to a power greater than the dimension of the space. In the latter case, the semigroup is indeed generated by $L_{A,b}$ in the classical sense that the closure of $(L_{A,b}, C_0^\infty)$ in $L^1(\mu)$ is the generator of the semigroup.

We recall that a bounded operator T on $L^1(\mu)$ is called *sub-Markovian* if one has $0 \le Tf \le 1$ whenever $f \in L^\infty(\mu)$ and $0 \le f \le 1$. If, in addition, $T1 = 1$, then T is called *Markovian*.

In general, the operators T_t^μ in the associated semigroup constructed below are sub-Markovian, not Markovian. In addition, μ is only sub-invariant for them, i.e.,

$$\int_X T_t^\mu f \, d\mu \le \int_X f \, d\mu, \quad f \in L^\infty(\mu), \ f \ge 0.$$

We observe that if $(T_t)_{t\ge0}$ is a semigroup on $L^1(\mu)$ and takes $L^\infty(\mu)$ to $L^\infty(\mu)$, then by the interpolation theorem it takes $L^p(\mu)$ to $L^p(\mu)$ for all $p \in [1, +\infty)$, and its adjoint semigroup also maps each $L^p(\mu)$ into itself.

First we discuss the case $A = \mathrm{I}$ studied in [**4**], where the proofs can be found.

1.5.1. Theorem. *Suppose that $A = \mathrm{I}$, $|b| \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ for some $p > d \ge 2$, and μ is a probability measure on $\mathbb{R}^d$ such that $L^*\mu = 0$, where $L = L_{\mathrm{I},b}$ and $\varrho := d\mu/dx$ is such that $|b - \nabla\varrho/\varrho| \in L^1(\mu)$. Then the following assertions are true.*

(i) *There exists exactly one strongly continuous semigroup $(T_t^\mu)_{t>0}$ on $L^1(\mu)$ such that its generator $(L_\mu, D(L_\mu))$ extends L, i.e., we have $C_0^\infty(\mathbb{R}^d) \subset D(L_\mu)$ and $L_\mu = L$ on $C_0^\infty(\mathbb{R}^d)$. In addition, $(L_\mu, D(L_\mu))$ is the closure of $(L, C_0^\infty(\mathbb{R}^d))$ on $L^1(\mu)$.*

(ii) *The semigroup $(T_t^\mu)_{t>0}$ is Markovian.*

(iii) *If $(G_\alpha)_{\alpha>0}$ denotes the corresponding resolvent, then $G_\alpha f$ has a unique continuous μ-version for all $f \in L^{dp/(d+p)}(\mu)$ and all $\alpha > 0$. In particular, $(G_\alpha)_{\alpha>0}$ is strongly Feller, i.e., $G_\alpha f$ has a continuous μ-version for all $f \in L^\infty(\mu)$, $\alpha > 0$.*

(iv) *For every $f \in C_0^\infty(\mathbb{R}^d)$ and $t > 0$, the function $T_t^\mu f$ has a unique continuous μ-version $\widetilde{T_t^\mu} f$ and μ is the only probability measure ν on $\mathbb{R}^d$ such that*

$$\int \widetilde{T_t^\mu f} \, d\nu = \int f \, d\nu \quad \textit{for all } f \in C_0^\infty(\mathbb{R}^d) \textit{ and all } t > 0.$$

1.5.2. Theorem. *Suppose that a probability measure μ on $\mathbb{R}^d$ satisfies the equation $L^*\mu = 0$, where $L := L_{\mathrm{I},b}$ and $|b| \in L^2_{\mathrm{loc}}(\mu)$. Let ϱ be the density of μ.*

(i) *Assume that*

$$|b - \nabla\varrho/\varrho| \in L^2(\mu). \tag{1.5.1}$$

Then there exists a strongly continuous semigroup $(T_t^\mu)_{t\ge0}$ on $L^1(\mu)$ with generator $(L_\mu, D(L_\mu))$ extending L. Furthermore, $(T_t^\mu)_{t\ge0}$ is Markovian and μ is $(T_t^\mu)_{t\ge0}$-invariant.

(ii) *Assume that $|b| \in L^2(\mu)$. Then $|b - \nabla\varrho/\varrho| \in L^2(\mu)$, so the conclusion of* (i) *holds.*

(iii) *Assume that $|b| \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ for some $p > d \ge 2$ and that*

$$|b - \nabla\varrho/\varrho| \in L^q(\mu) \quad \textit{for some } q \in [1, \infty].$$

Let $r := 2 - \frac{2}{q+1}$, *where* $\frac{1}{\infty} := 0$. *Then, the restriction of* $(T_t^\mu)_{t\geq 0}$ *to* $L^r(\mu)$ *is the unique strongly continuous semigroup on* $L^r(\mu)$ *whose generator extends* L. *This generator is the closure of* $(L, C_0^\infty(\mathbb{R}^d))$ *on* $L^r(\mu)$.

1.5.3. Proposition. *Suppose that* μ_1 *and* μ_2 *are two probability measures on* $\mathbb{R}^d$ *satisfying the equation* $L^*\mu = 0$ *with* $L = L_{\mathrm{I},b}$ *such that* μ_2 *is absolutely continuous with respect to* μ_1 *and* $|b|$ *is* μ_i*-square integrable for both measures. Let* $(T_t^{\mu_i})_{t\geq 0}$ *denote the corresponding strongly continuous semigroups on* $L^1(\mu_i)$ *whose generators* $(L^{\mu_i}, D(L^{\mu_i}))$ *extend* L, $i = 1, 2$, *which exist by the previous theorem. Then* $T_t^{\mu_1} f = T_t^{\mu_2} f$ μ_1*-a.e. for all* $t > 0$ *and all bounded Borel measurable functions* f *on* $\mathbb{R}^d$.

This proposition can be proved by a modification of an argument from [**119**] based on Duhamel's formula.

1.5.4. Remark. (i) We would like to point out that all the strongly continuous semigroups $(T_t^\mu)_{t\geq 0}$ above are automatically contractions on all the spaces $L^p(\mu)$, since they are sub-Markovian and since μ is T_t^μ-invariant.

(ii) In the situation of Theorem 1.5.2(i) there exists a "nice" Markov process on $\mathbb{R}^d$ whose transition probabilities are given by $(T_t^\mu)_{t\geq 0}$ and which is a weak solution to the stochastic equation $d\xi_t = dw_t + b(\xi_t)dt$. This follows from the paper [**168**], to which we refer for details and precise definitions.

Let us consider the symmetric case, i.e., the case where the operator $L_{\mathrm{I},b}$ is symmetric in $L^2(\mu)$; it turns out that this corresponds to that b is a logarithmic gradient; see Example 1.1.3 (a more precise description of this case is given below).

1.5.5. Proposition. *Let* $b = \nabla\varrho/\varrho$, *where* $\varrho \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ *is a probability density, and let* $\mu := \varrho\, dx$. *Then there exists a strongly continuous Markovian semigroup* $(T_t^\mu)_{t\geq 0}$ *on* $L^1(\mu)$ *whose generator* $(L^\mu, D(L^\mu))$ *extends* $L = L_{\mathrm{I},b}$ *and which has* μ *as a* $(T_t^\mu)_{t\geq 0}$*-invariant measure. Furthermore, the restriction of* $(T_t^\mu)_{t\geq 0}$ *to* $L^2(\mu)$ *consists of symmetric operators.*

1.5.6. Remark. (i) The *uniqueness* of $(T_t^\mu)_{t\geq 0}$ in Proposition 1.5.5 is ensured by Theorem 1.5.2(ii) provided that $|b| \in L^2(\mu)$. However, by [**151**, Theorem 3.1], we can relax the last condition to $|b| \in L^2_{\mathrm{loc}}(\mu)$ and still have uniqueness of $(T_t^\mu)_{t\geq 0}$ among all strongly continuous *symmetric* Markovian semigroups on $L^2(\mu)$ whose generators extend L (but we do not know whether the symmetry condition is essential here). We also note that in the present symmetric case we can take $r = 2$ in Theorem 1.5.2(iii), which was first shown in [**32**, Corollary 8]). We note that without extra assumptions the measure μ in this proposition can fail to be a unique probability measure satisfying the equation $L^*_{\mathrm{I},b}\mu = 0$, even among measures with a logarithmic gradient b (see Example 1.6.1 below).

(ii) In the situation of Proposition 1.5.5 it follows by its proof and the general theory in [**125**] that there exists a diffusion process on $\mathbb{R}^d$ whose transition semigroup is $(T_t^\mu)_{t\geq 0}$.

In his celebrated work [**105**] Kolmogorov studied the following problem. Let $(\xi_t)_{t\geq 0}$ be a diffusion in a finite dimensional Riemannian manifold X (in Kolmogorov's case it was compact) with generator $L = (\Delta + b)/2$, where b is a smooth vector field on X. When is the process $\eta_t = \xi_{T-t}$ governed by the same equation, i.e., when does it have the same generator L? The answer found by Kolmogorov says: if and only if b is the gradient of a function. Earlier this question had been considered by Schrödinger [**154**] in the one dimensional-case. In the paper [**105**], Kolmogorov considered only solutions of the Fokker–Planck equations as densities of transition probabilities; in those days no stochastic integration was developed (the corresponding stochastic equation on $\mathbb{R}^d$ would have the form $d\xi_t = dw_t + 2^{-1}b(\xi_t)dt$). Moreover, it was mostly the case of compact X that was considered, when invariant probability measures always exist. In this case the property studied by Kolmogorov is equivalent to the symmetrizability of $(\xi_t)_{t\geq 0}$. Thus, Kolmogorov's result is a criterion for symmetrizability of a diffusion in a compact manifold. In the noncompact case, additional conditions are required in order to ensure the existence of invariant probability measures (see the previous section). The difference between existence of an invariant measure $\mu = \varrho\, dx$ and symmetrizability of the diffusion $(\xi_t)_{t\geq 0}$ with generator $L = (\Delta + b)/2$ can be seen from Corollary 1.8.2 below, which extends Kolmogorov's theorem to general drifts from $L^2(\mu)$. According to this corollary, drifts of symmetrizable diffusions in $\mathbb{R}^d$ are logarithmic gradients of measures (an analogous fact is valid also for manifolds, see [**48**]). If $|b| \in L^p_{\rm loc}(\mathbb{R}^d)$ with some $p > d$, then we obtain an exact analog since μ has a positive density ϱ and $b = \nabla \ln \varrho$, i.e., b is indeed the gradient of a function.

We now consider the case of nonconstant A studied in [**46**], [**47**], where the proofs can be found. Let us fix an open subset Ω of a Riemannian manifold M of dimension d. Up to the end of this section we assume that A and b satisfy the following conditions with some $p > d$:

$$
\begin{aligned}
&\text{(A1)}\quad a^{ij} \in C(\Omega) \cap W^{p,1}_{\rm loc}(\Omega),\ \det A(x) \neq 0,\\
&\text{(A2)}\quad b^i \in L^p_{\rm loc}(\Omega).
\end{aligned}
\tag{1.5.2}
$$

We shall consider a measure $\mu \geq 0$ on Ω that satisfies the equation $L^*_{A,b}\mu = 0$; in some results we deal with the equation $\mathcal{L}^*_{A,b}\mu = 0$. Let ϱ be the density of μ. Let

$$\beta_{\mu,A} := (\beta^i_{\mu,A})_{i=1}^d, \quad \beta^i_{\mu,A} := \partial_j a^{ij} + a^{ij}\frac{\partial_j \varrho}{\varrho}.$$

Then $\beta^i_{\mu,A} \in L^p_{\rm loc}(\Omega)$. One can write

$$L_{A,b}\varphi = L_{A,\beta_{\mu,A}} + \langle b - \beta_{\mu,A}, \nabla\varphi\rangle, \quad \varphi \in C_0^\infty(\Omega).$$

In the divergence form case, the mapping $A\beta_\mu$ will be employed instead of $\beta_{\mu,A}$. The operator $L_{A,\beta_{\mu,A}}$ is symmetric on $L^2(\Omega,\mu)$, i.e.,

$$\int_\Omega L_{A,\beta_{\mu,A}}\varphi\,\psi\, d\mu = \int_\Omega \varphi\, L_{A,\beta_{\mu,A}}\psi\, d\mu, \quad \forall\, \varphi,\psi \in C_0^\infty(\Omega). \tag{1.5.3}$$

Indeed, by the integration by parts formula, both sides of (1.5.3) are equal to

$$-\int_\Omega \langle A\nabla\varphi, \nabla\psi\rangle\, d\mu.$$

Similarly, for divergence form operators one has

$$\int_\Omega \psi \mathcal{L}_{A,A\beta_\mu}\varphi\, d\mu = \int_\Omega \varphi\, \mathcal{L}_{A,A\beta_\mu}\psi\, d\mu.$$

Note that if $A = \mathrm{I}$, then $b_{\mu,\mathrm{I}} = \beta_\mu$ and

$$L_{\mathrm{I},b}\varphi = \mathcal{L}_{\mathrm{I},b}\varphi = \Delta\varphi + \langle b, \nabla\varphi\rangle.$$

If μ is a nonnegative measure on Ω with a density $\varrho \in W^{1,1}_{\rm loc}(\Omega)$, A and b satisfy conditions (A1) and (A2) in (1.5.2), and β_μ and $\beta_{\mu,A}$ are defined as above, then the equality $L^*_{A,b}\mu = 0$ is equivalent to the identity

$$\int_\Omega \langle b - \beta_{\mu,A}, \nabla\varphi\rangle \, d\mu = 0, \quad \varphi \in C_0^\infty(\Omega), \tag{1.5.4}$$

or shortly,

$$\mathrm{div}_\mu (b - \beta_{\mu,A}) = 0. \tag{1.5.5}$$

Indeed, it suffices to note that by the integration by parts formula, one has

$$\int_\Omega [\partial_j a^{ij}\partial_i\varphi\, \varrho + a^{ij}\partial_i\varphi\partial_j\varrho]\, dx = -\int_\Omega a^{ij}\partial_j\partial_i\varphi\, \varrho\, dx.$$

The divergence form analogue of (1.5.4) is

$$\int_\Omega \langle b - A\beta_\mu, \nabla\varphi\rangle \, d\mu = 0, \quad \forall\, \varphi \in C_0^\infty(\Omega).$$

The vector field

$$\widehat{b} := 2\beta_{\mu,A} - b \tag{1.5.6}$$

is called the dual drift (in the theory of diffusion processes, this is the drift of the time-reversed process). In the divergence form case, the dual drift is defined by the equality

$$\widetilde{b} := 2A\beta_\mu - b.$$

Clearly, one has

$$L_{A,\widehat{b}} = L_{A,\beta_{\mu,A}} - (b^i - \beta^i_{\mu,A})\partial_i \quad \text{on } C_0^\infty(\Omega)$$

and obviously $L_{A,\widehat{b}}$ is a formal adjoint to $L_{A,b}$, i.e., one has

$$\int_\Omega \psi\, L_{A,b}\varphi\, d\mu = \int_\Omega \varphi\, L_{A,\widehat{b}}\psi\, d\mu, \quad \varphi,\psi \in C_0^\infty(\Omega).$$

Similarly,

$$\int_\Omega \psi\, \mathcal{L}_{A,b}\varphi\, d\mu = \int_\Omega \varphi\, \mathcal{L}_{A,\widetilde{b}}\psi\, d\mu, \quad \varphi,\psi \in C_0^\infty(\Omega).$$

We observe that by (1.5.3) and (1.5.5) one has

$$\mu \in \mathcal{M}_{\rm ell}^{A,\beta_{\mu,A}} \cap \mathcal{M}_{\rm ell}^{A,\widehat{b}}.$$

1.5.7. Theorem. (i) *Let Ω be an open subset of $\mathbb{R}^d$ and assume that conditions* (A1) *and* (A2) *in* (1.5.2) *hold. Let $\mu \in \mathcal{M}_{\rm ell}^{A,b}$. Then there exists a closed extension $\big(L^\mu_{A,b}, D(L^\mu_{A,b})\big)$ of $\big(L_{A,b}, C_0^\infty(\Omega)\big)$ that generates a sub-Markovian contractive C_0-semigroup $(T^\mu_t)_{t\ge 0}$ on $L^1(\Omega,\mu)$ with the following properties*:

(a) *the above mentioned natural extension of the adjoint semigroup to $L^1(\Omega,\mu)$ has the generator $L^\mu_{A,\widehat{b}}$ which coincides with $L_{A,\widehat{b}}$ on $C_0^\infty(\Omega)$*;

(b) *for any bounded measurable function f with compact support, the function $(I - L^\mu_{A,b})^{-1}f$ is the limit in $L^1(\Omega,\mu)$ of the functions u_n that are solutions to the Dirichlet problems $(I - L_{A,b})u_n = f$ with zero boundary conditions on domains B_n with compact closures $\overline{B_n} \subset B_{n+1}$ and smooth boundaries ∂B_n such that $\Omega = \bigcup_{n=1}^\infty B_n$. Furthermore, μ is sub-invariant for $(T^\mu_t)_{t\ge0}$.*

(ii) *The same is true if Ω is an open subset of a complete Riemannian manifold M and the operator $L_{A,b}$ is replaced by the operator $\mathcal{L}_{A,b}$ in the form*

$$\mathcal{L}_{A,b}\varphi = \mathrm{div}(A\nabla\varphi) + \langle b, \nabla\varphi\rangle, \quad \varphi \in C_0^\infty(\Omega),$$

where b is a vector field on M and $A(x)$ is a positive operator on T_xM such that the hypotheses of (i) *are fulfilled in local coordinates.*

(iii) *Under the hypotheses of* (i) *or* (ii)*, the semigroup $(T_t^\mu)_{t\geq 0}$ has the following property: for every $\psi \in C_0^\infty(\Omega)$ and every $t \geq 0$, the function $T_t^\mu\psi$ has a continuous modification $\widetilde{T_t^\mu\psi}$ such that, for every compact set $K \subset \Omega$, one has $\lim_{t\to 0}\widetilde{T_t^\mu\psi}(x) = \psi(x)$ uniformly with respect to $x \in K$.*

The semigroup $(T_t^\mu)_{t\geq 0}$ constructed in this theorem will play a very important role below; in general, it is not the only semigroup associated with $L_{A,b}$ in the sense explained at the beginning of this section, but it is unique among associated semigroups with property (b). Below the super-index μ at T_t^μ will indicate this concrete semigroup. Such a semigroup exists also under weaker assumptions on A and b (see [**168**]).

1.5.8. Remark. In assertion (i) of the above theorem, since $\mu \in \mathcal{M}_{\rm ell}^{A,\widehat{b}}$, also the "formally adjoint" $\bigl(L_{A,\widehat{b}}, C_0^\infty(\Omega)\bigr)$ of $\bigl(L_{A,b}, C_0^\infty(\Omega)\bigr)$ has a closed extension $\bigl(L^\mu_{A,\widehat{b}}, D(L^\mu_{A,\widehat{b}})\bigr)$ generating a sub-Markovian C_0-semigroup $(\widehat{T}_t^\mu)_{t\geq 0}$ on $L^1(\Omega,\mu)$. By [**168**, Remark 1.7(ii)] one has

$$\int_\Omega g\, T_t^\mu f\, d\mu = \int_\Omega f\, \widehat{T}_t^\mu g\, d\mu, \quad f, g \in L^\infty(\Omega,\mu). \tag{1.5.7}$$

The same relation holds for the corresponding resolvents $(G_\alpha^\mu)_{\alpha>0}$ and $(\widehat{G}_\alpha^\mu)_{\alpha>0}$. Equality (1.5.7), in particular, immediately implies that μ is $(T_t^\mu)_{t\geq 0}$-invariant if and only if for every $t \geq 0$ one has $\widehat{T}_t^\mu 1 = 1$. Hence, since both semigroups are sub-Markovian, this is the case if and only if $T_t^\mu 1 = 1$ for all $t \geq 0$, which in turn is equivalent to the invariance of μ for $(\widehat{T}_t^\mu)_{t\geq 0}$. The equality $T_t^\mu 1 = 1$ is equivalent to the inclusion $1 \in D(L^\mu_{A,b})$ (or $1 \in D(L^\mu_{A,\widehat{b}})$), which is not always fulfilled in spite of the equality $L_{A,b}1 = L_{A,\widehat{b}}1 = 0$.

It is worth noting that for a symmetric operator $L_{A,b}$ (or $\mathcal{L}_{A,b}$) the closed extension mentioned in the theorem is the Friedrichs extension. The condition of symmetry of the semigroup $(T_t^\mu)_{t\geq 0}$ is seen from (1.5.6). The so called time reversal for finite and infinite-dimensional diffusions (i.e., the process ξ_{T-t} for a given diffusion ξ_t) is discussed in [**94**], [**83**], [**133**], [**134**].

We do not know whether a sub-Markovian strongly continuous semigroup whose generator extends $L_{A,b}$ is unique (i.e., whether property (b) in (i) holds automatically). As we shall see below, this is true under certain additional assumptions.

Let us recall that an operator L on a dense domain $\mathcal{D}$ in a Banach space X is called *dissipative* if, for every $u \in \mathcal{D}$, there exists $l_u \in X^*$ such that $\|l_u\|_{X^*} = \|u\|_X$, $l_u(u) = \|u\|_X^2$ and $l_u(Lu) \leq 0$. A dissipative operator L is called *essentially m-dissipative* if it satisfies the following additional condition:

$$\overline{(L-\lambda I)(\mathcal{D})} = X \ \forall\, \lambda > 0,$$

where $\overline{E}$ denotes the closure of E. In fact, it suffices that the above condition hold for some $\lambda > 0$.

1.5.9. Lemma. *Let $\mu \geq 0$ satisfy the equation $L^*_{A,b}\mu = 0$ with coefficients satisfying* (A1) *and* (A2) *in* (1.5.2)*. Then the operator $L_{A,b}$ is dissipative on the domain $C_0^\infty(\Omega)$ in the space $L^1(\Omega,\mu)$. In particular, it is closable. The same is true for the divergence form operator $\mathcal{L}_{A,b}$ in the manifold case.*

Hence $L_{A,b}$ is essentially m-dissipative on the domain $C_0^\infty(\Omega)$ in $L^1(\Omega,\mu)$ if and only if the set $(L_{A,b}-\lambda I)(C_0^\infty(\Omega))$ is dense in $L^1(\Omega,\mu)$ for some (and then for all) $\lambda>0$. In this case the operator $\bigl(L_{A,b}, C_0^\infty(\Omega)\bigr)$ is called $L^1(\Omega,\mu)$-unique; see §1.7 for sufficient conditions for this.

Let $\bigl(\overline{L}^\mu_{A,b}, D(\overline{L}^\mu_{A,b})\bigr)$ denote the closure of the operator $\bigl(L_{A,b}, C_0^\infty(\Omega)\bigr)$ in the space $L^1(\Omega,\mu)$. The closure $\bigl(\overline{\mathcal{L}}^\mu_{A,b}, D(\overline{\mathcal{L}}^\mu_{A,b})\bigr)$ of the operator $\mathcal{L}_{A,b}$ is defined analogously.

1.5.10. Proposition. *Let $\mu\in\mathcal{M}^{A,b}_{\rm ell}$, where A and b satisfy* (A1) *and* (A2) *in* (1.5.2). *The following assertions are equivalent*:

(i) $\bigl(\overline{L}^\mu_{A,b}, D(\overline{L}^\mu_{A,b})\bigr)$ *generates a C_0-semigroup $(T_t)_{t\ge0}$, i.e., a strongly continuous semigroup of bounded operators T_t on $L^1(\Omega,\mu)$*;

(ii) *for some* (*hence for all*) $\lambda>0$, *the set*

$$\bigl(L_{A,b}-\lambda I\bigr)\bigl(C_0^\infty(\Omega)\bigr)$$

is dense in $L^1(\Omega,\mu)$ (*equivalently, $\bigl(L_{A,b}, C_0^\infty(\Omega)\bigr)$ is essentially m-dissipative on $L^1(\Omega,\mu)$*);

(iii) *there exists exactly one C_0-semigroup on $L^1(\Omega,\mu)$ which has a generator extending the operator $\bigl(L_{A,b}, C_0^\infty(\Omega)\bigr)$*.

If any (*hence each*) *of the assertions* (i)–(iii) *is true, then the semigroups $(T^\mu_t)_{t\ge0}$ and $(\widehat{T}^\mu_t)_{t\ge0}$ are Markovian and μ is invariant for them. Finally, the same is true for the operator $\mathcal{L}_{A,b}$ in divergence form.*

1.5.11. Remark. For a bounded domain Ω (with smooth boundary), assertion (iii) (hence also (i) and (ii)) in Proposition 1.5.10 is not valid even if $A=\mathrm{I}$, $b\equiv0$. So we shall be mainly interested in the case when $\Omega=M$. As we shall see below, assertions (i)–(iii) are equivalent to the invariance of μ with respect to the semigroup $(T^\mu_t)_{t\ge0}$.

We shall see below that the operator $L_{A,b}$ on $C_0^\infty(\Omega)$ can have different closed extensions (even generating strongly continuous semigroups). Hence its closure can fail to generate a strongly continuous semigroup (i.e., it may not be a generator). This occurs even if $A=\mathrm{I}$ and b is infinitely differentiable.

The next theorem gives useful information about the domain of generator of $(T^\mu_t)_{t\ge0}$ on $L^p(\Omega,\mu)$. It is known that, for every $r\in[1,\infty)$, the restriction of $(T^\mu_t)_{t\ge0}$ to $L^r(\Omega,\mu)$ is a strongly continuous semigroup on $L^r(\Omega,\mu)$. Its generator will be denoted by $\bigl(L^{\mu,r}_{A,b}, D(L^{\mu,r}_{A,b})\bigr)$. It is not difficult to verify that

$$D(L^{\mu,r}_{A,b})=\Bigl\{f\in D(L^\mu_{A,b})\cap L^r(\Omega,\mu)\colon\ L^\mu_{A,b}f\in L^r(\Omega,\mu)\Bigr\}.$$

1.5.12. Theorem. (i) *In the situation of Theorem* 1.5.7, *one has*

$$\bigl(L^{\mu,p}_{A,b}, D(L^{\mu,p}_{A,b})\bigr)\subset\Bigl\{f\in L^p(\Omega,\mu)\cap H^{p,2}_{\rm loc}(\Omega)\colon\ L_{A,b}f\in L^p(\Omega,\mu)\Bigr\}\tag{1.5.8}$$

and $L^{\mu,p}_{A,b}f=L_{A,b}f$ for all $f\in D(L^{\mu,p}_{A,b})$. The same is true for any extension $\bigl(L,D(L)\bigr)$ of $\bigl(L_{A,b}, C_0^\infty(\Omega)\bigr)$ with the following property: it is the generator of a strongly continuous sub-Markovian semigroup $(T_t)_{t\ge0}$ on $L^1(\Omega,\mu)$ such that the adjoint semigroup $(T'_t)_{t\ge0}$ on $L^{p'}(\Omega,\mu)$ (*which is defined after one extends $(T_t)_{t\ge0}$ to $L^p(\Omega,\mu)$ as explained above*) *has a generator which coincides with $L_{A,\widehat{b}}$ on $C_0^\infty(\Omega)$. If one has an equality in* (1.5.8), *then μ is invariant for $(T^\mu_t)_{t\ge0}$* (*equivalently, $T^\mu_t1=1$*).

(ii) *If the operator* $\big(L_{A,\widehat{b}}, C_0^\infty(\Omega)\big)$ *is essentially* m*-dissipative on the space* $L^{p'}(\Omega,\mu)$*, i.e., the set* $(L_{A,\widehat{b}}-\lambda I)\big(C_0^\infty(\Omega)\big)$ *is dense in* $L^{p'}(\Omega,\mu)$ *for some* $\lambda>0$*, then one has an equality in* (1.5.8).

Finally, the same assertions are true for $\mathcal{L}_{A,b}$.

The reader should be warned that Theorem 1.5.7 does not assert that a sub-Markovian semigroup whose generator extends $L_{A,b}$ is unique: we do not know whether this is true under the indicated hypotheses, and without the sub-Markovian property (i.e., with the only requirement of strong continuity) this is not true in general. This is why we always specify the semigroup $(T_t^\mu)_{t\geq 0}$. Also, the measure μ may be only sub-invariant for the semigroup and not invariant. We already know that these two phenomena are closely related: Proposition 1.5.10 shows that the uniqueness of an associated strongly continuous semigroup is equivalent to the essential m-dissipativity of $L_{A,b}$.

We introduce the following subset of $\mathcal{M}_{\rm ell}^{A,b}$:

$$\mathcal{M}_{\rm ell,md}^{A,b} := \Big\{\mu\in\mathcal{M}_{\rm ell}^{A,b}\colon\ \overline{(L_{A,b}-I)\big(C_0^\infty(\Omega)\big)} = L^1(\Omega,\mu)\Big\}. \tag{1.5.9}$$

The same notation is used for the operator $\mathcal{L}_{A,b}$.

The next result from [**47**] gives a convenient technical characterization of the essential m-dissipativity of $L_{A,b}$ and shows that it is equivalent to the invariance of μ with respect to the associated semigroup $(T_t^\mu)_{t\geq 0}$ from Theorem 1.5.7. In the case $\Omega=\mathbb{R}^d$, this result was proved in [**168**, Proposition 1.9], under more general assumptions on A, b, and μ (the validity of those assumptions in our case follows from the results discussed in §1.2).

1.5.13. Theorem. *Suppose that conditions* (A1) *and* (A2) *in* (1.5.2) *hold. Let* $\mu\in\mathcal{M}_{\rm ell}^{A,b}$. *Then the following assertions are equivalent*:

(i) $\mu\in\mathcal{M}_{\rm ell,md}^{A,b}$;

(ii) μ *is invariant for* $(T_t^\mu)_{t\geq 0}$;

(iii) *there exist functions* $\chi_n\in W_{\rm loc}^{2,1}(\Omega,\mu)$ *and* $\alpha>0$ *such that* $(1-\chi_n)^+\in L^\infty(\Omega,\mu)$ *and* $(1-\chi_n)^+=0$ *outside some compact sets,* $\lim\limits_{n\to\infty}\chi_n(x)=0$ μ*-a.e., and for* $\eta=1$ *or* $\eta=-1$ *one has*

$$\int\langle A\nabla\chi_n,\nabla\varphi\rangle\,d\mu+\alpha\int\chi_n\varphi\,d\mu+\eta\int\langle b-\beta_{\mu,A},\nabla\chi_n\rangle\,\varphi\,d\mu\geq 0$$

for all nonnegative $\varphi\in C_0^\infty(\Omega)$ *and all* $n\in\mathbb{N}$*, with the corresponding condition in the case of* $\mathcal{L}_{A,b}$ *taking the form*

$$\int\langle A\nabla\chi_n,\nabla\varphi\rangle\,d\mu+\alpha\int\chi_n\varphi\,d\mu+\eta\int\langle b-A\beta_{\mu},\nabla\chi_n\rangle\,\varphi\,d\mu\geq 0;$$

(iv) $\mu\in\mathcal{M}_{\rm ell,md}^{A,\widehat{b}}$ (*respectively,* $\mu\in\mathcal{M}_{\rm ell,md}^{A,\widetilde{b}}$ *in the case of* $\mathcal{L}_{A,b}$).

1.5.14. Remark. (i) The operator $L_{A,b}$ is not essentially m-dissipative on domain $C_0^\infty(\Omega)$ in $L^p(\Omega,\mu)$ for some $p\in[1,\infty)$ precisely when there exists a nonzero function $h\in L^{p'}(\Omega,\mu)$ with $p'=p/(p-1)$ such that the measure $h\cdot\mu$ satisfies the equation $(L_{A,b}-1)^*(h\cdot\mu)=0$. It follows from the proof of Theorem 1.5.12 that (under our assumptions on A and b, of course) this is equivalent to the following: there exists a nonzero function $h\in L^{p'}(\Omega,\mu)\cap W_{\rm loc}^{p,2}(\Omega)$ such that $L_{A,\widehat{b}}h=h$ a.e.

(ii) The proof of this theorem employs some properties of our special semigroup $(T_t^\mu)_{t\geq 0}$, and we do not know whether in assertion (ii) one can use any associated semigroup.

Our next goal is to study relations between infinitesimal invariance and invariance. As we shall see, these two concepts are different, but under some additional assumptions they coincide. First we mention a useful technical result which shows that any reasonable invariant measure ν of the semigroup $(T_t^\mu)_{t\geq 0}$ on $L^1(\Omega,\mu)$ associated (as explained in Theorem 1.5.7 above) with a probability measure μ satisfying the equation $L_{A,b}^*\mu = 0$ also satisfies this equation.

1.5.15. Proposition. *Let Ω be an open set in M. Assume that conditions* (A1) *and* (A2) *in* (1.5.2) *are fulfilled and $\mu \in \mathcal{M}_{\rm ell}^{A,b}$. Let $(T_t^\mu)_{t\geq 0}$ be the associated semigroup specified in Theorem* 1.5.7. *Suppose that ν is a probability measure on Ω such that $\nu \ll \mu$ and*

$$\int T_t^\mu f\, d\nu = \int f\, d\nu \quad \forall\, f \in C_0^\infty(\Omega),\ \forall\, t > 0.$$

Assume, in addition, that $b^i \in L_{\rm loc}^q(\Omega,\nu)$ for some $q > 1$. Then $\nu \in \mathcal{M}_{\rm ell}^{A,b}$.

In particular, if $|b|$ is locally bounded, then any absolutely continuous measure ν that is invariant for $(T_t^\mu)_{t\geq 0}$ satisfies our elliptic equation, hence has a positive continuous density and is equivalent to μ.

It should be noted that this proposition is not true without the assumption that $\nu \ll \mu$. Indeed, let $x_0 \in \Omega$ be fixed. Let us consider versions of $T_t^\mu f$, where $f \in C_0^\infty(\Omega)$, such that $T_t^\mu f(x_0) = f(x_0)$. Then Dirac's measure at x_0 is invariant for T_t^μ. We do not know, however, how essential the assumption that $b^i \in L^q(\Omega,\nu)$ is. It will be explained below that in the case when $p > d+2$, there exist uniquely defined sub-probability kernels $K_t(\,\cdot\,,\,\cdot\,)$ such that $K_t f$ is a version of $T_t^\mu f$ for each $f \in L^1(\Omega,\mu)$ and the assertion of this proposition is true for any invariant measure ν of the semigroup $(K_t)_{t\geq 0}$.

The next result, slightly extending [**4**, Proposition 2.6(ii)], gives a sufficient condition for a solution of the elliptic equation to be invariant for our special associated semigroup.

1.5.16. Proposition. *Let $\mu \in \mathcal{M}_{\rm ell,md}^{A,b}$ and let $(T_t^\mu)_{t\geq 0}$ be the corresponding semigroup specified in Theorem* 1.5.7. *Suppose that ν is a bounded Borel measure on Ω such that $L_{A,b}^*\nu = 0$ and the function $\varrho := d\nu/d\mu$ is bounded. Then ν is $(T_t^\mu)_{t\geq 0}$-invariant. The same is true in the case of $\mathcal{L}_{A,b}$.*

A criterion for invariance of infinitesimally invariant measures in terms of martingale problems is obtained in [**73**].

The next proposition shows that the semigroup $(T_t^\mu)_{t\geq 0}$ cannot have invariant measures with positive densities if μ itself is not invariant.

1.5.17. Proposition. *Suppose that T is a sub-Markovian operator on $L^1(\Omega,\mu)$, where μ is a probability measure on Ω that is sub-invariant with respect to T. Let ν be a probability measure on Ω equivalent to μ. If ν is invariant for T, then μ is invariant as well.*

1.5.18. Remark. When applied to $(T_t^\mu)_{t\geq 0}$ in the case of locally bounded $|b|$, this result (along with Proposition 1.5.15 and subinvariance of μ with respect to $(T_t^\mu)_{t\geq 0}$ taken into account) shows that if μ is not invariant for $(T_t^\mu)_{t\geq 0}$, then no measure equivalent to μ can be invariant for $(T_t^\mu)_{t\geq 0}$. In §1.7 we return to this question.

For related results, see also [**2**], [**3**]. Concerning the existence of invariant measures for finite-dimensional diffusions, see [**18**], [**19**], [**93**], [**135**], [**180**]. General problems relating to ergodicity of diffusions and convergence to invariant measures are considered in [**9**], [**51**], [**60**], [**177**], [**110**], [**132**], [**141**], [**142**], [**164**], [**166**], [**175**], [**176**], [**180**]. Properties of diffusion semigroups in L^p-spaces with respect to invariant measures and related problems for elliptic operators are studied in [**66**], [**81**], [**122**], [**131**]. Hypercontractivity of diffusion semigroups on finite-dimensional spaces, Poincaré, log-Sobolev, and other related inequalities for them are studied in [**12**], [**54**], [**87**], [**112**], [**149**]. In these works one can find additional references.

1.6. On non-uniqueness of solutions

The problem of uniqueness for solutions of elliptic equations in the class of all probability measures on the whole space will be addressed in the next section. Here we present some negative results in the case $A = \mathrm{I}$ and infinitely differentiable b. It is easy to construct such examples for singular drifts b.

1.6.1. Example. Let ϱ be a smooth probability density on the real line such that $\varrho(0) = 0$ and $\varrho(x) > 0$ if $x \neq 0$. Let $b = \nabla\varrho/\varrho$ away from the origin and let $b(0) = 0$. Then the probability measure $\varrho\, dx$ satisfies the equation $L^*_{1,b}\mu = 0$, but this equation has another solution $c\varrho I_{[0,+\infty)}\, dx$, where c is a normalization constant. The function ϱ'/ϱ serves as a logarithmic gradient also for this second solution since $(-\infty, 0]$ has measure zero for it.

In dimension 1, the singularity of b is the only reason for non-uniqueness.

1.6.2. Proposition. *Suppose that b is locally Lebesgue integrable on the real line. Then the equation $L^*_{1,b}\mu = 0$ can have at most one solution in the class of probability measures.*

Proof. According to Proposition 1.1.2 any solution is given by a density

$$\left(k_1 + k_2 \int_0^x \frac{1}{\psi(s)}\, ds\right)\psi(x), \quad \psi(x) := \int_0^x b(t)\, dt.$$

Suppose that we have two linearly independent solutions. Hence there are two linearly independent vectors (k_1, k_2) for which the corresponding density is integrable over the real line. Then it is seen from the above formula that $\psi \in L^1(\mathbb{R}^1)$. Therefore, $1/\psi$ is not integrable on $(-\infty, 0]$ and $[0, +\infty)$. Since $1/\psi > 0$, the indefinite integral of $1/\psi$ tends to $+\infty$ as $x \to +\infty$ and tends to $-\infty$ as $x \to -\infty$. This shows that for nonnegative solutions we must have $k_2 = 0$, so there is at most one solution in the class of probability measures. □

However, in any dimension $d > 1$ there are examples of non-uniqueness with smooth b. It is not easy to find such examples, since, as we shall see in the next section, the existence results presented above always produce unique solutions. For several years the problem remained open until the following simple example was constructed in [**46**], [**47**].

1.6.3. Example. Let

$$b^i(x) = -x_i - 2x_{\sigma(i)} e^{(x_i^2 - x_{\sigma(i)}^2)/2},$$

where $\sigma\colon \{1, 2, 3, \ldots, d\} \to \{1, 2, 3, \ldots, d\}$ is one-to-one and such that $\sigma(i) \neq i$. Then our equation has at least two solutions: one is the standard Gaussian measure μ on $\mathbb{R}^d$ and another is the measure $\nu = v \cdot \mu$ with

$$v(x) = c_d \sum_{i=1}^d \int_{-\infty}^{x_i} e^{-s^2/2}\, ds,$$

where c_d is a normalizing constant.

More generally, let $f \in C^2(\mathcal{R}^1)$ be bounded, with $f, f' > 0$ and $f', f'' \in L^1(\mathcal{R}^1)$. Define $b = (b^i)\colon \mathcal{R}^d \to \mathcal{R}^d$ by

$$b^i(x) := \frac{f''(x_i)}{f'(x_i)} + 2\frac{f''(x_{\sigma(i)})}{f'(x_{\sigma(i)})}, \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d,$$

and set

$$\mu := c_1 \prod_{i=1}^d f'(x_i)\,dx, \quad \nu := c_2 \sum_{i=1}^d f(x_i)\mu(dx),$$

where $c_1, c_2 > 0$ are normalizing constants. Then μ and ν are two different elements in $\mathcal{M}_{\rm ell}^{\rm I,b}$.

However, even in this explicit example it remained unknown whether there exist other linearly independent probabilistic solutions. The phenomenon of non-uniqueness was investigated by S.V. Shaposhnikov [**158**], [**159**], who obtained the following results.

Until the end of this section we assume that $A = \mathrm{I}$ and $b^i \in C^\infty(\mathbb{R}^d)$ for each $1 \le i \le d$. Then any solution of the equation $L^*_{I,b}\mu = 0$ has a density $\varrho \in C^\infty(\mathbb{R}^d)$ and the equation can be written as the following equation for ϱ:

$$\mathrm{div}\big(\nabla\varrho - b\varrho\big) = 0.$$

Let us set

$$L := L_{\mathrm{I},b}, \quad a := b\varrho - \nabla\varrho.$$

Then $a \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and

$$\mathrm{div}\ a = 0.$$

If ϱ is a probability density, then we know that $\varrho > 0$, hence the coefficient b is expressed in the following way:

$$b = \frac{\nabla\varrho}{\varrho} + \frac{a}{\varrho}. \tag{1.6.1}$$

Let us seek another solution of the equation $L^*\mu = 0$ in the form $\nu = v\cdot\mu$. The measure $\nu = v\cdot\mu$ satisfies the same equation if and only if the function v satisfies the equation

$$L_\mu v := \mathrm{div}\big(\varrho\nabla v - av\big) = 0. \tag{1.6.2}$$

Certainly, every constant will be a solution to equation (1.6.2). We would like to find a sufficient condition for the existence of a bounded positive solution which is not constant. By analogy with [**181**] let us introduce the following bilinear skew-symmetric form on $C_0^\infty(\mathbb{R}^d)$:

$$[f, g] := \int_{\mathbb{R}^d} \langle a, \nabla f\rangle g\,dx.$$

Note that $[f, g]$ is defined if g is bounded and $\langle a, \nabla f\rangle$ is integrable, but it can fail to be skew-symmetric. The next theorem gives sufficient conditions for the existence of a bounded positive solution to (1.6.2) which is not constant.

1.6.4. THEOREM. *Assume that there exists a function $\varphi \in C_b^2(\mathbb{R}^d)$ such that $\langle a, \nabla\varphi\rangle \in L^1(\mathbb{R}^d)$,*

$$[\varphi, 1] = 0, \quad \textit{and} \quad [\varphi, \varphi] < 0. \tag{1.6.3}$$

Then equation (1.6.2) *has a bounded positive solution which is not constant.*

Obviously, by multiplying v by a positive constant we obtain a probability measure $v\cdot\mu$ satisfying the equation $L^*(v\cdot\mu) = 0$.

1.6.5. Remark. For the verification of the conditions of Theorem 1.6.4 it is useful to keep in mind the following expressions for $[\varphi, \varphi]$ and $[\varphi, 1]$. Let Ω_n be increasing domains with piecewise smooth boundaries and $\mathbb{R}^d = \bigcup_{n=1}^{\infty} \Omega_n$. Since div $a = 0$, we have

$$[\varphi, \varphi] = \int_{\mathbb{R}^d} \langle a, \nabla\varphi\rangle \varphi \, dx = \lim_{n\to\infty} \int_{\Omega_n} \langle a, \nabla\varphi\rangle \varphi \, dx = \lim_{n\to\infty} \frac{1}{2} \int_{\partial\Omega_n} \langle a, \nu_n\rangle \varphi^2 \, ds,$$

$$[\varphi, 1] = \int_{\mathbb{R}^d} \langle a, \nabla\varphi\rangle \, dx = \lim_{n\to\infty} \int_{\Omega_n} \langle a, \nabla\varphi\rangle \, dx = \lim_{n\to\infty} \int_{\partial\Omega_n} \langle a, \nu_n\rangle \varphi \, ds,$$

where ν_n is the outward normal on $\partial\Omega_n$. Consequently, in order to ensure (1.6.3) it is enough to have

$$\lim_{n\to\infty} \int_{\partial\Omega_n} \langle a, \nu_n\rangle \varphi^2 \, ds < 0, \quad \lim_{n\to\infty} \int_{\partial\Omega_n} \langle a, \nu_n\rangle \varphi \, ds = 0.$$

To get an example of equation $L^*\mu = 0$ with at least two different probability solutions, it is sufficient to do the following. First of all, we find a smooth vector field a with $\operatorname{div} a = 0$ and a function φ satisfying the conditions of Theorem 1.6.4. Next, we fix an arbitrary infinitely differentiable positive function ϱ with $\|\varrho\|_{L^1(\mathbb{R}^d)} = 1$. Finally, we take the vector field b given by (1.6.1). Then the equation $L^*\mu = 0$ with this coefficient b has at least two different probability solutions: one is the measure $\mu = \varrho\, dx$ and another is the measure $\nu = c_1 v \cdot \mu$, where c_1 is a normalizing constant, and the function v is a (non constant) solution of equation (1.6.2), which exists by Theorem 1.6.4. Let us present some examples of a and φ such that div $a = 0$ and conditions of Theorem 1.6.4 are fulfilled.

1.6.6. Example. Let $d = 2$. Let us take odd functions q, ψ, $\sigma \in C_b^2(\mathbb{R}^1)$ such that

$$q,\ q\psi,\ \sigma' \in L^1(\mathbb{R}^1), \quad \lim_{n\to\infty} \sigma(n) = 1,$$

and $q\psi \geq 0$ does not vanish identically. Clearly, this is possible. Let

$$a(x, y) := (0, -q(x)), \quad \varphi(x, y) := \psi(x) + \sigma(y).$$

Clearly, $\operatorname{div} a = 0$. Let us verify the conditions of Theorem 1.6.4 using Remark 1.6.5. Let Ω_n be the square with vertices at the points (n, n), $(-n, n)$, $(n, -n)$, and $(-n, -n)$. Then

$$\int_{\partial\Omega_n} \langle a, \nu_n\rangle \varphi^2 \, ds = -(\sigma(n) - \sigma(-n)) \int_{-n}^{n} q(x)(2\psi(x) + \sigma(n) + \sigma(-n)) \, dx,$$

$$\int_{\partial\Omega_n} \langle a, \nu_n\rangle \varphi \, ds = -(\sigma(n) - \sigma(-n)) \int_{-n}^{n} q(x) \, dx.$$

Consequently,

$$[\varphi, \varphi] = \lim_{n\to\infty} \frac{1}{2} \int_{\partial\Omega_n} \langle a, \nu_n\rangle \varphi^2 \, ds = -2 \int_{-\infty}^{+\infty} q(x)\psi(x) \, dx < 0,$$

$$[\varphi, 1] = \lim_{n\to\infty} \int_{\partial\Omega_n} \langle a, \nu_n\rangle \varphi \, ds = -2 \int_{-\infty}^{+\infty} q(x) \, dx = 0.$$

Hence the conditions of Theorem 1.6.4 are fulfilled. Therefore, choosing an arbitrary strictly positive smooth probability density ϱ, we can construct a drift b (as explained in the remark above) such that the corresponding equation $L^*\mu = 0$ is satisfied by at least two probability measures, one of which is the given probability measure $\mu = \varrho\, dx$.

1.6.7. Example. Let $d = 2$ and let functions q, ψ, $\sigma \in C_b^2(\mathbb{R}^1)$ not vanish identically and satisfy the following conditions:

$$q,\, \sigma' \in L^1(\mathbb{R}^1),\;\; q > 0,\;\; \lim_{n\to\infty} \sigma(n) = 1,\;\; \lim_{n\to\infty} \sigma(-n) = 0.$$

Set

$$a(x,y) := (0, -q(x)), \quad \varphi(x,y) := \psi(x)\sigma(y).$$

Again, $\operatorname{div} a = 0$. We calculate $[\varphi, 1]$ and $[\varphi,\varphi]$ by using Remark 1.6.5. Let Ω_n be the square with vertices at the points (n,n), $(-n,n)$, $(n,-n)$, $(-n,-n)$. Then

$$\int_{\partial\Omega_n} \langle a, \nu_n\rangle \varphi^2\, ds = -(\sigma^2(n) - \sigma^2(-n)) \int_{-n}^{n} q(x)\psi^2(x)\, dx,$$

$$\int_{\partial\Omega_n} \langle a, \nu_n\rangle \varphi\, ds = -(\sigma(n) - \sigma(-n)) \int_{-n}^{n} q(x)\psi(x)\, dx.$$

Consequently,

$$[\varphi,\varphi] = -\frac{1}{2}\int_{-\infty}^{+\infty} q(x)\psi^2(x)\, dx,\;\; [\varphi, 1] = -\int_{-\infty}^{+\infty} q(x)\psi(x)\, dx.$$

To satisfy the conditions of Theorem 1.6.4, it is sufficient to require the orthogonality of the functions ψ and 1 in $L^2(\mathbb{R}^1, q\, dx)$. Again, Remark 1.6.5 enables us to construct an equation $L^*\mu = 0$ with different probability solutions, one of which is a given measure $\mu = \varrho\, dx$.

It is easy to extend the last example to the case $d \geq 2$.

1.6.8. Example. Set $x' := (x_1, x_2, \ldots, x_{d-1})$. Let q, $\psi \in C_b^2(\mathbb{R}^{d-1})$, and $\sigma \in C_b^2(\mathbb{R}^1)$ not vanish identically and satisfy the following conditions:

$$q \in L^1(\mathbb{R}^{d-1}),\;\; \sigma' \in L^1(\mathbb{R}^1),\;\; q > 0,\;\; \lim_{n\to\infty} \sigma(n) = 1,\;\; \lim_{n\to\infty} \sigma(-n) = 0.$$

Set

$$a^i(x) := 0 \;\text{ if } 1 \leq i \leq d-1,\;\; a^d(x) := -q(x'),\;\; \varphi(x) := \psi(x')\sigma(x_d).$$

Then $\operatorname{div} a = 0$ and

$$[\varphi,\varphi] = -\frac{1}{2}\int_{\mathbb{R}^{d-1}} q(x')\psi^2(x')\, dx',\;\; [\varphi, 1] = -\int_{\mathbb{R}^{d-1}} q(x')\psi(x')\, dx'.$$

To satisfy the conditions of Theorem 1.6.4 it is sufficient to require the orthogonality of the functions ψ and 1 in $L^2(\mathbb{R}^{d-1}, q\, dx')$.

We fix a with $\operatorname{div} a = 0$ and two different functions φ_1 and φ_2 satisfying the conditions of Theorem 1.6.4, and we construct two solutions v_1 and v_2 according to this theorem. This theorem guarantees that $1, v_1$ and $1, v_2$ are pairs of linearly independent functions. Under what conditions on φ_1 and φ_2 will the three functions 1, v_1 and v_2 be linearly independent? The following theorem answers this question and gives some additional information.

1.6.9. Theorem. *Let $n \geq 1$. Assume that there exist functions φ_1, φ_2, …, φ_{n+1} in $C_b^2(\mathbb{R}^d)$ satisfying the conditions of Theorem* 1.6.4. *Let v_1, v_2, …, v_{n+1} be solutions of equation* (1.6.2) *generated by these functions according to Theorem* 1.6.4. *Assume also that the functions 1, v_1, …, v_n are linearly independent and that for every vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ the following inequality holds*:

$$\Big[\varphi_{n+1} - \sum_{k=1}^{n} \alpha_k\varphi_k, \varphi_{n+1} - \sum_{k=1}^{n} \alpha_k\varphi_k\Big] < 0. \tag{1.6.4}$$

Then the functions $1, v_1, \ldots, v_n, v_{n+1}$ are linearly independent.

1.6.10. Remark. Let $\Phi = (\Phi_{ij})_{i,j\le n}$ and $h = (h_i)_{i\le n}$, where

$$\Phi_{ij} = ([\varphi_i, \varphi_j] + [\varphi_j,\ \varphi_i])/2,\ \ h_i = [\varphi_i, \varphi_{n+1}] + [\varphi_{n+1}, \varphi_i],\ \ 1 \le i, j \le n.$$

Let $h_0 := [\varphi_{n+1}, \varphi_{n+1}]$. Then inequality (1.6.4) can be written as

$$\langle \Phi\alpha, \alpha\rangle - \langle h, \alpha\rangle + h_0 < 0.$$

Consequently, to ensure condition (1.6.4) it is enough to have the following: the matrix Φ is strictly negative and

$$4h_0 < \langle \Phi^{-1}h, h\rangle. \tag{1.6.5}$$

In particular, if $n = 1$, then inequality (1.6.5) has the following very simple form:

$$\bigl([\varphi_1, \varphi_2] + [\varphi_2, \varphi_1]\bigr)^2 < 4[\varphi_1, \varphi_1][\varphi_2, \varphi_2].$$

1.6.11. Remark. To verify condition (1.6.4) it is useful to keep in mind the following expression for $[\varphi_i, \varphi_j]+[\varphi_j, \varphi_i]$. Let Ω_n be increasing domains with piecewise smooth boundaries and $\mathbb{R}^d = \bigcup_{n=1}^{\infty} \Omega_n$. Since div $a = 0$, we have

$$[\varphi_i, \varphi_j] + [\varphi_j, \varphi_i] = \int_{\mathbb{R}^d} \langle a, \nabla\varphi_i\rangle \varphi_j\, dx + [\varphi_j, \varphi_i] =$$
$$\lim_{n\to\infty} \int_{\Omega_n} \langle a, \nabla\varphi_i\rangle \varphi_j\, dx + [\varphi_j, \varphi_i] = \lim_{n\to\infty} \int_{\partial\Omega_n} \langle a, \nu_n\rangle \varphi_i\varphi_j\, ds,$$

where ν_n is the outward normal on $\partial\Omega_n$.

1.6.12. Remark. Suppose that for a given smooth vector field a with div $a = 0$ there exist functions $\varphi_1, \ldots, \varphi_{n+1}$ such that the conditions of Theorem 1.6.9 hold. Assume that we are given a strictly positive infinitely differentiable function ϱ with $\|\varrho\|_{L^1(\mathbb{R}^d)} = 1$. Then the equation $L^*\mu = 0$ with the coefficient b that is expressed via a and ϱ by formula (1.6.1) has at least $n + 1$ linearly independent probability solutions, one of which is the measure $\mu = \varrho\, dx$, and n others are the measures $\nu_i = c_i v_i \cdot \mu$, where c_i are normalizing constants and the functions v_i are nonconstant solutions of equation (1.6.2) generated by the functions φ_i.

Let us present an explicit example of a, φ_1, and φ_2 such that the conditions of Theorem 1.6.9 are fulfilled.

1.6.13. Example. Set $x' := (x_1, x_2, \ldots, x_{d-1})$. Let $q, \psi_1, \psi_2 \in C_b^2(\mathbb{R}^{d-1})$, and $\sigma \in C_b^2(\mathbb{R}^1)$ not vanish identically. Assume that

$$q \in L^1(\mathbb{R}^{d-1}),\ q > 0,\ \sigma' \in L^1(\mathbb{R}^1),\ \lim_{n\to\infty} \sigma(n) = 1,\ \lim_{n\to\infty} \sigma(-n) = 0.$$

Let

$$a^k(x) := 0 \ \text{ if } 1 \le k \le d-1,\ \ a^d(x) := -q(x'),$$
$$\varphi_1(x) := \psi_1(x')\sigma(x_d), \quad \varphi_2(x) := \psi_2(x')\sigma(x_d).$$

Then div $a = 0$ and, whenever $1 \le i, j \le 2$, we have

$$[\varphi_i, \varphi_j] + [\varphi_j, \varphi_i] = -\int_{\mathbb{R}^{d-1}} \psi_i(x')\psi_j(x')q(x')\, dx',$$

$$[\varphi_i, 1] = -\int_{\mathbb{R}^{d-1}} \psi_i(x')q(x')\, dx'.$$

To satisfy the conditions of Theorem 1.6.9 it is enough to require the orthogonality of the functions $1, \psi_1, \psi_2$ in $L^2(\mathbb{R}^{d-1}, q\, dx')$.

This example can be easily extended to the case of an arbitrary number of functions φ_i. Moreover, we can give an example of an equation $L^*\mu = 0$ which has a countable sequence of linearly independent probability solutions. In particular, the space of solutions to such an equation in the class of bounded measures is infinite-dimensional. It is enough to find a sequence of positive bounded solutions $\{v_i\}_{i\geq 1}$ to (1.6.2) such that the functions $1, \{v_i\}_{i\geq 1}$ are linearly independent. According to Theorem 1.6.9 and Remark 1.6.10, it suffices to find a vector field a with div $a = 0$ and a sequence of functions $\{\varphi_i\}_{i\in\mathbb{N}}$ satisfying the conditions of Theorem 1.6.4 such that, for each n, the functions $\varphi_1, \varphi_2, \ldots, \varphi_{n+1}$ satisfy condition (1.6.5).

1.6.14. Example. Set $x' := (x_1, x_2, \ldots, x_{d-1})$. Let $q, \psi_i \in C_b^2(\mathbb{R}^{d-1})$, where $i \in \mathbb{N}$, and let $\sigma \in C_b^2(\mathbb{R}^1)$. Assume that

$$q \in L^1(\mathbb{R}^{d-1}),\ q > 0,\ \sigma' \in L^1(\mathbb{R}^1),\ \lim_{n\to\infty} \sigma(n) = 1,\ \lim_{n\to\infty} \sigma(-n) = 0.$$

Let

$$a^k(x) := 0 \ \text{ if } 1 \leq k \leq d-1, \quad a^d(x) := -q(x'), \quad \varphi_i(x) := \psi_i(x')\sigma(x_d),\ i \in \mathbb{N}.$$

Then, for any $i, j \geq 1$, we have

$$[\varphi_i, \varphi_j] + [\varphi_j, \varphi_i] = -\int_{\mathbb{R}^{d-1}} \psi_i(x')\psi_j(x')q(x')\, dx',$$

$$[\varphi_i, 1] = -\int_{\mathbb{R}^{d-1}} \psi_i(x')q(x')\, dx'.$$

Let $1, \{\psi_i\}_{i\in\mathbb{N}}$ be an orthonormal system in $L^2(\mathbb{R}^{d-1}, q\, dx')$. Then, for each n, condition (1.6.5) holds for the functions $1, \varphi_1, \ldots, \varphi_{n+1}$ because the matrix Φ is diagonal with $\Phi_{ii} = -1/2$, hence is negative, $h = 0$, and $h_0 = [\psi_{n+1}, \psi_{n+1}] < 0$.

1.6.15. Example. We have already mentioned the example in [**46**], [**47**], where the equation $L^*\mu = 0$ has at least two different probability solutions. We can now show that it actually has a countable sequence of linearly independent solutions that are probability measures. We have

$$b^k(x) = -x_k - 2x_{\sigma(k)} \exp\Bigl((x_k^2 - x_{\sigma(k)}^2)/2\Bigr), \quad \varrho(x) = (2\pi)^{-d/2} \exp(-|x|^2/2),$$

where $\sigma\colon \{1, 2, 3, \ldots, d\} \to \{1, 2, 3, \ldots, d\}$ is one-to-one such that $\sigma(k) \neq k$. Then

$$a^k(x) = -2(2\pi)^{d/2} x_{\sigma(k)} \exp\Bigl(-x_{\sigma(k)}^2 - 2^{-1}\sum_{i\neq k} x_i^2\Bigr).$$

Let $\mathbb{R}_+^{d-1} := \mathbb{R}^{d-1} \bigcap \{x_{\sigma(d)} > 0\}$ and $x' := (x_1, x_2, \ldots, x_{d-1})$. Let us take functions $\omega, \psi_i^{**} \in C_0^\infty(\mathbb{R}_+^{d-1})$, where $i \in \mathbb{N}$, with disjoint supports such that

$$-\int_{\mathbb{R}_+^{d-1}} \omega(x')a^d(x')\, dx' = 1.$$

Let

$$\psi_i^*(x') := \psi_i^{**}(x') + \omega(x') \int_{\mathbb{R}_+^{d-1}} \psi_i^{**}(y')a^d(y')\, dy', \quad x' \in \mathbb{R}_+^{d-1}.$$

Note that $\operatorname{div} a = 0$ and

$$\int_{\mathbb{R}_+^{d-1}} \psi_i^*(x')a^d(x')\, dx' = 0 \ \text{ for each } \ i \geq 1.$$

It is easy to see that the functions ψ_i^* are linearly independent. We apply the orthogonalization process in the space $L^2(\mathbb{R}_+^{d-1}, -a^d(x')\, dx')$ to the system $\{\psi_i^*\}_{i\geq 1}$

and obtain functions $\{\psi_i\}_{i\geq 1}$. We observe that the obtained functions have the following properties: $\lim\limits_{|x'|\to\infty} \psi_i(x') = 0$ and

$$\int_{\mathbb{R}^{d-1}_+} \psi_i(x')a^d(x')\,dx' = 0, \quad \int_{\mathbb{R}^{d-1}_+} \psi_i(x')\psi_j(x')a^d(x')\,dx' = 0 \ \text{ for all } \ i,j \geq 1.$$

Let us extend ψ_i to the whole space $\mathbb{R}^{d-1}$ by zero outside $\mathbb{R}^{d-1}_+$. Clearly, we obtain functions from $C_0^\infty(\mathbb{R}^{d-1})$ since ω and ψ^{**} are of compact support in $\mathbb{R}^{d-1}_+$. Let σ be a smooth function such that $\sigma' \in L^1(\mathbb{R}^1)$, $\lim\limits_{n\to\infty} \sigma(n) = 1$, and $\lim\limits_{n\to\infty} \sigma(-n) = 0$. Set $\varphi_i(x) := \psi_i(x')\sigma(x_d)$. According to Theorem 1.6.4, we can construct nonconstant solutions corresponding to the functions φ_i. Similarly to the previous example, we obtain a sequence of solutions $1, v_1, v_2, \ldots$ that are linearly independent.

Finally, we present one more sufficient condition for the existence of a nonconstant positive bounded solution of equation (1.6.2).

1.6.16. Proposition. *Assume that there exists a function* $\varphi \in C_b^2(\mathbb{R}^d)$ *such that*

$$2 \sup_{x\in\mathbb{R}^d} |\varphi(x)| \int_{\mathbb{R}^d} |L_\mu\varphi|\,dx < \int_{\mathbb{R}^d} |\nabla\varphi|^2 \varrho\,dx.$$

Then there exists a bounded positive solution of equation (1.6.2) *which is not constant.*

The following example demonstrates an application of this proposition.

1.6.17. Example. Let $d = 2$ and let $\varrho_1, \varrho_2 \in C^\infty(\mathbb{R}^1) \bigcap L^1(\mathbb{R}^1)$ be positive functions such that $\|\varrho_1\|_{L^1} = \|\varrho_2\|_{L^1} = 1$. Set $\varrho(x,y) := \varrho_1(x)\varrho_2(y)$,

$$\varphi(x,y) := \int_{-\infty}^x \varrho_1(s)\,ds + \int_{-\infty}^y \varrho_2(s)\,ds,$$

$$a^1(x,y) := -2\varrho_2'(y)\varrho_2(y) + c(y)\varrho_2(y), \ \ a^2(x,y) := -2\varrho_1'(x)\varrho_2(x) + d(x)\varrho_1(x).$$

Then div $a = 0$ and

$$L_\mu\varphi(x,y) = (d(x) + c(y))\varrho(x,y).$$

To satisfy the conditions of Proposition 1.6.16 it is enough to have the following estimate:

$$4\int_{\mathbb{R}^2} |d(x) + c(y)|\varrho(x,y)\,dx\,dy < \int_{\mathbb{R}^2} \Big[\varrho_1(x)^2 + \varrho_2(y)^2\Big]\varrho(x,y)\,dx\,dy.$$

1.6.18. Remark. Under the assumptions of Proposition 1.6.16 we have

$$\int_{\mathbb{R}^d} \operatorname{div}(\varrho\nabla\varphi)\,dx = 0.$$

Hence the estimate in the condition of that proposition can be replaced by the following one:

$$L_\mu\varphi \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |\nabla\varphi|^2\varrho\,dx + 2\sup_{x\in\mathbb{R}^d}|\varphi(x)|\int_{\mathbb{R}^d}(a,\nabla\varphi)\,dx > 0.$$

It should be noted that it remains unknown whether the equation $L^*\mu = 0$ can have only finitely many (but more than one) linearly independent probability solutions in the case of smooth b. For singular b such examples can be easily constructed on the real line.

1.7. Uniqueness problems

Here we give sufficient conditions on A and b that ensure that $\mathcal{M}_{\rm ell}^{A,b}$ contains at most one element (the sets $\mathcal{M}_{\rm ell}^{A,b}$ and $\mathcal{M}_{\rm ell,md}^{A,b}$ are defined in (1.1.2) and (1.5.9), respectively). For the best possible results it is necessary to separate the issue of uniqueness from that of existence. We have seen above that in the case $d=1$ for any locally Lebesgue integrable function b the equation $L_{{\rm I},b}^*=0$ has at most one solution in the class of probability measures. First we mention a result from [**4**].

1.7.1. Theorem. *Let $|b|\in L^p_{\rm loc}(\mathbb{R}^d)$ for some $p>d\ge 2$. Then there exists at most one measure $\mu\in\mathcal{M}_{\rm ell}^{{\rm I},b}$ such that its density ϱ has the property that*

$$|b\varrho-\nabla\varrho|\in L^1(\mathbb{R}^d).$$

No examples of non-uniqueness are known for the case $A={\rm I}$ if b is smooth and $|b|\in L^1(\mu)$.

There are also results that do not use any assumptions on the logarithmic gradient of a solution.

1.7.2. Theorem. *Assume that conditions* (A1) *and* (A2) *in* (1.5.2) *are fulfilled and that Ω is connected. Then*

$$\mathcal{M}_{\rm ell,md}^{A,b}\ne\emptyset \implies \#\mathcal{M}_{\rm ell}^{A,b}=1.$$

We note that the converse result is false for $d=1$ in any case (see Remark 1.7.19 below).

The importance of this result is seen from the fact that the uniqueness of solution is ensured by information on *some* solution.

1.7.3. Corollary. *Under the assumptions of Theorem* 1.7.2*, if there exists a measure $\mu\in\mathcal{M}_{\rm ell}^{A,b}$ invariant with respect to the corresponding semigroup $(T_t^\mu)_{t\ge0}$, then $\#\mathcal{M}_{\rm ell}^{A,b}=1$.*

1.7.4. Remark. Suppose that Ω is connected. Then, under the assumptions of Theorem 1.7.2, $\#\mathcal{M}_{\rm ell}^{A,b}\le 1$ if and only if for any $\mu,\nu\in\mathcal{M}_{\rm ell}^{A,b}$, one has $|\mu-\nu|\in\mathcal{M}_{\rm ell}^{A,b}$. Indeed, the indicated condition yields that $\mu=\nu$, since otherwise $|\mu-\nu|$ must have a strictly positive continuous density. The converse is trivial.

This leads us to the following question: if a signed measure μ satisfies the equation $L_{A,b}^*\mu=0$, then does $|\mu|$ also satisfy the same equation? In general, this is not true even if $A={\rm I}$ and b is smooth. Indeed, as explained above, in the situation where any probability measure satisfying the equation $L_{A,b}^*\mu=0$ possesses a positive continuous density, which is the case if $a^{ij}\in W^{p,1}_{\rm loc}(\mathbb{R}^d)$, $|b|\in L^p_{\rm loc}(\mathbb{R}^d)$ with $p>d$, and A is nondegenerate, the non-uniqueness of solutions to this equation in $\mathcal{P}(\mathbb{R}^d)$ always yields signed solutions whose absolute values are not solutions. Of course, this is not surprising for locally integrable solutions. For example, the absolute value of a harmonic function may not be harmonic, but for globally integrable solutions this phenomenon is more interesting. It is worth noting here that if μ is an invariant measure for a semigroup $(T_t)_{t\ge0}$ whose generator extends $(L_{A,b},C_0^\infty)$, then $|\mu|$ is also an invariant measure, which again exhibits some difference between invariant measures for semigroups and infinitesimally invariant measures. As we shall see below, under our typical assumptions on A and b involving Lyapunov functions, the equation $L_{A,b}^*\mu=0$ has one solution in $\mathcal{P}(\mathbb{R}^d)$. However, we do not know whether in such a case the space of all solutions in the class of bounded signed measures is one-dimensional.

We know from the previous section that it can really happen for $\Omega = \mathbb{R}^d$ that $\#\mathcal{M}_{\rm ell}^{A,b} > 1$. The above results give conditions on A and b under which $\#\mathcal{M}_{\rm ell}^{A,b} = 1$. Other results will be given below. However, the following question (already touched upon in Remark 1.5.18) arises:

Can the semigroup $(T_t^\mu)_{t\ge 0}$ have invariant measures if μ is not invariant or can it have invariant measures distinct from μ in the case when μ is invariant?

The following result from [**33**] answers this question in the case $p > d+2$ (this result complements an earlier result from [**4**] and under only local assumptions gives an affirmative answer to a question posed by S.R.S. Varadhan in [**174**]). The same is likely to hold with our standard assumption that $p > d$.

1.7.5. THEOREM. *Suppose that $\mu \in \mathcal{M}_{\rm ell}^{A,b}$ and that* (A1) *and* (A2) *in* (1.5.2) *are fulfilled. Assume also that $p > d+2$. Then, there exist sub-probability kernels $K_t(\,\cdot\,, dy)$, $t > 0$, on Ω such that*

$$K_t(x, dy) = p_{A,b}(t, x, y)\, dy,$$

where $p_{A,b}(t, x, y)$ is a locally Hölder continuous nonnegative function defined on the set $(0, +\infty) \times \Omega \times \Omega$, and for every $f \in L^1(\Omega, \mu)$, the function

$$x \mapsto K_t f(x) := \int_\Omega f(y)\, p_{A,b}(t, x, y)\, dy$$

is a μ-version of $T_t^\mu f$ such that the function $(t, x) \mapsto K_t f(x)$ is continuous on the set $(0, +\infty) \times \Omega$. The function $p_{A,b}$ is positive if Ω is connected.

In addition, if Ω is connected and ν is a bounded Borel measure on Ω that is invariant for $(K_t)_t \ge 0$, i.e.,

$$\nu = K_t^* \nu(dy) := \int_\Omega K_t(x, dy)\, \nu(dx) \quad \forall\, t \ge 0,$$

then $\nu = c\mu$ for some constant c. In particular, if $\nu \ne 0$, then μ is also invariant. Hence $(K_t)_{t\ge 0}$ cannot have invariant probability measures different from μ.

1.7.6. REMARK. By this theorem, the semigroup $(T_t^\mu)_{t\ge 0}$ is strong Feller: it takes bounded Borel functions to continuous functions. In addition, it is stochastically continuous in the sense that $\lim\limits_{t\to 0} T_t^\mu I_{B(z,r)}(x) = 1$ for every $z \in \Omega$ and $r > 0$ such that $B(z, r) \subset \Omega$. This follows from the last assertion in Theorem 1.5.7 and the estimate $T_t^\mu I_{B(z,r)} \ge T_t^\mu \psi$, where $\psi \in C_0^\infty(\Omega)$ is such that $0 \le \psi \le 1$, $\psi(z) = 1$ and $\psi = 0$ outside $B(z, r)$. If Ω is connected, we have the equivalence of all measures

$$B \mapsto T_t^\mu I_B(x) = K_t^* \delta_x(B) = \int_B p_{A,b}(t, x, y)\, dy.$$

Therefore, if μ is invariant for $(T_t^\mu)_{t\ge 0}$ (which is not automatically fulfilled in our situation!), then, by Doob's theorem (see [**69**] or [**67**, §4.2]), for every Borel set $B \subset \Omega$ we obtain

$$\lim_{t\to\infty} T_t^\mu I_B(x) = \mu(B) \quad \forall\, x \in \Omega,$$

where the jointly continuous version of $T_t^\mu I_B(x)$ is considered. Certainly, this yields that $\lim\limits_{t\to\infty} \|T_t^\mu f - f\|_{L^p(\mu)} = 0$ for all $f \in L^p(\mu)$.

1.7.7. REMARK. Thus, one of the main results of this section is the following alternative under local assumptions (A1) and (A2), in case when the equation $L_{A,b}^* \mu = 0$ has a solution μ in the class of probability measures: either μ is a unique probability measure satisfying this equation and its associated semigroup $(T_t^\mu)_{t\ge 0}$ is unique, and if μ is invariant for $(T_t^\mu)_{t\ge 0}$ (which is not implied automatically by the uniqueness of μ even on the real line, see Remark 1.7.19), then there are no other invariant probability measures for $(T_t^\mu)_{t\geq 0}$; or no probability measure μ satisfying this equation is invariant for its semigroup $(T_t^\mu)_{t\geq 0}$, and then for every such measure μ necessarily there are different associated semigroups in $L^1(\mu)$. An important qualitative conclusion is that, under reasonable local assumptions on A and b, one has uniqueness of invariant probability measures for our special associated semigroups, but not uniqueness for the elliptic equation, and that better smoothness of the coefficients does not help to obtain uniqueness of solutions for the elliptic equation.

Let us make a simple observation on extreme points of the convex set $\mathcal{M}_{\rm ell}^{A,b}$. Let ext$\mathcal{M}_{\rm ell}^{A,b}$ be the set of all $\mu \in \mathcal{M}_{\rm ell}^{A,b}$ which cannot be written as a nontrivial convex combination of two other elements of $\mathcal{M}_{\rm ell}^{A,b}$.

1.7.8. Proposition. *Let* $\mu \in \mathcal{M}_{\rm ell}^{A,b}$. *Then the following are equivalent*:
(i) $\mu \in {\rm ext}\mathcal{M}_{\rm ell}^{A,b}$;
(ii) *if* $\varrho \in L^\infty(\Omega,\mu)$ *and* $\varrho\cdot\mu \in \mathcal{M}_{\rm ell}^{A,b}$, *then* $\varrho = 1$ μ*-a.e.*

1.7.9. Corollary. *Assume that conditions* (A1) *and* (A2) *in* (1.5.2) *hold. Let* $\mu \in {\rm ext}\mathcal{M}_{\rm ell}^{A,b}$. *Then for all* $\nu \in \mathcal{M}_{\rm ell}^{A,b}\setminus\{\mu\}$ *the function* $d\nu/d\mu$ *is unbounded.*

Clearly, it can happen that $\mathcal{M}_{\rm ell}^{A,b} = \emptyset$ (for instance, if $\Omega = \mathbb{R}^d$, $A = {\rm I}$, $b \equiv 0$). But even in the case when $\Omega = \mathbb{R}^d$, $A = {\rm I}$, and b is infinitely differentiable, it can also happen that $\#\mathcal{M}_{\rm ell}^{A,b} > 1$, and, therefore, $\mathcal{M}_{\rm ell,md}^{A,b}$ is empty (see Example 1.6.3).

We now consider some examples borrowed from [**47**].

1.7.10. Example. Let conditions (A1) and (A2) in (1.5.2) be fulfilled and let Ω be connected. Suppose that there exist an unbounded compact function $V \in C^2(\Omega)$, a number $\alpha > 0$, and a compact set K such that

$$L_{A,b}V(x) \leq \alpha V(x) \quad \text{for a.e. } x \in \Omega\backslash K. \tag{1.7.1}$$

Then $\#\mathcal{M}_{\rm ell}^{A,b} \leq 1$. The analogous result holds for the operator $\mathcal{L}_{A,b}$.

For example, considering the function $V(x) = \ln(|x|^2+1)$ on $\mathbb{R}^d$ and noting that

$$L_{A,b}V(x) = \frac{2{\rm trace}A(x)}{|x|^2+1} - \frac{4\langle A(x)x,x\rangle}{(|x|^2+1)^2} + \frac{2\langle b(x),x\rangle}{|x|^2+1},$$

we arrive at the following result.

1.7.11. Example. Let conditions (A1) and (A2) in (1.5.2) be fulfilled. Suppose that $\Omega = \mathbb{R}^d$ and that there exists a number $C > 0$ such that a.e. outside some ball one has

$$-\frac{2}{1+|x|^2}\langle A(x)x,x\rangle + {\rm trace}A(x) + \langle b(x),x\rangle \leq C|x|^2\ln|x|.$$

Then $\#\mathcal{M}_{\rm ell}^{A,b} \leq 1$. In particular, if A is uniformly bounded, then it suffices to have the estimate

$$\langle b(x),x\rangle \leq C|x|^2\ln|x| \quad \text{a.e. outside some ball.}$$

For example, an estimate

$$|b(x)| \leq c + c|x|\ln|x|$$

is sufficient. However, an estimate $|b(x)| \leq c + c|x|(\ln(2+|x|))^r$ with $r > 1$ is not enough: in Example 1.6.3 we take a smooth function f with $f'(s) = |s|^{-1}(\ln s)^{-r}$ outside $[-2,2]$. Here one can take any positive integrable function f', which will give an example of non-uniqueness with $|b(x)| \leq C/f'(|x|)$.

1.7.12. Remark. (i) The same reasoning as in [**168**, Remark 1.11(ii)] shows that if there exist a bounded function $u \in C^2(\mathbb{R}^d)$ and a number $\alpha > 0$ such that $u > 0$ and $L_{A,b}u \geq \alpha u$, then μ is not invariant for $(T_t^\mu)_{t\geq 0}$.

(ii) Let A satisfy condition (A1) and let $b^i \in W^{p,1}_{loc}(\Omega)$. Let $V \in C^3(\Omega)$ be an unbounded compact function and let $\theta(x) = |L_{A,b}V(x)| + 1$. Then the operator $L_{A/\theta,b/\theta} = \theta^{-1}L_{A,b}$ satisfies conditions (A1) and (A2) and condition (1.7.1), since $\theta^{-1}L_{A,b}V \leq 1$. This shows that condition (1.7.1) can always be obtained by a proper scaling of a given operator (with reasonable coefficients). In addition, we see that there exists at most one probability density $f \in L^1(\Omega,\mu)$ such that $f\cdot\mu \in \mathcal{M}^{A,b}_{\text{ell}}$ and $\theta f \in L^1(\Omega,\mu)$. Indeed, $(\theta^{-1}L_{A,b})^*(\theta f \cdot \mu) = 0$.

In the next example, some information about μ itself is used to conclude that μ is the only element in $\mathcal{M}^{A,b}_{\text{ell}}$. This result gives weaker sufficient conditions than [**168**, Proposition 1.10(a)] (where the proof requires some corrections).

1.7.13. Example. Let conditions (A1) and (A2) in (1.5.2) be fulfilled. Assume that $\Omega = \mathbb{R}^d$, $\mu \in \mathcal{M}^{A,b}_{\text{ell}}$, and

$$\lim_{j\to\infty} \int_{j\leq |x|\leq j+1} \bigl[|a^{ik}| + |b - \beta_{\mu,A}|\bigr]\, d\mu = 0$$

for all i, k. Then $\#\mathcal{M}^{A,b}_{\text{ell}} = 1$. In particular, it suffices to have the integrability of a^{ik} and $|b - \beta_{\mu,A}|$ with respect to μ.

In the case of the operator $\mathcal{L}_{A,b}$ on a connected complete Riemannian manifold M, a sufficient condition for μ to be a unique measure in $\mathcal{M}^{A,b}_{\text{ell}}$ is the relation

$$\lim_{j\to\infty} \int_{B_{j+1}\setminus B_j} \bigl[\|A\| + |b - A\beta_\mu|\bigr]\, d\mu = 0.$$

where B_j is the geodesic ball of radius j centered at a fixed point $o \in M$. In particular, it suffices to have the integrability of $\|A\|$ and $|b - A\beta_\mu|$ with respect to μ.

Finally, we single out a special case of the above example, in which, however, knowledge of β_μ is not necessary.

1.7.14. Example. Let $\Omega = \mathbb{R}^d$ and $\mu \in \mathcal{M}^{A,b}_{\text{ell}}$. Assume that A and A^{-1} are uniformly bounded, A is globally Lipschitzian, and let $|b| \in L^p_{loc}(\Omega)$, where $p > d$. Let $|b| \in L^2(\mathbb{R}^d,\mu)$. Then $\#\mathcal{M}^{A,b}_{\text{ell}} = 1$.

1.7.15. Remark. In the last example, one cannot omit the assumption that $|b| \in L^p_{loc}(\Omega, dx)$, $p > d$, as is clear from Example 1.6.1. If in that example we take $\mu = \varrho\, dx$ with $\varrho(x) = Cx^2\exp(-x^2)$, where C is a normalizing constant, then μ satisfies the equation $L^*_{1,b}\mu = 0$ with $b(x) = 2x^{-1} - 2x$. The probability measure with density g such that $g(x) = f(x)/2$ if $x < 0$ and $g(x) = 3f(x)/2$ if $x \geq 0$ is another solution. For both solutions the operator $L_{1,b}$ is even symmetric and the coefficient b is square-integrable with respect to both measures. One can verify that these two measures are not merely solutions to the same elliptic equation, but are also invariant for the same Markovian semigroup associated to the operator $L_{1,b}$ (in the case under consideration such a semigroup is unique).

The results presented above have the following character: either it is asserted that $\#\mathcal{M}^{A,b}_{\text{ell}} \leq 1$ or it is asserted that $\#\mathcal{M}^{A,b}_{\text{ell}} = 1$, but the existence of some solution is part of the hypotheses. However, the existence results presented in §1.4 in terms of Lyapunov functions yield also the uniqueness of solutions, as is clear from the results of this section.

1.7.16. Example. Let $\Omega = \mathbb{R}^d$, $A = \mathrm{I}$, $b^i \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ and

$$\limsup_{|x|\to\infty} |x|^{\gamma-1}\bigl[2(\gamma-1) + d + \langle b(x), x\rangle\bigr] = -\infty$$

for some $\gamma \ge 1$. Then $\#\mathcal{M}^{A,b}_{\mathrm{ell}} = 1$.

1.7.17. Example. Suppose that $\Omega = \mathbb{R}^1$, A is locally absolutely continuous and positive, and b is locally Lebesgue integrable. If

$$\int_{-\infty}^{0} \frac{1}{\sqrt{A(s)}}\, ds = \infty, \qquad \int_{0}^{+\infty} \frac{1}{\sqrt{A(s)}}\, ds = \infty, \tag{1.7.2}$$

then $\#\mathcal{M}^{A,b}_{\mathrm{ell}} \le 1$.

1.7.18. Remark. Let A be a positive locally absolutely continuous function on $\mathbb{R}^1$ such that (1.7.2) does not hold. Then one can show that there exists a locally integrable function b such that $\#\mathcal{M}^{A,b}_{\mathrm{ell}} = \infty$.

1.7.19. Remark. Let $\Omega = \mathbb{R}^1$, $A(x) = 1$, and $b(x) = -2x - 6e^{x^2}$. Then $\mathcal{M}^{A,b}_{\mathrm{ell}} = \{\mu\}$ with $\mu(dx) = \pi^{-1/2}e^{-x^2}\,dx$, but according to [**168**, Example 1.12] the operator $\bigl(L_{A,b}, C_0^\infty(\mathbb{R}^1)\bigr)$ is not essentially m-dissipative on $L^1(\mathbb{R}^1,\mu)$. Hence the converse to Theorem 1.7.2 is not true in the case $d = 1$.

The presented uniqueness results assumed the Sobolev differentiability of a^{ij}. The case of a general nondegenerate measurable diffusion coefficient has not been studied. For example, the uniqueness of solutions to the equation $L^*_{A,b}\mu = 0$ with uniformly bounded Borel A, A^{-1}, b has not been studied. It is worth noting that if a^{ij} and b^i are locally Hölder continuous, $\det A > 0$, and a probability measure μ is invariant for the semigroup $(T^\mu_t)_{t\ge 0}$ associated with $L_{A,b}$, then it has a density locally uniformly separated from zero and hence is a unique invariant measure (see, e.g., [**122**, §8.1]).

Let us briefly discuss the so-called symmetric case, that is, the case $b = \beta_{A,\mu}$ (or $b = A\beta_\mu$ for operators in divergence form). If $A = \mathrm{I}$, then b is just $\nabla\varrho/\varrho$. This case has attracted particular attention in the literature since the operator $L = L_{\mathrm{I},b}$ becomes symmetric nonpositive. In addition to the L^1-uniqueness of this operator, its essential self-adjointness on $C_0^\infty(\Omega)$ has also been intensively studied. Various results can be found in [**32**], [**53**], [**71**], [**72**], [**103**], [**117**], [**119**], [**151**], [**168**], and in the references therein. We merely note that, as shown in [**32**], if $\Omega = \mathbb{R}^d$ and $A = \mathrm{I}$, then our usual assumption $|b| \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ with $p > d$ is sufficient for the essential self-adjointness (if $d > 3$, then a somewhat weaker local condition is shown to be sufficient in [**117**]; for instance, in terms of L^p_{loc} it suffices that $p = d$). Further improvements are obtained in [**103**], where the following result is established. Let $\varrho \ge 0$ be such that $\sqrt{\varrho} \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^d)$ and $\nabla\varrho/\varrho \in L^4_{\mathrm{loc}}(\mathbb{R}^d, \varrho\,dx)$. Suppose additionally that for every ball B_0 there is $\varepsilon > 0$ such that

$$\sup_{B\subset B_0} \Bigl(|B|^{-1}\int_B \varrho(x)^{1+\varepsilon}\,dx\Bigr)\Bigl(|B|^{-1}\int_B \varrho(x)^{-1-\varepsilon}\,dx\Bigr) < \infty,$$

where sup is taken over all balls $B \subset B_0$ and $|B|$ is the volume of B. Then the operator $L_{\mathrm{I},b}$ with $b = \nabla\varrho/\varrho$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d) \subset L^2(\varrho\,dx)$. It is still unknown whether the single condition $\nabla\varrho/\varrho \in L^4_{\mathrm{loc}}(\mathbb{R}^d, \varrho\,dx)$ is sufficient (but such a global condition is sufficient). It was shown in [**40**] that in the symmetric case, conditions (A1) and (A2) are sufficient for the essential self-adjointness of $L_{A,b}$ on $C_0^\infty(M)$, provided that M is complete with respect to the metric generated by A^{-1}. It was also shown in [**40**] that under the same assumptions the stronger L^p-uniqueness holds for the symmetric operator $L_{A,b}$.

Suppose now that in the symmetric case we have $A = \mathrm{I}$. Let b be a fixed Borel measurable dx-version of the mapping $\nabla\varrho_0/\varrho_0$ for some probability density $\varrho_0 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$, where $\nabla\varrho_0/\varrho_0 := 0$ on the set $\{\varrho_0 = 0\}$, i.e., we take a dx-version of the logarithmic gradient β^{μ_0} of the measure $\mu_0 := \varrho_0\, dx$. Then $\mu_0 := \varrho_0\, dx \in \mathcal{M}^{\mathrm{I},b}_{\mathrm{ell}}$. Our precise specification of the character of a version is due to the fact that the measure μ_0 need not be equivalent to Lebesgue measure, so a μ_0-version of b may fail to be a dx-version. Clearly, $|b| \in L^1_{\mathrm{loc}}(\mu_0)$. In this case we have the following result (see [**4**]) with a much weaker local condition on b than in Theorem 1.7.1, but again we need the global condition (1.5.1).

1.7.20. Theorem. *Let b be of the indicated form and let $|b| \in L^2_{\mathrm{loc}}(\mu_0)$.*

(i) *Let $\mu \in \mathcal{M}^{\mathrm{I},b}_{\mathrm{ell}}$ with $|b| \in L^2_{\mathrm{loc}}(\mu)$ be such that $\varrho := d\mu/dx$ satisfies* (1.5.1). *Then*

$$\frac{\nabla\varrho}{\varrho} = b \quad \mu\text{-a.e.}$$

and $(L, C_0^\infty(\mathbb{R}^n))$ is symmetric on $L^2(\mu)$.

(ii) *If $|b| \in L^1_{\mathrm{loc}}(U)$ for some connected open set $U \subset \mathbb{R}^d$ whose complement has Lebesgue measure zero, then μ_0 is the only measure $\mu \in \mathcal{M}^{\mathrm{I},b}_{\mathrm{ell}}$ such that $|b| \in L^2_{\mathrm{loc}}(\mu)$ and $\varrho = d\mu/dx$ satisfies* (1.5.1).

Assertion (i) gives conditions under which infinitesimally invariant measures are symmetrizing. However, the following question arises: suppose that a probability measure μ solves the equation $L^*_{\mathrm{I},b} = 0$, where $b = \nabla V$ for some $V \in C^\infty(\mathbb{R}^d)$; is it true that b coincides with the logarithmic gradient of μ and then $\mu = c\exp V\, dx$? As the following example suggested by S.V. Shaposhnikov shows, this is not true without additional assumptions.

1.7.21. Example. Let us take the following smooth function on the plane:

$$V(x,y) = -\Bigl(\ln(1+x^2) + \ln(1+y^2) + x + x^3/3 + y + y^3/3\Bigr).$$

Then the measure μ with density $\varrho(x) = (1+x^2)^{-1}(1+y^2)^{-1}$ satisfies the equation $L^*_{\mathrm{I},b}\mu = 0$ with $b = \nabla V$. Indeed, this equation can be written as $\operatorname{div}(\nabla\varrho - \varrho b) = 0$, and for the indicated ϱ we find that $\nabla\varrho - \varrho b = ((y^2+1)^{-1}, (x^2+1)^{-1})$, which obviously has divergence zero.

However, one can show that the answer is positive if either $|\nabla V| \in L^1(\mu)$ or $\exp V \in L^1(\mathbb{R}^d)$.

Concerning uniqueness problems, see also [**2**], [**179**].

1.8. Global properties of densities

We now proceed to some global estimates related to the regularity of invariant measures. Unlike Example 1.2.15, the assumptions on the order of integrability of the drift may be considerably weakened in the case of the global integrability. The following result was obtained in [**31**]. Suppose that a mapping A with values in the space of positive symmetric matrices is uniformly bounded, uniformly Lipschitzian, and that there is a number $\alpha > 0$ such that

$$\langle A(x)h, h\rangle \geq \alpha\langle h, h\rangle, \quad \forall\, x, h \in \mathbb{R}^d.$$

Set

$$a := (a^1, \dots, a^d), \quad a^j := \sum_{i=1}^{d} \partial_i a^{ij}.$$

1.8.1. THEOREM. *Let μ be a Borel probability measure on $\mathbb{R}^d$, let b be a Borel vector fields such that $|b| \in L^2(\mu)$, and let $L_{A,b}^*\mu = 0$. Then*

(1) *$\mu = \varrho\, dx$, where $\varrho = \varphi^2$ and $\varphi \in W^{2,1}(\mathbb{R}^d)$, in particular, $\varrho \in L^{d/(d-1)}(\mathbb{R}^d)$;*

(2) *one has*

$$\frac{1}{4}\int_{\mathbb{R}^d}\Big|\frac{\nabla\varrho}{\varrho}\Big|^2 \varrho\, dx = \int_{\mathbb{R}^d} |\nabla\varphi|^2\, dx \le \frac{1}{4\alpha^2}\int |b+a|^2\, d\mu;$$

(3) *the mapping $\nabla p/p$ coincides μ-a.e. with the orthogonal projection of the vector field $A^{-1}(b-a)$ onto the closure of the set $\{\nabla u |\ u \in C_0^\infty(\mathbb{R}^d)\}$ in the space $L^2(\mu, \mathbb{R}^d)$ equipped with the inner product $\langle\cdot,\cdot\rangle_2$ defined by*

$$\langle F, G\rangle_2 := \int_{\mathbb{R}^d} \langle AF, G\rangle\, d\mu.$$

In particular, if $A = \mathrm{I}$, then these assertions hold with $\alpha = 1$ and $a = 0$.

1.8.2. COROLLARY. *Let a probability measure μ on $\mathbb{R}^d$ satisfy the equation $L_{\mathrm{I},b}^*\mu = 0$, where $|b| \in L^2(\mu)$. The operator $L_{\mathrm{I},b}$ on domain $C_0^\infty(\mathbb{R}^d)$ in $L^2(\mu)$ is symmetric precisely when b coincides μ-a.e. with the logarithmic gradient of μ.*

This theorem increases the global integrability of ϱ, but obviously cannot ensure the uniform boundedness of ϱ. The latter will be obtained below from the inclusion $\varrho \in W^{p,1}(\mathbb{R}^d)$ with $p > d$ under additional assumptions, but already now we can use local estimates in §1.2 to establish a uniform bound on ϱ.

1.8.3. THEOREM. *Let μ be a probability measure on $\mathcal{B}(\mathbb{R}^d)$ such that $L_{A,b}^*\mu = 0$, where A satisfies the conditions listed before the previous theorem and b satisfies the following condition with some $p > d$:*

$$\textit{either}\ \ \sup\nolimits_{x\in\mathbb{R}^d} \|b\|_{L^p(B(x,1))} < \infty \quad \textit{or}\ \ \sup\nolimits_{x\in\mathbb{R}^d} \|b\|_{L^p(B(x,1),\mu)} < \infty.$$

Then the continuous version ϱ of the density of μ is uniformly bounded.

If $|b| \in L^p(\mu)$, then $\varrho \in W^{p,1}(\mathbb{R}^d)$.

PROOF. It follows from §1.2 that the maximum of ϱ on $B(z,1)$ is estimated by a number that depends only on $\sup_x \Big[\|A(x)\| + \|A(x)^{-1}\|\Big]$, the Lipschitzian constant for A, and $\|b\|_{L^p(B(z,1))}$ (or $\|b\|_{L^p(B(z,1),\mu)}$ in the second case), hence is majorized by a number independent of z. If $|b| \in L^p(\mu)$, then the $W^{p,1}$-norm of ϱ on a cube K with unit edge length is estimated by $C(\|\varrho\|_{L^1(Q)} + \|b\|_{L^p(Q,\mu)})$, where Q is the cube with the same center and doubled edge, and C does not depend on the center. Hence the $W^{p,1}$-norm of ϱ is finite on the whole space. □

The last assertion will be strenthened below.

Our next global elliptic regularity result employs the following uniform local condition on A. For $a^{ij} \in W_{\mathrm{loc}}^{1,1}(\mathbb{R}^d)$ we set

$$\Theta_A(x) := \sum_{j=1}^{d}\Big|\sum_{i=1}^{d} \partial_{x_i} a^{ij}(x)\Big|.$$

For given $p > 1$ and $\gamma > 0$ let

$$q = q(d,p,\gamma) = \begin{cases} d & \text{if} \quad p > d/(d-1), \\ d+\gamma & \text{if} \quad p = d/(d-1), \\ p' = p/(p-1) \ \ (> d) & \text{if} \quad p < d/(d-1). \end{cases}$$

We say that A satisfies condition (C1) for $p > 1$ if $a^{ij} \in W^{1,1}_{\rm loc}(\mathbb{R}^d)$ and

$$\lim_{r\to 0} \sup_{z\in\mathbb{R}^d} \int_{B(z,r)} \Theta_A^q(x)\, dx = 0, \tag{1.8.1}$$

where q is defined above (in the case $p = d/(d-1)$ this equality must be fulfilled with $q = d + \gamma$ for some $\gamma > 0$).

We observe that this condition is weaker than

$$\lim_{r\to 0} \sup_{z\in\mathbb{R}^d} \int_{B(z,r)} \widetilde{\Theta}_A^q(x)\, dx = 0, \tag{1.8.2}$$

where

$$\widetilde{\Theta}_A(x) := \sum_{j=1}^d \sum_{i=1}^d |\partial_{x_i} a^{ij}(x)|.$$

It is clear that if there is $p_0 > d > 1$ such that

$$\sup_{z\in\mathbb{R}^d} \int_{B(z,1)} \sum_{i,j=1}^d |\nabla a^{ij}(x)|^{p_0}\, dx < \infty, \tag{1.8.3}$$

then A satisfies condition (C1) (as well as (1.8.2)) for any $p \in (1, p_0)$ and is uniformly continuous (even uniformly Hölder continuous) on all of $\mathbb{R}^d$. In particular, both properties hold if A is uniformly Lipschitzian.

It is worth noting that although in most of our results we assume that $a^{ij} \in W^{p,1}_{\rm loc}(\mathbb{R}^d)$, hence one can write $\mathcal{L}_{A,b}$ as L_{A,b_0} with $b_0^i := b^i + \partial_{x_j} a^{ij}$, the case of $\mathcal{L}_{A,b}$ does not always reduce to that of $L_{A,b}$, because the global integrability conditions on $|b|$ and $|\nabla a^{ij}|$ are different. In some situations, it is easier to deal with divergence form operators, in others the standard form is more convenient. In the manifold case, usually divergence form operators lead to more natural geometric objects. Apparently, the most natural setting for most of the problems discussed should appeal to the geometry related to A and to weighted Sobolev spaces. However, the corresponding techniques, in particular, embedding theorems, is less developed than the classical Sobolev theory.

1.8.4. Theorem. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be such that $L^*_{A,b}\mu = \nu \in W^{p,-1}(\mathbb{R}^d)$ for some $p \in (1, \frac{d}{d-1})$, $|b| \in L^1(|\mu|)$. Suppose that A is uniformly continuous and $c_1 \cdot \mathrm{I} \le A(x) \le c_2 \cdot \mathrm{I}$ for some constants $c_1, c_2 > 0$. Then μ has a density in $L^r(\mathbb{R}^d)$ for every $r \in [1,p]$.*

In the case of $\mathcal{L}_{A,b}$ the same is true under the additional assumption that $a^{ij} \in W^{1,1}_{\rm loc}(\mathbb{R}^d)$ for all i, j and $\partial_{x_i} a^{ij} \in L^1(|\mu|)$ for every j.

The next result is a generalization of [**129**, Theorem 3.1] and a partial generalization of a result in [**31**]. We impose weaker assumptions than in [**129**], where A^{-1} is bounded and $|b| \in L^2(\mu)$ (in addition, in [**129**] the same local assumptions as below are imposed along with a condition which is a bit stronger than (1.8.4)); as compared to [**31**] (where A is uniformly Lipschitzian, A and A^{-1} are uniformly bounded, and $|b| \in L^2(\mu)$), we weaken the assumptions on A, but add an extra local condition on b. That extra condition is not needed if we know in advance that μ has a locally bounded density in $W^{2,1}_{\rm loc}(\mathbb{R}^d)$. It should be noted that unlike most other results in this section, this theorem deals with probability measures and fails for signed measures.

1.8.5. Theorem. *Suppose that* $\mu \in \mathcal{P}(\mathbb{R}^d)$ *satisfies the equation* $\mathcal{L}^*_{A,b}\mu = 0$, *where the mapping* A *is continuous,* $\det A > 0$, $a^{ij} \in W^{p,1}_{\text{loc}}(\mathbb{R}^d)$ *with some* $p > d$, *and* $|b| \in L^p_{\text{loc}}(\mu)$. *Suppose in addition that* $|A^{-1/2}b| \in L^2(\mu)$ *and that*

$$\liminf_{r\to\infty} \int_{r\le|x|\le 2r} \Bigl[r^{-2}\|A(x)\| + r^{-1}\Theta_A(x)\Bigr]\, \mu(dx) = 0. \tag{1.8.4}$$

Then μ *has a density* $\varrho \in W^{2,1}_{\text{loc}}(\mathbb{R}^d)$ *such that*

$$\int_{\mathbb{R}^d} \Bigl|\frac{\sqrt{A}\nabla\varrho}{\varrho}\Bigr|^2\, d\mu \le \int_{\mathbb{R}^d} |A^{-1/2}b|^2\, d\mu. \tag{1.8.5}$$

In particular, under the additional assumption that $A \ge \varepsilon\cdot \mathrm{I}$, $\varepsilon > 0$, *one has* $\sqrt{\varrho} \in W^{2,1}(\mathbb{R}^d)$, $\varrho \in L^{d/(d-2)}(\mathbb{R}^d)$ *if* $d > 2$ *and* $\varrho \in L^s(\mathbb{R}^d)$ *for all* $s \in [1,\infty)$ *if* $d = 2$.

1.8.6. Remark. (i) Condition (1.8.4) is fulfilled if

$$|\nabla a^{ij}(x)| \le C_0 + C_1|x|.$$

If μ is known to have finite first moment, i.e., $|x| \in L^1(\mu)$, then quadratic growth of $|\nabla a^{ij}|$ is allowed.

(ii) Condition (1.8.4) can be replaced by the assumption that for some $r > 0$ one has

$$\liminf_{R\to\infty} \int_{R\le|x|\le R+r} \Bigl[\|A(x)\| + \Theta_A(x)\Bigr]\, \mu(dx) = 0.$$

This condition is weaker on the part of Θ_A, but is stronger on the part of $\|A\|$; for uniformly bounded A, it is weaker.

(iii) Note also that if A is uniformly bounded and satisfies (C1) with (1.8.1), then (1.8.4) is ensured by the assumption that $\liminf\limits_{r\to\infty} r^{d-1}\mu\bigl(\{|x|\ge r\}\bigr) = 0$, which is fulfilled, e.g., if $|x|^{d-1} \in L^1(\mu)$. The latter can be effectively verified in terms of A and b by the Lyapunov function method.

Estimate (1.8.5) can be regarded as the estimate

$$\int \Bigl\langle \frac{\nabla\varrho}{\varrho}, \frac{\nabla\varrho}{\varrho}\Bigr\rangle\, d\mu \le \int \langle b, b\rangle\, d\mu$$

with respect to the Riemannian geometry generated by A. Such an estimate was indeed obtained in [**48**], [**49**] for a broad class of Riemannian manifolds (concerning diffusion operators on manifolds, see also [**115**]).

1.8.7. Theorem. *Let* M *be a Riemannian manifold with Riemannian volume measure* λ *such that the Ricci curvature is bounded from below and the Riemannian volumes of balls of any fixed positive radius are bounded away from zero. Let* μ *be a Borel probability measure on* M *such that* $L^*\mu = 0$, *where* $Lf = \Delta f + \langle b, \nabla f\rangle$ *and* $|b| \in L^2(\mu)$. *Then* $\mu = \varrho\cdot\lambda$, *where* $\sqrt{\varrho} \in W^{2,1}(M)$ *and*

$$\int_M \frac{|\nabla\varrho|^2}{\varrho^2}\, d\mu \le \int_M |b|^2\, d\mu.$$

If in place of $|b| \in L^2(\mu)$ *we have* $|b| \in L^2(\lambda)$, *then*

$$\int_M \frac{|\nabla\varrho|^2}{\varrho^2}\, d\lambda \le \int_M |b|^2\, d\lambda.$$

Actually, the technical conditions imposed on M in [**48**] are even broader and are expressed in terms of the heat semigroup. However, as noted in [**49**, Remark 2.5(ii)], this estimate may fail for general Riemannian manifolds even if $b = 0$. Namely, there exist complete connected Riemannian manifolds on which there are nonconstant positive integrable harmonic functions; such a function defines a measure satisfying our equation with $b = 0$, and the above estimates fail for it. In the situation of Theorem 1.8.5, we do not know whether the natural estimate (1.8.5) holds without any extra local assumptions on b and without (1.8.4). However, there is an important special case when (1.8.4) is not needed.

1.8.8. THEOREM. *Let A be continuous, $\det A > 0$, $a^{ij} \in W^{p,1}_{\rm loc}(\mathbb{R}^d)$, and $|b| \in L^p_{\rm loc}(\mathbb{R}^d)$ for some $p > d$. Suppose that there exists a quasi-compact function $V \in W^{2,2}_{\rm loc}(\mathbb{R}^d)$ such that*

$$\mathcal{L}_{A,b}V(x) \to -\infty \quad \text{as } |x| \to +\infty.$$

Assume also that there are numbers $c_1, c_2 > 0$ such that

$$\mathcal{L}_{A,b}V \le c_1 - c_2|A^{-1/2}b|^2$$

outside some ball. Then there exists a measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ with a positive density $\varrho \in W^{p,1}_{\rm loc}(\mathbb{R}^d)$ such that $\mathcal{L}^*_{A,b}\mu = 0$ and $|\sqrt{A}\nabla\varrho|^2/\varrho \in L^1(\mathbb{R}^d)$.

If, in addition, there exists a positive Borel function θ on $[0, +\infty)$ such that $\lim\limits_{t\to\infty} \theta(t) = +\infty$ and

$$\mathcal{L}_{A,b}V \le c_1 - c_2\theta(|A^{-1/2}b|)|A^{-1/2}b|^2$$

outside some ball, then $|A^{-1/2}b| \in L^2(\mu)$ and (1.8.5) holds.

1.8.9. THEOREM. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be such that $\mathcal{L}^*_{A,b}\mu = \nu$. Suppose that*

(a) *$A \ge \varepsilon \mathrm{I}$ with some $\varepsilon > 0$,*

$$a^{ij} \in W^{\alpha,1}_{\rm loc}(\mathbb{R}^d), \text{ and either } |b| \in L^\alpha_{\rm loc}(\mathbb{R}^d) \text{ or } |b| \in L^\alpha_{\rm loc}(|\mu|), \text{ where } \alpha > d,$$

(b) *$|b|, \operatorname{trace} A \in L^\beta(|\mu|)$, where $\beta > 1$,*

(c) *A satisfies condition* (C1) (*see* (1.8.1)) *for the number β in* (b) *and is uniformly continuous.*

Assume also that the density ϱ of μ belongs to $L^{\beta_0}(\mathbb{R}^d)$ for some $\beta_0 > 1$, which is automatically the case in (i)–(iii) *below if A is bounded and $\partial_{x_i} a^{ij} \in L^\beta(|\mu|)$ for each j.*

(i) *Let $1 < \beta < d$ and let $\nu \in W^{\theta,-1}$ for all $\theta \in \bigl(1, \frac{d}{d-\beta+1}\bigr)$. Then $\varrho \in W^{r,1}(\mathbb{R}^d)$ for all $r \in \bigl(1, \frac{d}{d-\beta+1}\bigr)$. Moreover, if μ is nonnegative, then the same is true for $r = \frac{d}{d-\beta+1}$.*

(ii) *Let $\beta = d$ and $\nu \in W^{\theta,-1}(\mathbb{R}^d)$ for all $\theta \in (1, d)$. Then $\varrho \in W^{r,1}(\mathbb{R}^d)$ for all $r \in (1, d)$.*

(iii) *Let $d < \beta \le \alpha$ and $\nu \in W^{\theta,-1}(\mathbb{R}^d)$ for all $\theta \in (1, \beta]$. Then $\varrho \in W^{r,1}(\mathbb{R}^d)$ for any $r \in (1, \beta]$. In particular, $\varrho \in L^\infty(\mathbb{R}^d)$.*

The same is true in the case of $L_{A,b}$ provided that one has, in addition, $\partial_{x_i} a^{ij} \in L^\beta(|\mu|)$ for each j.

1.8.10. COROLLARY. *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ satisfy the equation $\mathcal{L}^*_{A,b}\mu = 0$. Suppose that there is a number $\alpha > d$ such that $|b|, \operatorname{trace} A \in L^\alpha(\mu)$, $a^{ij} \in W^{\alpha,1}_{\rm loc}(\mathbb{R}^d)$ are uniformly continuous,* (C1) *and* (1.8.4) *hold, and $A \ge \varepsilon \mathrm{I}$, $\varepsilon > 0$. Then $\mu = \varrho\, dx$, where $\varrho \in W^{\alpha,1}(\mathbb{R}^d)$. In particular, $\varrho \in L^\infty(\mathbb{R}^d)$.*

In particular, the conclusion holds true if one has (1.8.3) *and*

$$A \ge \varepsilon \mathrm{I},\ |b| \in L^\alpha(\mu),\ \operatorname{trace} A \in L^\alpha(\mu),\ (1 + |x|)^{-1}\Theta_A \in L^1(\mu).$$

The case $\beta = 1$ in the above theorem has not been studied so far. In particular, it is unknown whether $\varrho \in W^{1,1}(\mathbb{R}^d)$, i.e., $|\nabla\varrho/\varrho| \in L^1(\mu)$, if $|b| \in L^1(\mu)$ and μ is a probability measure satisfying the equation $L^*_{\mathrm{I},b}\mu = 0$. If $|b| \in L^2(\mu)$, then this is true by Theorem 1.8.1. In addition, it is unknown whether in the same case we have $|\nabla\varrho/\varrho| \in L^p(\mu)$ if $|b| \in L^p(\mu)$ and $p \neq 2$. Additional sufficient conditions for this are given in §1.9. So, without any additional assumptions such as the existence of Lyapunov functions or bounds on the coefficients, the inclusion $\varrho \in W^{1,1}(\mathbb{R}^d)$ is established so far only under the assumption that $|b| \in L^2(\mu)$.

It is clear that ϱ may not belong to the class $W^{p,2}$ unless we require certain regularity of b. The following theorem extends [**129**, Theorem 4.7], where somewhat stronger assumptions on A were employed.

1.8.11. Theorem. *Suppose that* $\alpha \geq 2d$, $A \geq \varepsilon\mathrm{I}$ *with* $\varepsilon > 0$, $a^{ij}, b^i \in W^{\alpha,1}_{\mathrm{loc}}(\mathbb{R}^d)$, *$A$ is uniformly continuous and satisfies condition* (C1) *for* α. *Let* $\mu \in \mathcal{P}(\mathbb{R}^d)$ *satisfy* $\mathcal{L}^*_{A,b}\mu = \nu$, *where* $|b|, \operatorname{trace} A \in L^\alpha(\mu)$ *and* $\nu \in L^r(\mathbb{R}^d)$ *for all* $r \in (1,\alpha]$. *Assume also that* $\operatorname{div} b \in L^\alpha(\mu)$ *and* $|\nabla a^{ij}| \in L^\theta(\mu)$, *where* $\theta \geq \max(2d^2, \alpha)$. *Then* $\mu = \varrho\, dx$, *where* $\varrho \in W^{r,2}(\mathbb{R}^d)$ *for all* $r \in (1, \alpha/2)$. *If* $\alpha > 2d$, *then* $\varrho \in W^{\alpha/2,2}(\mathbb{R}^d)$ *and* $|\nabla\varrho| \in L^\infty(\mathbb{R}^n)$.

If $a^{ij}, |\nabla a^{ij}| \in L^\infty(\mathbb{R}^d)$, *then these assertions are true for any solution* μ *from the class of all measures of bounded variation.*

We now turn to pointwise bounds of solutions. The idea is simple: in order to show that $|\varrho(x)| \leq C\Psi(x)$ for some positive function Ψ, one has to consider the measure μ_0 with density ϱ/Ψ and verify that this measure satisfies an equation of the type considered in Theorem 1.8.9. This idea was employed in [**129**] in the case of exponential functions. For further developments we refer to the paper [**34**], where the following results were obtained. Case (iii) of Example 1.8.13 below gives the bound from [**129**] under slightly weaker assumptions.

1.8.12. Theorem. *Suppose that* μ *is a probability measure satisfying the equation* $\mathcal{L}^*_{A,b}\mu = 0$, *where A and b satisfy hypotheses* (a)–(b) *of Theorem* 1.8.9 *for some* $\alpha = \beta > d$. *Let* $\Phi \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ *be a positive function such that for some* $\theta > d$ *and all* $j = 1, \ldots, d$ *one has*

$$\Phi \in L^1(\mu),\ |\nabla\Phi| \in L^\theta(\mu),\ \partial_{x_i} a^{ij} \in L^d(\mu).$$

Then the density ϱ *of* μ *satisfies the estimate* $\varrho(x) \leq C\Phi(x)^{-1}$ *with some constant* C.

We shall now see that the existence of polynomial or exponential moments of solutions yield a corresponding decay of densities at infinity.

1.8.13. Example. Suppose that $\mu \in \mathcal{P}(\mathbb{R}^d)$ satisfies the equation $\mathcal{L}^*_{A,b}\mu = 0$ and that A is uniformly Lipschitzian and A and A^{-1} are uniformly bounded. Furthermore, assume that $|b| \in L^p(\mu)$ for some $p > d$.

(i) If $\Psi \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ is a positive function such that

$$\Phi \in L^1(\mu),\ |\nabla\Phi| \in L^\theta(\mu), \quad \theta > d,$$

then $\varrho(x) \leq C\Phi(x)^{-1}$.

(ii) Let $k > 1$ and suppose that $|x|^r \in L^1(\mu)$ for some $r > (k-1)d$. Then, letting $\Phi(x) = |x|^k$, we obtain $\varrho(x) \leq C|x|^{-k}$.

(iii) Suppose that

$$\exp(\alpha|x|^\beta) \in L^1(\mu), \quad |b(x)| \leq C_0 + C_1\exp(\alpha_0|x|^\beta),$$

where $\alpha, \beta, C_0, C_1 > 0$, and $\alpha_0 < \alpha/d$. Then, for any $\kappa < \beta/d$, there is a number $C > 0$ such that one has $\varrho(x) \leq C\exp(-\kappa|x|^\beta)$.

It is surprising that the above estimates, very rough at first glance, are in fact sufficiently sharp. Below we shall see that there are lower bounds of the same order.

In a similar manner one obtains upper bounds on $|\nabla\varrho|$.

1.8.14. Proposition. *Suppose that in Theorem* 1.8.12 *we have additionally* $\Phi \in W^{1,2}_{\text{loc}}(\mathbb{R}^d)$ *and*

$$\varrho|\nabla\Phi| \in L^\infty(\mathbb{R}^d), \quad |b|, |\nabla\Phi|, \partial_{x_i} a^{ij}\partial_{x_j}\Phi, L_A\Phi, |A\nabla\Phi| \in L^r(\mu),\ r > 2d.$$

Then $|\nabla\varrho(x)| \le C\Phi(x)^{-1}$.

1.8.15. Example. Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ satisfy the equation $\mathcal{L}^*_{A,b}\mu = 0$, where A is uniformly Lipschitzian, A and A^{-1} are uniformly bounded, and $|b|, \operatorname{div} b \in L^p(\mu)$ for some $p > 2d$.

(i) Let $\Phi(x) = |x|^k$, $k \ge 1$, and let $|x|^m \in L^1(\mu)$, where $m > 2d(k-1)$. Then

$$|\nabla\varrho(x)| \le C(1+|x|)^{-k}.$$

(ii) Let $\Phi(x) = \exp(K|x|^\beta)$ and let $\exp(M|x|^\beta) \in L^1(\mu)$, where $M > 2dK$. Then

$$|\nabla\varrho(x)| \le C\exp(-K|x|^\beta).$$

By using the method of Lyapunov functions, one can give effective conditions for the existence of polynomial or exponential moments for μ. For example, if $A(x) \le \Lambda\mathrm{I}$ and $\langle b(x), x\rangle \le -K < -\Lambda d$ outside some ball, then letting $V(x) = \langle x, x\rangle^\gamma$ with $1 < \gamma < 1 + (K - \Lambda d)/2$, we obtain outside some ball

$$L_{A,b}V(x) \le 2\gamma\langle x, x\rangle^{\gamma-1}\big[\Lambda d + 2(\gamma-1) + \langle b(x), x\rangle\big] \le -\kappa|x|^{2\gamma-2},$$

where $\kappa > 0$. Hence $|x|^{2\gamma-1} \in L^1(\mu)$. Stronger decay of $\langle b(x), x\rangle$, e.g., the estimate $\langle b(x), x\rangle \le c_1 - c_2|x|^r$, implies the exponential integrability (see [**39**], [**129**]). Certainly, the required integrability of the coefficients can be also deduced from such estimates provided we know certain bounds on the coefficients. It is worth noting that some of the conditions on A employed above can be relaxed by using the results in [**107**], [**108**], [**109**].

1.9. Lower estimates

In this section, following [**42**], where the proofs are given, we discuss lower bounds for densities of solutions to elliptic equations of the form

$$\mathcal{L}^*_{A,b}\mu = 0 \tag{1.9.1}$$

for Borel measures μ on $\mathbb{R}^d$, where $\mathcal{L}$ is an elliptic second order operator of divergence form

$$\mathcal{L}\varphi(x) := \partial_{x_i}(a^{ij}(x)\partial_{x_j}\varphi(x)) + b^i(x)\partial_{x_i}\varphi(x).$$

The interpretation of this equation is as usual: the functions a^{ij} and b^i must be integrable on every compact set in $\mathbb{R}^d$ with respect to the measure μ and, for every $\varphi \in C_0^\infty(\mathbb{R}^d)$, we must have the equality

$$\int_{\mathbb{R}^d} \mathcal{L}\varphi\, d\mu = 0.$$

However, the latter can be understood in one of the following two ways.

(I) One has $a^{ij} \in W^{1,1}_{\text{loc}}(\mathbb{R}^d)$, the functions a^{ij}, $\partial_{x_i}a^{ij}$, and b^i are Borel measurable and locally integrable with respect to $|\mu|$, and

$$\int_{\mathbb{R}^d} [a^{ij}\partial_{x_i}\partial_{x_j}\varphi + \partial_{x_i}a^{ij}\partial_{x_j}\varphi + b^j\partial_{x_j}\varphi]\, d\mu = 0. \tag{1.9.2}$$

(II) The measure μ possesses a density ϱ in the class $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ such that the functions $a^{ij}\partial_{x_i}\varrho$ and $b^i\varrho$ are locally Lebesgue integrable and

$$\int_{\mathbb{R}^d} [-a^{ij}\partial_{x_i}\varrho\partial_{x_j}\varphi + b^i\partial_{x_i}\varphi\varrho]\,dx = 0. \tag{1.9.3}$$

Clearly, if the coefficients a^{ij} are locally Sobolev and the functions $\partial_{x_i}a^{ij}\varrho$ are locally integrable, then (1.9.3) can be written as (1.9.2). The divergence form of operators is used only for convenience of formulations; under our standard assumptions on A all the main results of this section can be easily rewritten for the operator $L_{A,b}$.

As above, we assume that $a^{ij} = a^{ji}$ and $A(x)$ is positive.

In the next theorem we suppose that a nonnegative locally bounded measure μ on $\mathbb{R}^d$ has a density ϱ such that $\varrho \in W^{2,1}(U)$ for every ball $U \subset \mathbb{R}^d$. Let the measure μ satisfy equation (1.9.1) on $\mathbb{R}^d$ in the sense of (1.9.3), i.e., in Case (II), where

$$\mathcal{L} = \partial_{x_i}(a^{ij}\partial_{x_j}) + b^i\partial_{x_i},$$

the matrix-valued mapping $A = (a^{ij})_{1\le i,j\le d}$ is measurable, the functions $\|A(x)\|$ and $\|A(x)^{-1}\|$ are locally bounded, and the coefficient $b = (b^i)_{i\le d}$ is a measurable locally bounded vector field. We recall that in the case where $a^{ij} \in W^{p,1}_{\mathrm{loc}}(\mathbb{R}^d)$ and $b^i \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ for some $p > d$ and $A(x)^{-1}$ is locally bounded, any solution of (1.9.1) in the sense of (1.9.2) automatically has a density in $W^{p,1}_{\mathrm{loc}}(\mathbb{R}^d)$, hence it also satisfies (1.9.3).

Let V be a continuous increasing function on $[0,\infty)$ with $V(0) > 0$.

1.9.1. THEOREM. *Let $|b(x)| \le V(|x|/\theta)$, where $\theta > 1$. Set*

$$\alpha(r) := \sup_{|x|\le r} \|A(x)^{-1}\|, \quad \gamma(r) := \sup_{|x|\le r} \|A(x)\|.$$

Then there exists a positive number $K(d)$ depending only on d such that the continuous version of the function ϱ satisfies the inequality

$$\varrho(x) \ge \varrho(0)\exp\bigl\{-K(d)(\theta-1)^{-1}\alpha(\theta|x|)^{-1}\bigl(\gamma(\theta|x|) + V(|x|)|x|\bigr)\bigr\}.$$

In particular, if $\|A(x)\| \le \gamma$ and $\|A(x)^{-1}\| \le \alpha$, then there exists a positive number $K = K(d,\alpha,\gamma,\theta)$ such that the continuous version of the function ϱ satisfies the inequality

$$\varrho(x) \ge \varrho(0)\exp\bigl\{-K\bigl(1 + V(|x|)|x|\bigr)\bigr\}.$$

1.9.2. EXAMPLE. Suppose that

$$\sum_{i,j} |a^{ij}(x)|^2 \le \gamma^2 \quad \text{and} \quad A(x) \ge \alpha\cdot \mathrm{I} \quad \text{for all } x\in\mathbb{R}^d.$$

If, for some numbers $c_1, c_2 > 0$, for almost all x one has the estimate

$$|b(x)| \le c_1|x|^\beta + c_2,$$

then there exists a constant K such that the following inequality holds:

$$\varrho(x) \ge \varrho(0)\exp\bigl\{-K\bigl(1+|x|^{\beta+1}\bigr)\bigr\}.$$

If we have

$$\sup_{x,i,j}\Bigl[\|A(x)\| + \|A(x)^{-1}\| + |\nabla a^{ij}(x)|\Bigr] < \infty,$$

$$|b(x)| \le c_1|x|^\beta + c_2, \quad \limsup_{|x|\to\infty} |x|^{-\beta-1}(b(x),x) < 0,$$

then we obtain the following two-sided estimate:

$$\exp\bigl\{-K_1\bigl(1+|x|^{\beta+1}\bigr)\bigr\} \le \varrho(x) \le \exp\bigl\{-K_2\bigl(1+|x|^{\beta+1}\bigr)\bigr\}.$$

The upper estimate holds if

$$\exp(M|x|^\beta)\in L^1(\mu),\quad |b(x)|\le C_0+C_1\exp(M_0|x|^\beta),\quad 0\le M_0<d^{-1}M,$$

$$\langle b(x),x\rangle\le c_0-c_1|x|^\beta,\quad c_1>M\beta\sup_x\|A(x)\|,$$

where $M>0$ is sufficiently small. For example, if $A=\mathrm{I}$ and $b^i(x)=x_i$, then the measure with density $\varrho(x)=\exp(-|x|^2/2)$ is a solution. The aforementioned results ensure the estimate

$$\exp(-K_1(1+|x|^2))\le\varrho(x)\le\exp(-K_2(1+|x|^2))$$

with some numbers $K_1,K_2>0$, which gives a sufficiently adequate description of the decay at infinity, although it does not yield the precise asymptotics.

It should be noted that the hypothesis that $\limsup_{|x|\to\infty}|x|^{-\beta-1}\langle b(x),x\rangle<0$ is only needed to ensure the integrability of $\exp(M|x|^\beta)$ and can be replaced by the latter (however, its advantage is that it is expressed explicitly in terms of A and b).

The presented results generalize the results obtained in [**129**], where the theory of nonlinear equations (in particular, well-known results of Bernstein) was employed, which required certain additional assumptions on A.

By using the obtained estimates we can give effectively verified conditions for the membership of the logarithmic gradient $\nabla\varrho/\varrho$ of the measure μ in $L^p(\mu)$. In the case $p=2$ simple sufficient conditions were obtained in [**37**], [**31**] and presented above. The first general result for $p>2$ was established in [**129**] (a special was considered in [**61**]). The condition from [**42**] presented in the theorem below improves this result since we do not require the differentiability of the drift coefficient and assume lower regularity of the diffusion coefficient (it is assumed in [**129**] that $a^{ij}\in C^3(\mathbb{R}^d)$ and $b\in C^2(\mathbb{R}^d)$). This weakening of the conditions on the coefficients became possible due to the fact that, unlike in [**129**], the proof in [**42**] did not use methods of the theory of nonlinear equations.

In the next theorem and its corollaries we suppose that μ is a probability measure on $\mathbb{R}^d$ with a continuous positive density ϱ satisfying the elliptic equation (1.9.1) in the sense of (1.9.2), i.e., we deal with Case (I). In particular, the weighed Sobolev class $W^{p,1}(\mu)$ is well-defined.

1.9.3. Theorem. *Let $a^{ij}\in C^{0,\delta}(\mathbb{R}^d)\cap W^{p_0,1}_{\mathrm{loc}}(\mathbb{R}^d)$, $\alpha\cdot\mathrm{I}\le A\le\gamma\cdot\mathrm{I}$, where $\alpha,\gamma,\delta>0$ and $p_0>d$, and let $\lim_{r\to0}\sup_x\|\partial_{x_i}a^{ij}\|_{L^d(U(x,r))}=0$ (the latter holds if A is Lipschitzian). Let a positive continuous function $\Phi\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^1)$ increase on $[0,+\infty)$ such that $\Phi(N+1)\le C\Phi(N)^{1+\varepsilon}$ for some $C,\varepsilon>0$, and let the functions $\Phi(|x|)$ and $\Phi'(|x|)^{p_1}$ with some $p_1>d$ be integrable with respect to the measure μ on $\mathbb{R}^d$. Suppose also that there exist numbers $C_0>0$, $\theta>1$, $p>1$, and $\gamma\in[0,1/d)$ such that*

$$|b(x)|\le C_0\Phi(|x|-\theta)^\gamma,\quad |\nabla a^{ij}(x)|^d\le C_0\Phi(|x|),$$

$$\sum_{N=1}^\infty N^{d-1}\Phi(N)^{-q}<\infty,\quad \textit{where } q:=1-\gamma(p+\varepsilon d)>0.$$

Then $\ln\varrho\in W^{p,1}(\mu)$.

1.9.4. Corollary. *Let $a^{ij}\in C^{0,\delta}(\mathbb{R}^d)\cap W^{p_0,1}_{\mathrm{loc}}(\mathbb{R}^d)$, $\alpha\cdot\mathrm{I}\le A\le\gamma\cdot\mathrm{I}$, where $\alpha,\gamma,\delta>0$ and $p_0>d>1$, and let $\lim_{r\to0}\sup_x\|\partial_{x_i}a^{ij}\|_{L^d(U(x,r))}=0$. Let $p>1$*

be given. Suppose that for some $M > 0$ *and* $\beta > 0$ *the function* $\exp(M|x|^\beta)$ *is integrable with respect to the measure* μ *and that*

$$|b(x)| \le C_0 \exp\{\kappa|x|^\beta\},\ |\nabla a^{ij}(x)| \le C_0 \exp\{\kappa|x|^\beta\},\ \text{where } 0 < \kappa d \max(p, d) < M.$$

Then $\ln \varrho \in W^{p,1}(\mu)$. *In particular, if for every* $\kappa > 0$ *there is a number* $C(\kappa)$ *such that*

$$|b(x)| + |\nabla a^{ij}(x)| \le C(\kappa) \exp\{\kappa|x|^\beta\},$$

then $\ln \varrho \in W^{p,1}(\mu)$ *for all* $p \in [1, +\infty)$.

Taking $\Phi(r) = r^\beta + 1$, one proves the following result.

1.9.5. Corollary. *Let* $a^{ij} \in C^{0,\delta}(\mathbb{R}^d) \cap W^{p_0,1}_{\mathrm{loc}}(\mathbb{R}^d)$, $\alpha \cdot \mathrm{I} \le A \le \gamma \cdot \mathrm{I}$, *where* $\alpha, \gamma, \delta > 0$ *and* $p_0 > d > 1$, *and let* $\lim\limits_{r\to 0} \sup\limits_x \|\partial_{x_i} a^{ij}\|_{L^d(U(x,r))} = 0$. *Let* $p > 1$. *Suppose that for some* $\beta > d$ *the function* $|x|^{\beta d}$ *is integrable with respect to the measure* μ *and that*

$$|b(x)| \le C_0 + C_0|x|^{\beta\gamma},\ |\nabla a^{ij}(x)| \le C_0 + C_0|x|^{\beta/d},\ \text{where } 0 < \gamma < d^{-1}, \gamma < 1 - d\beta^{-1}.$$

Then $\ln \varrho \in W^{p,1}(\mu)$ *for every* $p \in \Big[1, (\beta - d)\beta^{-1}\gamma^{-1}\Big)$.

We now consider lower bounds without assumptions on the growth of the drift coefficient, but using instead a certain integrability of the drift with respect to the solution. Until the end of this section we assume that the matrix $A(x) = (a^{ij}(x))_{1\le i,j\le d}$ is symmetric and satisfies the following condition:

(G1) for some $p > d$ the functions a^{ij} belong to the class $W^{p,1}_{\mathrm{loc}}(\mathbb{R}^d)$ and there exist numbers $m, M > 0$ such that for all $x, y \in \mathbb{R}^d$ we have

$$m|y|^2 \le \sum_{1\le i,j\le d} a^{ij}(x) y_i y_j \le M|y|^2.$$

If in addition to Condition (G1) we have $b^i \in L^p_{\mathrm{loc}}(\mu)$ (or $b^i \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$), then μ is given by a continuous density $\varrho \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$, which we shall deal with. Equation (1.9.1) can be written as the equality

$$\partial_{x_i}\big(a^{ij}\partial_{x_j}\varrho\big) - \partial_{x_i}(b^i\varrho) = 0,$$

understood in the weak sense.

The method of obtaining lower bounds discussed above is not applicable in the case of locally unbounded b. It has been shown in [**44**] that without any restrictions on the growth of b one can obtain estimates of the form

$$\varrho(x) \ge e^{-f(c_1|x| + c_2)}, \tag{1.9.4}$$

where c_1, c_2 are some positive numbers and the function $f \in C^2\big([0,\infty)\big)$ satisfies the conditions

(H1) $f(z) > 0$, $f'(z) > 0$, $f''(z) > 0$ if $z > 0$;

(H2) the function $e^{-f(z)}$ is convex (that is, $(e^{-f})'' \ge 0$) on the set $z > z_0$ for some $z_0 \ge 0$ and it decreases to 0 as $z \to +\infty$.

Namely, for obtaining estimate (1.9.4) it suffices, to require the following conditions in addition to (G1):

(G2) $|b|\exp(\psi(|b|)) \in L^p(\mu)$, where $p > \min\{2, d\}$ and ψ is a nonnegative strictly increasing continuous function mapping $[0,\infty)$ onto $[0,\infty)$ such that

(H3) $\psi^{-1}(z) \le N f'(f^{-1}(z))$ for some $N > 0$ and all $z > 0$.

Let us give several typical examples of the functions f and ψ. Let $\delta > 0$ be a given number. If $f(z) = e^z$, then one can take $\psi(z) = \delta \cdot z$. In this case we obtain the estimate

$$\varrho(x) \geq \exp(-\widetilde{c_2}\exp(\widetilde{c_1}|x|)).$$

If $f(z) = z^{r/(r-1)}$ with $r > 1$, then $\psi(z) = \delta \cdot z^r$ is suitable. Then

$$\varrho(x) \geq \widetilde{c_2}\exp(-\widetilde{c_1}|x|^{r/(r-1)}).$$

In the case where $d = 1$, $A = 1$, and $b = \varrho'/\varrho$ such estimates were obtained in [**140**] (and extended to the case $d > 1$ in [**126**] still with the assumption that $A = \mathrm{I}$ and $b = \nabla\varrho/\varrho$). It follows from (1.9.4) that the solution density has no zeros under a condition weaker than the exponential integrability of $|b|$ (sufficiency of the latter condition was proved in [**35**]). For example, if we set $f(z) = e^{e^z}$ and $\psi(z) = \delta \cdot \dfrac{z}{|\ln z|^\kappa}$ for $z > 2$ and $0 < \kappa < 1$, then we obtain a condition that is sufficient for positivity but is weaker than the exponential integrability of $|b|$. If $d = 1$, $A = 1$, and $b = \varrho'/\varrho$, then this new sufficient condition for positivity is close to the one obtained in [**153**], and in a sense the latter cannot be improved.

Let $V = e^f/f'$.

Since $(e^{-f})'' = [(f')^2 - f'']e^{-f} \geq 0$ on $[z_0, +\infty)$, we have

$$V' = [(f')^2 - f'']e^{-f}(f')^{-2} \geq 0 \quad \text{on } [z_0, +\infty).$$

In addition, V increases to $+\infty$ since the function $1/V = f'e^{-f}$ cannot be separated from zero on $[0, +\infty)$. It follows from conditions (H1) and (H3) that $f'(y) \to +\infty$ as $y \to +\infty$. Therefore, there exists $y_0 > \max\{z_0, 1\}$ such that $f'(y) \geq 1$ and $V(y) \geq e^{\psi(0)}$ whenever $y > y_0$. Let $\tau_0 := \exp\{-f(\ln y_0)\}$. Then $0 < \tau_0 < 1$.

We fix a cub Q of unit edge. Let

$$\Lambda := \min\{\tau_0(2\|\varrho\|_{L^\infty(\mathbb{R}^d)})^{-1}, 1\}.$$

1.9.6. Theorem. *Let $\mu = \varrho\,dx$ be a solution of equation* (1.9.1)*, where the coefficients a^{ij}, b^i satisfy conditions* (G1)*,* (G2) *and let conditions* (H1)*,* (H2)*, and* (H3) *be fulfilled. Then there exist numbers $C > 0$ and $\alpha > 0$ such that for every measurable subset $E \subset Q$ one has*

$$\sup_{x\in Q}\exp(f^{-1}(|\ln(\Lambda\varrho)|)) \leq C\Bigl(\int_E \exp(-\alpha f(|\ln(\Lambda\varrho)|)\,dx)\Bigr)^{-1/\alpha},$$

where the numbers C and α depend only on the following quantities:

$$p,\ N,\ N_1,\ \tau_0,\ m,\ M,\ d,\ \|\varrho\|_{L^\infty(\mathbb{R}^d)},\ \int_{\mathbb{R}^d}|b|^p e^{p\psi(|b|)}\varrho\,dx.$$

1.9.7. Theorem. *Let $\mu = \varrho\,dx$ be a solution of equation* (1.9.1)*, where the coefficients a^{ij}, b^i satisfy conditions* (G1)*,* (G2) *and let conditions* (H1)*,* (H2)*, and* (H3) *be fulfilled. Then there exist numbers $c_1 > 0$ and $c_2 > 0$ such that*

$$\varrho(x) \geq e^{-f(c_1|x| + c_2)}, \quad x \in \mathbb{R}^d.$$

This result gives lower bounds for the density of the stationary measure of the diffusion process with diffusion coefficient $\sqrt{2A}$ and drift b.

1.9.8. Example. Let condition (G1) be fulfilled and let a number $r > 1$ be given.

(i) In order to obtain the estimate

$$\varrho(x) \geq \widetilde{c_2}\exp(-\widetilde{c_1}|x|^{r/(r-1)}), \tag{1.9.5}$$

it suffices to have $\exp(\delta|b|^r) \in L^1(\mu)$ for some $\delta > 0$.

Indeed, the function $\psi(z)=\delta z^r/(2p)$ satisfies condition (H3) with

$$f(z)=z^{r/(r-1)}.$$

There exists a number $C(\delta)>0$ such that $|z|\le C(\delta)\exp(\delta|z|^r/2)$. Then

$$\bigl(|b|\exp(\delta|b|^r/(2p))\bigr)^p \le C(\delta)^p\exp(\delta|b|^r)$$

and so $|b|\exp(\delta|b|^r/(2p))\in L^p(\mu)$, that is, condition (G2) is fulfilled.

(ii) In order to obtain the estimate

$$\varrho(x)\ge \exp(-\widetilde{c_2}\exp(\widetilde{c_1}|x|)), \tag{1.9.6}$$

it suffices that $\exp(\delta|b|)\in L^1(\mu)$ for some $\delta>0$.

Indeed, whenever $0<\delta_1<\delta$, the functions $\psi(z)=\delta_1\cdot z$ and $f(z)=e^z$ satisfy (H3) with $N=1/\delta_1$ and (G2) is fulfilled as well.

1.9.9. Example. Let $\mu=\varrho\,dx$ be a probability measure and let $\varrho\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$. Then μ obviously satisfies equation (1.9.1) with $A=\mathrm{I}$ and $b=\nabla\varrho/\varrho$, where $b(x):=0$ if $\varrho(x)=0$. Therefore, to get estimate (1.9.5) it suffices that $\exp(\delta|\nabla\varrho/\varrho|^r)\in L^1(\mu)$ for some $\delta>0$, and estimate (1.9.6) follows from the inclusion $\exp(\delta|\nabla\varrho/\varrho|)\in L^1(\mu)$ for some $\delta>0$.

For $d=1$ the assertion in the last example was obtained in [**140**] (where in the case $r=1$ the formulation contains a minor inaccuracy: $\widetilde{c_1}$ is replaced by 1; but the function $\varrho(x)=\exp\bigl(-\exp(2|x|)\bigr)$ shows that one cannot get rid of $\widetilde{c_1}$). For $d>1$ and $r=1$ the assertion of the last example is given in [**22**, Exercise 6.8.4]; in [**126**] the case $r>1$ is considered. However, the methods of [**140**] and [**126**] employ in a very essential way the fact that b is $\nabla\varrho/\varrho$.

CHAPTER 2

Parabolic equations for measures

2.1. A priori estimates

In this chapter we consider parabolic equations of the form

$$L^*\mu = 0 \tag{2.1.1}$$

for Borel measures μ on $\mathbb{R}^d \times (0,1)$. Here L is a second order parabolic operator

$$Lu(x,t) := \frac{\partial u(x,t)}{\partial t} + a^{ij}(x,t)\partial_{x_i}\partial_{x_j}u(x,t) + b^i(x,t)\partial_{x_i}u(x,t),$$

where $A(x,t) := (a^{ij}(x,t))_{i,j\le d}$ is a nonnegative symmetric matrix of dimension d and $b(x,t) := (b^i(x,t))_{i\le d}$ is a vector in $\mathbb{R}^d$, and the interpretation of our equation is as follows. We shall say that a family of Radon measures $\mu = (\mu_t)_{t\in(0,1)}$ on $\mathbb{R}^d$ satisfies the weak parabolic equation (2.1.1) if the functions a^{ij} and b^i are integrable on every compact set in $\mathbb{R}^d\times(0,1)$ with respect to the measure $\mu(dt\,dx) := \mu_t(dx)\,dt$ on $\mathbb{R}^d \times (0,1)$ (below by μ we denote also the measure $\mu_t(dx)\,dt$) and, for every function $u \in C_0^\infty(\mathbb{R}^d \times (0,1))$, one has

$$\int_0^1\int_{\mathbb{R}^d} Lu(x,t)\,\mu_t(dx)\,dt = 0. \tag{2.1.2}$$

Therefore, the interpretation is the same as in the elliptic case. For divergence form operators

$$\mathcal{L}u(x,t) := \frac{\partial u(x,t)}{\partial t} + \partial_{x_i}(a^{ij}(x,t)\partial_{x_j}u(x,t)) + b^i(x,t)\partial_{x_i}u(x,t)$$

the equation

$$\mathcal{L}^*\mu = 0$$

is defined similarly (here, as in §1.9, two cases are possible).

We shall say that μ satisfies the initial condition $\mu_0 := \nu$ at $t = 0$ if ν is a measure on $\mathbb{R}^d$ and

$$\lim_{t\to 0}\int_{\mathbb{R}^d}\zeta(x)\,\mu_t(dx) = \int_{\mathbb{R}^d}\zeta(x)\,\nu(dx) \tag{2.1.3}$$

for all $\zeta \in C_0^\infty(\mathbb{R}^d)$. In this case we write $\mu = (\mu_t)_{t\in[0,1)}$.

The same definitions are introduced in the case where $\mathbb{R}^d$ is replaced by an open set $\Omega \subset \mathbb{R}^d$ or by an open set in a Riemannian manifold. In particular, in (2.1.2) we take $u \in C_0^\infty(\Omega\times(0,1))$ and in (2.1.3) we take $\zeta \in C_0^\infty(\Omega)$.

Equation (2.1.1) is satisfied for the transition probabilities of the diffusion process with the diffusion matrix $\sqrt{2A}$ and drift b provided such a diffusion exists and the coefficients A and b satisfy certain conditions (see, e.g., [**170**, Chapters 2, 3]; the conditions there can be further relaxed on the basis of recent progress in the theory of equations with VMO coefficients, see, e.g., [**107**], [**108**], [**109**]). This diffusion process may exist in different settings, for example, as a suitable solution to the stochastic differential equation

$$d\xi_t = \sigma(\xi_t)dw_t + b(\xi_t)dt, \quad A = \frac{1}{2}\sigma\sigma^*. \tag{2.1.4}$$

However, (2.1.1) can be considered regardless of any probabilistic assumptions. Moreover, a study of this equation in a purely analytic setting may be useful for constructing an associated diffusion (see [**168**], [**169**]).

It is worth noting that (2.1.2) can be written as

$$\frac{\partial\mu}{\partial t} = \partial_{x_i}\partial_{x_j}(a^{ij}\mu) - \partial_{x_i}(b^i\mu)$$

in the sense of distributions on $\mathbb{R}^d \times (0,1)$. It turns out that under mild restrictions on A and b specified below, any solution μ admits a density ϱ possessing certain Sobolev regularity with respect to x such that this equality can be further rewritten in terms of classical weak solutions.

For a function u on $(0,1)\times\mathbb{R}^d$, we set

$$\partial_t u(x,t) := \partial u(x,t)/\partial t, \quad \nabla u(x,t) = (\partial_{x_1}u(x,t), \ldots, \partial_{x_d}u(x,t)).$$

This section is devoted to some a priori estimates of solutions; the proofs can be found in [**26**].

2.1.1. Lemma. *If* $\mu = (\mu_t)_{t\in[0,1)}$ *satisfies* (2.1.2) *and* (2.1.3), *then, for every* $\zeta \in C_0^\infty(\mathbb{R}^d)$, *for almost all* $t \in [0,1)$ *one has*

$$\int_{\mathbb{R}^d} \zeta(x)\,\mu_t(dx) - \lim_{\varepsilon\to 0}\int_\varepsilon^t \int_{\mathbb{R}^d} L\zeta(x,s)\,\mu_s(dx)\,ds = \int_{\mathbb{R}^d}\zeta(x)\,\nu(dx). \tag{2.1.5}$$

If, for each $\zeta \in C_0^\infty(\mathbb{R}^d)$, *the function* $t \mapsto \int_{\mathbb{R}^d}\zeta(x)\,\mu_t(dx)$ *is continuous on* $[0,1)$, *then* (2.1.5) *holds for all* $t\in[0,1)$ *and is equivalent to* (2.1.1) *and* (2.1.3). *The same is true in the case when our equation is considered on an open set.*

Note that if every μ_t is a probability measure and there is a μ-integrable function Θ such that $L\zeta(x,s) \le \Theta(x,s)$ μ-a.e., then the function

$$h\colon\ s \mapsto \int_{\mathbb{R}^d} L\zeta(x,s)\,\mu_s(dx)$$

is integrable on $[0,t]$ (so that the limit of the integrals over $[\varepsilon,t]$ equals the integral over $[0,t]$). Indeed, in this case the function h, which coincides with the derivative of the continuous version of the function

$$f(s) := \int_{\mathbb{R}^d}\zeta(x)\,\mu_s(dx)$$

on $(0,1)$, is majorized by the integrable function $s \mapsto \int_{\mathbb{R}^d}\Theta(x,s)\,\mu_s(dx)$. Since f is bounded, this implies that the continuous version of f has a finite limit at 0 and is absolutely continuous on $[0,1]$. Certainly, all this is true if the functions a^{ij} and b^i are μ-integrable on every set $B\times[0,1]$, where B is a ball in $\mathbb{R}^d$.

It is worth noting that one of the reasons why we require below that all the s measures μ_t (and not just almost all) be probabilities is that this is the case when one deals with transition probabilities. From the analytic point of view, this is not essential, of course. Another reason is that, as we shall see, this assumption simplifies certain technical issues.

The following lemma from [**26**] is a straightforward extension of [**24**, Lemma 2.2], where $M = 0$ and Θ is a constant.

2.1.2. Lemma. *Let $\mu = (\mu_t)_{t\in[0,1)}$ be a family of probability measures on $\mathbb{R}^d$ satisfying* (2.1.1) *and* (2.1.3), *where ν is a probability measure on $\mathbb{R}^d$. Suppose that there exist a μ-integrable function Θ, a nonnegative function $\Psi \in C^2(\mathbb{R}^d)$, and a number M such that $\Psi \in L^1(\nu)$, $\lim\limits_{|x|\to\infty} \Psi(x) = +\infty$, and*

$$L\Psi(x,t) \le \Theta(x,t) + M\Psi(x) \quad \mu_t\, dt\text{-a.e.} \tag{2.1.6}$$

Then, for a.e. $t \in [0,1)$, one has

$$\begin{aligned}\int_{\mathbb{R}^d} \Psi\, d\mu_t \le \int_{\mathbb{R}^d} \Psi\, d\nu + \int_0^t \int_{\mathbb{R}^d} \Theta\, d\mu_s\, ds \\ + M\exp(Mt)\int_0^t \exp(-Ms)\Big[\int_{\mathbb{R}^d} \Psi\, d\nu + \int_0^s \int_{\mathbb{R}^d} \Theta\, d\mu_r\, dr\Big]\, ds \\ \le (Me^M + 1)\big[\|\Psi\|_{L^1(\nu)} + \|\Theta\|_{L^1(\mu)}\big].\end{aligned} \tag{2.1.7}$$

If $M = 0$ and $\Theta = K$ is constant, then, for a.e. $t \in [0,1)$, one has

$$\int_{\mathbb{R}^d} \Psi(x)\, \mu_t(dx) \le tK + \int_{\mathbb{R}^d} \Psi(x)\, \nu(dx). \tag{2.1.8}$$

Furthermore, if the functions

$$t \mapsto \int \zeta(x)\, \mu_t(dx), \quad \text{where } \zeta \in C_0^\infty(\mathbb{R}^d),$$

are continuous on $[0,1)$, then (2.1.7) *holds for all $t \in [0,1)$, and in the case $M = 0$ so does* (2.1.8).

If (2.1.1) *and* (2.1.3) *are fulfilled on the open set $\{\Psi < c\}$, then the same assertions hold with $\mathbb{R}^d$ replaced by $\{\Psi < c\}$. Finally, the assumption that every μ_t is a probability measure can be replaced by the assumption that $\mu_t \ge 0$ and $\|\mu_t\| \le \|\nu\|$.*

2.1.3. Corollary. *Let $\mu = (\mu_t)_{t\in[0,1)}$ be a family of probability measures on $\mathbb{R}^d$ satisfying* (2.1.1) *and* (2.1.3), *where ν is a probability measure on $\mathbb{R}^d$. Let $\Psi \in C^2(\mathbb{R}^d)$ be a nonnegative function such that*

$$\lim_{|x|\to\infty} \Psi(x) = +\infty \quad \text{and} \quad L\Psi(x,t) \le C + M\Psi(x) \quad \mu_t\, dt\text{-a.e.},$$

where $C \ge 0$ and $M \ge 0$ are constants. Then one can find a nonnegative function $\Psi_0 \in C^2(\mathbb{R}^d)$ such that

$$\Psi_0 \in L^1(\nu),\ \lim_{|x|\to\infty} \Psi_0(x) = +\infty \quad \text{and} \quad L\Psi_0(x,t) \le C + M \quad \mu_t\, dt\text{-a.e.}$$

Moreover, if $\mathfrak{M}$ is a uniformly tight family of probability measures on $\mathbb{R}^d$ and for every $\nu \in \mathfrak{M}$ there exists a solution $\mu^\nu = (\mu_t^\nu)_{t\in[0,1)}$ of the problem $L_\nu^*\mu^\nu = 0$, $\mu_0^\nu = \nu$ in the sense of (2.1.1) *and* (2.1.3), *where each operator L_ν satisfies the same conditions as L, and for some nonnegative compact function $\Psi \in C^2(\mathbb{R}^d)$ one has*

$$L_\nu\Psi(x,t) \le C + M\Psi(x) \quad \mu_t^\nu\, dt\text{-a.e.},$$

then one can find a function Ψ_0 as above such that

$$\sup_{\nu\in\mathfrak{M}} \operatorname*{esssup}_{t\in[0,1)} \int_{\mathbb{R}^d} \Psi_0\, d\mu_t^\nu \le C + M + \sup_{\nu\in\mathfrak{M}} \int_{\mathbb{R}^d} \Psi_0\, d\nu < \infty.$$

If the functions $t \mapsto \int \zeta\, d\mu_t$, where $\zeta \in C_0^\infty(\mathbb{R}^d)$, are continuous on $[0,1]$, then esssup *can be replaced by* sup. *The same assertions are true in the case where $\mathbb{R}^d$ is replaced by $\{\Psi < c\}$.*

Let us consider examples of how (2.1.6) can be verified in terms of the coefficients of L.

2.1.4. EXAMPLE. (i) Suppose that

$$|a^{ij}(x,t)| \le c_1 + c_2|x|^2, \quad \langle b(x,t), x\rangle \le c_3 + c_4|x|^2$$

for some constants c_i. Then, letting $\Psi(x) := |x|^{2k}$, $k > 0$, we obtain $L\Psi \le C + C\Psi$ for a sufficiently large number $C > 0$. Consequently, if a solution μ exists and $|x|^{2k} \in L^1(\mu_0)$, then, for a.e. t, we have

$$\int_{\mathbb{R}^d} |x|^{2k}\,\mu_t(dx) \le e^C \int_{\mathbb{R}^d} |x|^{2k}\,\mu_0(dx) + Ce^C.$$

(ii) Suppose that

$$|a^{ij}(x,t)| \le c_1 + c_2 \ln(|x|^2+1), \quad \langle b(x,t), x\rangle \le c_3 + c_4|x|^2 + c_5|x|^2\ln(|x|^2+1),$$

for some constants c_i. Then, letting $\Psi(x) := \ln(|x|^2+1)$, we find that

$$\partial_{x_i}\Psi(x) = 2x_i(|x|^2+1)^{-1},$$
$$\partial_{x_j}\partial_{x_i}\Psi(x) = 2\delta_{ij} - 4x_ix_j(|x|^2+1)^{-2},$$

which yields

$$\begin{aligned} L\Psi(x,t) &= 2\,\mathrm{trace}\,A(x,t) - 4(|x|^2+1)^{-2}\langle A(x,t)x, x\rangle + 2(|x|^2+1)^{-1}\langle b(x,t), x\rangle \\ &\le C + C\Psi(x) \end{aligned}$$

for a sufficiently large number $C > 0$. Consequently, if a solution μ exists and $\ln(|x|^2+1) \in L^1(\mu_0)$, then, for a.e. t, we have

$$\int_{\mathbb{R}^d} \ln(|x|^2+1)\,\mu_t(dx) \le e^C \int_{\mathbb{R}^d} \ln(|x|^2+1)\,\mu_0(dx) + Ce^C.$$

Moreover, letting $\Psi(x) = |\ln(|x|^2+1)|^2$, we also have $L\Psi \le C + C\Psi$, hence

$$\int_{\mathbb{R}^d} |\ln(|x|^2+1)|^2\,\mu_t(dx) \le e^C \int_{\mathbb{R}^d} |\ln(|x|^2+1)|^2\,\mu_0(dx) + Ce^C.$$

(iii) Suppose that

$$\langle A(x,t)x, x\rangle \le \gamma_1 + \alpha|x|^{2\beta}, \quad \langle b(x,t), x\rangle \le \gamma_2 - (2\alpha ck + \varepsilon)|x|^{2k+2\beta-2}$$

with some positive constants $\gamma_1, \gamma_2, \alpha, \beta, c, k, \varepsilon$. Let $\Psi(x) = \exp\bigl(c|x|^{2k}\bigr)$. Then

$$\begin{aligned} L\Psi(x,t) &= 2ck\,\mathrm{trace}\,A(x,t)|x|^{2k-2}\Psi(x) + 4ck(k-1)\langle A(x,t)x, x\rangle|x|^{2k-4}\Psi(x) \\ &\quad + (2ck)^2\langle A(x,t)x, x\rangle|x|^{4k-4}\Psi(x) + 2ck|x|^{2k-2}\Psi(x)\langle b(x,t), x\rangle \\ &\le c_0 - \varepsilon|x|^{2k+2\beta-2}\Psi(x) \end{aligned}$$

with some constant c_0. Hence, if $\beta \ge 1$ and $\Psi \in L^1(\mu_0)$, then

$$\operatorname*{esssup}_{t\in[0,1)} \int_{\mathbb{R}^d} \exp\bigl(c|x|^{2k}\bigr)\,\mu_t(dx) < \infty.$$

Let us introduce the following conditions on A, b, $p \in [1, +\infty)$, and a bounded open set $B \subset \mathbb{R}^d$:

(CP1) *there exist two numbers $M_1 = M_1(B) > 0$ and $M_2 = M_2(B)$ such that for all i, j one has*

$$A(x,t) \ge M_1 \cdot \mathrm{I} \quad \forall\, (x,t) \in B \times (0,1), \qquad \sup_{t\in(0,1)} \|a^{ij}(\,\cdot\,,t)\|_{W^{p,1}(B)} \le M_2.$$

(CP2) *there exists $M_3 = M_3(B)$ such that for all i one has*

$$\sup_{t\in(0,1)} \|b^i(\,\cdot\,,t)\|_{L^p(B)} \le M_3.$$

It follows from (CP1) and the Sobolev embedding theorem that if $p > d$, then every function a^{ij} has a jointly measurable version such that all functions $x \mapsto a^{ij}(x,t)$, $t \in (0,1)$, are Hölder continuous of order $1 - d/p$ and bounded on B uniformly with respect to t (their Hölder and sup-norms on B are estimated by a constant depending on p, d, B, and M_2). Below we use the same notation a^{ij} for these particular versions.

The main existence result presented below is based on the following lemma.

Let

$$b_0 := A^{-1/2}(b - \Gamma), \quad \Gamma := (\Gamma^1, \ldots, \Gamma^d), \quad \Gamma^j = \partial_{x_i} a^{ij}.$$

2.1.5. Lemma. *Let Ω be a bounded open set in $\mathbb{R}^d$ with a C^1-boundary and volume $|\Omega|$ and let the functions a^{ij} and b^i be uniformly bounded on $\overline{\Omega} \times (0,1)$ along with their first and second derivatives in the first argument. Suppose that $\mu = \varrho(x,t)\,dx\,dt$, where every function $\varrho(\,\cdot\,,t)$, $t > 0$, is nonnegative on Ω with bounded second order derivatives in x on $\Omega \times (0,1)$, and satisfies*

$$\frac{\partial\varrho(x,t)}{\partial t} = \partial_{x_i}\partial_{x_j}(a^{ij}\varrho) - \partial_{x_i}(b^i\varrho)$$

in $\Omega\times(0,1)$. Suppose also that the functions ϱ and $\partial_{x_i}\varrho$ are continuous on $\overline{\Omega}\times[0,1]$, $\varrho(x,0) = \varrho_0(x)$, where $\varrho_0 \in C_0^1(\Omega)$, and

$$\langle A\nabla\varrho(x,t) + [\Gamma(x,t) - b(x,t)]\varrho(x,t), \mathrm{n}_{\partial\Omega}(x)\rangle = 0, \quad (x,t) \in \partial\Omega \times (0,1),$$

where $\mathrm{n}_{\partial\Omega}$ is the outward unit normal on $\partial\Omega$. Then the following inequalities hold:

$$\int_\Omega \varrho(x,t)^2\,dx + \int_0^t \int_\Omega |\sqrt{A}\nabla\varrho(x,s)|^2\,dx\,ds \le \int_\Omega \varrho(x,0)^2\,dx + \int_0^t \int_\Omega |b_0(x,s)|^2 \varrho(x,s)\,dx\,ds, \quad (2.1.9)$$

$$\int_\Omega \varrho(x,t)^2\,dx + \int_0^t \int_\Omega |\sqrt{A}\nabla\varrho(x,s)|^2\,dx\,ds \le e^{t/2}\int_\Omega \varrho(x,0)^2\,dx + \frac{1}{2}e^{t/2}\|b_0\|^4_{L^4(\Omega\times[0,1])}. \quad (2.1.10)$$

If $A(x,t) \ge \alpha\cdot \mathrm{I}$ for some number $\alpha > 0$ and each $\varrho(\,\cdot\,,t)$, $t \in [0,1]$, is a probability density or, more generally, $0 < \mu_t(\Omega) \le \mu_0(\Omega)$, then

$$\int_0^1 \int_\Omega |\sqrt{A}\nabla\varrho(x,t)|^2 \varrho(x,t)^{-1}\,dx\,dt \le 2\int_0^1 \int_\Omega |b_0(x,t)|^2\, \varrho(x,t)\,dx\,dt + 2\int_\Omega \varrho_0(x)\ln\varrho_0(x)\,dx + 2|\Omega|. \quad (2.1.11)$$

2.1.6. Corollary. *In the situation of the above lemma, there is a constant $C(\Omega)$ such that*

$$\int_0^1 \biggl(\int_\Omega |\varrho(x,t)|^{d/(d-2)}\,dx\biggr)^{(d-2)/d} dt \le \frac{C(\Omega)}{\alpha}\biggl(\int_0^1 \int_\Omega |b_0(x,t)|^2\, \varrho(x,t)\,dx\,dt + \int_\Omega \varrho_0(x)\ln\varrho_0(x)\,dx + |\Omega|\biggr) + 2|\Omega|^{(2d-2)/d}$$

if $d > 2$. *In the case* $d \le 2$ *a similar estimate holds with any* $r < \infty$ *in place of* $d/(d-2)$.

Finally, for every $p > d$, *there is a constant* $C(\Omega, p)$ *such that*

$$\int_0^1 \int_\Omega |\sqrt{A}\nabla \varrho(x,t)|^2 \varrho(x,t)^{-1}\, dx\, dt$$

$$\le C(\Omega,p) \sup_{t\in(0,1)} \biggl(\int_\Omega |b_0(x,t)|^p\, dx\biggr)^{2d/(p-d)} + 2\int_\Omega \varrho_0(x)\ln \varrho_0(x)\, dx + C(\Omega,p).$$

2.1.7. Corollary. *Suppose that in Lemma* 2.1.5 *there exist a nonnegative function* $\Psi \in C^2(\mathbb{R}^d)$ *and a constant* M *such that* $\Omega = \{x \in \mathbb{R}^d\colon\ \Psi(x) < c\}$ *and*

$$L\Psi(x,t) \le M + M\Psi(x) \quad \text{and} \quad |b_0(x,t)|^2 \le \Psi(x).$$

Then we have

$$\int_\Omega \varrho(x,t)^2\, dx + \int_0^1 \int_\Omega |\sqrt{A}\nabla\varrho(x,t)|^2\, dx\, dt$$

$$\le 2(Me^M+1)\int_\Omega \Psi(x)\varrho_0(x)\, dx + 2M(Me^M+1) + \int_\Omega \varrho_0(x)^2\, dx,$$

$$\int_0^1 \int_\Omega |\sqrt{A}\nabla \varrho(x,t)|^2 \varrho(x,t)^{-1}\, dx\, dt$$

$$\le 2(Me^M+1)\int_\Omega \Psi(x)\varrho_0(x)\, dx + 2M(Me^M+1) + 2\int_\Omega \varrho_0(x)\ln\varrho_0(x)\, dx + 2|\Omega|.$$

In particular, this is true for $\Psi(x) = |x|^{2k}$ *with* $k \ge 1$ *provided that*

$$\operatorname{trace} A(x,t) \le C + C|x|^2, \quad |b_0(x,t)|^2 \le C + C|x|^{2k}, \quad \langle b(x,t), x\rangle \le C + C|x|^2.$$

2.2. Local regularity

Let J be an interval and let U be an open set in $\mathbb{R}^d$. Let $\mathbb{H}^{p,s}(U,J)$ denote the space of all measurable functions u on $U\times J$ such that $u(\,\cdot\,,t) \in H^{p,s}(U)$ and the norm

$$\|u\|_{\mathbb{H}^{p,s}(U,J)} = \biggl(\int_J \|u(\,\cdot\,,t)\|^p_{H^{p,s}(U)}\, dt\biggr)^{1/p}$$

is finite. The space $\mathbb{H}^{p,s}_0(U,J)$ is defined similarly with $H^{p,s}_0(U)$ instead of $H^{p,s}(U)$, and $\mathbb{H}^{p',-s}(U,J)$ denotes its dual. In connection with parabolic equations, it is useful to introduce also the following spaces. Let $\mathcal{H}^{p,1}(U,J)$ be the space of all functions $u \in \mathbb{H}^{p,1}(U,J)$ with $\partial_t u \in \mathbb{H}^{p,-1}(U,J)$ and finite norm

$$\|u\|_{\mathcal{H}^{p,1}(U,J)} = \|\partial_t u\|_{\mathbb{H}^{p,-1}(U,J)} + \|u\|_{\mathbb{H}^{p,1}(U,J)}.$$

We denote by B_R an open ball of radius $R > 0$ centered at some point in $\mathbb{R}^d$ (in the case where the center is important we use the notation $B(a,R)$). Let $H^{p,2;1}_0(B_R,[0,T])$ denote the closure of the space of smooth functions u on the cylinder $B_{R,T} := B_R \times [0,T]$ that vanish on $(\partial B_R \times [0,T])\bigcup(B_R\times\{0\})$ (i.e., have zero limits on this part of the boundary) with respect to the norm

$$\|u\|_{H^{p,2;1}_0(B_R,[0,T])} = \|u\|_{L^p(B_R\times[0,T])} + \|\partial_t u\|_{L^p(B_R\times[0,T])}$$

$$+ \|\nabla_x u\|_{L^p(B_R\times[0,T])} + \sum_{i,j=1}^d \|\partial_i\partial_j u\|_{L^p(B_R\times[0,T])}.$$

In this section, we extend the local regularity results in the previous chapter to the parabolic case. The proofs of the presented results are given in [**33**]. Let us start again with the existence of densities.

Let $\Omega_T = \Omega \times (0,T)$, $T > 0$, and let $A(\,\cdot\,,\,\cdot\,) = \big(a^{ij}(\,\cdot\,,\,\cdot\,)\big)_{i,j=1}^{d}$ be a Borel mapping on Ω_T with values in the space of nonnegative symmetric operators on $\mathbb{R}^d$.

2.2.1. THEOREM. *Let μ be a locally finite Borel measure on Ω_T such that one has $a^{ij} \in L^1_{\text{loc}}(\Omega_T, \mu)$ and*

$$\int_{\Omega_T} \Big[\partial_t\varphi + a^{ij}\partial_i\partial_j\varphi\Big]\, d\mu \le C(\sup_{\Omega_T}|\varphi| + \sup_{\Omega_T}|\nabla_x\varphi|)$$

for all nonnegative $\varphi \in C_0^\infty(\Omega_T)$. Then the following assertions are true.

(i) *If μ is nonnegative, then $(\det A)^{1/(d+1)}\mu = \varrho\, dx\, dt$, where $\varrho \in L^{(d+1)'}_{\text{loc}}(\Omega_T)$.*

(ii) *If, on every compact set in Ω_T, A is uniformly bounded, uniformly nondegenerate, and Hölder continuous in x uniformly with respect to t, then $\mu = \varrho\, dx\, dt$, where $\varrho \in L^r_{\text{loc}}(\Omega_T)$ for every $r \in [1, (d+2)')$.*

2.2.2. COROLLARY. *Let μ be a locally finite Borel measure on Ω_T such that a^{ij}, b^i, $c \in L^1_{\text{loc}}(\Omega_T, \mu)$, and*

$$\int_{\Omega_T} \Big[\partial_t\varphi + a^{ij}\partial_i\partial_j\varphi + b^i\partial_i\varphi + c\varphi\Big]\, d\mu = 0 \qquad \forall\, \varphi \in C_0^\infty(\Omega_T). \tag{2.2.1}$$

Then statements (i) *and* (ii) *of Theorem* 2.2.1 *are true. In addition, in case* (ii)*, if $J = [T_0, T_1] \subset (0,T)$, B is a ball with compact closure in Ω, and W is a neighborhood of $\overline{B} \times J$ with compact closure, then, for each $r < (d+2)'$, one has*

$$\|\varrho\|_{L^r(B\times J)} \le C(d,r,A,W)\Big(|\mu|(W) + \|c\|_{L^1(W,\mu)} + \|b\|_{L^1(W,\mu)}\Big),$$

where $C(d,r,A,W)$ depends only on d, r, the Hölder norms of a^{ij} with respect to x on W, $\inf\limits_W \det A$, $\sup\limits_{i,j}\sup\limits_W |a^{ij}|$, and the distance from $B \times J$ to ∂W. An analogous statement is true in case (i)*.*

2.2.3. REMARK. Assume that in the situation of Corollary 2.2.2, one has, in addition, that $|b| + |c| \in L^p_{loc}(\Omega_T)$, where $p > r'$. Then one has

$$\|\varrho\|_{L^r(B\times J)} \le C(d,r,A,W)\Big(|\mu|(W) + (\|c\|_{L^p(W)} + \|b\|_{L^p(W)})\|\varrho\|_{L^{p'}(W)}\Big).$$

2.2.4. REMARK. (i) If there exists a diffusion process $\xi = (\xi_t^{s,x})$ governed by the stochastic differentiable equation

$$d\xi_t^{s,x} = \sqrt{2A(\xi_t^{s,x}, t)}dW_t + b(\xi_t^{s,x}, t)dt, \quad \xi_s^{s,x} = x,$$

then the above results apply to the transition probabilities $P(s,x;t,dy)$ of the diffusion ξ. Namely, for any fixed (x,s), the measure $\mu = P(s,x;t,dy)\,dt$ satisfies (2.2.1) with $c = 0$. Hence, for almost every t, the measure $P(s,x;t,dy)$ is absolutely continuous. This fact is well known for locally bounded b (see [**106**, Ch. II, §2], [**170**, Ch. 7, Ch. 9]). However, the measure $P(s,x;t,dy)$ can fail to be absolutely continuous for all t. In [**80**], an example is constructed such that $b = 0$ and $A(x,t)$ is uniformly continuous, uniformly bounded, and uniformly positive, but, for some fixed t, the measures $P(s,x;t,dy)$ are purely singular with respect to Lebesgue measure for all $s \in (0,t)$ and all x (a similar example is constructed in [**152**]).

(ii) It is worth noting that Portenko [**145**] employed analogous assumptions on A and b in his study of generalized diffusion processes. In particular, it is shown in [**145**, Chapter II] that if A is uniformly bounded, uniformly positive, and uniformly Hölder continuous, and b is a measurable vector field on $\mathbb{R}^n$ such that $|b| \in L^p(\mathbb{R}^d)$ for some $p > d+2$, then there exists a continuous function $G(s,x,t,y)$, $0 \le s < t \le T$, $x, y \in \mathbb{R}^d$, that is the transition probability density for a continuous Markov process $\bigl(x(t), \mathcal{M}_t^s, P_{s,x}\bigr)$, where $P_{s,x}$ is a probability measure generated on the σ-field $\mathcal{M}_t^s$ on the space Ω of continuous paths $x(\,\cdot\,)\colon [0,+\infty) \to \mathbb{R}^d$ by the evaluation mappings $x(\,\cdot\,) \mapsto x(u)$ with $u \in [s,t]$, $P_{s,x}\bigl\{x(s) = x\bigr\} = 1$, and $P_{s,x}$-almost surely

$$x(t) - x(s) = \int_s^t b\bigl(x(\tau), \tau\bigr)\, d\tau + \int_s^t \sqrt{A\bigl(x(\tau), \tau\bigr)}\, dw_s(\tau)$$

with a certain Wiener process $\bigl(w_s(t), \mathcal{M}_t^s, P_{s,x}\bigr)$.

(iii) We also note that parabolic equations considered in Corollary 2.2.2 have been used in [**55**] for the study of certain flows of probability measures. The above results yield the absolute continuity of such measures. Stronger regularity properties will be established for them below under certain additional assumptions on the coefficients.

We shall now assume that the functions $a^{ij}(x,t)$ are continuous in x uniformly with respect to t, i.e., one has

$$\lim_{\delta\to 0} \sup_t \sup_{|x-y|\le\delta} |a^{ij}(x,t) - a^{ij}(y,t)| = 0.$$

Note that a^{ij} has a modification with such a property provided that

$$\sup_t \|a^{ij}(\,\cdot\,,t)\|_{W^{p,1}(B_R)} < \infty, \qquad \text{where } p > d. \tag{2.2.2}$$

2.2.5. Theorem. *Let $d \ge 2$, $p > d$, $q \in [p', +\infty)$. Let A and A^{-1} be uniformly bounded and let* (2.2.2) *hold. Suppose that μ is a finite measure on $B_{R,T}$ such that, for some $N > 0$ one has*

$$\left| \int \Bigl[\partial_t\varphi + a^{ij}\partial_i\partial_j\varphi\Bigr]\, d\mu \right| \le N \|\nabla_x \varphi\|_{L^q(B_{R,T})} \quad \forall\, \varphi \in C_0^\infty(B_{R,T}). \tag{2.2.3}$$

Then $\mu \in \mathbb{H}^{q',1}(B_{R'}, [t_0,t_1])$ and $\mu \in \mathcal{H}^{q',1}(B_{R'}, [t_0,t_1])$ for every $R' < R$ and $[t_0,t_1] \subset (0,T)$.

It should be noted that the proof of this result in [**33**, Theorem 2.7] contains a certain inaccuracy: it is asserted at the very beginning that by multiplying a solution by a compactly supported smooth function ζ one can easily pass to the case of solutions with compact support. In principle, this is true but is not straightforward and requires some extra work, because the inequality for the product will be of somewhat different type (in particular, a term with the integral of $\langle \nabla\zeta, \nabla\varphi\rangle$ with respect to μ will appear). However, the case of a priori compact support is sufficient for subsequent applications of this result in [**33**]. Nevertheless, if one wishes to justify the result as it is stated, it is necessary first to apply the previous results and the parabolic embedding theorem in order to ensure that the density ϱ of the solution belongs to $L^{q'}_{\rm loc}$ (this is done in several iterations) and only then multiply by a compactly supported function; once the required integrability of ϱ is established, the term with $\langle \nabla\zeta, \nabla\varphi\rangle$ is easily estimated by Hölder's inequality, which gives an estimate of type (2.2.3) for $\zeta \cdot \mu$.

2.2.6. Theorem. *Let A and A^{-1} be locally bounded on $B_{R,T}$ and let* (2.2.2) *hold, where now $p > d+2$. Let μ be a finite Borel measure on $B_{R,T}$ such that $\mu \in$*

$L^r(B_{R,T})$ *with some* $r > p'$. *Let* $\beta \in L^p(B_{R,T})$. *Suppose that for all* $\varphi \in C_0^\infty(B_{R,T})$ *one has*

$$\left|\int \Big[\partial_t \varphi + a^{ij}\partial_i\partial_j\varphi\Big]\, d\mu\right| \le \int \Big(|\varphi| + |\nabla_x \varphi|\Big)|\beta\mu|\, dx.$$

Then μ *has a density that is locally Hölder continuous on* $B_R \times (0,T)$ *and belongs to the classes* $\mathbb{H}^{p,1}(B_{R'}, [T_0,T_1])$ *and* $\mathcal{H}^{p,1}(B_{R'}, [T_0,T_1])$ *for all* $R' < R$ *and any interval* $[T_0,T_1] \subset (0,T)$.

2.2.7. Corollary. *Let* $p > d+2$ *and let* A *and* A^{-1} *be locally bounded on* Ω_T, *and let* (2.2.2) *hold with* $p > d+2$ *for every ball* B_R *with closure in* Ω. *Let* b^i, $c \in L^p_{\rm loc}(\Omega_T)$. *Assume that* μ *is a locally finite signed Borel measure on* Ω_T *such that* b^i, $c \in L^1_{loc}(\Omega_T, \mu)$ *and*

$$\int_{\Omega_T} \Big[\partial_t\varphi + a^{ij}\partial_i\partial_j\varphi + b^i\partial_i\varphi + c\varphi\Big]\, d\mu = 0 \quad \forall\, \varphi \in C_0^\infty(\Omega_T). \tag{2.2.4}$$

Then μ *has a locally Hölder continuous density that belongs to the spaces* $\mathbb{H}^{p,1}(U,J)$ *and* $\mathcal{H}^{p,1}(U,J)$ *for every interval* J *and open set* U *such that* $U \times J$ *has compact closure in* Ω_T.

It is worth noting that if μ is nonnegative, then, by Harnack's inequality, the continuous version of its density is strictly positive in every component of Ω_T in which it is not identically zero. This will be considered in more detail in §2.4.

2.2.8. Corollary. *Assume that, for all* $\varphi \in C_0^\infty(\Omega_T)$, *one has*

$$\int_{\Omega_T} \Big[\partial_t\varphi + a^{ij}\partial_i\partial_j\varphi + b^i\partial_i\varphi + c\varphi\Big]\, d\mu = \int_{\Omega_T} f^i \partial_i\varphi\, dx\, dt, \tag{2.2.5}$$

where $f^i \in L^p_{\rm loc}(\Omega_T)$ *and* a^{ij}, b^i, c *satisfy the same hypotheses as in Corollary* 2.2.7. *Then the assertion of Corollary* 2.2.7 *with* (2.2.4) *replaced by* (2.2.5) *is still true.*

2.2.9. Corollary. *Let* μ *satisfy the hypotheses of Corollary* 2.2.7, *let* B_{R_0} *be an open ball with closure in* Ω, *and let* $[t_1,t_2] \subset (0,T)$. *Then, for every closed interval* $[\tau_1,\tau_2] \subset (t_1,t_2)$ *and any* $R < R_0$, *there exists a constant* N *depending on*

$$t_1,\, t_2,\, \tau_1,\, \tau_2,\, R_0,\, R,\, \|c\|_{L^p(B_{R_0}\times[t_1,t_2])},$$

$$\inf_{B_{R_0}\times[t_1,t_2]} \det A,\ \sup_{t\in[t_1,t_2]} \|a^{ij}(\,\cdot\,,t)\|_{W^{p,1}(B_{R_0})},\ \|b^i\|_{L^p(B_{R_0}\times[t_1,t_2])},$$

such that N *is a locally bounded function of the indicated quantities and one has*

$$\|\mu\|_{\mathbb{H}^{p,1}(B_R,[\tau_1,\tau_2])} \le N\|\mu\|_{L^{p'}(B_{R_0}\times[t_1,t_2])}.$$

Moreover, one can choose N *so that if* μ *is nonnegative, then*

$$\|\mu\|_{\mathbb{H}^{p,1}(B_R,[\tau_1,\tau_2])} \le N\|\mu\|_{L^1(B_{R_0}\times[t_1,t_2])}.$$

2.2.10. Remark. The assumption that $p > d+2$ was used in the proof of Theorem 2.2.6 in order to improve the initial integrability of a solution and also to guarantee its Hölder continuity (by an appropriate embedding theorem). It is clear from the first step of that proof (or from Theorem 2.2.5) that if we assume only that $p > d$, then we obtain that $\mu \in \mathbb{H}^{s,1}(B_{R'}, [T_0,T_1])$ and $\mu \in \mathcal{H}^{s,1}(B_{R'}, [T_0,T_1])$ with $s = pr/(p+r) > d$ provided that $r > pd/(p-d)$.

We now give a modification of Theorem 2.2.6 obtained in [**43**], which differs in that the integrability condition on the coefficient β_2 (which was absent in the above theorem) is expressed in terms of the measure μ and not in terms of Lebesgue measure.

2.2.11. Theorem. *Let A and A^{-1} be locally bounded on $B_{R,T}$ and let* (2.2.2) *hold, where we assume now that $p > d+2$. Let μ be a finite Borel measure on $B_{R,T}$ with a density $\varrho \in L^r(B_{R,T})$ with some $r > p'$. Let $\beta_1 \in L^p_{\text{loc}}(B_{R,T})$ and $\beta_2 \in L^p_{\text{loc}}(\mu)$. Suppose that for all $\varphi \in C_0^\infty(B_{R,T})$ one has*

$$\left|\int_{B_{R,T}} [\partial_t\varphi + a^{ij}\partial_{x_i}\partial_{x_j}\varphi]\, d\mu\right| \le \int_{B_{R,T}} (|\varphi| + |\nabla_x\varphi|)(|\beta_1\varrho| + |\beta_2\varrho|)\, dx\, dt.$$

Then ϱ has a version that is locally Hölder continuous on $B_R \times (0,T)$ and belongs to the classes $\mathbb{H}^{p,1}(B_{R'},[T_0,T_1])$ and $\mathcal{H}^{p,1}(B_{R'},[T_0,T_1])$ for every $R' < R$ and every interval $[T_0,T_1] \subset (0,T)$.

2.2.12. Corollary. *Let $p > d+2$ and let A and A^{-1} be locally bounded on Ω_T and let* (2.2.2) *hold with some $p > d+2$ for every ball B_R with closure in Ω. Assume that μ is a locally finite signed Borel measure on Ω_T such that $b^i, c \in L^p_{\text{loc}}(\mu)$ and*

$$\int_{\Omega_T} [\partial_t\varphi + a^{ij}\partial_{x_i}\partial_{x_j}\varphi + \partial_{x_i}a^{ij}\partial_{x_j}\varphi + b^i\partial_{x_i}\varphi + c\varphi]\, d\mu = 0 \quad \forall\, \varphi \in C_0^\infty(\Omega_T).$$

Then μ has a locally Hölder continuous density that belongs to the spaces $\mathbb{H}^{p,1}(U,J)$ and $\mathcal{H}^{p,1}(U,J)$ for every interval J and every open set U such that $U \times J$ has compact closure in Ω_T.

Let us return to the stochastic equation (2.1.4). It is known that if A and b are bounded, A has two bounded derivatives and is nondegenerate on $\mathbb{R}^d$, and b has a bounded derivative, then there exists a diffusion process ξ_t^x that satisfies (2.1.4) with $\xi_0^x = x$, $x \in \mathbb{R}^d$. Its transition probabilities $P(t,x,dy)$ have densities $p(t,x,y)$ satisfying the following equations:

$$\frac{\partial}{\partial t}p(t,x,y) = a^{ij}\frac{\partial^2}{\partial x_i\partial x_j}p(t,x,y) + b^i\frac{\partial}{\partial x_i}p(t,x,y), \tag{2.2.6}$$

$$\frac{\partial}{\partial t}p(t,x,y) = \frac{\partial^2}{\partial y_i\partial y_j}\Bigl(a^{ij}p(t,x,y)\Bigr) - \frac{\partial}{\partial y_i}\Bigl(b^i p(t,x,y)\Bigr) \tag{2.2.7}$$

for all $t > 0$, $x, y \in \mathbb{R}^d$; in addition, $P(0,x,dy) = \delta_x$. This means that the measures $P(t,x,dy)\,dt$ satisfy (2.1.2). It should be noted that these equations bear many names in the literature; in particular, they are called the Fokker–Planck equations, the Fokker–Planck–Kolmogorov equations, the forward and backward Kolmogorov equations, and so on (Kolmogorov himself in [**105**] refer to both as the Fokker–Planck equations). To avoid confusion, in this survey we call (2.2.6) the Kolmogorov equation and call (2.2.7) the Fokker–Planck equation. The transition semigroup $(T_t)_{t\ge 0}$ of ξ_t^x is defined on $C_b(\mathbb{R}^d)$ by the formula

$$T_t f(x) = \int f(y)\, P(t,x,dy).$$

If a probability measure μ is invariant for $(T_t)_{t\ge0}$, then μ satisfies the elliptic equation $L^*_{A,b}\mu = 0$, and, conversely, if μ satisfies this stationary equation, then it is invariant for $(T_t)_{t\ge0}$. As we know from §§1.5, 1.6, there is no such equivalence in the case of unbounded coefficients even if we have our semigroup $(T_t^\mu)_{t\ge0}$. Let us explain how we can consider analogs of equations (2.2.6)–(2.2.7) for this semigroup in the situation of Theorem 1.7.5. In order to obtain a function of three arguments (t,x,y), we take $K_t^*\delta_x$, where δ_x is Dirac's measure at x. The next result shows that $K_t^*\delta_x = p_{A,b}(t,x,y)dy$ if $t > 0$, where $p_{A,b}$ is the function from Theorem 1.7.5, and that an analog of equation (2.2.7) holds.

2.2.13. Theorem. *Suppose that in the situation of Theorem* 1.7.5 *we are given a bounded measure* μ_0 *on* Ω *and*

$$K_t^*\mu_0(dy) := \int_\Omega K_t(x,dy)\,\mu_0(dx) = \int_\Omega p_{A,b}(t,x,y)\,\mu_0(dx)\,dy.$$

Then the measure $K_t^*\mu_0(dy)\,dt$ *satisfies* (2.1.2) *for all* $T>0$. *In particular,* $K_t^*\mu_0$ *has a positive continuous density in* $W^{p,1}_{\rm loc}(\Omega)$.

2.2.14. Remark. It follows from the proof in [**33**] that for every compact set $E\subset\Omega\times(0,+\infty)$, there exists a constant $C(E)$ such that

$$\sup_{(x,t)\in E}\sup_{y\in\Omega} p_t(x,y) \le C(E),$$

where $p_t(x,y)$ is the jointly continuous version of the Radon–Nikodym density of the measure $K_t(x,dy)$ with respect to the measure $\mu=\varrho\,dx$ from Theorem 1.7.5, i.e., $p_{A,b}(t,x,y)=p_t(x,y)\varrho(y)$. Hence the continuous density of the measure $K_t^*\mu_0(dy)$ is given by

$$\int_\Omega p_t(x,y)\,\mu_0(dx)\,\varrho(y).$$

In [**130**], under stronger assumptions on the coefficients A and b, useful global estimates and inclusions in parabolic Sobolev classes are obtained for the transition density $p_{A,b}(t,x,y)$ from Theorem 1.7.5. In [**124**] the situation where $\nabla_x\varrho(x,t)$ exists even for not necessarily differentiable coefficient A is studied.

Certainly, once the existence of densities is established, in many cases one can apply the results known for functions (but taking special care of the form of the equation), in particular, the results from [**108**], [**111**], [**116**], [**123**].

2.3. Upper estimates for densities

Here we present upper bounds on densities in the case of an unbounded drift. On the one hand, the assumptions and techniques applied in this situation differ considerably from those used for obtaining Gaussian decay of densities in the case of bounded drifts or zero drifts (see [**75**], [**143**], [**144**], [**77**], [**161**], and also [**8**], where some assumptions on $\operatorname{div} b$ are used). On the other hand, the ideas and methods used for bounded coefficients turn out to be very useful also in the situation under consideration. First we discuss the case when the initial distribution has a sufficiently nice density, and then briefly comment on the case of a Dirac initial distribution (i.e., fundamental solutions or transition densities) studied in [**84**], [**130**], [**167**] under some additional assumptions. For the proofs of the results below, see [**41**].

We shall say that a nonnegative measure μ_0 has finite entropy if μ_0 has a density ϱ_0 with respect to Lebesgue measure such that $\varrho_0\ln\varrho_0\in L^1(\mathbb{R}^d)$, where we set $0\ln 0:=0$.

2.3.1. Theorem. *Suppose that* μ, *where each* μ_t *is a probability measure, satisfies* (2.1.1) *and* (2.1.3) *and assume the following hypotheses*:

(i) *the mapping* A *be uniformly bounded with* $A(x,t)\ge\alpha\cdot\mathrm{I}$ *for some constant* $\alpha>0$, *and let the functions* $x\mapsto a^{ij}(x,t)$ *be Lipschitzian with constant* λ,

(ii) $|b|\in L^2(\mu)$.

Assume also that the function $\Lambda(x):=\ln\max(|x|,1)$ *is in* $L^2(\mu)$ (*which is the case if, e.g.,* $\langle b(x,t),x\rangle\le C_1|x|^2\Lambda(x)+C_2$ *with some constants* C_1 *and* C_2 *and* $\Lambda\in$

$L^2(\mu_0)$). If μ_0 has finite entropy, then $\mu_t = \varrho(\,\cdot\,,t)\,dx$, where $\varrho(\,\cdot\,,t) \in W^{1,1}_{\rm loc}(\mathbb{R}^d)$, and for each $\tau < 1$ one has

$$\int_0^\tau \int_{\mathbb{R}^d} \frac{|\nabla \varrho(x,t)|^2}{\varrho(x,t)}\,dx\,dt < \infty. \tag{2.3.1}$$

In particular, we have $\sqrt{\varrho} \in \mathbb{H}^{2,2}(\mathbb{R}^d \times [0,\tau])$ and $\varrho \in L^{d/(d-2),1}(\mathbb{R}^d \times [0,\tau])$ if $d > 2$, and $\varrho \in L^{s,1}(\mathbb{R}^d \times [0,\tau])$ for all $s \in [1,\infty)$ if $d = 2$.

If

$$\limsup_{t\to 1} \int_{\mathbb{R}^d} \varrho(x,t)\Lambda(x)\,dx < \infty,$$

which is the case, e.g., if $\langle b(x,t),x\rangle \le C_1|x|^2 + C_2$ with some constants C_1 and C_2 and $\Lambda \in L^1(\mu_0)$, then (2.3.1) *is true for $\tau = 1$.*

The proof in [**41**] yields a useful estimate

$$\begin{aligned}\int_0^\tau \int_{\mathbb{R}^d} \frac{|\nabla \varrho|^2}{\varrho}\,dx\,dt &\le \alpha^{-2}\Bigl(\|b\|_{2,\mu} + \lambda d^{3/2}\sqrt{\gamma}\Bigr)^2 + 2\alpha^{-1}\ln 2 + 2\alpha^{-1}\int_{\mathbb{R}^d} \varrho_0(x)\ln\varrho_0(x)\,dx \\ &\quad + 2\alpha^{-1}(d+1)\int_{\mathbb{R}^d} \varrho(\tau,x)\Lambda(x)\,dx. \end{aligned} \tag{2.3.2}$$

2.3.2. Remark. It is seen from the proof in [**41**] that in place of the integrability of the function $\varrho(x,0)\ln\varrho(x,0)$ it suffices to require only the integrability of $\varrho(x,0)\max(0,\ln\varrho(x,0))$. This leads to the effect that in estimate (2.3.2) one obtains $\varrho(x,0)\max(0,\ln\varrho(x,0))$ in place of the function $\varrho(x,0)\ln\varrho(x,0)$. However, the estimates obtained and (2.1.10) show that if we keep all other assumptions, then the entropy of $\varrho(x,0)$ is finite anyway. But if no μ-integrability of Λ is required, then the situation may change. For example, if $d = 1$, $b = 0$, and $a = 1/2$, then for any initial distribution μ_0, the solution is given by the convolution $\mu_0 * g_t$, where $g_t(x) = (2\pi t)^{-1/2}\exp(-x^2/(2t))$. If μ_0 has a density ϱ_0 such that $|\varrho_0'|^2/\varrho_0 \in L^1(\mathbb{R}^1)$, but the function $\varrho_0\ln\varrho_0$ is not integrable, then for all t the solution $\varrho(x,t)$ does not have finite entropy, although the quantities

$$\int |\partial_x \varrho(x,t)|^2 \varrho(x,t)^{-1}\,dx$$

are uniformly bounded. The same example shows that for the validity of estimate (2.3.1) certain conditions on the initial distribution are necessary. It suffices to take μ_0 to be Dirac's measure at the origin. Then the function $|\partial_x\varrho|^2/\varrho$ is not integrable on $\mathbb{R}^1 \times (0,1)$. Some sufficient conditions on A and b ensuring finite entropy of $\varrho(\,\cdot\,,t)$ for $t > 0$ and Dirac's initial distribution are mentioned at the end of the section.

In Example 2.3.9 below one can find conditions on the coefficients A and b ensuring that $|b| \in L^2(\mu)$.

Estimate (2.3.2) can be improved under additional hypotheses on A and b.

Let

$$b_0 := (b_0^j), \quad b_0^j = b^j - \partial_{x_i} a^{ij}.$$

2.3.3. Theorem. *Suppose that μ satisfies* (2.1.1), (2.1.3), *where $\nu = \varrho_0\,dx$, and ϱ_0 has finite entropy and is locally Hölder continuous. Let A and b satisfy* (CP1) *and* (CP2) *with some* $p > d+2$. *Suppose that* $|A^{-1/2}b_0| \in L^2(\mu)$, $\ln(1+|x|) \in L^4(\mu)$, *and*

$$\liminf_{r\to\infty} \int_0^1 \int_{r\le|x|\le 2r} \Big[r^{-4}\|A(x,t)\|^2 + r^{-2}\Theta_A(x,t)^2\Big]\,\mu_t(dx)\,dt = 0.$$

Then $\varrho(\,\cdot\,,t) \in W^{p,1}_{\rm loc}(\mathbb{R}^d)$, *for almost all* $\tau \in [0,1]$ *one has*

$$\int_0^\tau \int_{\mathbb{R}^d} \Big|\frac{\sqrt{A}\nabla\varrho}{\varrho}\Big|^2\,d\mu \\ \le \int_0^\tau \int_{\mathbb{R}^d} |A^{-1/2}b_0|^2\,d\mu + 2\int_{\mathbb{R}^d} [\varrho(x,0)\ln\varrho(x,0) - \varrho(x,\tau)\ln\varrho(x,\tau)]\,dx,$$

and the right-hand side is finite. Under the extra assumption that $A \ge \alpha\cdot \mathrm{I}$ *for some* $\alpha > 0$, *one has* $\sqrt{\varrho} \in \mathbb{H}^{2,2}(\mathbb{R}^d\times[0,1])$, $\varrho \in L^{d/(d-2),1}(\mathbb{R}^d\times[0,1])$ *if* $d > 2$, *and* $\varrho \in L^{s,1}(\mathbb{R}^d\times[0,1])$ *for all* $s\in[1,\infty)$ *if* $d = 2$.

2.3.4. Remark. If A is uniformly bounded, then the assumption that $\ln(1+|x|) \in L^4(\mu)$ can be relaxed to $\ln(1+|x|) \in L^2(\mu)$.

2.3.5. Theorem. *Under the hypotheses of Theorem* 2.3.1 *suppose that also*

$$\sup_{t\in[0,1]} \|b(\,\cdot\,,t)\|_{L^d(\mu_t)} < \infty$$

and $\mu_0 = \varrho(\,\cdot\,,0)\,dx$, *where* $\varrho(\,\cdot\,,0)\in L^p(\mathbb{R}^d)$ *for all* $p\in[1,+\infty)$. *Then*

$$\int_0^\tau \Big(\int_{\mathbb{R}^d} |\varrho(x,t)|^p\,dx\Big)^{q/p} dt < \infty$$

for all $p,q \in [1,+\infty)$ *and* $\tau\in(0,1)$.

2.3.6. Theorem. *Under the hypotheses of Theorem* 2.3.1 *suppose that* $|b| \in L^\beta(\mu)$ *for some* $\beta > d+2$ *and* $\varrho(\,\cdot\,,0)\in L^\infty(\mathbb{R}^d)$. *Assume that either*

$$\sup_{t\in[0,1]} \|b(\,\cdot\,,t)\|_{L^d(\mu_t)} < \infty,$$

or $\varrho \in L^p(\mathbb{R}^d\times[0,\tau])$ *for all* $\tau < 1$ *for some* $p > 1$. *Then* $\varrho\in L^\infty(\mathbb{R}^d\times[0,\tau])$ *for every* $\tau < 1$.

2.3.7. Remark. (i) In view of Remark 2.3.2, there is no need to require the integrability of the function $\varrho(x,0)\ln\varrho(x,0)$ in the last two theorems, since the integrability of the function $\varrho(x,0)\max(0,\ln\varrho(x,0))$ follows by the inclusion $\varrho(\,\cdot\,,0)\in L^p(\mathbb{R}^d)$ with $p > 1$.

(ii) The assumption in Theorem 2.3.5 and Theorem 2.3.6 that the integrals of $\varrho(x,t)$ with respect to x equal 1 can be replaced by the assumption that these integrals are uniformly bounded.

(iii) If in Theorem 2.3.6 it is given in advance that $\varrho\in L^p(\mathbb{R}^d\times[0,\tau])$ for some $p>1$, then we need not require the integrability of the function $|\ln(1+|x|)|^2\varrho(x,t)$, but the boundedness of $\varrho(x,0)$ is important.

These theorems enable one to obtain global upper bounds on ϱ. As in the elliptic case considered above, the idea is this: in order to obtain a pointwise estimate $\varrho(x,t)\le\Phi(x,t)^{-1}$, one has to consider the measure $\nu := \Phi\cdot\mu$ and establish the boundedness of its density. We shall consider functions Φ that do not depend on t. If Φ has locally bounded first and second order derivatives, then the measure ν satisfies the equation

$$L^*\nu = (a^{ij}\partial_{x_i}\partial_{x_j}\Phi)\varrho + 2\partial_{x_i}\Phi\partial_{x_j}(a^{ij}\varrho) - b^i\partial_{x_i}\Phi\varrho = -L\Phi\cdot\varrho + 2\partial_{x_j}(a^{ij}\partial_{x_i}\Phi\varrho)$$

understood in the same sense as (2.1.1).

2.3.8. Theorem. *Suppose that all hypotheses of Theorem* 2.3.6 *are fulfilled and we are given a function* $\Phi \geq c > 0$ *on* $\mathbb{R}^d$ *with locally bounded second order derivatives such that* $\varrho(x,0) \leq C\Phi(x)^{-1}$, $\Phi \in L^1(\mu_0)$, *and*

$$\Phi^{1+\varepsilon},\ |L\Phi|^{\beta/2}\Phi^{1-\beta/2},\ |A\nabla\Phi|^{\beta}\Phi^{1-\beta} \in L^1(\mu), \qquad \sup_{t\in[0,1]} \int_{\mathbb{R}^d} \Phi(x)\varrho(x,t)\,dx < \infty$$

for some $\varepsilon > 0$. *Then for every* $\tau < 1$ *there is a number* C_τ *such that*

$$\varrho(x,t) \leq C_\tau \Phi(x)^{-1} \quad \text{for almost all } (x,t) \in \mathbb{R}^d \times [0,\tau].$$

2.3.9. Example. Suppose that A and A^{-1} are uniformly bounded, A is uniformly Lipschitzian in x, and for some $\beta > d+2$, $r > 0$, $\varepsilon > 0$, $K > 0$ one has

$$|b| \in L^{\beta}(\mu),\ \exp[(2K+\varepsilon)|x|^r] \in L^1(\mu), \tag{2.3.3}$$

$$\sup_{t\in[0,1]} \int_{\mathbb{R}^d} \exp(K|x|^r)\varrho(x,t)\,dx < \infty.$$

Let $\sup_{t\in[0,1]} \|b(\,\cdot\,,t)\|_{L^d(\mu_t)} < \infty$. Finally, let the function $\exp(K|x|^r)\varrho(x,0)$ be bounded and integrable on $\mathbb{R}^d$. Then for every $\tau < 1$ there is a number $C(\tau)$ such that

$$\varrho(x,t) \leq C(\tau)\exp(-K|x|^r), \quad (x,t) \in \mathbb{R}^d \times [0,\tau].$$

To ensure condition (2.3.3) and the stated assumptions on b and $\varrho(\,\cdot\,,0)$ it suffices to have the estimates

$$|b(x,t)| \leq C\exp(2K\beta^{-1}|x|^r), \quad \varrho(x,0) \leq C\exp(-K'|x|^r)$$

with some $K' > K$ and the estimate

$$\langle x, b(x,t)\rangle \leq c_1 - c_2|x|^r,\ c_2 > 2rK\sup_{t,x}\|A(x,t)\|. \tag{2.3.4}$$

Indeed, let $\Phi \in C^2(\mathbb{R}^d)$ be such that $\Phi(x) = \exp(K|x|^r)$ if $|x| \geq 1$. All hypotheses of Theorem 2.3.8 hold. Under condition (2.3.4) we pick $\delta \in (0,\varepsilon)$ such that

$$r(2K+\delta)\sup_{x,t}\|A(x,t)\| < c_2,$$

and take a function $V \in C^2(\mathbb{R}^d)$ that equals $\exp[(2K+\delta)|x|^r]$ if $|x| \geq 1$. Then, for some c, we have the estimate $LV \leq c$. It follows from the existence results in §2.6 below that a solution exists and the norms $\|V\varrho(\,\cdot\,,t)\|_{L^1(\mathbb{R}^d)}$ are uniformly bounded. The remaining assumptions of Theorem 2.3.8 also hold. In a similar way, one can obtain a power bound under weaker conditions.

Under some additional assumptions, time-dependent bounds for solutions are obtained in the recent papers [**84**], [**130**], [**167**]. For example, the following result is proved in [**130**].

2.3.10. Theorem. *Suppose that the mappings* A *and* A^{-1} *are uniformly bounded,* $a^{ij} \in C^{1+\alpha}(\mathbb{R}^d)$, $b^i \in C^{\alpha}_{\mathrm{loc}}(\mathbb{R}^d)$, *and*

$$\limsup_{|x|\to\infty} |x|^{-\beta}\langle x, b(x)\rangle \leq -c, \quad |b(x)| \leq c_1\exp(c_2|x|^{\beta-\varepsilon}),$$

where $c, c_1, c_2 > 0$, $\varepsilon > 0$, *and* $\beta > 2$. *Then the following estimate holds for the fundamental density* $p_{A,b}(t,x,y)$ *in Theorem* 1.7.5:

$$p_{A,b}(t,x,y) \leq c_3\exp\Bigl(c_4 t^{-\beta/(\beta-2)}\Bigr)\exp(-\gamma|y|^{\beta}), \quad t \in (0,1), x, y \in \mathbb{R}^d.$$

If $|b(x)| \le c(1+|x|^q)$, *then*

$$p_{A,b}(t,x,y) \le ct^{-\sigma}(1+|y|)^{-\theta}$$

for some constants $c, \sigma, \theta > 0$.

Analogous estimates are established also for $\partial_t p_{A,b}$, $D_y p_{A,b}$ and $D_y^2 p_{A,b}$ in the case where $a^{ij} \in C_b^2(\mathbb{R}^d)$, $b^i \in C^2(\mathbb{R}^d)$, and $|Db(x)|$ and $|D^2 b(x)|$ admit the same bound as $|b|$. Some of these estimates are refined in [**167**] by using Lyapunov functions depending on t. For example, the previous estimate is refined to

$$p_{A,b}(t,x,y) \le c_3 t^{-\kappa} \exp(-\delta t^{\alpha}|y|^{\beta}),$$

where $\theta = \alpha k(\beta-1)/\beta - 1$, $k > d+2$, $\alpha > \beta/(\beta-2)$, $\delta < c/(\Lambda\beta)$, $\Lambda = \sup_x \|A(x)\|$. These results have recently been strengthened by S.V. Shaposhnikov.

It would be interesting to study the behavior of solutions at infinity under the above assumptions on the coefficients in the case where the initial distribution is a measure; there are many results in this direction for solutions in various classes of functions and typically in the case of bounded coefficients (see the survey [**68**]).

2.4. Harnack's inequality for parabolic equations

We now consider Harnack's inequality for the parabolic equation. Let Ω be a bounded domain in $\mathbb{R}^d$, let $Q = \Omega \times (0,1)$, and let $A = (a^{ij})_{1\le i,j\le d}$ be a measurable matrix-valued mapping on Q with $a^{ij} = a^{ji}$ such that there exist constants $\gamma \ge 0$ and $\alpha > 0$ such that

$$\sum_{i,j} |a^{ij}(x,t)|^2 \le \gamma^2 \quad \text{and} \quad A(x,t) \ge \alpha \cdot \mathrm{I} \quad \text{for all } (x,t) \in Q. \tag{2.4.1}$$

In addition, let $b\colon\ Q \to \mathbb{R}^d$ be a measurable vector field such that

$$\sup_{(x,t)\in Q} |b(x,t)| \le B < \infty. \tag{2.4.2}$$

Suppose that a nonnegative function $u \in \mathbb{H}^{2,1}(Q)$ satisfies the equation

$$\partial_t u = \partial_{x_i}(a^{ij}\partial_{x_j} u - b^i u), \tag{2.4.3}$$

i.e., for every function $\varphi \in C_0^1(Q)$, one has the equality

$$\int\!\!\int_Q \Big[-\partial_t\varphi u + \partial_{x_i}\varphi\big(a^{ij}\partial_{x_j}u - b^i u\big)\Big]\,dx\,dt = 0.$$

It follows from the general theory of parabolic equations (see, e.g., Theorem 8.1 in §8 and Theorem 10.1 in §10 in Chapter 3 of the book [**111**]) that under our assumptions any solution u has a version that is locally Hölder continuous.

Let us fix a point $(\bar{x},\bar{t}) \in \Omega \times (0,1]$. Let $R(\bar{x},r)$ be the open cube with the edge length r centered at the point $\bar{x}$. Let

$$Q(r) = R(\bar{x},r) \times (\bar{t}-r^2,\bar{t}), \quad Q^*(r) = R(\bar{x},r) \times (\bar{t}-8r^2,\bar{t}-7r^2).$$

The following classical theorem is true (see the paper [**10**, Theorem 3], generalizing a result of Moser [**137**]). Concerning methods of obtaining Harnack's inequality, see also [**143**] and the references there.

2.4.1. Theorem. *Let* $Q(3r) \subset Q$. *Then, for the continuous version of the function* u *satisfying equation* (2.4.3)*, we have*

$$\sup_{(x,t)\in Q^*(r)} u(x,t) \le C \inf_{(x,t)\in Q(r)} u(x,t),$$

where the number $C = C(d,\alpha,\gamma,Br)$ *depends only on* d, α, γ*, and* Br.

As in the elliptic case, we are interested in a more precise form of dependence of C on the indicated parameters from (2.4.1), (2.4.2). The results below are proved in [**42**].

2.4.2. Theorem. *Let $Q(3r) \subset Q$. Then the following inequality holds for the continuous version of the function u:*

$$\sup_{(x,t)\in Q^*(r)} u(x,t) \le C \inf_{(x,t)\in Q(r)} u(x,t),$$

where

$$C := C(d,\alpha,\gamma,B,r) := \exp\Bigl\{c(d)\bigl[1+\alpha^{-1}+(\alpha^{-1/2}+\alpha^{-1})(Br+\gamma)\bigr]^2\Bigr\}.$$

Our next result further refines the obtained estimate with respect to dependence on r. Its advantage as compared to the previous theorem is that now B appears in the estimate without the factor r.

2.4.3. Theorem. *Suppose that $B(z_0,\theta r) \subset \Omega$ for some $\theta > 1$ and $r > 0$. Then, whenever $0 < s < t < 1$ and $x, y \in B(z_0, r)$, the following inequality holds for the continuous version of u:*

$$u(y,s) \le u(x,t) \exp\Bigl\{K\Bigl(\frac{|x-y|^2}{t-s} + (B+1)^2\frac{t-s}{\delta^2} + 1\Bigr)\Bigr\},$$

where $\delta = \min\{(\theta-1)r, \sqrt{s}\}$ and the number K depends only on d, α, and γ as follows, with some number $c(d)$ depending only on d:

$$K := c(d)\Big|1+\alpha^{-1}+(\alpha^{-1}+\alpha^{-1/2})\gamma\Big|^2.$$

2.5. Lower bounds in the parabolic case

We proceed to lower bounds for densities of solutions to parabolic equations for measures. As we shall see, there are lower bounds similar to the upper bounds discussed in §2.3; however, the difference between the upper and lower bounds is more significant than in the elliptic case. The proofs are given in [**42**].

Let $A = (a^{ij})_{1\le i,j\le d}$ be a measurable matrix-valued mapping on $\mathbb{R}^d \times (0,1)$ such that $A(x,t)$ is positive definite and let b be a measurable vector field on $\mathbb{R}^d \times (0,1)$ with values in $\mathbb{R}^d$. As in the elliptic case, in our study of lower bounds it is more convenient to consider the divergence form operators $\mathcal{L} = \mathcal{L}_{A,b}$.

A Borel measure μ on $\mathbb{R}^d \times (0,1)$ satisfies the weak parabolic equation (2.1.1) if the functions a^{ij} and b^i are integrable on every compact set in $\mathbb{R}^d \times (0,1)$ with respect to μ and, for every function $\varphi \in C_0^\infty(\mathbb{R}^d \times (0,1))$, we have the equality

$$\int_{\mathbb{R}^d\times(0,1)} \mathcal{L}\varphi\, d\mu = 0,$$

which is understood in one of the following two ways.

(I) For every compact interval $J \subset (0,1)$ and every ball $U \subset \mathbb{R}^d$, the functions a^{ij} belong to the class $\mathbb{H}^{1,1}(U \times J)$, the functions a^{ij}, $\partial_{x_i} a^{ij}$, and b^i are Borel measurable and locally integrable with respect to $|\mu|$, and

$$\int_{\mathbb{R}^d\times(0,1)} [\partial_t\varphi + a^{ij}\partial_{x_i}\partial_{x_j}\varphi + \partial_{x_i}a^{ij}\partial_{x_j}\varphi + b^j\partial_{x_j}\varphi]\, d\mu = 0. \qquad (2.5.1)$$

(II) For every compact interval $J \subset (0,1)$ and every ball $U \subset \mathbb{R}^d$, the restriction of the measure μ to $U \times J$ has a density ϱ in the class $\mathbb{H}^{1,1}(U \times J)$ such that the functions $a^{ij}\partial_{x_i}\varrho$ and $b^i\varrho$ are locally Lebesgue integrable and

$$\int_{\mathbb{R}^d\times(0,1)} [\partial_t\varphi\varrho - a^{ij}\partial_{x_i}\varrho\partial_{x_j}\varphi + b^i\partial_{x_i}\varphi\varrho]\,dx\,dt = 0. \tag{2.5.2}$$

Below we always specify which case we are considering by referring to (2.5.1) in Case (I) and to (2.5.2) in Case (II). Our assumption that the matrices $A(x,t) = (a^{ij}(x,t))_{i,j\le d}$ are symmetric and strictly positive guarantees the absolute continuity of μ in Case (I). For this reason, we consider measures μ represented in the form $\mu(dt\,dx) = \mu_t(dx)\,dt$ by means of a family of Borel measures $(\mu_t)_{t\in(0,1)}$ on $\mathbb{R}^d$. In this case (2.1.1) can be written as

$$\int_0^1\int_{\mathbb{R}^d} \mathcal{L}\varphi(x,t)\mu_t(dx)\,dt = 0,$$

which is understood in one of the two ways described above. It should be noted that the alternative assumption in Case (II) that μ has a locally Sobolev density is fulfilled automatically if, in Case (I), the coefficients A and b satisfy certain additional assumptions specified in §2.2. For this reason, in some results we consider solutions a priori possessing locally Sobolev densities and make no assumptions on the regularity of A; however, in other results, which deal with applications to transition probabilities, we impose a suitable local Sobolev regularity of A in order to guarantee that all solutions are Sobolev regular.

Let V be a continuous increasing function on $[0,\infty)$ with $V(0) > 0$.

2.5.1. Theorem. *Let* $\sup\limits_{t\in(0,1)} |b(x,t)| \le V(|x|/\theta)$ *for almost all* $x\in\mathbb{R}^d$, *where* $\theta > 1$. *Let*

$$\alpha(r) := \sup_{t\in(0,1),|x|\le r} \|A(x,t)^{-1}\|, \quad \gamma(r) := \sup_{t\in(0,1),|x|\le r} \|A(x,t)\|.$$

Let μ *be a nonnegative measure with a density* ϱ *on* $\mathbb{R}^d\times(0,1)$ *such that*

$$\varrho \in \mathbb{H}^{2,1}(U\times J) \tag{2.5.3}$$

for any ball $U\subset\mathbb{R}^d$ *and any closed interval* J *in* $(0,1)$. *Suppose that* μ *satisfies equation* (2.1.1) *in the sense of* (2.5.2), *i.e., we deal with Case* (II). *Then, there exists a positive number* $K = K(d)$ *such that the continuous version of the function* ϱ *satisfies the inequality*

$$\varrho(x,t) \ge \varrho(0,s)\exp\Bigl\{-K(d)\bigl|1+\alpha(\theta|x|)^{-1} + \bigl[\alpha(\theta|x|)^{-1}+\alpha(\theta|x|)^{-1/2}\bigr]\gamma(\theta|x|)\bigr|^2\times$$
$$\times\Bigl(1+\frac{t-s}{s}V(|x|)^2 + \frac{1}{t-s}|x|^2\Bigr)\Bigr\},$$

where $0<s<t<1$, $x\in\mathbb{R}^d$. *In particular, if* $\|A(x,t)\|\le\gamma$ *and* $\|A(x,t)^{-1}\|\le\alpha$, *then there exists a positive number* $K = K(d,\alpha,\gamma,\theta)$ *such that the continuous version of the function* ϱ *satisfies the inequality*

$$\varrho(x,t) \ge \varrho(0,s)\exp\Bigl\{-K\Bigl(1+\frac{t-s}{s}V(|x|)^2 + \frac{1}{t-s}|x|^2\Bigr)\Bigr\}, \quad 0<s<t<1,\ x\in\mathbb{R}^d.$$

2.5.2. Corollary. *Suppose that in the situation of Theorem* 2.5.1 *one has*

$$\|A(x,t)\|\le\gamma \quad \text{and} \quad \|A(x,t)^{-1}\|\le\alpha$$

and that for almost all $t\in(0,1)$ *the function* $x\mapsto\varrho(x,t)$ *does not vanish identically. Then, for every closed interval* $[\tau_1,\tau_2]$ *in* $(0,1)$, *there exists a number*

$K = K(d,\alpha,\gamma,\theta,\tau_1,\tau_2) \geq 0$ *such that for all* $t \in [\tau_1,\tau_2]$ *and* $x \in \mathbb{R}^d$ *the following inequality holds:*

$$\exp\Bigl(-K\bigl(1+V(|x|)^2+|x|^2\bigr)\Bigr) \leq \varrho(x,t) \leq \exp\Bigl(K\bigl(1+V(|x|)^2+|x|^2\bigr)\Bigr).$$

2.5.3. Example. Suppose that in the situation of Theorem 2.5.1 the matrices $A(x,t)$ and $A(x,t)^{-1}$ are uniformly bounded and for some constants $c_1 > 0$ and $c_2 > 0$ the inequality

$$\sup_{t\in(0,1)} |b(x,t)| \leq c_1|x|^\beta + c_2$$

holds for almost all x. Then there exists a positive number K such that

$$\varrho(x,t) \geq \varrho(0,s)\exp\Bigl\{-K\Bigl(1+\frac{t-s}{s}|x|^{2\beta}+\frac{1}{t-s}|x|^2\Bigr)\Bigr\}, \quad s,t\in(0,1), s<t.$$

For example, if

$$L = \partial_t + \frac{1}{2}\Delta,$$

then the measure $(2\pi t)^{-1/2}e^{-|x|^2/2t}\,dx\,dt$ is a solution. For any $\delta > 0$, our results yield a number $K(\delta) > 0$ such that $\varrho \geq e^{-K(\delta)|x|^2/t}$ in the strip $\mathbb{R}^d\times(\delta,1)$. Similarly, our lower estimate is sharp in the case of a linear drift coefficient, but it becomes less precise in the case of quadratic growth of $|b|$; e.g., if $\varrho(x,t) = C\exp(-|x|^3)$, then $\exp(-K|x|^4)$ appears in our lower bound.

Let us give conditions on the coefficients A and b ensuring two-sided exponential estimates of the density of the solution in the parabolic case.

2.5.4. Example. Suppose that $A(x,t)$ is symmetric and positive, $A(x,t)$ and $A(x,t)^{-1}$ are uniformly bounded, the functions $x \mapsto a^{ij}(x,t)$ are uniformly Lipschitzian with a common constant, and that for some $r > 1$, $\sigma \geq 0$, $K > 0$, and $K' > K$ we have

$$|b(x,t)| \leq C + C|x|^{r-1+\sigma}, \quad \varrho(x,0) \leq C\exp(-K'|x|^r),$$
$$\langle x, b(x,t)\rangle \leq c_1 - c_2|x|^r, \quad c_2 > 2rK\sup_{x,t}\|A(x,t)\|.$$

Suppose that a probability measure μ on $\mathbb{R}^d\times(0,1)$ satisfies equation (2.1.1) in the sense of (2.5.1), i.e., we deal with Case (I). Then μ has a continuous density ϱ such that, for every closed interval $[\tau_1,\tau_2] \subset (0,1)$, there exist numbers C_1, C_2, and K_0 for which

$$C_1\exp\Bigl(-K_0|x|^{2r+2\sigma-2} - K_0|x|^2\Bigr) \leq \varrho(x,t) \leq C_2\exp(-K|x|^r),\ (x,t)\in\mathbb{R}^d\times[\tau_1,\tau_2].$$

The upper bound follows from §2.3, and the lower bound follows from the above results. Unlike the elliptic case, here there is no coincidence of the powers of $|x|$ in the lower and upper bounds. We observe that the indicated conditions also give the existence of a solution $\mu = \mu_t\,dt$, where every μ_t is a probability measure, for an arbitrary initial distribution (see §2.6). The uniqueness problem is also considered in §2.6.

One more application of our results is concerned with the proof of the existence of finite entropy for any solution with respect to the space variable at any positive t for every initial distribution. The existence of finite entropy, which is useful in many respects, is necessary for applying those results of §2.3 which give integrability of $|\nabla\varrho(x,t)|^2/\varrho(x,t)$.

Let W be a continuous increasing function on the half-line $[0,\infty)$ such that $W(0) > 0$ and $\lim_{r\to\infty} W(r) = +\infty$.

2.5.5. Proposition. *Suppose that the matrices $A(x,t)$ are positive definite and uniformly bounded along with the matrices $A(x,t)^{-1}$ and that, for some $\theta > 1$, the following inequality holds*:

$$\sup_{t\in(0,1)} |b(x,t)|^2 \le W(|x|/\theta), \quad x \in \mathbb{R}^d.$$

Let μ be a measure of the form $\mu = \mu_t\,dt$, where every μ_t is a probability measure on $\mathbb{R}^d$, and let μ satisfy condition (2.5.3) *and equation* (2.1.1) *in the sense of* (2.5.2). *Suppose that*

$$\int_0^1 \int_{\mathbb{R}^d} [|x|^2 + W(|x|)]\,\mu_t(dx)\,dt < \infty.$$

Then, for every closed interval $[\tau_1, \tau_2] \subset (0,1)$, we have

$$\int_{\tau_1}^{\tau_2} \int_{\mathbb{R}^d} \varrho(x,s)|\ln \varrho(x,s)|\,dx\,ds < \infty.$$

2.5.6. Corollary. *Suppose that $a^{ij} \in \mathbb{H}^{2,1}_{\rm loc}(\mathbb{R}^d \times (0,1))$, $A(x,t)$ is positive definite, and that $A(x,t)$ and $A(x,t)^{-1}$ are uniformly bounded. In addition, suppose that for some $\theta > 1$ the following inequality holds*:

$$\sup_{t\in(0,1)} |b(x,t)|^2 \le W(|x|/\theta), \quad x \in \mathbb{R}^d.$$

Let μ be a measure of the form $\mu = \mu_t\,dt$, where every μ_t is a probability measure, and let μ satisfy condition (2.5.3). *Assume that μ satisfies equation* (2.1.1) *in the sense of* (2.5.2) *and has the initial distribution μ_0 such that the function $W_0(x) := W(|x|)$ is integrable with respect to μ_0. Finally, let*

$$W(r) \ge c_1 + c_2 r^2 \quad \text{and} \quad \mathcal{L}W_0(x) \le C$$

for some constants $c_1, c_2, C > 0$. Then, for the continuous version ϱ of the density of μ we have

$$\sup_{s\in(0,1)} \int_{\mathbb{R}^d} \varrho(x,s)|\ln \varrho(x,s)|\,dx < \infty.$$

We emphasize once again that if the functions $x \mapsto a^{ij}(x,t)$ are locally Lipschitzian uniformly with respect to $t \in (0,1)$, then in Theorem 2.5.1, Corollary 2.5.2, Example 2.5.3, Proposition 2.5.5, and Corollary 2.5.6 any solution μ of (2.1.3) in the sense of (2.5.1) is a solution in the sense of (2.5.2) as well.

2.5.7. Example. Suppose that in Corollary 2.5.6 it is known additionally that the functions $x \mapsto a^{ij}(x,t)$ are uniformly Lipschitzian with a common constant. Then, for every closed interval $[\tau_1, \tau_2] \subset (0,1)$, according to §2.3 we have

$$\int_{\tau_1}^{\tau_2} \int_{\mathbb{R}^d} \frac{|\nabla \varrho(x,t)|^2}{\varrho(x,t)}\,dx\,dt < \infty.$$

2.5.8. Example. Suppose that the matrices $A(x,t)$ and $A(x,t)^{-1}$ are uniformly bounded, the functions $x \mapsto a^{ij}(x,t)$ are uniformly Lipschitzian with a common constant and that there exist numbers $c, c_0, c_1, c_2, r > 0$ such that

$$|b(x,t)| \le c_0 \exp(c|x|^r), \quad (b(x,t),x) \le c_1 - c_2|x|^r, \quad c_2 > 2cr \sup_{x,t} \|A(x,t)\|.$$

Let μ be a probability measure on $\mathbb{R}^d \times (0,1)$ satisfying equation (2.1.1) in the sense of (2.5.1) with an initial condition μ_0 such that the function $\exp(c|x|^r)$ is integrable with respect to μ_0. Then

$$\int_{\tau_1}^{\tau_2}\int_{\mathbb{R}^d} \frac{|\nabla \varrho(x,t)|^2}{\varrho(x,t)}\,dx\,dt < \infty$$

for every closed interval $[\tau_1, \tau_2] \subset (0,1)$. To prove this, it suffices to take $W_0(z) = \exp(M|z|^r)$, where $M > 2c$ is sufficiently close to $2c$.

We now turn to lower bounds without conditions on the growth of the drift coefficient; the proofs can be found in [**160**]. Until the end of this section we assume that the matrix $A(x,t) = (a^{ij}(x,t))_{1\le i,j\le d}$ is symmetric and satisfies the following conditions:

(GP1) there is a constant $\lambda > 0$ such that for all $x, y \in \mathbb{R}^d$ and $t \in (0,1)$ one has

$$|a^{ij}(x,t) - a^{ij}(y,t)| \le \lambda |x-y|,$$

(GP2) there are constants $m, M > 0$ such that for all $x, y \in \mathbb{R}^d$ and $t \in (0,1)$ one has

$$m|y|^2 \le \sum_{1\le i,j\le d} a^{ij}(x,t) y_i y_j \le M|y|^2.$$

If, in addition to (GP1) and (GP2) we have $b \in L^p_{\rm loc}(\mu)$ for some $p > d+2$, then the solution μ has a continuous density ϱ belonging to the class $\mathbb{H}^{p,1}(U,J)$ for each ball U and each interval $J \subset (0,1)$.

It turns out that without any assumptions of local boundedness or local integrability of the coefficient b with respect to Lebesgue measure one can obtain estimates of the form

$$\varrho(x,t) \ge e^{-f(c_1|x|^2 + c_2)}, \quad x \in \mathbb{R}^d, t \in J, \tag{2.5.4}$$

where $J = [\tau_1, \tau_2] \subset (0,1)$ is any closed interval, $c_1 = c_1(J), c_2 = c_2(J)$ are some constants, and $f \in C^2\bigl([0,+\infty)\bigr)$ satisfies the following conditions:

(HP1) $f(z) > 0$ $f'(z) > 0$ and $f''(z) > 0$ for $z > 0$;

(HP2) for some $0 < \gamma < 1$ the function $e^{-(1-\gamma)f(z)}$ is convex (its second derivative is nonnegative) on $(z_0, +\infty)$ for some $z_0 \ge 0$.

For obtaining estimate (2.5.4) in addition to conditions (GP1) and (GP2) it suffices to require the following conditions:

(GP3) $|b|\exp(\psi(|b|)) \in L^p(\mu)$, where $p > d+2$ and ψ is a nonnegative strictly increasing continuous function on $[0,\infty)$ such that for some $N > 0$ one has

(HP3) $\psi^{-1}(z) \le N\sqrt{f'(f^{-1}(z))}$ for all $z > 0$.

According to (HP1) and (HP3), $f'(y) \to \infty$. Hence there exists a number $y_0 > \max\{z_0, 1\}$ such that $f'(y) \ge 1$ and $(f'(y))^{-1/2} e^{f(y)} \ge e^{\psi(0)}$ whenever $y > y_0$. Let $\omega_0 = e^{-f(\ln y_0)}$. Then $0 < \omega_0 < 1$. Let

$$\Lambda = \min\{\omega_0 (2\|\varrho\|_{L^\infty})^{-1}, 1\}.$$

Let us fix a cube $Q = Q(y, 1/2)$ and numbers $s_1 < s_2 < t_1 < t_2$ such that $s_1, s_2, t_1, t_2 \in [T_0, T_1]$, where $[T_0, T_1] \subset (0,1)$. Let

$$K^- := Q \times [s_1, s_2], \quad K^+ := Q \times [t_1, t_2].$$

2.5.9. Theorem. *Let $\mu = \varrho\,dx\,dt$ satisfy equation* (2.1.1)*, where the coefficients a^{ij}, b^i satisfy conditions* (GP1)–(GP3) *and let conditions* (HP1)–(H3) *hold. Then*

$$\sup_{K^+} e^{f^{-1}(|\ln(\Lambda\varrho)|)} \le C\Bigl(\int_{K^-} e^{-\lambda f^{-1}(|\ln(\Lambda\varrho)|)}\,dx\,dt\Bigr)^{-1/\lambda},$$

where C depends only on the following quantities:

$$s_1, s_2, t_1, t_2, p, N, \omega_0, m, M, \gamma, d, \|\varrho\|_{L^\infty(\mathbb{R}^d\times[T_0,T_1])},$$
$$\int_{T_0}^{T_1}\int_{\mathbb{R}^d} |b|^p \exp\{p\psi(|b|)\}\varrho\, dx\, dt.$$

2.5.10. Theorem. *Let $\mu = \varrho\, dx\, dt$ satisfy equation* (2.1.1)*, where the coefficients a^{ij}, b^i satisfy conditions* (GP1)–(GP3) *and let conditions* (HP1)–(HP3) *hold. Let us fix an interval $J = [\tau_1, \tau_2] \subset (T_0, T_1)$. Then there exist positive numbers c_1 and c_2 such that*

$$\varrho(x,t) \ge e^{-f(c_1|x|^2 + c_2)}, \quad x \in \mathbb{R}^d, t \in J.$$

2.6. Fokker–Planck–Kolmogorov equations

Here we discuss sufficient conditions for the existence and uniqueness of solutions to parabolic equations for measures. Our presentation follows [**30**] and [**26**].

We shall say that a compact function $\Psi \in C^2(\mathbb{R}^d)$ is nondegenerate if there is a sequence of numbers $c_n \to +\infty$ such that the level sets $\Psi^{-1}(c_n)$ are C^1-surfaces (in fact, we only need that Lemma 2.1.5 be applicable to the domains $\{\Psi < c_n\}$). For example, if Ψ is convex, then it is nondegenerate; the same is true if $\Psi(x) = \psi_0(|x|^2)$, where $\psi_0 \in C^2([0, +\infty)$ is increasing to $+\infty$.

2.6.1. Theorem. *Let $p > d + 2$ and let A and b satisfy the conditions*

$$\sup_{t\in(0,1)} [\|a^{ij}(\,\cdot\,,t)\|_{W^{p,1}(B)} + \|b^i(\,\cdot\,,t)\|_{L^p(B)}] < \infty,$$
$$A(x,t) \ge M(B)\cdot \mathrm{I} \quad \forall (x,t) \in B\times(0,1) \tag{2.6.1}$$

for every ball B with some $M(B) > 0$. Assume that there exist a nondegenerate nonnegative compact function $\Psi \in C^2(\mathbb{R}^d)$ and a constant $C \ge 0$ such that

$$L\Psi(x,t) \le C + C\Psi(x) \quad \text{a.e. in } \mathbb{R}^d \times (0,1). \tag{2.6.2}$$

Then, for every Borel probability measure ν on $\mathbb{R}^d$, there exists a family $\mu = (\mu_t)_{t\in[0,1)}$ of probability measures on $\mathbb{R}^d$ satisfying (2.1.1) *and* (2.1.3) *such that*

$$t \mapsto \int_{\mathbb{R}^d} \zeta\, d\mu_t$$

is continuous on $[0,1)$ for every $\zeta \in C_0^\infty(\mathbb{R}^d)$.

Let us note that this theorem (established in [**26**]) is stronger than the analogous result of [**24**] in all respects except that here the Lyapunov function Ψ is supposed to be nondegenerate. However, this is a rather mild restriction, which is fulfilled in all interesting cases.

2.6.2. Remark. (i) In the case when the functions b^i are bounded on bounded subsets of $\mathbb{R}^d \times (0,1)$, the nondegeneracy condition on A can be slightly relaxed as follows: it suffices to have

$$\inf_{(x,t)\in K\times[\tau_1,\tau_2]} \det A(x,t) > 0$$

for every $[\tau_1, \tau_2] \subset (0,1)$ and every compact set $K \subset \mathbb{R}^d$.

(ii) Condition (2.6.2) can be relaxed as follows: there exists a compact set $K \subset \mathbb{R}^d$ such that $L\Psi(x,t) \le C$ a.e. in $(0,1)\times(\mathbb{R}^d\backslash K)$.

(iii) For almost every t, the measure μ_t has a density in the Sobolev class $W^{p,1}_{\mathrm{loc}}(\mathbb{R}^d)$. This is true for any solution of (2.1.1) under our local assumptions on

A and b. Hence, under these assumptions, equation (2.1.1) can be written in the classical weak form after integrating by parts in the term with $\partial_{x_i}\partial_{x_j}u$. Below we consider more general equations whose solutions do not have such a property.

2.6.3. Corollary. *Suppose that there is a constant C such that*

$$\|A(x,t)\| \le C + C\ln(|x|^2+1), \quad (x,t)\in(0,1)\times\mathbb{R}^d, \tag{2.6.3}$$

and, for every compact set $K\subset\mathbb{R}^d$ and every $[\tau_1,\tau_2]\subset(0,1)$, one has

$$\inf_{(x,t)\in K\times[\tau_1,\tau_2]} \det A(x,t) > 0, \quad \sup_{(x,t)\in K\times(0,1)} |b(x,t)| < \infty.$$

Assume also that there is a constant M such that

$$\langle b(x,t),x\rangle \le M(1+|x|^2)\ln(|x|^2+1), \quad (x,t)\in\mathbb{R}^d\times(0,1). \tag{2.6.4}$$

Then, for every Borel probability measure ν on $\mathbb{R}^d$, there exists a family $(\mu_t)_{t\in[0,1)}$ of probability measures on $\mathbb{R}^d$ satisfying (2.1.1) *and* (2.1.3) *such that*

$$t\mapsto \int_{\mathbb{R}^d} \zeta\, d\mu_t$$

is continuous on $[0,1)$ for every $\zeta\in C_0^\infty(\mathbb{R}^d)$. The same is true if we replace (2.6.3) *and* (2.6.4) *by*

$$\|A(x,t)\| \le C + C|x|^2, \quad \langle b(x,t),x\rangle \le C + C|x|^2, \quad (x,t)\in\mathbb{R}^d\times(0,1).$$

If the functions b^i and a^{ij} are continuous in x for a.e. fixed t, then the same is true without the assumption that $\det A$ *is strictly positive.*

2.6.4. Corollary. *Suppose that the functions $x\mapsto a^{ij}(x,t)$ and $x\mapsto b^i(x,t)$ are continuous for each $t\in(0,1)$ and are bounded on bounded sets in $\mathbb{R}^d\times(0,1)$. In addition, suppose that, for every fixed ball $U\subset\mathbb{R}^d$, the functions $x\mapsto a^{ij}(x,t)$, $t\in(0,1)$, are equicontinuous on U. Finally, assume that there exist a nondegenerate nonnegative compact function $\Psi\in C^2(\mathbb{R}^d)$ and a constant $C\ge 0$ such that*

$$L\Psi(x,t) \le C + C\Psi(x).$$

Then, for every Borel probability measure ν, there exists a family $\mu=(\mu_t)_{t\in[0,1)}$ of probability measures on $\mathbb{R}^d$ satisfying (2.1.1) *and* (2.1.3) *such that*

$$t\mapsto \int_{\mathbb{R}^d} \zeta\, d\mu_t$$

is continuous on $[0,1)$ for every $\zeta\in C_0^\infty(\mathbb{R}^d)$.

Moreover, if $\det A$ *is bounded away from zero on compact subsets in $\mathbb{R}^d\times(0,1)$, then the continuity of b in x is not needed.*

We observe that these results are applicable to degenerate A, in particular, to $A=0$.

2.6.5. Corollary. *If $\nu=\varrho_0\,dx$, then the solution from Theorem* 2.6.1 *satisfies inequalities* (2.1.9), (2.1.10), *and* (2.1.11) *for any domain $\Omega\subset\mathbb{R}^d$ such that the right-hand sides are finite. If $|b(x,t)|\le C\Psi(x)$, $\Psi\in L^1(\nu)$, and $\varrho_0\in L^2(\mathbb{R}^d)$, then it satisfies* (2.4.3) *as well.*

Under appropriate assumptions it is possible to construct a solution defined for t in the whole real line.

2.6.6. Corollary. *Suppose that the coefficients A and b are defined on the whole space $\mathbb{R}^d \times \mathbb{R}^1$ and satisfy condition* (2.6.1) *on every bounded interval in place of* $(0,1)$. *Suppose also that there exist a nondegenerate nonnegative compact function $\Psi \in C^2(\mathbb{R}^d)$ and positive numbers c_1 and c_2 such that*

$$L\Psi(x,t) \le c_1 - c_2\Psi(x) \quad \textit{for almost all } (x,t).$$

Then, there exists a family $\mu = (\mu_t)_{t\in\mathbb{R}^1}$ of probability measures on $\mathbb{R}^d$ satisfying (2.1.1) *such that μ_t has a density $\varrho(\,\cdot\,,t)$ and ϱ is jointly continuous and positive.*

Under some stronger assumptions, there are similar uniqueness results. Let us introduce the following conditions.

(HU1) *There exists $p > d+2$ such that for every open ball $B \subset \mathbb{R}^d$ one has*
(a) $\inf_{(x,t)\in B\times[0,T]} \det A(x,t) > 0$, $\sup_{t\in[0,T]:\ 1\le i,j\le d} \|a^{ij}(\,\cdot\,,t)\|_{W^{p,1}(B)} < \infty$,
(b) $\displaystyle\int_0^T \int_B |b(x,t)|^p \,dxdt < \infty$.

We introduce the following set of measures on $\mathbb{R}^d \times (0,T)$:

$$\begin{aligned}\mathcal{M}_{\rm par}^{A,b,\nu} := \bigl\{\mu |\ & \mu(dx,dt) = \mu_t(dx)dt,\ \mu_t \in \mathcal{P}(\mathbb{R}^d)\ \ \forall\, t \in (0,T) \text{ and } \mu \\ & \text{satisfies (2.1.1), (2.1.3), where } |b| \in L^1(B\times(0,T),\mu) \text{ for every ball } B \subset \mathbb{R}^d\bigr\}.\end{aligned}$$

2.6.7. Theorem. *Assume* (HU1). *Suppose, in addition, that the following condition holds*:
(HU2) *each a^{ij} is Hölder continuous in $t \in [0,T]$ locally uniformly with respect to $x \in \mathbb{R}^d$.*
Let $\mathcal{K} \subset \mathcal{M}_{\rm par}^{A,b,\nu}$ be such that $\mathcal{K}$ is convex and for all $\mu \in \mathcal{K}$

$$(1-L)(C_0^\infty(\mathbb{R}^d \times [0,T))) \textit{ is dense in } L^1(\mathbb{R}^d \times (0,T),\mu).$$

Then $\#\mathcal{K} \le 1$.

We now give constructive conditions on the coefficients which enable us to use the above result.

Define the logarithmic gradient $\beta_\mu = (\beta_\mu^1,\ldots,\beta_\mu^d)$ of μ with respect to the metric given by A as follows:

$$\beta_\mu^i := \sum_{j=1}^d (\partial_{x_j} a^{ij} + a^{ij}\varrho^{-1}\partial_{x_j}\varrho), \quad i = 1,\ldots,d.$$

2.6.8. Proposition. *Assume* (HU1) *and* (HU2) *and define $\mathcal{K}$ to be the set of all measures $\mu \in \mathcal{M}_{\rm par}^{A,b,\nu}$ satisfying the following three conditions for all $1 \le i,j \le d$:*
(i) $\partial_{x_j} a^{ij} \in L^1(B\times(0,T),\mu)$ *for all open balls* $B \subset \mathbb{R}^d$,
(ii) $a^{ij} \in L^1(\mathbb{R}^d \times (0,T),\mu)$,
(iii) $b^i - \beta_\mu^i \in L^1(\mathbb{R}^d \times (0,T),\mu)$.
Then $\#\mathcal{K} \le 1$.

2.6.9. Proposition. *Assume* (HU1) *and* (HU2). *Let $V \in C^{1,2}(\mathbb{R}^d \times [0,T])$ be such that $\lim\limits_{|x|\to\infty} V(x,t) = +\infty$ uniformly with respect to $t \in [0,T]$. Let $\mathcal{K}$ be the set of all measures $\mu \in \mathcal{M}_{\rm par}^{A,b,\nu}$ satisfying condition* (i) *in Proposition* 2.6.8 *and such that for some $\alpha_0 = \alpha_0(\mu) \in (0,\infty)$ one has*

$$L_{A,2\beta_\mu - b}V - \partial_t V \le \alpha_0 V.$$

Then $\#\mathcal{K} = 1$.

2.6.10. Theorem. *Assume* (HU1) *and* (HU2). *Suppose that the following global conditions hold for A, b, and ν:*

(iv) *the measure ν has finite entropy, i.e., $\nu = \varrho_0\,dx$ and $\varrho_0 \ln \varrho_0 \in L^1(\mathbb{R}^d)$,*

(v) *there exists $\varepsilon \in (0,\infty)$ such that*

$$\varepsilon \mathrm{I} \le A(x,t) \le \varepsilon^{-1}\mathrm{I} \quad \textit{for all } (x,t) \in \mathbb{R}^d \times [0,T],$$

(vi) *there exists $\Lambda \in (0,\infty)$ such that for all $x, y \in \mathbb{R}^d$ one has*

$$\sup\{|a^{ij}(x,t) - a^{ij}(t,y)|\colon\ t \in [0,T],\ 1 \le i \le j \le d\} \le \Lambda|x-y|,$$

(vii) *there exists $c \in (0,\infty)$ such that for all $(x,t) \in \mathbb{R}^d \times [0,T]$ one has*

$$\langle b(x,t), x\rangle \le c(1+|x|^2),$$

and either for some $k \in \mathbb{N}$ one has

$$|b(x,t)| \le c(1+|x|^{2k}) \quad \textit{and} \quad \int_{\mathbb{R}^d} |x|^{2k}\,\nu(dx) < \infty$$

or there exist numbers $\alpha, \gamma, \delta, c, k \in (0,\infty)$ such that for all $(x,t) \in \mathbb{R}^d \times [0,T]$

$$\langle b(x,t), x\rangle \le \gamma - (2\varepsilon^{-1}ck + \delta)|x|^{2k},$$

with ε as in (v), *and*

$$|b(x,t)| \le \alpha \exp\left(\frac{c}{2}\,|x|^{2k}\right), \quad \int_{\mathbb{R}^d} \exp\left(\frac{c}{2}\,|x|^{2k}\right)\,\nu(dx) < \infty.$$

Then there exists a unique family $\{\mu_t, t \in (0,T]\}$ of probability measures on $\mathbb{R}^d$ solving (2.1.1), (2.1.3).

The equations for measures considered above are the Fokker–Planck equations according to the terminology indicated above. We do not discuss here the closely related Kolmogorov equations for functions (see, e.g., [**29**], [**28**], [**62**]). Finally, we note that similar existence results have been recently obtained for nonlinear equations for measures in [**45**].

CHAPTER 3

Infinite-dimensional case

3.1. Equations for measures on infinite-dimensional spaces

The problems considered in the previous two chapters arise also in the infinite-dimensional case; as it was noted in the Introduction, the infinite-dimensional case even served as one of motivations for the study of such problems. We shall briefly discuss elliptic equations because a detailed account of the whole area deserves a separate survey. Moreover, our discussion will be restricted to problems directly connected with the finite-dimensional case. First of all, it should be noted that many of the principal objects considered above such as stochastic equations, transition probabilities, stationary distributions, Fokker–Planck and Kolmogorov equations are defined on very general spaces and bear no finite-dimensional features. However, in the infinite-dimensional case new phenomena arise. Partly this is due to the absence of exact analogs of Lebesgue measure, but there are also other reasons. One of the starting points is again a Markov process with a state space X whose transition semigroup $(T_t)_{t\geq 0}$ defined on a suitable function space $\mathcal{F}$ (such as $C_b(X)$ or $L^p(\mu)$). If this process has an invariant probability measure μ on X, i.e.,

$$\int_X f(x)\,\mu(dx) = \int_X T_t f(x)\,\mu(dx), \quad f \in \mathcal{F},$$

then, under broad assumptions, $(T_t)_{t\geq 0}$ extends to a strongly continuous semigroup on $L^1(\mu)$ which has a generator L on some domain $\mathcal{D}$, and μ satisfies the equation

$$L^*\mu = 0 \tag{3.1.1}$$

in the sense that the integral of Lf vanishes for all $f \in \mathcal{D}$. Typically, there is a smaller class $\mathcal{D}_0 \subset \mathcal{D}$ of functions for which Lf is defined explicitly without any reference to the semigroup and the domain of its generator. Then we can consider equation (3.1.1) as the identity

$$\int_X Lf(x)\,\mu(dx) = 0, \quad f \in \mathcal{D}_0. \tag{3.1.2}$$

Therefore, questions arise about the relation between (3.1.2) and the invariance of μ with respect to the semigroup, as well as about the properties of solutions to (3.1.2), i.e., the same questions that we have discussed in the finite-dimensional case. For example, let μ be an invariant measure of a diffusion process in l^2 or in the space $\mathbb{R}^\infty$ of all real sequences governed by the stochastic differential equation

$$d\xi_t = \sqrt{2}dW_t + b(\xi_t)dt, \tag{3.1.3}$$

where W_t is a Wiener process (in a suitable sense, see [**21**, Chapter 7]) and b is a Borel vector field. Then one can take for $\mathcal{D}_0$ the class $\mathcal{FC}_b^\infty$ of all functions of the form

$$f(l_1(x), \dots, l_n(x)), \tag{3.1.4}$$

where $f \in C_b(\mathbb{R}^n)$ and $l_1, \ldots, l_n$ are continuous linear functionals (e.g., just functions of $x_1, \ldots, x_n$). Then L has the form

$$Lf = \operatorname{trace} f'' + \langle f', b\rangle.$$

However, as we shall see below in concrete examples, in the infinite-dimensional case it is often desirable to have a broader setting where b cannot be interpreted as a vector field with values in the original space (e.g., when dealing with l^2 it may happen that v takes values in $\mathbb{R}^\infty$). For this reason, the following more general framework is used.

Suppose that X is a locally convex space with dual X^* and that $H \subset X$ is a dense separable Hilbert space such that the embedding operator is continuous. The inner product in H is denoted by $(u, v)_H$. This embedding defines the embedding $j_H\colon X^* \to H$ since for each $l \in X^*$ there is a unique element $j_H(l) \in H$ with

$$\langle l, h\rangle = (j_H(l), h)_H \quad \forall\, h \in H.$$

Suppose also that a family $\{l_i\} \subset X^*$ is given such that the vectors $e_n := j_H(l_n)$ form an orthonormal basis in H. Let $\mathcal{FC}_b^\infty(\{l_i\})$ denote the class of all functions of the form (3.1.4). Suppose also that we are given Borel functions a^{ij}, b^i on X. The infinite matrix with entries $a^{ij}(x)$ will be denoted by $A(x)$ and the collection of scalar functions b^i will be denoted by b, although we do not assume that $A(x)$ and $b(x)$ correspond to some operator or vector. Then we can consider the elliptic operator

$$L_{A,b} f(x) = \sum_{i,j} a^{ij}(x)\partial_{e_i}\partial_{e_j} f(x) + \sum_i b^i(x)\partial_{e_i} f(x). \tag{3.1.5}$$

We shall say that a Radon measure μ on X satisfies equation (3.1.1) with respect to the class $\mathcal{FC}_b^\infty(\{l_i\})$ if $a^{ij}, b^i \in L^1(\mu)$ for all i, j and (3.1.2) is fulfilled with $\mathcal{D}_0 = \mathcal{FC}_b^\infty(\{l_i\})$. The parabolic operators and equations are defined similarly in complete analogy with the finite-dimensional case.

Infinite-dimensional equations for measures have been considerably less studied so far as compared to the finite-dimensional case. There are sufficient conditions for the existence and sometimes for the uniqueness of solutions, but not much is known about their properties and connections between infinitesimal invariance and proper invariance with respect to the associated semigroups (the very existence of such semigroups has also been less studied). There exists an extensive literature on stationary distributions of infinite-dimensional diffusions (see, e.g., the book [**67**]), especially connected with stochastic partial differential equations (see [**11**], [**52**], [**57**], [**58**], [**62**], [**63**], [**70**], [**74**], [**79**], [**82**], [**90**], [**91**], [**128**], [**150**], [**163**]), infinite gradient systems, Gibbs measures, and stochastic quantization (see [**5**], [**6**], [**56**], [**83**], [**85**], [**86**], [**96**], [**138**]); in these works numerous additional references can be found. Standard methods of proving the existence of stationary distributions are based on Prohorov's theorem and Lyapunov functions combined with a priori estimates (see, e.g., [**59**], [**113**], [**114**]) or on convergence of transition probabilities (which in turn employs various assumptions of dissipativity and Lyapunov functions). The same techniques are used for the proof of existence of solutions to elliptic or parabolic equations. Generally speaking, existence results for elliptic equations can be obtained under broader assumptions on the coefficients, since they do not assume the existence of the corresponding diffusions (the latter is usually more stringent). However, there are cases where there is no direct proof of the solvability of the elliptic equation for measures, and one has to construct a process and analyze its transition semigroup.

For solutions to elliptic and parabolic equations for measures on infinite-dimensional spaces we can consider various properties of their finite-dimensional projections, their mutual absolute continuity or singularity, and directional properties such as continuity or differentiability. We recall that a Radon measure μ on a locally convex space X is continuous along a vector $h \in X$ if $\lim\limits_{t\to 0} \|\mu_{th} - \mu\| = 0$, where $\mu_{th}(B) := \mu(B + th)$ for every Borel set B. If, for every Borel set B, the function $t \mapsto \mu(B + th)$ is differentiable, then μ is called Fomin differentiable along h. This is equivalent to the existence of a function $\beta_h^\mu \in L^1(\mu)$, called the logarithmic derivative of μ along h, such that the following integration by parts formula holds for all functions $f \in \mathcal{FC}_b^\infty(X)$:

$$\int_X \partial_h f(x)\, \mu(dx) = -\int_X f(x)\beta_h^\mu(x)\, \mu(dx).$$

If $X = \mathbb{R}^d$, then a measure μ is Fomin differentiable along all vectors precisely when it has a density $\varrho \in W^{1,1}(\mathbb{R}^d)$; then $\beta_h^\mu = \langle \nabla\varrho, h\rangle$. In the infinite-dimensional case, only the zero measure is differentiable along all vectors. A detailed account of the theory of differentiable measures is given in [**20**], [**22**]. There is also a reasonable infinite-dimensional analog of the logarithmic gradient $\nabla\varrho/\varrho$, although neither ϱ nor $\nabla\varrho$ exist separately. Namely, suppose that we a have a continuously and densely embedded Hilbert space $H \subset X$ generating the embedding $j\colon X^* \to X$ as explained above. If there exists a μ-measurable mapping $\beta_H\colon X \to X$ such that for each $l \in X^*$ the measure μ is differentiable along $j(l)$ and

$$\beta_{j(h)}^\mu = \langle l, \beta_H\rangle,$$

then β_H is called a logarithmic gradient of μ associated with H. However, it can happen that the measure μ is differentiable along all vectors $j(l)$, where $l \in X^*$, but there is no logarithmic gradient. Concerning this object, see [**22**, Chapter 7].

Let $\{l_i\} \subset X^*$ be such that for the vectors $e_i := j(l_i)$ we have $l_j(e_i) = \delta_{ij}$. We shall assume additionally that the functionals l_i separate points in X. Then, under mild assumptions on X (see [**21**, Chapter 7]) there is a process W_t in X, called a Wiener process in X associated with H, such that the scalar processes $\langle l_i, W_t\rangle$ are independent Wiener processes. It is known (see [**7**], [**22**, Chapter 12]) that under broad assumptions, for a given probability measure μ possessing a logarithmic gradient β_H, there is a diffusion process in X governed by the stochastic differential equation (3.1.3) with $b = \beta_H$ such that μ is its invariant measure and the corresponding transition semigroup $(T_t)_{t\geq 0}$ is symmetric on $L^2(\mu)$; its generator is given by

$$Lf = \sum_i^\infty [\partial_{e_i}^2 f + \langle l_i, \beta_H\rangle \partial_{e_i} f]$$

on functions $f \in \mathcal{FC}_b^\infty(\{l_i\})$, where $f_0 \in C_b^\infty(\mathbb{R}^n)$. An efficient method of constructing more general processes is based on perturbations of the drift β_H.

As in the finite-dimensional case, the symmetry of the operator $L_{\mathrm{I},b}$ is equivalent to b being in some sense of gradient type. The exact result is as follows.

3.1.1. Proposition. *Let $b^i \in L^2(\mu)$. The operator*

$$Lf = \sum_{i=1}^\infty [\partial_{e_i}^2 f + b^i \partial_{e_i} f]$$

is symmetric on $\mathcal{FC}_b^\infty(\{l_i\}) \subset L^2(\mu)$ precisely when the measure μ is differentiable along each vector $e_n = j(l_n)$ and $\beta_{e_n}^\mu = b^n$ μ-a.e.

Proof. If $\beta^\mu_{e_n}$ exists, then

$$\int_X g(x)\partial^2_{e_n} f(x)\,\mu(dx) = -\int_X [\partial_{e_n} g(x)\partial_{e_n} f(x) + g(x)\partial_{e_n} f(x)\beta^\mu_{e_n}(x)]\,\mu(dx),$$

whence we obtain the equality of the integrals of gLf and fLg for all functions $f, g \in \mathcal{FC}_b^\infty(\{l_i\})$. Conversely, suppose that L is symmetric on $\mathcal{FC}_b^\infty(\{l_i\})$. Let $f = \exp(il)$, $g = \exp(itl_n)$, where l is a finite linear combination of the functionals l_i. Let $\langle l, b\rangle := \sum_{k=1}^\infty l(e_k)b^k$, where the sum is finite since only finitely many numbers $l(e_k)$ are nonzero. Then

$$Lf = -\sum_{k=1}^\infty l(e_k)^2 e^{il} + i\langle l, b\rangle e^{il}, \quad Lg = -t^2 e^{itl_n} + itb^n e^{itl_n},$$

whence by the symmetry of L we obtain the equality

$$\int_X e^{il+itl_n}\Bigl[-\sum_{k=1}^\infty l(e_k)^2 + i\langle l, b\rangle\Bigr]\,\mu(dx) = \int_X e^{il+itl_n}\Bigl[-t^2 + itb^n\Bigr]\,\mu(dx).$$

Letting $\xi := l + tl_n$ and replacing l by $\xi - tl_n$, we can write this as follows:

$$\int_X e^{i\xi}\Bigl[-\sum_{k=1}^\infty \xi(e_k)^2 + i\langle \xi, b\rangle\Bigr]\,\mu(dx) = 2\int_X e^{i\xi}\Bigl[-t\xi(e_n) + itb^n\Bigr]\,\mu(dx).$$

Since the left-hand side is independent of t, the right-hand side must vanish. Therefore, we obtain

$$-i\xi(e_n)\int_X e^{i\xi}\,\mu(dx) = \int_X e^{i\xi} b^n\,\mu(dx)$$

for any functional ξ that is a finite linear combination of the functionals l_i. Since μ is a Radon measure and $\{l_i\}$ separates points in X, it follows that the same is true for any $\xi \in X^*$. This identity yields that μ is differentiable along e_n and $\beta^\mu_{e_n} = b^n$ (see [**22**, Theorem 3.6.7]). Note that without the assumption that $\{l_i\}$ separates points in X we could obtain the same assertion for μ on the σ-algebra generated by $\{l_i\}$. □

The case of non-constant A is studied similarly in [**121**].

The case of gradient-type drifts (this concept in turn admits different interpretations in infinite dimensions) has deep and interesting connections with the study of Gibbs measures, i.e., measures with given conditional distributions. For example, for a broad class of models, constructing a Gibbs measure is equivalent to constructing a measure with a given logarithmic gradient; the latter problem is studied in [**5**], [**6**], [**20**], [**22**], [**99**], [**100**], [**101**], [**102**]; in particular, the method of reconstructing a measure with a given logarithmic gradient by means of Lyapunov functions was initiated by A.I. Kirillov. For general elliptic equations for measures this method was developed in [**37**], [**39**], [**48**]. The uniqueness problem for elliptic equations for measures becomes especially difficult in infinite dimensions. There are simple examples where there is no uniqueness even for $A = \mathrm{I}$ and drifts b that are bounded linear operators on Hilbert spaces (see [**37**], [**22**, §7.6]). In the framework of Gibbs measures, such examples arise in cases of phase transitions, i.e., distinct Gibbs distributions with equal conditional distributions. Concerning existence and uniqueness of solutions to parabolic equations, see [**27**], [**29**], [**127**].

Let us give a number of typical examples, in which the described framework is applied to the study of stochastic partial differential equations of the type of stochastic porous media equations, reaction-diffusion equations, and Burgers and Navier–Stokes equations. The precise formulations and additional references can be found in [**39**], [**25**]. A general plan of studying such equations is as follows. The primary object is a nonlinear partial differential equation of the form

$$\partial\xi(u,t)/\partial t = \Delta\big[\Psi(\xi)\big](u,t) + \Phi(\xi)(u,t),$$

where Ψ and Φ are some functions on the real line, e.g., polynomials. The analysis of this equation usually turns out to be very complicated (e.g., the Navier–Stokes equation). However, it turns out that adding a stochastic noise in the right-hand side leads to substantial simplifications (it is even possible that the stochastic equation has at least the same physical significance). This stochastic partial differential equation is heuristically written as

$$d\xi_t = \sqrt{2}dW_t + \Big(\Delta[\Psi(\xi_t)] + \Phi(\xi_t)\Big)\,dt.$$

But a rigorous interpretation in the case of nonlinear functions Ψ and Φ is not obvious. One possible approach to this problem is to consider the associated infinite-dimensional elliptic operator L on a suitable domain, find an infinitesimally invariant measure μ for L, construct a Markov semigroup on $L^2(\mu)$ with μ as an invariant measure in such a way that the generator of this semigroup extends L, and finally construct a Markov process solving the martingale problem corresponding to this operator. One can also try to solve the parabolic equation for the transition probabilities of the expected process. The case $\Psi(s) = s^m$ and $\Phi = 0$ corresponds to the porous media equation and the case $\Psi(s) = s$ and $\Phi(s) = s^m$ to the reaction-diffusion equation.

Let $D \subset \mathbb{R}^d$ be a bounded open domain with smooth boundary, let $\{e_n\}$ be the orthonormal basis in $L^2(D)$ formed by the eigenfunctions of the Laplacian Δ with Dirichlet boundary condition, and let λ_1 be the minimal (in absolute value) eigenvalue. Let Ψ be a C^1-function with $\Psi(0) = 0$ such that for some positive numbers κ_0, C_0, κ_1, and $r \geq 1$ we have

$$\kappa_0|s|^{r-1} \leq \Psi'(s) \leq C_0 + \kappa_1|s|^{r-1} \quad \text{for all } s \in \mathbb{R}^1,$$

and let Φ be a continuous function satisfying the condition

$$|\Phi(s)| \leq C + \delta|s|^r,$$

where $0 < \delta < 4\kappa_0\lambda_1(r+1)^{-2}$ and C is a constant. For example, it suffices that $|\Phi(s)| \leq \kappa_2 + \kappa_3|s|^q$, where $q \in (0,r)$, $\kappa_2, \kappa_3 \in (0,+\infty)$. We are interested in the existence of infinitesimally invariant measures for the infinite-dimensional elliptic operator L which is informally given by

$$Lf := \Delta_Q f + \langle b, \nabla f\rangle, \quad b(x) = \Delta\Psi(x) + \Phi(x)$$

on smooth cylindrical functions defined on $X := L^2(D)$ or on the negative Sobolev space $H^{2,-1}(D)$. A rigorous interpretation is as follows. Let

$$b^i(x) := \int_D \Big[\Psi(x(u))\Delta e_i(u) + \Phi(x(u))e_i(u)\Big]\,du, \quad x \in L^r(D).$$

Let $q_i > 0$ be such that $S := \sum_{i=1}^\infty q_i < \infty$. The operator

$$Lf := \sum_{i=1}^\infty \big[q_i\partial_{e_i}^2 f + b^i\partial_{e_i} f\big]$$

where ∂_{e_i} denotes the partial derivative along e_i, is well-defined on $\mathcal{FC}_b^\infty(\{l_i\})$, where $l_i(x) = (x, e_i)_2$ and $(x,y)_2$ is the inner product in $L^2(D)$. The second order part in L can be regarded as trace$\,(QD^2 f)$, where Q is the operator on X defined by $Qe_i = q_i e_i$. The operator Q is the covariance operator of a Wiener process W_t

in the indicated stochastic equation. A minor nuance is that the functions b^i, hence also Lf, are defined not on all of X, but only on $L^r(D)$.

3.1.2. THEOREM. *Under the stated assumptions, there exists a Borel probability measure on μ on $L^r(D)$ that is infinitesimally invariant for L.*

For example, if $\Psi(t) = t^r$, where r is an odd number, then we can take for Φ any polynomial of degree r with a sufficiently small leading coefficient (the smallness of which depends on λ_1, in particular, one can take $\Phi(x) = x^r$ provided that λ_1 is sufficiently large). Similarly, one considers the parabolic equation with a time-dependent drift b formally given by $b(x,t)(u) = \Delta_u[\Psi(x(u),t)] + \Phi(x(u),t)$, where Ψ and Φ are real functions on $\mathbb{R}^1 \times [0,1]$. Set

$$b^i(x,t) := \int_D \Big[\Psi(x(u),t)\Delta e_i(u)\,du + \Phi(x(u),t)e_i(u)\Big]\,du, \quad x \in L^r(D).$$

The corresponding parabolic operator L is given by

$$Lf = \partial_t f + \sum_{i=1}^{\infty} q_i \partial_{e_i}^2 f + \sum_{i=1}^{\infty} b^i \partial_{e_i} f.$$

Suppose that Ψ and Φ are continuous functions, Ψ has a continuous partial derivative $\partial_s\Psi(s,t)$, and

$$\kappa_0|s|^{r-1} \le \partial_s\Psi(s,t) \le C_1 + \kappa_1|s|^{r-1}, \ |\Phi(s,t)| \le C_2 + \kappa_2|s|^r,$$

where $\kappa_0, \kappa_1, \kappa_2, C_1, C_2 \in (0,+\infty)$ are some constants and $r \ge 1$. Under these assumptions there exists a probability measure μ on $L^r(D) \times [0,1)$ satisfying the parabolic equation $L^*\mu = 0$ with a suitable initial data.

A stochastic Navier–Stokes type equation is considered in the space X of all $\mathbb{R}^d$-valued mappings $\xi = (\xi^1, \ldots, \xi^d)$ such that $\xi^j \in W_0^{2,1}(D)$ and $\operatorname{div}\xi = 0$. The space X is equipped with a Hilbert norm $\|\xi\|_0$ defined by $\|\xi\|_0^2 := \sum_{j=1}^d \|\nabla\xi^j\|_{L^2}^2$. The equation is written formally as

$$d\xi(x,t) = \sqrt{2}dW(x,t) + \Big[\Delta_x\xi(x,t) - \sum_{j=1}^d \xi^j(x,t)\partial_{x_j}\xi(x,t) + F(x,\xi(x,t),t)\Big]dt,$$

where W is a suitable Wiener process in X and $F\colon D \times \mathbb{R}^d \times (0,1) \to \mathbb{R}^d$ is a bounded continuous mapping. Since the Laplacian Δ is not defined on all of X, this equation requires some interpretation. Our approach suggests the following procedure. Let $\{\eta_n\}$ be the eigenbasis in the closure of X in $L^2(D,\mathbb{R}^d)$ formed by the eigenfunctions of Δ, where $\eta_n \in X$. Let us introduce the functions

$$b^n(\xi,t) := (\xi, \Delta\eta_n)_2 - \sum_{j=1}^d (\partial_{x_j}\xi, \xi^j\eta_n)_2 + \Big(F(\,\cdot\,,\xi(\,\cdot\,),t),\eta_n\Big)_2.$$

These functions are defined already on the whole X. It is easily verified that they are continuous on all balls in X with respect to the topology from $L^2(D,\mathbb{R}^d)$. Choosing an appropriate Wiener process we arrive at the parabolic operator

$$Lf(\xi,t) = \partial_t f(\xi,t) + \sum_{n=1}^{\infty} \alpha_n \partial_{\eta_n}^2 f(\xi,t) + \sum_{n=1}^{\infty} b^n(\xi,t)\partial_{\eta_n} f(\xi,t).$$

Here we also have a probability measure μ on $X \times [0,1)$ satisfying the parabolic equation $L^*\mu = 0$ with any initial distribution μ_0 for which $\|\xi\|_0^2\|\xi\|_2^k \in L^1(\mu_0)$ for all k. If F does not depend on t, then one can consider similarly the elliptic equation and establish the existence of infinitesimally invariant measures of the stochastic Navier–Stokes equation.

3.2. Properties of solutions

The results of Chapter 1 yield certain properties of the finite-dimensional projections of a measure μ, satisfying an infinite-dimensional equation, generated by the mappings $P_n x = (l_1(x), \ldots, l_n(x))$. We shall assume that we are given an embedding $H \subset X$ generating the embedding $j\colon X^* \to H$, and the functionals l_i are such that the vectors $e_i := j(l_i)$ have the property $l_i(e_j) = \delta_{ij}$. Let μ be a probability measure satisfying the equation $L^*_{A,b}\mu = 0$ with $A = (a^{ij})_{i,j\ge 1}$ and $b = (b^i)_{i\ge 1}$ in the sense explained above.

Let E_n denote the conditional expectation with respect to the σ-field generated by P_n and let
$$\sigma_n^{ij} := \mathrm{E}_n\sigma^{ij}, \quad b_n^i := \mathrm{E}_n b^i.$$
We consider the elliptic operator
$$L_n f := \sum_{i,j=1}^n a_n^{ij}(x)\partial_{x_i}\partial_{x_j} f(x) + \sum_{i=1}^n b_n^i(x)\partial_{x_i} f(x)$$
on $\mathbb{R}^n$. For the measures $\mu_n := \mu\circ P_n^{-1}$ on $\mathbb{R}^n$, where $\mu\circ P_n^{-1}(B) := \mu(P_n^{-1}(B))$, we obtain
$$L_n^*\mu_n = 0.$$
Indeed, if $f(x) = f(l_1(x), \ldots, l_n(x))$, then
$$\begin{aligned}\int_{\mathbb{R}^n} L_n\varphi(y)\,\mu_n(dy) &= \int_X \Big[\sum_{i,j\le n} a_n^{ij}(x)\partial_{e_i}\partial_{e_j}\varphi(l_1(x),\ldots,l_n(x)) \\ &\quad + \sum_{i\le n} b_n^i(x)\partial_{e_i}\varphi(l_1(x),\ldots,l_n(x))\Big]\,\mu(dx) \\ &= \int_X \Big[\sum_{i,j\le n} a^{ij}(x)\partial_{e_i}\partial_{e_j}\varphi(l_1(x),\ldots,l_n(x)) \\ &\qquad + \sum_{i\le n} b^i(x)\partial_{e_i}\varphi(l_1(x),\ldots,l_n(x))\Big]\,\mu(dx) = 0.\end{aligned}$$

3.2.1. Example. Suppose that the functions a^{ij} are constant and the matrices $(a^{ij})_{i,j\le n}$ are positive and that $b^i \in L^p(\mu)$ for all $p < \infty$. Then μ_n has a bounded continuous density ϱ_n of the class $W^{p,1}(\mathbb{R}^n)$ for all n.

If we have only that $b^i \in L^2(\mu)$, then μ_n has a density $\varrho_n \in W^{1,1}(\mathbb{R}^n)$ and $|\nabla\varrho_n/\varrho_n|^2 \in L^2(\mu_n)$.

If we have $\exp(c_i|b^i|) \in L^1(\mu)$ for some $c_i > 0$, then the continuous density ϱ_n is positive.

Other applications of the results in the first two chapters to finite-dimensional projections of solutions of infinite-dimensional equations for measures are given in [**43**]. For some special equations, there are other results, e.g., certain uniform local estimates for measures of balls and densities, see [**14**], [**1**], [**163**]. However, the following question remains open. Let a probability measure μ on a separable Hilbert space satisfy the equation $L^*_{I,b} = \mu = 0$ with a continuous or even locally Lipschitzian drift b; is it then positive on all balls of positive radius? In a number of special cases positive results are known; for example, this is the case if the measure μ is equivalent to some measure possessing the indicated property (say, Gaussian).

It is more difficult to obtain infinite-dimensional properties of solutions which could be regarded as infinite-dimensional analogs of absolute continuity and differentiability. There are results asserting that, under rather special assumptions, a stationary distribution of an infinite-dimensional diffusion or a solution to an elliptic equation is absolutely continuous with respect to a given measure (typically Gaussian) and its density belongs to some Sobolev class; see [**23**], [**31**], [**36**], [**37**], [**64**], [**65**], [**67**], [**78**], [**88**], [**90**], [**95**], and the references in [**22**, Chapter 12]. It should be noted that Tolmachev [**171**] constructed an example of an infinite-dimensional diffusion with smooth coefficients and bounded and uniformly nondegenerate diffusion coefficient such that its transition probabilities and stationary distribution have no directions of continuity (in the sense defined above). A long-standing open problem is whether this can happen if $A = \mathrm{I}$. Some positive results for special drifts b can be found in the works cited above, here we mention only one typical example, which is a result in [**37**] solving a problem raised by Shigekawa [**162**]. For notational simplicity we formulate this result for the space $\mathbb{R}^\infty$, although it is valid for general locally convex spaces.

3.2.2. Theorem. *Let $v = (v^i)$ be a Borel vector field on $X = \mathbb{R}^\infty$ with values in $H = l^2$ and let μ be a Borel probability measure on $\mathbb{R}^\infty$ satisfying equation* (3.1.1) *with $A = \mathrm{I}$ and $b(x) = -x + v(x)$, where $|v|_H \in L^2(\mu)$ and $l_i \in L^2(\mu)$, $l_i(x) = x_i$. Then μ is absolutely continuous with respect to the Gaussian measure γ that is the countable power of the standard Gaussian measure on the real line.*

Related problems for infinite-dimensional manifolds are studied in [**48**], [**49**], [**146**], [**147**], [**148**], [**97**]. Concerning various problems connected with essential selfadjointness and uniqueness properties of infinite-dimensional elliptic operators and their generated semigroups, see [**13**], [**118**], [**139**], and also [**22**, Chapter 12], and the references in these works. All these problems will be considered in detail in a separate survey devoted entirely to the infinite-dimensional case.

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