

ANISOTROPIC YOUNG DIAGRAMS AND INFINITE-DIMENSIONAL DIFFUSION PROCESSES WITH THE JACK PARAMETER

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ABSTRACT. We construct a family of Markov processes with continuous sample trajectories on an infinite-dimensional space, the Thoma simplex. The family depends on three continuous parameters, one of which, the Jack parameter, is similar to the beta parameter in random matrix theory. The processes arise in a scaling limit transition from certain finite Markov chains, the so called up-down chains on the Young graph with the Jack edge multiplicities. Each of the limit Markov processes is ergodic and its stationary distribution is a symmetrizing measure. The infinitesimal generators of the processes are explicitly computed; viewed as selfadjoint operators in the L^2 spaces over the symmetrizing measures, the generators have purely discrete spectrum which is explicitly described.

For the special value 1 of the Jack parameter, the limit Markov processes coincide with those of the recent work by Borodin and the author (Prob. Theory Rel. Fields 144 (2009), 281–318). In the limit as the Jack parameter goes to 0, our family of processes degenerates to the one-parameter family of diffusions on the Kingman simplex studied long ago by Ethier and Kurtz in connection with some models of population genetics.

The techniques of the paper are essentially algebraic. The main computations are performed in the algebra of shifted symmetric functions with the Jack parameter and rely on the concept of anisotropic Young diagrams due to Kerov.

Keywords: Diffusion processes; up-down Markov chains; Thoma’s simplex; Jack symmetric functions; Young diagrams; z -measures; Kerov interlacing coordinates; shifted symmetric functions; Poisson–Dirichlet distribution; Selberg integral

Mathematics Subject Classification (2000): 60J60; 60C05; 60J10; 05E05

At various stages of work, the present research was supported by the RFBR grants 07-01-91209 and 08-01-00110, by the project SFB 701 (Bielefeld University), and by the grant “Combinatorial Stochastic Processes” (Utrecht University).

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1. INTRODUCTION

1.1. Motivation and general description of the work. This work can be viewed as a continuation of the project started by Borodin and myself, see [BO6], [BO7], [BO8], but it can be also read independently. The goal of the project is to study Markovian stochastic dynamics in certain infinite-dimensional models that originate from random partitions. The main difference of this paper from the previous ones is in introducing a new parameter θ , the so-called Jack parameter, which is analogous to the β -parameter (inverse temperature) in the log-gas systems ($\theta = \beta/2$). Exact statements of our results can be found in the next subsection. Meanwhile, we would like to explain some ideas behind this work and to describe our motivation.

Several classes of infinite-dimensional Markov processes are known. A large part of the literature on those deals with interacting particle systems on a lattice, and also with similar systems in \mathbb{R}^d that are related to Gibbs measures. Statistical mechanics serves as the main motivational source for such models. Another source of infinite-dimensional Markov processes is population genetics. A very interesting but not very well studied problem is construction of dynamics for particle systems with nonlocal interaction of log-gas type; such systems naturally arise as large N limits of N -particle random matrix type ensembles.

The model that we study in this paper is of a different origin — it came up in the asymptotic representation theory of symmetric groups. Nevertheless, it turns out to be somewhat similar to log-gas systems on one hand, and on the other hand it is closely connected to one of the well-known models from population genetics [EK1].

Constructing Markov dynamics on an infinite-dimensional state space often constitutes a nontrivial problem. For example, it may be difficult to assign rigorous meaning to an intuitive definition of the infinitesimal generator of the Markov process. See, e. g., the paper by Spohn [Sp], where the problem of the justification of the large N limit transition for Dyson's log-gas systems is discussed. In order to construct a Markov generator, one often uses Dirichlet forms. This is a very effective yet technically demanding analytic method. In this paper we follow a more direct approach in which analysis is largely replaced by algebra and combinatorics. I hope that some ideas below may be useful for studying the dynamics in log-gas systems as well.

Let us now describe (in very general terms) the model considered below. Denote by \mathbb{Y}_n the set of partitions of the natural number n . We identify partitions $\lambda \in \mathbb{Y}_n$ and Young diagrams with n boxes. For any $n = 1, 2, \dots$ we introduce a probability distribution $M_{\theta, z, z'}^{(n)}$ on \mathbb{Y}_n that depends on three continuous parameters θ, z, z' . The resulting ensemble of random partitions can be compared to N -particle random matrix ensembles; the role of the parameter N

is played by n . Let me not give an exact expression for the weights $M_{\theta,z,z'}^{(n)}(\lambda)$ here, as it requires a fairly long discussion (cf. [BO5]). Instead, let me note that this expression can be represented in the form very much reminiscent of the joint probability density for random matrix β -ensembles:

$$M_{\theta,z,z'}^{(n)}(\lambda) = \text{const} \exp(-2\theta W(\lambda_1, \dots, \lambda_\ell)),$$

where ℓ is the length of the partition λ , and W is a function on partitions that can be split into the sum of one-particle and two-particle “interaction potentials”,

$$W(\lambda_1, \dots, \lambda_\ell) = \sum_{i=1}^{\ell} W_1(\lambda_i) + \sum_{1 \leq i < j \leq \ell} W_2(\lambda_i, \lambda_j).$$

The two-particle potential on large distances is asymptotically equivalent to the logarithmic one:

$$W_2(\lambda_i, \lambda_j) \sim \log \frac{1}{\lambda_i - \lambda_j}, \quad \lambda_i - \lambda_j \gg 0.$$

The analogy between random partitions and random matrices that looked startling 10 years ago (see, e.g., [BO1], [BO4]), is nowadays viewed as commonplace [Ok].

The key property of the distributions $M_{\theta,z,z'}^{(n)}$ is the fact that they are related to each other via a certain canonical chain of Markovian transition functions $\mathbb{Y}_n \rightarrow \mathbb{Y}_{n-1}$ ($n = 1, 2, \dots$), that depend only on θ , and that are defined in terms of the Jack symmetric functions corresponding to the parameter θ . According to a general theorem proved in [KOO], this coherency property implies the existence of the limiting probability distribution

$$M_{\theta,z,z'} = \lim_{n \rightarrow \infty} M_{\theta,z,z'}^{(n)},$$

which lives on the infinite-dimensional compact space ¹

$$\Omega = \{(\alpha; \beta) : \alpha = (\alpha_1, \alpha_2, \dots), \quad \beta = (\beta_1, \beta_2, \dots), \\ \alpha_1 \geq \alpha_2 \geq \dots \geq 0, \quad \beta_1 \geq \beta_2 \geq \dots \geq 0, \quad \sum_i \alpha_i + \sum_j \beta_j \leq 1\}.$$

The space Ω is called the *Thoma simplex*, and the limit measures $M_{\theta,z,z'}$ are called the (boundary) *z-measures*.

The above-mentioned theorem from [KOO] claims that there exists a one-to-one correspondence $M \leftrightarrow \{M^{(n)}\}$ between the probability distributions M on

¹The appearance of the *double* set of coordinates in Ω is related to the fact that Young diagrams are *two-dimensional* objects — they have rows and columns that play equal roles in our model.

Ω and the *coherent families* $\{M^{(n)}\}$, that is, sequences of probability distributions related by the Markovian (θ -dependent) transition functions $\mathbb{Y}_n \rightarrow \mathbb{Y}_{n-1}$ mentioned above. In fact, this theorem contains more: it provides a possibility of constructing a *canonical* Markov dynamics on Ω that preserves the given distribution M .

The idea is the following. There exists a simple and natural way of constructing, for any coherent family $\{M^{(n)}\}$, a sequence of reversible Markov chains $\mathbb{Y}_n \rightarrow \mathbb{Y}_n$ with stationary distributions $M^{(n)}$. We call those the *up-down chains*. Now it is natural to raise a question whether the up-down chains converge, as $n \rightarrow \infty$, to a Markov process on Ω that has M as its stationary distribution.² In the concrete case of z -measures M we can answer this question in the affirmative.³

Although the results of this paper are stated in probabilistic terms, the main content of the paper is algebraic, and it can be phrased as follows. Consider the algebra Λ of symmetric functions, and identify it with the algebra of polynomials in countably many generators p_1, p_2, \dots (the power sums). Further, denote by T_n the Markov operator for the n th up-down chain. Originally T_n is defined as an operator in the space of functions on the finite set \mathbb{Y}_n . However, we show that there exists a uniformly defined (for all $n = 1, 2, \dots$) representation of the operators T_n in terms of certain operators in the algebra Λ (essentially we carry the operators T_n over to a common space). The most difficult part of the work is the computation of this representation in the form of a differential operator with respect to formal variables p_1, p_2, \dots . The techniques that we apply here are discussed in Subsection 1.5 below.

After bringing T_n 's to a suitable form, we find the pre-generator of the Markov process on Ω as the limit (in a certain rigorous sense)

$$A = \lim_{n \rightarrow \infty} n^2(T_n - 1).$$

The factor n^2 corresponds to scaling time — one step of the Markov chain with large number n is equated to a small time interval of size $\Delta t = n^{-2}$. The justification of the limit transition is performed using standard techniques (Trotter-type theorems, see the book [EK2]), as well as some ideas from the paper [EK1]. This paper and its relation to the present work is further discussed in Subsection 1.2 below.

²I got this idea from a conversation with my friend Sergei Kerov that took place in the nineties. Kerov had never written about up-down chains, and for the first time they seem to have appeared in the literature in the paper by Fulman [Fu1], who used them for different purposes (cf. also his subsequent publications [Fu2], [Fu3]). Borodin and I implicitly applied the up-down chains to constructing Markov dynamics in [BO6] and [BO7], and then explicitly in [BO8].

³I do not know what can be done for arbitrary measures on Ω .

1.2. Ethier–Kurtz’s diffusions and statement of the main results. In the remarkable paper [EK1] published in 1981, Ethier and Kurtz studied a one-parameter family of diffusions ⁴ on the space

$$\bar{\nabla}_\infty = \{\alpha = (\alpha_1, \alpha_2, \dots) : \alpha_1 \geq \alpha_2 \geq \dots \geq 0, \sum_{i=1}^{\infty} \alpha_i \leq 1\}. \quad (1.1)$$

The space $\bar{\nabla}_\infty$ is compact in the topology of coordinatewise convergence and can be regarded as an infinite-dimensional simplex. We call it the *Kingman simplex*. The diffusions are determined by infinitesimal generators acting on an appropriate space of functions on $\bar{\nabla}_\infty$ and can be written as second order differential operators

$$\sum_{i,j=1}^{\infty} \alpha_i (\delta_{ij} - \alpha_j) \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} - \tau \sum_{i=1}^{\infty} \alpha_i \frac{\partial}{\partial \alpha_i}, \quad (1.2)$$

where $\tau > 0$ is the parameter. ⁵ Each of the diffusions has a unique stationary distribution and is reversible and ergodic. This is a nice example of infinite-dimensional Markov processes, especially interesting because the stationary distributions are the famous Poisson-Dirichlet distributions (about them, see, e.g. [Ki2]).

The 3-parameter family of Markov processes on the Thoma simplex Ω , constructed in the present paper, is a wider model of infinite-dimensional diffusions, containing Ethier–Kurtz’s diffusions as a limit case. In our model, in contrast to that of Ethier–Kurtz, the infinitesimal generator cannot be written as a differential operator in natural coordinates $(\alpha; \beta)$ on the state space Ω . Nevertheless, the restriction of the generator on an appropriate invariant core \mathcal{F} admits an explicit expression.

Specifically, the core \mathcal{F} is the algebra of polynomials $\mathbb{R}[q_1, q_2, \dots]$, where q_1, q_2, \dots are the following functions on Ω

$$q_k(\alpha; \beta) = \sum_{i=1}^{\infty} \alpha_i^{k+1} + (-\theta)^k \sum_{i=1}^{\infty} \beta_i^{k+1}, \quad k = 1, 2, \dots, \quad (1.3)$$

and $\theta > 0$ is the Jack parameter mentioned above. ⁶ These functions are continuous and algebraically independent. We call them the *moment coordinates* for the following reason: Let us embed Ω into the space of probability

⁴By a diffusion we mean a strong Markov process with continuous sample trajectories.

⁵In [EK1], the parameter is denoted as θ . We use another symbol because of a conflict of notations. We have also omitted the factor of $\frac{1}{2}$ in the formula of [EK1].

⁶The functions (1.3) are the Jack deformation of the *supersymmetric* power sums in coordinates α_i and $-\beta_j$, cf. [Ma]. About the link between supersymmetry and Young diagrams, see [VK], [KeO], [KOO].

measures on the closed interval $[-\theta, 1] \subset \mathbb{R}$ by assigning to an arbitrary point $(\alpha; \beta) \in \Omega$ the atomic measure

$$\nu_{\alpha; \beta} = \sum_{i=1}^{\infty} \alpha_i \delta_{\alpha_i} + \sum_{i=1}^{\infty} \beta_i \delta_{-\theta \beta_i} + \gamma \delta_0, \quad \gamma := 1 - \sum \alpha_i - \sum \beta_i, \quad (1.4)$$

where δ_x stands for the Dirac measure at $x \in \mathbb{R}$. Then $q_k(\alpha; \beta)$ is equal to the k th moment of the measure $\nu_{\alpha; \beta}$.

Even though the moment coordinates are not true coordinates on Ω in the conventional differential-geometric sense, they allow us to define the Markov pre-generator as a second order differential operator acting in the polynomial algebra \mathcal{F} and depending on the Jack parameter θ and two additional continuous parameters z and z' :

$$\begin{aligned} & \sum_{i, j \geq 1} (i+1)(j+1)(q_{i+j} - q_i q_j) \frac{\partial^2}{\partial q_i \partial q_j} \\ & + \sum_{i \geq 1} (i+1) [((1-\theta)i + (z+z'))q_{i-1} - (i + \theta^{-1}zz')q_i] \frac{\partial}{\partial q_i} \\ & + \theta \sum_{i, j \geq 0} (i+j+3)q_i q_j \frac{\partial}{\partial q_{i+j+2}}, \end{aligned} \quad (1.5)$$

where $q_0 \equiv 1$. Now are in a position to state the main results of the paper.

Theorem 1.1. *Let $C(\Omega)$ be the Banach space of continuous real-valued functions on the Thoma simplex Ω . Regard the differential operator (1.5) as an operator in $C(\Omega)$ with dense invariant domain $\mathcal{F} \subset C(\Omega)$.*

(i) *The operator (1.5) is closable in $C(\Omega)$ and its closure serves as the infinitesimal generator of a diffusion process on Ω .*

(ii) *The process has a unique stationary distribution and is reversible and ergodic.*

(iii) *The closure of the pre-generator (1.5) in the L^2 space with respect to the stationary distribution is a self-adjoint operator with purely discrete spectrum*

$$\{0\} \cup \{-m(m-1 + \theta^{-1}zz') : m = 2, 3, \dots\},$$

where the multiplicity of the eigenvalue $-m(m-1 + \theta^{-1}zz')$ equals the number of partitions of m without parts equal to 1.

(iv) *The stationary distribution is the z -measure $M_{\theta, z, z'}$.*

About the z -measures see the next subsection. The restrictions on parameters (z, z') are indicated below in Proposition 5.3.

In the limit regime as

$$z \rightarrow 0, \quad z' \rightarrow 0, \quad \theta \rightarrow 0, \quad \theta^{-1}zz' \rightarrow \tau > 0, \quad (1.6)$$

our model degenerates to the Ethier–Kurtz model with parameter τ . Let me explain this informally:

Observe that as $\theta \rightarrow 0$, the interval $[-\theta, 1]$ shrinks to $[0, 1]$ so that the β -coordinates disappear from (1.4). This explains why the Thoma simplex Ω degenerates to the Kingman simplex (1.1). Next, in the regime (1.6), the expression (1.5) degenerates to

$$\sum_{i,j \geq 1} (i+1)(j+1)(q_{i+j} - q_i q_j) \frac{\partial^2}{\partial q_i \partial q_j} + \sum_{i \geq 1} (i+1)[i q_{i-1} - (i+\tau)q_i] \frac{\partial}{\partial q_i}. \quad (1.7)$$

Finally, it is not difficult to show that (1.7) is merely another form of (1.2), provided that the moment coordinates in (1.7) are viewed as functions on the subspace $\{(\alpha, 0) \subset \Omega\} = \overline{\nabla}_\infty$.⁷ Note also that in the regime (1.6), the z -measures weakly converge to the Poisson-Dirichlet distribution with parameter τ .

1.3. Z-measures and Selberg integrals. The z -measures form a distinguished family of probability measures on the Thoma simplex Ω . These measures first emerged in the note [KOV1] in connection with the problem of harmonic analysis on the infinite symmetric group, see also [KOV2] for a detailed exposition and [B2], [BO1], [BO2]. All these papers concerned the special case $\theta = 1$. Then the z -measures are the spectral measures governing the decomposition of some analogs of the regular representation; here it is worth noting that Ω is a kind of dual space to the infinite symmetric group. The case $\theta = \frac{1}{2}$ is also related to a problem of harmonic analysis (see [Str]), while (as was already mentioned above) the limit case $\theta = 0$ corresponds to the Poisson-Dirichlet distributions. For general $\theta > 0$, the z -measures were defined in [Ke4] (see also [BO3] for a different approach). The idea of building a theory valid for all $\theta > 0$ is similar to Dyson's idea of introducing the beta parameter into random matrix theory (see [Dy])⁸ or to Heckman–Opdam's idea of generalizing harmonic analysis on symmetric spaces to root systems with formal root multiplicities (see Heckman's lectures in [HS]).

A formal definition of the z -measures can be given in the following way. Consider the algebra Λ of symmetric functions and its basis $\{\mathcal{P}_{\lambda;\theta}\}$ of Jack symmetric functions with parameter θ ; here the index λ ranges over the set \mathbb{Y} of Young diagrams.⁹ Identify Λ with $\mathbb{R}[p_1, p_2, \dots]$ where p_k 's are the Newton power sums, and consider the algebra morphism $\Lambda \rightarrow C(\Omega)$ defined by

$$p_1 \rightarrow 1, \quad p_2 \rightarrow q_1, \quad p_3 \rightarrow q_2, \dots,$$

⁷The fact was established in [Sch], see also [Pe1].

⁸In our notation, Dyson's β corresponds to 2θ .

⁹See, e.g., [Ma]. Our parameter θ is inverse to Macdonald's parameter α and coincides with Kadell's [Ka] parameter k .

where $q_1 = q_1(\alpha; \beta)$, $q_2 = q_2(\alpha; \beta), \dots$ are the moment coordinates (1.3). Then each Jack function $\mathcal{P}_{\lambda; \theta}$ turns into a continuous function $\mathcal{P}_{\lambda; \theta}^\circ(\alpha; \beta)$ on Ω , which may be viewed as a version of *supersymmetric* Jack function in α and β .¹⁰ The z -measure with parameters (θ, z, z') , denoted as $M_{\theta, z, z'}$, can be characterized by the integrals

$$M_{\theta, z, z'}^{(n)}(\lambda) := [p_1^n : \mathcal{P}_{\lambda; \theta}] \int_{\Omega} \mathcal{P}_{\lambda; \theta}^\circ(\alpha; \beta) M_{\theta, z, z'}(d\alpha d\beta) \quad (1.8)$$

for which there is a nice multiplicative formula (here n equals $|\lambda|$, the number of boxes in the diagram λ , and $[p_1^n : \mathcal{P}_{\lambda; \theta}]$ stands for the coefficient of $\mathcal{P}_{\lambda; \theta}$ in the expansion of p_1^n in the basis of Jack functions). For any fixed λ , $M_{\theta, z, z'}^{(n)}(\lambda)$ is a rational function in z , z' , and θ , whose explicit expression can be found in [BO5].

For any fixed $\theta > 0$, the set of admissible values of (z, z') (for which the z -measure is well defined as a probability measure on Ω) is precisely the set of those (z, z') for which the quantities $M_{\theta, z, z'}^{(n)}(\lambda)$ are nonnegative for all diagrams λ (about the meaning of this condition, see the next subsection). Since $M_{\theta, z, z'}^{(n)}(\lambda)$ does not change under transposition $z \leftrightarrow z'$, one can equally well take as parameters $z + z'$ and zz' ; in these new coordinates, the set of admissible values becomes a closed subset of \mathbb{R}^2 with a nonempty interior. This set can be divided into two parts depending on whether the integrals (1.8) are strictly positive for all λ (the nondegenerate series) or vanish for some λ (the degenerate series). For more detail, see [BO5].

The present paper focuses on the nondegenerate series but I would like to give some comments on the degenerate series, because the degenerate z -measures demonstrate in miniature some features of the general z -measures.

The characteristic property of the degenerate series is that the corresponding z -measure $M_{\theta, z, z'}$ is supported by a finite-dimensional subset in Ω . The simplest example is

$$z = N\theta, \quad z' = (N-1)\theta + \sigma, \quad N = 1, 2, \dots, \quad \sigma > 0.$$

Then the support of the z -measure is the subset

$$\{(\alpha; \beta) : \alpha_{N+1} = \alpha_{N+2} = \dots = 0, \quad \beta_1 = \beta_2 = \dots = 0, \\ \alpha_1 \geq \dots \geq \alpha_N \geq 0, \quad \alpha_1 + \dots + \alpha_N = 1\} \subset \Omega,$$

which is a simplex of dimension $N-1$; the measure itself has the form

$$\text{const} \cdot \prod_{i=1}^N \alpha_i^{\sigma-1} \cdot \prod_{1 \leq i < j \leq N} (\alpha_i - \alpha_j)^{2\theta} \cdot d^\circ \alpha, \quad (1.9)$$

¹⁰About applications of such functions to integrable systems, see [SV].

where $d^\circ\alpha$ stands for the Lebesgue measure on the simplex.

It is worth noting that in this special case, the integrals (1.8) turn into Selberg-type integrals, see [Ke3], [Ke6]. More refined examples of degenerate z -measures involve both the α and β coordinates and provide super-analogs of the Selberg integral (see [BO3]). In a certain sense, the general z -measures on Ω can be viewed as an infinite-dimensional (super) generalization of the Selberg measures (1.9).¹¹

Closely related to (1.9) is the following probability measure on the cone $\alpha_1 \geq \dots \geq \alpha_N \geq 0$ in \mathbb{R}^N

$$\text{const} \cdot \prod_{i=1}^N \alpha_i^{\sigma-1} e^{-\alpha_i} \cdot \prod_{1 \leq i < j \leq N} (\alpha_i - \alpha_j)^{2\theta} \cdot d\alpha, \quad (1.10)$$

where $d\alpha$ is the Lebesgue measure on the cone.¹² Note that (1.10) determines the N -particle *Laguerre ensemble* with the beta parameter 2θ .

This example builds a bridge between the z -measures and random matrix type ensembles and illustrates the thesis that the Jack parameter θ plays the role of the beta parameter of random matrix theory.

1.4. Discrete approximation. Up-down Markov chains. As was already mentioned above, the diffusion processes of Theorem 1.1 arise as limits of some finite Markov chains. Here is an outline of the construction.

Recall that by \mathbb{Y}_n we denote the set of Young diagrams with n boxes, $n = 0, 1, 2, \dots$. Let us return to integrals (1.8) and write them in abstract form

$$M^{(n)}(\lambda) := [p_1^n : \mathcal{P}_{\lambda;\theta}] \int_{\Omega} \mathcal{P}_{\lambda;\theta}^\circ(\alpha; \beta) M(d\alpha d\beta), \quad \lambda \in \mathbb{Y}_n, \quad (1.11)$$

where M is an arbitrary probability measure on Ω . One can prove that the functions $\mathcal{P}_{\lambda;\theta}^\circ(\alpha; \beta)$ are nonnegative on Ω , which implies that the numbers $M^{(n)}(\lambda)$ are nonnegative, too. Furthermore, for any fixed n , one has

$$\sum_{\lambda \in \mathbb{Y}_n} M^{(n)}(\lambda) = 1,$$

so that $M^{(n)}(\cdot)$ is a probability measure on the finite set \mathbb{Y}_n . Next, one can prove that there exist embeddings $\iota_{\theta,n} : \mathbb{Y}_n \rightarrow \Omega$ such that the push-forwards $\iota_{\theta,n}(M^{(n)})$ converge to M in the weak topology. It is worth noting that the embeddings do not depend on the initial measure M .

¹¹About the history and various aspects and versions of the Selberg integral, see the recent survey [FW].

¹²The passage from (1.9) to (1.10) is similar to that from Euler's Beta integral to Euler's Gamma integral.

Thus, for each fixed $\theta > 0$, there exists an approximation of the compact space Ω by finite sets \mathbb{Y}_n providing an approximation of any probability measure M on Ω by some canonical sequence $\{M^{(n)}\}$ of probability measures on the sets \mathbb{Y}_n .¹³

The sequences $\{M^{(n)}\}$ coming from probability measures on Ω can be characterized by a system of relations,

$$M^{(n-1)}(\mu) = \sum_{\lambda \in \mathbb{Y}_n} M^{(n)}(\lambda) p_{\theta}^{\downarrow}(\lambda, \mu), \quad (1.12)$$

where n and $\mu \in \mathbb{Y}_{n-1}$ are arbitrary and

$$p_{\theta}^{\downarrow}(\lambda, \mu) := \frac{[p_1^{n-1} : \mathcal{P}_{\mu; \theta}][p_1 \mathcal{P}_{\mu; \theta} : \mathcal{P}_{\lambda; \theta}]}{[p_1^n : \mathcal{P}_{\lambda; \theta}]}. \quad (1.13)$$

Thus, (1.11) establishes a bijective correspondence between probability measures M on Ω and sequences $\{M^{(n)}\}$ of probability measures on the sets \mathbb{Y}_n , satisfying the relations (1.12). Such sequences are called *coherent systems*.¹⁴

The quantities (1.13) are nonnegative and, for fixed $\lambda \in \mathbb{Y}_n$,

$$\sum_{\mu \in \mathbb{Y}_{n-1}} p_{\theta}^{\downarrow}(\lambda, \mu) = 1,$$

so that they determine transition kernels from \mathbb{Y}_n to \mathbb{Y}_{n-1} , for each n . We call them the *down transition probabilities* (these are the Markovian transition functions mentioned above in Subsection 1.1). The relations (1.12) mean that the down transition kernels transform $M^{(n)}$ to $M^{(n-1)}$, for each n .

Now assume that M is nondegenerate in the sense that for the corresponding coherent system $\{M^{(n)}\}$, all quantities $M^{(n)}(\lambda)$ are strictly positive. This condition is fulfilled, for instance, if the topological support of M is the whole space Ω . Then one can define some *up transition probabilities* $p_{\theta, M}^{\uparrow}(\lambda, \nu)$ which determine transition kernels in the reverse direction, from \mathbb{Y}_n to \mathbb{Y}_{n+1} , and transform $M^{(n)}$ to $M^{(n+1)}$, for each n . Let us emphasize that the up transition probabilities depend not only on θ (as the down probabilities) but also of the coherent system, that is, of the initial measure M .

Taking the superposition of these two transition kernels we get, for each n , a transition kernel from \mathbb{Y}_n to itself,

$$\text{Prob}_n\{\lambda \rightarrow \kappa\} = \sum_{\nu \in \mathbb{Y}_{n+1}} p_{\theta, M}^{\uparrow}(\lambda, \nu) p_{\theta}^{\downarrow}(\nu, \kappa), \quad \lambda, \kappa \in \mathbb{Y}_n, \quad (1.14)$$

¹³In the special case $\theta = 1$ this fact is a refinement of Thoma's theorem [T], essentially due to Vershik and Kerov [VK]. For general $\theta > 0$ this is the result of [KOO]. A similar fact holds in the limit case $\theta = 0$, with Ω replaced by $\overline{\mathbb{V}}_{\infty}$: this is Kingman's theorem, see [Ki1], [Ke6].

¹⁴We follow the terminology adopted in [Ol]. In the limit case $\theta = 0$, coherent systems were earlier introduced by Kingman under the name of *partition structures*, see [Ki1].

It determines a reversible ergodic Markov chain on \mathbb{Y}_n which has $M^{(n)}$ as the stationary distribution. We call this chain the *n*th *up-down Markov chain* associated with M .

Thus, given M , we dispose not only of a canonical approximation $M^{(n)} \rightarrow M$ but also of a natural reversible Markov chain preserving the *n*th measure, for each n . This fact forms the basis of the work: the idea is to analyze the asymptotics of the up-down chains associated with a nondegenerate z -measure, as $n \rightarrow \infty$, and show that the chains have a scaling limit leading to a Markov process on Ω .

This idea was first realized for the special case $\theta = 1$ in the paper [BO8] by Borodin and myself.¹⁵ However, our computation of the limit pre-generator in the moment coordinates relied on a combinatorial result of Lascoux and Thibon [LT] for which no Jack analog is available. In the present paper I apply another method; its basic ideas and related concepts are described in the next subsection.

1.5. Shifted symmetric functions, interlacing coordinates, and anisotropic diagrams. Let $\text{Fun}(\mathbb{Y}_n)$ be the space of functions on the finite set \mathbb{Y}_n and $T_n : \text{Fun}(\mathbb{Y}_n) \rightarrow \text{Fun}(\mathbb{Y}_n)$ be the one-step operator of the *n*th up-down Markov chain, induced by the transition kernel (1.14):

$$(T_n F)(\lambda) = \sum_{\kappa \in \mathbb{Y}_n} \text{Prob}_n\{\lambda \rightarrow \kappa\} F(\kappa), \quad \lambda \in \mathbb{Y}_n.$$

We show that

$$\lim_{n \rightarrow \infty} n^2(T_n - 1) = A, \tag{1.15}$$

where A is the differential operator (1.5). Although the pre-limit operators live in varying spaces, one can give a sense to the limit transition by making use of the projections $C(\Omega) \rightarrow \text{Fun}(\mathbb{Y}_n)$, which are induced by the embeddings $\iota_{\theta,n} : \mathbb{Y}_n \rightarrow \Omega$ mentioned above. Here we employ a well-known formalism, described in [EK2]. Then, using a refined version of Trotter's theorem ([EK2, Theorem 7.5]), we show that A is closable and generates a Markov semigroup in $C(\Omega)$. The remaining claims of Theorem 1.1 are established in the same way as in [BO8].

The heart of the paper is the proof of (1.15). To handle the transition probabilities entering formula (1.14) we use an ingenious trick invented by Kerov [Ke4]: Kerov's idea was to consider *anisotropic Young diagrams* made of rectangular boxes of size $\theta \times 1$ and to parametrize such diagrams by pairs

¹⁵Note that Ethier–Kurtz's diffusions were obtained in two ways, both using an approximation procedure, but the motivation of [EK1] was quite different. Petrov [Pe1] showed that application of up-down chains makes it possible to re-derive Ethier–Kurtz's diffusions and also get their analogs corresponding to Pitman's two-parameter generalization of the Poisson-Dirichlet distributions.

of interlacing sequences, which encode the positions of the outer and inner corners (for more detail, see Section 4). This trick allows one to completely avoid the hard machinery related to Jack symmetric functions and reduce the proof of (1.15) to a computation in the algebra of θ -regular functions on the set \mathbb{Y} of Young diagrams.

This algebra, denoted as \mathbb{A}_θ , consists of θ -shifted symmetric functions in coordinates $\lambda_1, \lambda_2, \dots$. According to the original definition of the algebra \mathbb{A}_θ (see [KOO]), it is generated by the “ θ -shifted” analogs of power sums

$$p_m^*(\lambda) = \sum_{i=1}^{\infty} [(\lambda_i - \theta i)^m - (-\theta i)^m], \quad m = 1, 2, \dots, \quad \lambda \in \mathbb{Y}.$$

On the other hand, an important fact is that \mathbb{A}_θ also admits a nice description in terms of Kerov’s interlacing coordinates.

Note also that the computation of the limit operator in (1.15) substantially employs an asymptotic formula for θ -regular functions, established in [KOO] (see Theorem 9.5 below).

1.6. A variation: shifted Young diagrams and Schur’s Q-functions.

Schur’s Q-functions span a proper subalgebra in the algebra of symmetric functions. As well known, these functions play the same role in the theory of projective characters of symmetric groups as the ordinary Schur functions do for ordinary characters. An analog of z-measures related to Schur’s Q-functions was found in [B1], see also [BO3]. Replacing the ordinary Young diagrams by the so-called shifted Young diagrams (which correspond to strict partitions), one can define again the up-down Markov chains. Their scaling limits were studied by Petrov [Pe2]. The results he obtained are parallel to those of [BO8], but the computation leading to an analog of formula (1.5) for the pre-generator is based on the method of the present paper.

1.7. Organization of the paper. In Section 2 we discuss the general formalism of up-down Markov chains. In Section 3 we recall the definition of the Young graph with Jack edge multiplicities [KOO] and introduce the corresponding system of down probabilities. In Section 4 we explain what are Kerov’s anisotropic Young diagrams and their interlacing coordinates [Ke4]. Using these concepts, we give an alternative definition of the down probabilities, and then in Section 5 we describe the up probabilities associated to the z-measures. In Section 6 we present the necessary material about the algebra \mathbb{A}_θ of θ -regular functions on \mathbb{Y} . Here we also establish a link between \mathbb{A}_θ and the up and down transition functions. The long Section 7 contains the key computation. Its result, which is stated in the beginning of the section (Theorem 7.1), describes the top degree terms of the down and up operators in the algebra \mathbb{A}_θ . Proceeding from this computation, we find in Section 8 the top

degree term of the operator $T_n - 1$ (Theorem 8.2). Combining this with an asymptotic theorem from [KOO] we perform in Section 9 the limit transition from the up-down Markov chains to diffusion processes on Ω : the final results are Theorems 9.6, 9.7, 9.9, and 9.10.

2. MARKOV GROWTH OF YOUNG DIAGRAMS AND ASSOCIATED UP-DOWN MARKOV CHAINS

Let \mathbb{Y} denote the set of all Young diagrams, including the empty diagram \emptyset , and let $\mathbb{Y}_n \subset \mathbb{Y}$ be the subset of diagrams with n boxes, $n = 0, 1, 2, \dots$. Thus, \mathbb{Y} is the disjoint union of the finite sets $\mathbb{Y}_0, \mathbb{Y}_1, \dots$. By $|\lambda|$ we denote the number of boxes in a diagram λ . As in [Ma], we identify Young diagrams and the corresponding partitions of natural numbers, so that \mathbb{Y}_n is identified with the set of partitions of n . Using this identification we write Young diagrams in the partition notation: $\lambda = (\lambda_1, \lambda_2, \dots)$.

If λ and μ are two Young diagrams then we write $\mu \nearrow \lambda$ or, equivalently, $\lambda \searrow \mu$ if $\mu \subset \lambda$ and $|\lambda| = |\mu| + 1$ (that is, μ is obtained from λ by removing a box).

The *Young graph* is the graph with the vertex set \mathbb{Y} and the edges formed by arbitrary couples of diagrams, μ and λ , such that $\mu \nearrow \lambda$. This is a *graded graph*, in the sense that the vertex set \mathbb{Y} is partitioned into *levels* (the finite sets \mathbb{Y}_n) and only vertices of adjacent levels can be joined by an edge.

By an *infinite standard Young tableau* we mean an infinite sequence of Young diagrams, $\{\lambda(n)\}_{n=0,1,2,\dots}$, subject to the following condition: for any n , one has $\lambda(n) \in \mathbb{Y}_n$ and $\lambda(n) \nearrow \lambda(n+1)$. In other words, this is an infinite monotone path in the Young graph started at $\emptyset \in \mathbb{Y}_0$. Let \mathcal{T} denote the space of all infinite standard Young tableaux; it is a closed subset in the infinite product space $\prod \mathbb{Y}_n$ equipped with the product topology. Thus, \mathcal{T} is a compact topological space and we can define the sigma-algebra of Borel subsets in \mathcal{T} .

Assume we are given a probability Borel measure \mathcal{M} on the space \mathcal{T} . Then \mathcal{M} can be viewed as the law of a random sequence $\{\lambda(n)\}$ of Young diagrams. Let us say that \mathcal{M} is a *Markov measure* if $\{\lambda(n)\}$ possesses the Markov property. That is, conditioned on $\lambda(n) = \lambda$, the subsequences $\{\lambda(0), \dots, \lambda(n-1)\}$ and $\{\lambda(n+1), \lambda(n+2), \dots\}$ are independent from each other.

Definition 2.1 (Up and down transition probabilities). With any Markov measure \mathcal{M} on \mathcal{T} we associate the following objects: the *one-dimensional distributions* $M^{(n)}$, the *up transition probabilities* $p^\uparrow(\lambda, \nu)$, and the *down transition probabilities* $p^\downarrow(\lambda, \mu)$. Here $M^{(n)}$ is the probability measure on \mathbb{Y}_n defined by

$$M^{(n)}(\lambda) = \text{Prob}\{\lambda(n) = \lambda\}, \quad \lambda \in \mathbb{Y}_n.$$

Further, for $\lambda \in \mathbb{Y}_n$, $\nu \in \mathbb{Y}_{n+1}$, and $\mu \in \mathbb{Y}_{n-1}$, we define $p^\uparrow(\lambda, \nu)$ and $p^\downarrow(\lambda, \mu)$ as the conditional probabilities

$$\begin{aligned} p^\uparrow(\lambda, \nu) &= \text{Prob}\{\lambda(n+1) = \nu \mid \lambda(n) = \lambda\}, \\ p^\downarrow(\lambda, \mu) &= \text{Prob}\{\lambda(n-1) = \mu \mid \lambda(n) = \lambda\}. \end{aligned}$$

We view these probabilities as certain quantities associated to the oriented edges of the graph.

More precisely, the above definition makes sense if $M^{(n)}(\lambda) > 0$ for all n and all $\lambda \in \mathbb{Y}_n$. This assumption holds in the concrete situation studied in the paper. Note, however, that even if $M^{(n)}(\lambda)$ vanishes for some diagrams λ , one can still define the up and down transition probabilities on an appropriate subgraph of \mathbb{Y} .

Obviously, for any fixed λ ,

$$\sum_{\nu: \nu \searrow \lambda} p^\uparrow(\lambda, \nu) = 1, \quad \sum_{\mu: \mu \nearrow \lambda} p^\downarrow(\lambda, \mu) = 1,$$

and the measures $M^{(n)}$ are consistent with both the up and down transition probabilities in the following sense:

$$M^{(n+1)}(\nu) = \sum_{\lambda: \lambda \nearrow \nu} M^{(n)}(\lambda) p^\uparrow(\lambda, \nu), \quad (2.1)$$

$$M^{(n-1)}(\mu) = \sum_{\lambda: \lambda \searrow \mu} M^{(n)}(\lambda) p^\downarrow(\lambda, \mu). \quad (2.2)$$

Remark that the up transition probabilities $p^\uparrow(\lambda, \nu)$ determine the initial Markov measure \mathcal{M} uniquely. Indeed, this happens because there exists an initial ‘‘time moment’’, $n = 0$, and the state space for $n = 0$ is a singleton. Once we know the up transition probabilities, we can reconstruct from the recurrence (2.1) the one-dimensional marginals $M^{(n)}$ and, more generally, all finite-dimensional distributions. The up transition probabilities are well suited to represent $\{\lambda(n)\}$ as a model of *random Markov growth* of Young diagrams, where at each consecutive moment of time a single new box is appended.

The down transition probabilities $p^\downarrow(\lambda, \mu)$ do not possess the above property for the obvious reason that for reversed time n , which ranges from $+\infty$ to 0, there is no finite initial moment. In such a situation, for a given system of transition probabilities, a host of Markov measures satisfying the corresponding recurrence relations may exist. Specifically, the following abstract theorem holds:

Theorem 2.2. *Fix an arbitrary system $p^\downarrow = \{p^\downarrow(\lambda, \mu)\}$ of down transition probabilities on the edges of the Young graph. That is, assign to all downward*

oriented edges $\lambda \searrow \mu$ nonnegative numbers $p^\downarrow(\lambda, \mu)$ in such a way that

$$\sum_{\mu: \mu \nearrow \lambda} p^\downarrow(\lambda, \mu) = 1 \quad \text{for any fixed vertex } \lambda.$$

Then there exists a topological space $\Omega(p^\downarrow)$ and a function $\mathcal{K}(\lambda, \omega)$ on $\mathbb{Y} \times \Omega(p^\downarrow)$, continuous with respect to ω , taking values in $[0, 1]$, and such that the relation

$$M^{(n)}(\lambda) = \int_{\Omega(p^\downarrow)} \mathcal{K}(\lambda, \omega) M(d\omega), \quad \lambda \in \mathbb{Y}_n, \quad n = 1, 2, \dots, \quad (2.3)$$

establishes a bijective correspondence $\{M^{(n)}\} \leftrightarrow M$ between sequences of probability measures solving the recurrence relations (2.2) and probability measures on the space $\Omega(p^\downarrow)$.

This theorem is no more than an adaptation of well-known results concerning boundaries of Markov chains (or more general Markov processes). For a proof of the theorem, see [KOO]. For the concrete systems p^\downarrow considered in the present paper, the precise form of the space $\Omega(p^\downarrow)$ and the kernel $\mathcal{K}(\lambda, \omega)$ is indicated in Subsection 9.4 below.

The space $\Omega(p^\downarrow)$ is called the (minimal) *entrance boundary* for the couple $(\mathbb{Y}, p^\downarrow)$, and for the measure M we will use the term the *boundary measure* of $\{M^{(n)}\}$. (Note that the marginals $M^{(n)}$ together with the down transition probabilities already suffice to reconstruct the initial Markov measure \mathcal{M} .)

We will not use Theorem 2.2 in our arguments but it is useful for better understanding the constructions of the paper. Heuristically, the result of Theorem 2.2 can be explained as follows: If we be dealing with *finite* Markov sequences $\{\lambda(0), \dots, \lambda(n)\}$, then we could reconstruct the law of such a sequence from its down transition probabilities and the distribution $M^{(n)}$ on the uppermost level \mathbb{Y}_n . For infinite sequences, the boundary $\Omega(p^\downarrow)$ plays the role of the nonexistent uppermost level \mathbb{Y}_∞ , and the boundary measure M is a substitute of the nonexistent distribution $M^{(\infty)}$. It is not surprising that $\Omega(p^\downarrow)$ is obtained as a kind of limit of the sets \mathbb{Y}_n as $n \rightarrow \infty$. As for M , then at least for concrete down transition probabilities that are discussed below, M can also be obtained as a limit of the distributions $M^{(n)}$.

We will regard down transition probabilities as a tool for specifying a *class* of Markov measures on \mathcal{T} .

Definition 2.3 (Up-down Markov chains). Let \mathcal{M} be a Markov measure on \mathcal{T} and $\{M^{(n)}\}$ be the corresponding family of distributions on the sets \mathbb{Y}_n . To simplify the discussion assume that $M^{(n)}(\lambda) > 0$ for all n and all $\lambda \in \mathbb{Y}_n$, so that the transition probabilities $p^\uparrow = \{p^\uparrow(\lambda, \nu)\}$ and $p^\downarrow = \{p^\downarrow(\lambda, \mu)\}$ are well defined for all edges of the Young graph.

For each $n = 1, 2, \dots$, we define a Markov chain with the state space \mathbb{Y}_n in the following way. Given a diagram $\lambda \in \mathbb{Y}_n$ we apply first the up transition probabilities $p^\uparrow(\lambda, \nu)$ and get a random diagram $\nu \in \mathbb{Y}_{n+1}$. Then we come back to \mathbb{Y}_n by using the down transition probabilities $p^\downarrow(\nu, \kappa)$. The composition $\lambda \rightarrow \nu \rightarrow \kappa$ constitutes a single step of the chain.

In other words, for two diagrams $\lambda, \kappa \in \mathbb{Y}_n$, the probability of the one-step transition $\lambda \rightarrow \kappa$ is equal to

$$\text{Prob}\{\lambda \rightarrow \kappa\} = \sum_{\nu: \lambda \nearrow \nu \searrow \kappa} p^\uparrow(\lambda, \nu) p^\downarrow(\nu, \kappa). \quad (2.4)$$

We call this chain the *up-down Markov chain* of level n associated with the two systems p^\downarrow and p^\uparrow of transition probabilities

Likewise, one could introduce the down-up chains using the superposition in the inverse order, $p^\downarrow \circ p^\uparrow$, but we will not use them.

Definition 2.4 (The graph $\tilde{\mathbb{Y}}_n$). For any $n = 1, 2, \dots$, introduce the graph $\tilde{\mathbb{Y}}_n$ whose vertices are diagrams $\lambda \in \mathbb{Y}_n$ and whose edges are couples of distinct diagrams $\lambda, \kappa \in \mathbb{Y}_n$ for which there exists $\nu \in \mathbb{Y}_{n+1}$ such that $\lambda \nearrow \nu \searrow \kappa$. The latter condition is equivalent to saying that κ can be obtained from λ by displacing a single box to a new position. Note that this is a minimal possible transformation of a Young diagram preserving the number of boxes. One more equivalent formulation is as follows: Two diagrams λ and κ form an edge in the graph $\tilde{\mathbb{Y}}_n$ if their symmetric difference $\lambda \Delta \kappa$ consists of precisely two boxes.

The up-down chain of level n may be viewed as a nearest neighbor random walk on the graph $\tilde{\mathbb{Y}}_n$.

Proposition 2.5. *For any $n = 1, 2, \dots$, the up-down Markov chain on \mathbb{Y}_n determined by (2.4) has a unique stationary distribution, which is the measure $M^{(n)}$. Moreover, $M^{(n)}$ is the symmetrizing measure, so that the chain is reversible in the stationary regime.*

Proof. The fact that $M^{(n)}$ is a stationary distribution follows from the recurrence relations (2.1) and (2.2). Indeed, (2.1) shows that the transition $\lambda \rightarrow \nu$ transforms $M^{(n)}$ to $M^{(n+1)}$, and (2.2) shows that $\nu \rightarrow \kappa$ returns $M^{(n+1)}$ back to $M^{(n)}$.

It is easy to check that the graph $\tilde{\mathbb{Y}}_n$ is connected so that all the states of the chain are communicating. This proves the uniqueness statement.

The last statement means that

$$M^{(n)}(\lambda) \text{Prob}\{\lambda \rightarrow \kappa\} = M^{(n)}(\kappa) \text{Prob}\{\kappa \rightarrow \lambda\}$$

for any edge $\{\lambda, \kappa\}$ of the graph $\tilde{\mathbb{Y}}_n$. By virtue of (2.4), this can be written as

$$\sum_{\nu} M^{(n)}(\lambda) p^\uparrow(\lambda, \nu) p^\downarrow(\nu, \kappa) = \sum_{\nu} M^{(n)}(\kappa) p^\uparrow(\kappa, \nu) p^\downarrow(\nu, \lambda). \quad (2.5)$$

Observe that

$$M^{(n)}(\lambda)p^\uparrow(\lambda, \nu) = M^{(n+1)}(\nu)p^\downarrow(\nu, \lambda). \quad (2.6)$$

Indeed, by the very definition of the up and down probabilities, the both sides of (2.6) are equal to

$$\text{Prob}\{\lambda(n) = \lambda, \lambda(n+1) = \nu\}.$$

Now (2.6) and the similar equality with λ replaced by κ imply that the both sides of (2.5) are equal to

$$\sum_{\nu} M^{(n+1)}(\nu)p^\downarrow(\nu, \lambda)p^\downarrow(\nu, \kappa).$$

□

3. DOWN TRANSITION PROBABILITIES IN THE YOUNG GRAPH WITH JACK EDGE MULTIPLICITIES

In this section we introduce a special system p_θ^\downarrow of down transition probabilities, which are associated with the Jack symmetric functions. Here $\theta > 0$ is the “Jack parameter”, an arbitrary positive number.

Let us start with the particular case $\theta = 1$, when the down probabilities have a simple representation-theoretic meaning. Recall that the diagrams $\lambda \in \mathbb{Y}_n$ parametrize the irreducible representations of the group S_n . Let $\dim \lambda$ stand for the dimension of the corresponding representation of S_n . Then

$$p_1^\downarrow(\lambda, \mu) = \frac{\dim \mu}{\dim \lambda}, \quad \mu \nearrow \lambda. \quad (3.1)$$

The classic *Young rule* says that the restriction of the irreducible representation indexed by $\lambda \in \mathbb{Y}_n$ to the subgroup S_{n-1} splits into the multiplicity free direct sum of the irreducible representations indexed by the diagrams $\mu \nearrow \lambda$. Therefore, for any λ ,

$$\sum_{\mu: \mu \nearrow \lambda} \dim \mu = \dim \lambda,$$

which explains why the numbers (3.1) sum to 1.

Thus, one can say that the probabilities (3.1) reflect the *branching rule of irreducible representations* of symmetric groups.

We proceed to the definition of the down probabilities for general $\theta > 0$. We will present two equivalent formulations. The first one is stated in terms of the *Jack symmetric functions*; it explains the origin of the probabilities in question. The second one has the advantage of being completely elementary and will be used in the computations.

For more detail about the notions that will be used below, see [Ma].

Let Λ denote the algebra of symmetric functions over \mathbb{R} . It is isomorphic to the algebra of polynomials with countably many variables p_1, p_2, \dots which are

identified with the Newton power sums. The canonical grading of the algebra Λ is specified by setting $\deg p_i = i$. The n th homogeneous component of the algebra, denoted as Λ_n , has dimension equal to $|\mathbb{Y}_n|$.

All natural bases in Λ are indexed by partitions. Of particular importance for us is the basis $\{\mathcal{P}_{\mu;\theta}\}_{\mu \in \mathbb{Y}}$ of the Jack symmetric functions. These are homogeneous elements, the degree of $\mathcal{P}_{\mu;\theta}$ equals $|\mu|$. Recall that Macdonald [Ma] uses as the parameter the inverse quantity θ^{-1} .

The starting point of the definition is the simplest case of the *Pieri rule*: for any $\mu \in \mathbb{Y}$

$$\mathcal{P}_{\mu;\theta} \cdot p_1 = \sum_{\lambda: \lambda \searrow \mu} \varkappa_\theta(\mu, \lambda) \mathcal{P}_{\lambda;\theta},$$

where $\varkappa_\theta(\mu, \lambda)$ are certain strictly positive numbers called the *Jack formal edge multiplicities*. A *standard Young tableau* of shape λ is a finite monotone path in the Young graph, $\emptyset \nearrow \cdots \nearrow \lambda$, starting at \emptyset and ending at λ ; its *weight* is defined as the product of the formal multiplicities $\varkappa_\theta(\cdot, \cdot)$ of its edges. The θ -*dimension* $\dim_\theta \lambda$ of a diagram λ is defined as the sum of the weights of all standard tableaux of the shape λ . Now we are in a position to state the definition:

Definition 3.1. For $\mu \nearrow \lambda$ we set

$$p_\theta^\downarrow(\lambda, \mu) = \frac{\dim_\theta \mu \cdot \varkappa_\theta(\mu, \lambda)}{\dim_\theta \lambda}. \quad (3.2)$$

In words: Consider the finite set of all directed paths from \emptyset to λ and make it a probability space by assigning to each path the probability proportional to its weight; then $p_\theta^\downarrow(\lambda, \mu)$ is the probability that the random path passes through μ . Note that (3.2) is the same as (1.13).

Remark 3.2. The following duality relation holds :

$$p_\theta^\downarrow(\lambda, \mu) = p_{\theta^{-1}}^\downarrow(\lambda', \mu'), \quad 0 < \theta < +\infty,$$

where λ' and μ' are the transposed diagrams.

Remark 3.3. The construction of the present section first appeared in the joint paper [KOO]. However, the idea is implicitly contained in an earlier work by Kerov (see [Ke1, §7]).

The alternative definition of the down transition probabilities is given in the next section.

4. KEROV INTERLACING COORDINATES AND THE SECOND DEFINITION OF THE DOWN TRANSITION PROBABILITIES

The present section is essentially an extraction from Kerov's paper [Ke4], with minor modifications.

Let $\lambda \in \mathbb{Y}$ be a Young diagram. Recall that we draw Young diagrams according to the so-called “English picture” [Ma], where the first coordinate axis (the row axis) is directed downwards and the second coordinate axis (the column axis) is directed to the right. Consider the border line of λ as the directed path coming from $+\infty$ along the second (horizontal) axis, next turning several times alternately down and to the left, and finally going away to $+\infty$ along the first (vertical) axis. The corner points on this path are of two types: the *inner corners*, where the path switches from the horizontal direction to the vertical one, and the *outer corners* where the direction is switched from vertical to horizontal. Observe that the inner and outer corners always interlace and the number of inner corners always exceeds by 1 that of outer corners. Let $2d - 1$ be the total number of the corners and (r_i, s_i) , $1 \leq i \leq 2d - 1$, be their coordinates. Here the odd and even indices i refer to the inner and outer corners, respectively.

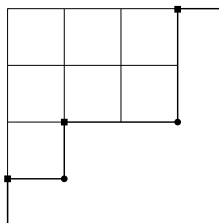


Figure 1. The corners of the diagram $\lambda = (3, 3, 1)$.

For instance, the diagram $\lambda = (3, 3, 1)$ shown on the figure has $d = 3$, three inner corners $(r_1, s_1) = (0, 3)$, $(r_3, s_3) = (2, 1)$, $(r_5, s_5) = (3, 0)$, and two outer corners $(r_2, s_2) = (2, 3)$, $(r_4, s_4) = (3, 1)$.

Fix $\theta > 0$. The numbers

$$x_1 := s_1 - \theta r_1, \quad y_1 := s_2 - \theta r_2, \dots$$

$$\dots, \quad y_{d-1} := s_{2d-2} - \theta r_{2d-2}, \quad x_d := s_{2d-1} - \theta r_{2d-1} \quad (4.1)$$

form two interlacing sequences of integers

$$x_1 > y_1 > x_2 > \dots > y_{d-1} > x_d$$

satisfying the relation

$$\sum_{i=1}^d x_i - \sum_{j=1}^{d-1} y_j = 0. \quad (4.2)$$

Definition 4.1. The two interlacing sequences

$$X = (x_1, \dots, x_d), \quad Y = (y_1, \dots, y_{2d-1})$$

as defined in (4.1) will be called the (θ -dependent) *Kerov interlacing coordinates* of a Young diagram λ . We will write $\lambda = (X; Y)$. (Note that the

original definition of the interlacing coordinates given in [Ke4] differs from the present one by a factor of θ^{-1} , because Kerov uses the homothetic transformation $s \mapsto \alpha s$ with $\alpha = \theta^{-1}$, while we prefer to transform the r -axis. This minor difference is inessential: all formulas in [Ke4] can be easily rewritten in our notation. The term ‘‘anisotropic diagram’’ employed in [Ke4] refers to the image of a Young diagram under a homothetic transformation of a coordinate axis.)¹⁶

Let u be a complex variable and consider the following expansion in partial fractions

$$\frac{\prod_{i=1}^d (u - x_i)}{\prod_{j=1}^{d-1} (u - y_j)} = u - \sum_{j=1}^{d-1} \frac{\pi_j^\downarrow}{u - y_j}. \quad (4.3)$$

Note that the constant term in the right-hand side vanishes because of (4.2). The coefficients $\pi_j^\downarrow = \pi_j^\downarrow(X; Y)$ are given by the formula

$$\pi_j^\downarrow = \pi_j^\downarrow(X; Y) = - \frac{\prod_{i=1}^d (y_j - x_i)}{\prod_{\substack{1 \leq l \leq d-1 \\ l \neq j}} (y_j - y_l)}. \quad (4.4)$$

They are strictly positive, and their sum is equal to the area of the shape λ in the modified coordinates $r' = \theta r$, $s' = s$:

$$\sum_{j=1}^{d-1} \pi_j^\downarrow = \theta |\lambda| = \text{Area}(X, Y) := \prod_{1 \leq i \leq j \leq d-1} (x_i - y_i)(y_j - x_{j+1}). \quad (4.5)$$

Observe that there is a natural bijective correspondence between the outer corners of $\lambda = (X; Y)$ and those boxes that may be removed from λ . Thus, we may associate these boxes with the coordinates y_j .

Proposition 4.2. *Let λ be a Young diagram, $(X; Y)$ be its θ -dependent Kerov interlacing coordinates, $\pi_j^\downarrow = \pi_j^\downarrow(X; Y)$ be the coefficients from (4.3), given by (4.4), and $\text{Area}(X; Y)$ be the quantity defined in (4.5). Let \square_j denote the corner box in λ associated with the j th coordinate y_j in Y . Then the θ -dependent down transition probabilities as defined in (3.2) are given by the*

¹⁶In the particular case $\theta = 1$, the interlacing coordinates $(X; Y)$ were introduced in earlier Kerov’s paper [Ke2] and further exploited in [Ke5]; see also [Ke6]. A somewhat similar parametrization was suggested by Stanley [Sta] and then employed in a number of recent publications. Stanley’s $(p; q)$ coordinates differ from the Kerov ($\theta = 1$) coordinates by a simple linear transformation.

following elementary expression

$$p_{\theta}^{\downarrow}(\lambda, \lambda \setminus \square_j) = \frac{\pi_j^{\downarrow}}{\text{Area}(X; Y)}. \quad (4.6)$$

Proof. See [Ke4, Section 7]. (Here and below I do not give more precise references to claims in [Ke4] because they are numbered differently in the journal version of the paper and its preprint version posted on arXiv.) \square

Note that the right-hand side of (4.6) is a rational fraction in the Kerov coordinates, and this function *does not depend on θ* ; the dependence on θ is hidden in the Kerov coordinates themselves.

Thus, we obtain an alternative description of the down transition probabilities.

Remark 4.3. There is an obvious relation between the set of Kerov coordinates with parameter θ of a diagram and the set of Kerov coordinates with reversed parameter θ^{-1} of the transposed diagram. Specifically, if the former set is $(X; Y) = \{x_i\} \cup \{y_j\}$ then the latter set is $(-\theta^{-1}X; -\theta^{-1}Y) = \{-\theta^{-1}x_i\} \cup \{-\theta^{-1}y_j\}$, with the reversed enumeration. This fact provides a simple proof of the duality stated in Remark 3.2.

5. THE UP TRANSITION PROBABILITIES OF THE Z-MEASURES

There is a host of Markov measures on \mathcal{T} consistent with the down probability system p_{θ}^{\downarrow} . In this section we exhibit a distinguished family of Markov measures which depend on θ and some additional parameters z and z' . We do this by specifying the corresponding up transition probabilities $p_{\theta, z, z'}^{\uparrow}$.

Let $\lambda \in \mathbb{Y}$ be a Young diagram and $(X; Y)$ be its Kerov interlacing coordinates as defined in (4.1). Reverse the expression in the left-hand side of (4.3) and expand it again in partial fractions:

$$\frac{\prod_{j=1}^{d-1} (u - y_j)}{\prod_{i=1}^d (u - x_i)} = \sum_{i=1}^d \frac{\pi_i^{\uparrow}}{u - x_i}. \quad (5.1)$$

Here the coefficients π_i^{\uparrow} are given by the formula

$$\pi_i^{\uparrow} = \pi_i^{\uparrow}(X; Y) = \frac{\prod_{j=1}^{d-1} (x_i - y_j)}{\prod_{\substack{1 \leq l \leq d \\ l \neq i}} (x_i - x_l)}, \quad i = 1, \dots, d. \quad (5.2)$$

Recall that the dependence of the right-hand side on θ is hidden in the definition (4.1) of the Kerov coordinates.

Those boxes that may be appended to λ are associated, in a natural way, with the inner corners of the boundary of λ . Consequently, we may also associate these boxes with the x 's: $\square_i \leftrightarrow x_i$.

Assume first that z and z' are arbitrary complex numbers such that $zz' + \theta n \neq 0$ for all $n = 0, 1, 2, \dots$, and set

$$p_{\theta, z, z'}^\uparrow(\lambda, \lambda \cup \square_i) = \frac{(z + x_i)(z' + x_i)}{zz' + \theta n} \cdot \pi_i^\uparrow, \quad n = |\lambda|, \quad (5.3)$$

where the coefficients π_i^\uparrow are the same as in (5.1), (5.2).

Proposition 5.1. *For any fixed $\lambda \in \mathbb{Y}_n$, these numbers sum up to 1.*

Proof. Since $(z + x_i)(z' + x_i) = zz' + (z + z')x_i + x_i^2$, the claim is equivalent to the following three equalities:

$$\sum_{i=1}^d \pi_i^\uparrow = 1, \quad \sum_{i=1}^d x_i \pi_i^\uparrow = 0, \quad \sum_{i=1}^d x_i^2 \pi_i^\uparrow = \theta n.$$

These equalities are verified directly from (5.1) using the relation (4.2) and the expression of θn through $(X; Y)$, see (4.5). For more detail, see [Ke4, Section 6]. \square

Let us define the numbers $M^{(n)}(\lambda) = M_{\theta, z, z'}^{(n)}(\lambda)$ from the recurrence relations (2.1) by setting $p^\uparrow = p_{\theta, z, z'}^\uparrow$ and using the initial condition $M^{(0)}(\emptyset) = 1$. Then, using Proposition 5.1 and induction on n , one sees that for all n ,

$$\sum_{\lambda \in \mathbb{Y}_n} M_{\theta, z, z'}^{(n)}(\lambda) = 1.$$

Proposition 5.2. *The numbers $M^{(n)}(\lambda) = M_{\theta, z, z'}^{(n)}(\lambda)$ just defined are consistent with the down probabilities $p_\theta^\downarrow(\lambda, \mu)$. That is, setting $p^\downarrow = p_\theta^\downarrow$, the relations (2.2) are satisfied.*

Proof. This is the main result of [Ke4]; it is established at the very end of that paper. Another proof is given in [BO3]. \square

Proposition 5.3. *The quantities $p_{\theta, z, z'}^\uparrow(\lambda, \nu)$ defined by (5.3) are strictly positive for all edges $\lambda \nearrow \nu$ of the Young graph if and only if one of the following two conditions holds:*

- (i) $z \in \mathbb{C} \setminus (\mathbb{Z}_{\leq 0} + \theta \cdot \mathbb{Z}_{\geq 0})$ and $z' = \bar{z}$.
- (ii) θ is rational and both z and z' are real numbers lying in one of the open intervals between two consecutive numbers from the lattice $\mathbb{Z} + \theta \cdot \mathbb{Z} \subset \mathbb{R}$.

In particular, the simple sufficient condition of strict positivity is that z and z' should be nonreal and conjugate to each other.

Proof. See [BO5, Proposition 2.2]. \square

Definition 5.4. We say that the couple (z, z') belongs to the *principal series* or to the *complementary series* if it satisfies (i) or (ii), respectively. Of course, the complementary series exists for rational θ only.

Let us summarize the results of this and preceding sections:

Let $\theta > 0$ and let (z, z') belong to the principal or complementary series. For each $n = 1, 2, \dots$ all the numbers $M_{\theta, z, z'}^{(n)}(\lambda)$, $\lambda \in \mathbb{Y}_n$, are strictly positive and sum up to 1, and hence they determine a probability measure $M_{\theta, z, z'}^{(n)}$ on \mathbb{Y}_n . By the very construction, these measures are consistent with the up transition probabilities $p_{\theta, z, z'}^{\uparrow}(\lambda, \nu)$ defined in (5.3). They are also consistent with the down transition probabilities $p_{\theta}^{\downarrow}(\lambda, \mu)$ defined in (4.6).

We call the measures $M_{\theta, z, z'}^{(n)}$ the *z-measures* with Jack parameter θ . An explicit expression for the weights $M_{\theta, z, z'}^{(n)}(\lambda)$ is given in [BO5], but in the present paper we will not need it.

6. THE ALGEBRA A_{θ} OF θ -REGULAR FUNCTIONS ON YOUNG DIAGRAMS

In this section we fix an arbitrary $\theta > 0$. For a set \mathfrak{X} , we will denote by $\text{Fun}(\mathfrak{X})$ the algebra of all real-valued functions on \mathfrak{X} . Below λ stands for an arbitrary Young diagram.

Let u be a complex variable. Set

$$\Phi(u; \lambda) = \prod_{i=1}^{\infty} \frac{u + \theta i}{u - \lambda_i + \theta i}$$

and observe that the product is actually finite because only finitely many of λ_i 's differ from 0, so that only finitely many factors differ from 1. Clearly, for λ fixed, $\Phi(u; \lambda)$ is a rational function in u taking value 1 at $u = \infty$. Therefore, $\Phi(u; \lambda)$ admits the Taylor expansion at $u = \infty$ with respect to the variable u^{-1} . Likewise, such an expansion also exists for $\log \Phi(u; \lambda)$.

Definition 6.1. Let $\mathbb{A}_{\theta} \subset \text{Fun}(\mathbb{Y})$ be the unital subalgebra generated by the coefficients of the Taylor expansion at $u = \infty$ of $\Phi(u; \lambda)$ (or, equivalently, of $\log \Phi(u; \lambda)$). We call \mathbb{A}_{θ} the *algebra of θ -regular functions on \mathbb{Y}* .

The Taylor expansion of $\log \Phi(u; \lambda)$ at $u = \infty$ has the form

$$\log \Phi(u; \lambda) = \sum_{m=1}^{\infty} \frac{p_m^*(\lambda)}{m} u^{-m}, \quad (6.1)$$

where, by definition,

$$p_m^*(\lambda) = \sum_{i=1}^{\infty} [(\lambda_i - \theta i)^m - (-\theta i)^m], \quad m = 1, 2, \dots, \quad \lambda \in \mathbb{Y}.$$

The above expression makes sense because the sum is actually finite. Thus, the algebra \mathbb{A}_θ is generated by the functions p_1^*, p_2^*, \dots . It is readily verified that these functions are algebraically independent, so that \mathbb{A}_θ is isomorphic to the algebra of polynomials in the variables p_1^*, p_2^*, \dots .

Definition 6.2. Using the isomorphism $\mathbb{A}_\theta \cong \mathbb{R}[p_1^*, p_2^*, \dots]$ we define a *filtration* in \mathbb{A}_θ by setting $\deg p_m^*(\cdot) = m$. In more detail, the m th term of the filtration, consisting of elements of degree $\leq m$, is the finite-dimensional subspace $\mathbb{A}_\theta^{(m)} \subset \mathbb{A}_\theta$ defined in the following way:

$$\mathbb{A}_\theta^{(0)} = \mathbb{R}1; \quad \mathbb{A}_\theta^{(m)} = \text{span}\{(p_1^*)^{r_1}(p_2^*)^{r_2} \dots : 1r_1 + 2r_2 + \dots \leq m\}, \quad m = 1, 2, \dots$$

Note that $p_1^*(\lambda) = |\lambda|$. The θ -regular functions on \mathbb{Y} (that is, elements of \mathbb{A}_θ) coincide with the θ -shifted symmetric polynomials in the variables $\lambda_1, \lambda_2, \dots$ as defined in [OO], [KOO].

Next, we set

$$\mathbf{H}(u; \lambda) = \frac{u \prod_{j=1}^{d-1} (u - y_j)}{\prod_{i=1}^d (u - x_i)}, \quad \widehat{\mathbf{E}}(u; \lambda) = \frac{-1}{\mathbf{H}(u; \lambda)} = -\frac{\prod_{i=1}^d (u - x_i)}{u \prod_{j=1}^{d-1} (u - y_j)}, \quad (6.2)$$

where $x_1, \dots, x_d, y_1, \dots, y_{d-1}$ are the θ -dependent Kerov interlacing coordinates of λ defined in (4.1). Consider the Taylor expansions of $\mathbf{H}(u; \lambda)$ and $\widehat{\mathbf{E}}(u; \lambda)$ at $u = \infty$:

$$\mathbf{H}(u; \lambda) = 1 + \sum_{m=1}^{\infty} \mathbf{h}_m(\lambda) u^{-m}, \quad \widehat{\mathbf{E}}(u; \lambda) = -1 + \sum_{m=1}^{\infty} \widehat{\mathbf{e}}_m(\lambda) u^{-m}. \quad (6.3)$$

Because of (4.2) we have

$$\mathbf{h}_1(\lambda) = \widehat{\mathbf{e}}_1(\lambda) \equiv 0.$$

Further, we set

$$\mathbf{p}_m(\lambda) = \sum_{i=1}^d x_i^m - \sum_{j=1}^{d-1} y_j^m, \quad m = 1, 2, \dots \quad (6.4)$$

Obviously,

$$\log \mathbf{H}(u; \lambda) = \sum_{m=1}^{\infty} \frac{\mathbf{p}_m(\lambda)}{m} u^{-m}, \quad \mathbf{p}_1(\lambda) \equiv 0. \quad (6.5)$$

Proposition 6.3. *The following relation holds*

$$\mathbf{H}(u; \lambda) = \frac{\Phi(u - \theta; \lambda)}{\Phi(u; \lambda)}. \quad (6.6)$$

For $\theta = 1$, another proof (due to Kerov) can be found in [IO, Proposition 3.6] (note that the definition of $\Phi(u, \lambda)$ in [IO] differs from our definition by a shift of the argument u).

Proof. We proceed by induction on $n = |\lambda|$. For $n = 0$ there exists only one diagram, the empty one. The relation (6.6) is satisfied because $\Phi(u, \emptyset) = \mathbf{H}(u; \emptyset) \equiv 1$.

Let us examine the transformation of the both sides of (6.6) when one appends a box $\square = (i, j + 1)$ to λ .

In terms of the row coordinates, this means that the coordinate $\lambda_i = j$ is increased by 1. Consequently,

$$\frac{\Phi(u; \lambda \cup \square)}{\Phi(u; \lambda)} = \frac{u - \lambda_i + \theta i}{u - \lambda_i - 1 + \theta i} = \frac{u - j + \theta i}{u - j - 1 + \theta i},$$

which implies that the right-hand side of (6.6) is multiplied by

$$\frac{(u - j - 1 + \theta i)(u - \theta - j + \theta i)}{(u - j + \theta i)(u - \theta - j - 1 + \theta i)}.$$

On the other hand, recall that there is a natural bijective correspondence between the boxes that may be appended to $\lambda = (X; Y)$ and the points in X . Observe that the point x corresponding to the box $\square = (i, j + 1)$ is $j - \theta(i - 1)$. Therefore, the above expression can be rewritten as

$$\frac{(u - x)(u - x + \theta - 1)}{(u - x - 1)(u - x + \theta)}.$$

Now, the lemma below implies that the left-hand side of (6.6) undergoes precisely the same transformation. This completes the induction step. \square

Lemma 6.4. *Let $\lambda = (X; Y)$ be a Young diagram and $\lambda \cup \square$ be the diagram obtained from λ by appending the box \square corresponding to a point $x \in X$. Then*

$$\frac{\mathbf{H}(u; \lambda \cup \square)}{\mathbf{H}(u; \lambda)} = \frac{(u - x)(u - x + \theta - 1)}{(u - x - 1)(u - x + \theta)}. \quad (6.7)$$

Proof. Let y' and y'' be the neighboring points to x in Y , $y' > x > y''$. If x is the greatest element of X then y' does not exist, and if x is the smallest element then y'' does not exist (these two extreme cases occur when \square lies in the first row or in the first column, respectively). Observe that if y' exists then the difference $y' - x$ is in $\{1, 2, 3, \dots\}$, and if y'' exists then the difference $x - y''$ is in $\{\theta, 2\theta, 3\theta, \dots\}$.

Write $\lambda \cup \square = (\bar{X}; \bar{Y})$. Consider first the generic case, when $y' - x \neq 1$ and $x - y'' \neq \theta$.¹⁷ It is readily checked that

$$\bar{X} = (X \setminus \{x\}) \cup \{x + 1, x - \theta\}, \quad \bar{Y} = Y \cup \{x + 1 - \theta\}.$$

¹⁷If y' does not exist then we formally set $y' - x = +\infty \neq 1$. Likewise, if y'' does not exist we formally set $x - y'' = +\infty \neq \theta$.

Then (6.7) follows directly from the definition of $\mathbf{H}(u; \lambda)$ in (6.2).

Let us examine now the remaining cases:

If $y' - x = 1$ while $x - y'' \neq \theta$ then

$$\bar{X} = (X \setminus \{x\}) \cup \{x - \theta\}, \quad \bar{Y} = (Y \setminus \{y'\}) \cup \{x + 1 - \theta\} = (Y \setminus \{x + 1\}) \cup \{x + 1 - \theta\}.$$

If $x - y'' = \theta$ while $y' - x \neq 1$ then

$$\bar{X} = (X \setminus \{x\}) \cup \{x + 1\}, \quad \bar{Y} = (Y \setminus \{y''\}) \cup \{x + 1 - \theta\} = (Y \setminus \{x - \theta\}) \cup \{x + 1 - \theta\}.$$

Finally, if both $y' - x = 1$ and $x - y'' = \theta$, then

$$\bar{X} = X \setminus \{x\}, \quad \bar{Y} = (Y \setminus \{y', y''\}) \cup \{x + 1 - \theta\} = (Y \setminus \{x + 1, x - \theta\}) \cup \{x + 1 - \theta\}.$$

Again, in each of these three cases one readily checks that (6.7) remains true. \square

Proposition 6.5. *The functions $\mathbf{p}_m(\lambda)$ defined in (6.4) belong to the algebra \mathbb{A}_θ . More precisely, we have*

$$\mathbf{p}_m = \theta \cdot m \cdot p_{m-1}^* + \dots, \quad m = 2, 3, \dots, \quad (6.8)$$

where dots stand for lower degree terms, which are a linear combination of elements p_l^* , with $1 \leq l \leq m - 2$.

Proof. Combining (6.1), (6.5), and (6.6) we get

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{\mathbf{p}_m(\lambda)}{m} u^{-m} &= \sum_{l=1}^{\infty} \frac{p_l^*(\lambda)}{l} ((u - \theta)^{-l} - u^{-l}) \\ &= \sum_{l=1}^{\infty} \frac{p_l^*(\lambda)}{l} u^{-l} ((1 - \theta \cdot u^{-1})^{-l} - 1) \\ &= \sum_{l=1}^{\infty} \frac{p_l^*(\lambda)}{l} u^{-l} \left(\theta \cdot l \cdot u^{-1} - \theta^2 \cdot \frac{l(l+1)}{2} u^{-2} + \theta^3 \cdot \frac{l(l+1)(l+2)}{2 \cdot 3} u^{-3} - \dots \right) \\ &= \sum_{l=1}^{\infty} p_l^*(\lambda) \left(\theta \cdot u^{-(l+1)} - \theta^2 \cdot \frac{l+1}{2} u^{-(l+2)} + \theta^3 \cdot \frac{(l+1)(l+2)}{2 \cdot 3} u^{-(l+3)} - \dots \right). \end{aligned}$$

Equating the coefficients we obtain the desired claim. More precisely:

$$\frac{\mathbf{p}_m}{m} = \theta \cdot p_{m-1}^* - \theta^2 \cdot \frac{l+1}{2} p_{m-2}^* + \theta^3 \cdot \frac{(l+1)(l+2)}{2 \cdot 3} p_{m-3}^* - \dots$$

\square

Corollary 6.6. *Each of the three families of functions $\{\mathbf{p}_2, \mathbf{p}_3, \dots\}$, $\{\mathbf{h}_2, \mathbf{h}_3, \dots\}$, $\{\hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3, \dots\}$ is a system of algebraically independent generators of the algebra \mathbb{A}_θ .*

Corollary 6.7. *Under the identification of \mathbb{A}_θ with any of the three algebras of polynomials*

$$\mathbb{R}[\mathbf{p}_2, \mathbf{p}_3, \dots], \quad \mathbb{R}[\mathbf{h}_2, \mathbf{h}_3, \dots], \quad \mathbb{R}[\hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3, \dots]$$

the filtration introduced in Definition 6.2 is determined by setting

$$\deg \mathbf{p}_m = m - 1, \quad \deg \mathbf{h}_m = m - 1, \quad \deg \hat{\mathbf{e}}_m = m - 1 \quad (m = 2, 3, \dots),$$

respectively.

Let Λ be the algebra of symmetric functions over the base field \mathbb{R} . Following Macdonald [Ma], we will denote by $\{p_1, p_2, \dots\}$ and $\{h_1, h_2, \dots\}$ the two systems of generators consisting of the Newton power sums and the complete homogeneous symmetric functions.

Definition 6.8. Define the *covering homomorphism* $\Lambda \rightarrow \mathbb{A}_\theta$ by the specialization

$$p_1 \rightarrow 0, \quad p_2 \rightarrow \mathbf{p}_2, \quad p_3 \rightarrow \mathbf{p}_3, \quad \dots \quad (6.9)$$

By virtue of Corollary 6.6, the covering homomorphism is surjective and its kernel is the principal ideal generated by p_1 . Note also that under (6.9) we have

$$h_1 \rightarrow 0, \quad h_2 \rightarrow \mathbf{h}_2, \quad h_3 \rightarrow \mathbf{h}_3, \quad \dots \quad (6.10)$$

Corollary 6.9. *If $F(\lambda)$ is a function from \mathbb{A}_θ then the function $\lambda \mapsto F(\lambda')$ is in $\mathbb{A}_{\theta-1}$.*

Proof. To make explicit the dependence on θ , introduce the more detailed notation $\mathbf{p}_{\theta,m}$ instead of \mathbf{p}_m . Using this notation and Remark 4.3, we have

$$\mathbf{p}_{\theta-1,m}(\lambda') = (-\theta)^{-m} \mathbf{p}_{\theta,m}(\lambda),$$

which implies the claim. \square

The next two lemmas will be used in Section 7 below.

Lemma 6.10. *Let $\lambda = (X; Y)$ be a Young diagram and $\lambda \setminus \square$ be the diagram obtained from λ by removing the box \square corresponding to a point $y \in Y$. Then*

$$\frac{\mathbf{H}(u; \lambda \setminus \square)}{\mathbf{H}(u; \lambda)} = \frac{(u - y + 1)(u - y - \theta)}{(u - y)(u - y - \theta + 1)}. \quad (6.11)$$

Proof. The argument is similar to that in the proof of Lemma 6.4. Write $\lambda \setminus \square = (\bar{X}; \bar{Y})$. Let x' and x'' be the neighboring points to y in X , $x' > y > x''$. In the generic case, when $x' > y + \theta$ and $x'' < y - 1$, we have

$$\bar{X} = X \cup \{y + \theta - 1\}, \quad \bar{Y} = (Y \setminus \{y\}) \cup \{y + \theta, y - 1\},$$

and the claim follows from the definition of $\mathbf{H}(u; \lambda)$. The remaining possible cases are examined as in the proof of Lemma 6.4. \square

Lemma 6.11. *Let λ be a Young diagram; $X = \{x_1, \dots, x_d\}$ and $Y = \{y_1, \dots, y_{d-1}\}$ be its Kerov interlacing coordinates; $\pi_1^\uparrow, \dots, \pi_d^\uparrow$ be the numbers associated to $(X; Y)$ according to formulas (5.1) and (5.2); $\pi_1^\downarrow, \dots, \pi_{d-1}^\downarrow$ be the numbers associated to $(X; Y)$ according to formulas (4.3) and (4.4).*

Then for $m = 0, 1, 2, \dots$

$$\sum_{i=1}^d \pi_i^\uparrow x_i^m = \mathbf{h}_m(\lambda), \quad \sum_{j=1}^{d-1} \pi_j^\downarrow y_j^m = \hat{\mathbf{e}}_{m+2}(\lambda). \quad (6.12)$$

Proof. Comparing the definition of $\mathbf{H}(u; \lambda)$ (see (6.2)) with (5.1), we get

$$\mathbf{H}(u; \lambda) = u \sum_{i=1}^d \frac{\pi_i^\uparrow}{u - x_i} = \sum_{i=1}^d \pi_i^\uparrow \left(1 + \frac{x_i}{u} + \frac{x_i^2}{u^2} + \dots \right).$$

Equating the coefficients in u^{-m} gives the first equality in (6.12).

Likewise, from the definition of $\hat{\mathbf{E}}(u; \lambda)$ (see (6.2)) and (4.3) it follows

$$\hat{\mathbf{E}}(u; \lambda) = -1 + \frac{1}{u} \sum_{j=1}^{d-1} \frac{\pi_j^\downarrow}{u - y_j} = -1 + \sum_{j=1}^{d-1} \pi_j^\downarrow \left(\frac{1}{u^2} + \frac{y_j}{u^3} + \frac{y_j^2}{u^4} + \dots \right),$$

which implies the second equality in (6.12). \square

Finally, consider one more set of generators in Λ , the elementary symmetric functions e_1, e_2, \dots .

Lemma 6.12. *Let $\mathbf{e}_1, \mathbf{e}_2, \dots$ denote the images of e_1, e_2, \dots under the covering homomorphism $\Lambda \rightarrow \mathbb{A}_\theta$, see (6.9) and (6.10).*

We have $\hat{\mathbf{e}}_1 = 0$ and $\hat{\mathbf{e}}_m = (-1)^{m-1} \mathbf{e}_m$ for $m = 2, 3, \dots$.

Proof. Consider the generating series

$$H(u) = 1 + \sum_{m=1}^{\infty} h_m u^{-m}, \quad E(u) = 1 + \sum_{m=1}^{\infty} e_m u^{-m}.$$

By the very definition, the covering homomorphism send $H(u)$ to $\mathbf{H}(u; \cdot)$. On the other hand, $E(u) = 1/H(-u)$, so that the covering homomorphism send $E(u)$ to $\mathbf{E}(u; \cdot) := 1/\mathbf{H}(-u; \cdot)$. Comparing this with the definition $\hat{\mathbf{E}}(u; \cdot) = -1/\mathbf{H}(u; \cdot)$ (see (6.2)) we conclude that $\hat{\mathbf{E}}(u; \cdot) = -\mathbf{E}(-u; \cdot)$. This implies the claim. \square

7. THE OPERATORS D AND U IN THE ALGEBRA \mathbb{A}_θ

In this section we fix a triple of parameters (θ, z, z') . We assume $\theta > 0$. The parameters z and z' may be arbitrary complex numbers such that right-hand side of formula (5.3) makes sense, so that the numbers $p_{\theta, z, z'}^\uparrow(\lambda, \nu)$ are well defined. Since we will be dealing with formal computations we will not

need to require these numbers to be positive. Thus, in this section, the only restriction on (z, z') is that $zz' + \theta n \neq 0$ for all $n = 0, 1, 2, \dots$.

7.1. Statement of the result. Let $D_{n+1,n} : \text{Fun}(\mathbb{Y}_n) \rightarrow \text{Fun}(\mathbb{Y}_{n+1})$ and $U_{n,n+1} : \text{Fun}(\mathbb{Y}_{n+1}) \rightarrow \text{Fun}(\mathbb{Y}_n)$ be the “down” and “up” operators acting on functions:

$$\begin{aligned} (D_{n+1,n}F)(\nu) &= \sum_{\lambda \in \mathbb{Y}_n} p_{\theta}^{\downarrow}(\nu, \lambda)F(\lambda), & F \in \text{Fun}(\mathbb{Y}_n), \quad \nu \in \mathbb{Y}_{n+1}, \\ (U_{n,n+1}G)(\lambda) &= \sum_{\nu \in \mathbb{Y}_{n+1}} p_{\theta,z,z'}^{\uparrow}(\lambda, \nu)G(\nu), & G \in \text{Fun}(\mathbb{Y}_{n+1}), \quad \lambda \in \mathbb{Y}_n. \end{aligned}$$

This action arises by dualizing the natural action of p_{θ}^{\downarrow} and $p_{\theta,z,z'}^{\uparrow}$ on measures, which explains the seeming contradiction: the “down” operator raises the level n while the “up” operator reduces the level.

In the formulation of Theorem 7.1 below we identify \mathbb{A}_{θ} with the polynomial algebra $\mathbb{R}[\mathbf{h}_2, \mathbf{h}_3, \dots]$. Recall that \mathbb{A}_{θ} is a filtered algebra (Definition 6.2) and that under the identification $\mathbb{A}_{\theta} = \mathbb{R}[\mathbf{h}_2, \mathbf{h}_3, \dots]$ the filtration is determined by setting $\deg \mathbf{h}_m = m - 1$ (Corollary 6.7). We say that an operator $\mathcal{A} : \mathbb{A}_{\theta} \rightarrow \mathbb{A}_{\theta}$ has degree $\leq m$, where $m \in \mathbb{Z}$, if for any $F \in \mathbb{A}_{\theta}$, $\deg(\mathcal{A}F) \leq \deg F + m$.

Observe that any operator in the algebra of polynomials (in finitely or countably many variables) can be written as a differential operator with polynomial coefficients, that is, as a formal infinite sum of differential monomials. This fact is well known and can be readily proved; we do not use it but it is helpful to take it in mind while reading the formulation and the proof of Theorem 7.1.

Given $F \in \mathbb{A}_{\theta}$, we denote by F_n the restriction of the function $F(\cdot)$ to $\mathbb{Y}_n \subset \mathbb{Y}$. It is readily checked that the subalgebra $\mathbb{A}_{\theta} \subset \text{Fun}(\mathbb{Y})$ separates points, which implies that for each n , the functions of the form F_n , with $F \in \mathbb{A}_{\theta}$, exhaust the space $\text{Fun}(\mathbb{Y}_n)$.

Theorem 7.1. (i) *There exists a unique operator $D : \mathbb{A}_{\theta} \rightarrow \mathbb{A}_{\theta}$ such that*

$$D_{n+1,n}F_n = \frac{1}{\theta(n+1)}(DF)_{n+1}, \quad \text{for all } n = 0, 1, \dots \text{ and all } F \in \mathbb{A}_{\theta}.$$

More precisely, the operator D has degree 1 with respect to the filtration of \mathbb{A}_θ , and its top degree terms look as follows

$$\begin{aligned}
D &= \mathbf{h}_2 + \frac{1}{2}\theta^2 \sum_{r,s \geq 2} (r-1)(s-1)\mathbf{h}_{r+s-2} \frac{\partial^2}{\partial \mathbf{h}_r \partial \mathbf{h}_s} \\
&\quad - \theta \sum_{r \geq 2} (r-1)\mathbf{h}_r \frac{\partial}{\partial \mathbf{h}_r} \\
&\quad + \frac{1}{2}\theta(1-\theta) \sum_{r \geq 3} (r-1)(r-2)\mathbf{h}_{r-1} \frac{\partial}{\partial \mathbf{h}_r} \\
&\quad + \frac{1}{2}\theta \sum_{r,s \geq 2} (r+s)\mathbf{h}_r \mathbf{h}_s \frac{\partial}{\partial \mathbf{h}_{r+s}} \\
&\quad + \text{terms of degree } \leq -2.
\end{aligned}$$

(ii) There exists a unique operator $U : \mathbb{A}_\theta \rightarrow \mathbb{A}_\theta$ such that

$$U_{n,n+1}F_{n+1} = \frac{1}{zz' + \theta n} (UF)_n, \quad \text{for all } n = 0, 1, \dots \text{ and all } F \in \mathbb{A}_\theta.$$

More precisely, the operator U has degree 1 with respect to the filtration of \mathbb{A}_θ , and its top degree terms look as follows

$$\begin{aligned}
U &= \mathbf{h}_2 + zz' + \theta zz' \frac{\partial}{\partial \mathbf{h}_2} \\
&\quad + \theta(z+z') \sum_{r \geq 3} (r-1)\mathbf{h}_{r-1} \frac{\partial}{\partial \mathbf{h}_r} \\
&\quad + \frac{1}{2}\theta^2 \sum_{r,s \geq 2} (r-1)(s-1)\mathbf{h}_{r+s-2} \frac{\partial^2}{\partial \mathbf{h}_r \partial \mathbf{h}_s} \\
&\quad + \theta \sum_{r \geq 2} (r-1)\mathbf{h}_r \frac{\partial}{\partial \mathbf{h}_r} \\
&\quad + \frac{1}{2}\theta(1-\theta) \sum_{r \geq 3} (r-1)(r-2)\mathbf{h}_{r-1} \frac{\partial}{\partial \mathbf{h}_r} \\
&\quad + \frac{1}{2}\theta \sum_{r,s \geq 2} (r+s-2)\mathbf{h}_r \mathbf{h}_s \frac{\partial}{\partial \mathbf{h}_{r+s}} \\
&\quad + \text{terms of degree } \leq -2.
\end{aligned}$$

Note that D depends only on θ while U depends on the whole triple (θ, z, z') .

The rest of the section is devoted to the proof. Since it is long, let us briefly describe its idea. Instead of dealing with individual elements of \mathbb{A}_θ it is more convenient to manipulate with generating series. We know that the series

$\mathbf{H}(u; \lambda)$ gathers the generators $\mathbf{h}_2(\lambda), \mathbf{h}_3(\lambda), \dots$ of the algebra \mathbb{A}_θ . Therefore, the products $\mathbf{H}(u_1; \lambda)\mathbf{H}(u_2; \lambda) \dots$ gather various products of the generators, which in turn constitute a linear basis in \mathbb{A}_θ . Thus, we know the action of our operators if we know how they transform products of generating series. Now, it turns out that the transformation of $\mathbf{H}(u_1; \lambda)\mathbf{H}(u_2; \lambda) \dots$ can be written down in a closed form. From this we can extract all the necessary information.

7.2. Action of D and U on products of generating series. We proceed to the detailed proof. Recall that

$$H(u) = 1 + \sum_{m=0}^{\infty} h_m u^{-m} \in \Lambda[[u^{-1}]]. \quad (7.1)$$

Let $\rho = (\rho_1, \rho_2, \dots)$ range over the set of partitions. Recall the standard notation h_ρ and m_ρ for the complete homogeneous symmetric functions and monomial symmetric functions, respectively, see [Ma]. Take a finite collection of variables u_1, u_2, \dots (we prefer to not indicate their number explicitly). Then

$$\prod_l H(u_l) = \sum_{\rho} m_{\rho}(u_1^{-1}, u_2^{-1}, \dots) h_{\rho}$$

summed over all ρ 's such that $\ell(\rho)$ (the number of nonzero parts in ρ) does not exceed the number of variables u_1, u_2, \dots .

Applying to (7.1) the covering homomorphism $\Lambda \rightarrow \mathbb{A}_\theta$ (Definition 6.8) and using (6.10) we get

$$\prod_l \mathbf{H}(u_l; \lambda) = \sum_{\rho} m_{\rho}(u_1^{-1}, u_2^{-1}, \dots) \mathbf{h}_{\rho}(\lambda). \quad (7.2)$$

Because $\mathbf{h}_1(\lambda) \equiv 0$, we may and do additionally assume that ρ does not contain parts equal to 1. Note that the set $\{\mathbf{h}_{\rho} : \rho_1, \rho_2, \dots \neq 1\}$ is a basis in \mathbb{A}_θ .

We regard the left-hand side of (7.2) as a generating series for the basis elements \mathbf{h}_{ρ} . Thus, the transformation of the left-hand side under the action of an operator acting on functions in λ is completely determined by its action on the functions $\mathbf{h}_{\rho}(\lambda)$ in the right-hand side.

Occasionally, it will be convenient to omit the argument λ in the notation $\mathbf{H}(u; \lambda)$. Recall also the notation $(\dots)_n$ for the operation of restriction to the subset $\mathbb{Y}_n \subset \mathbb{Y}$.

By the very definition of $U_{n,n+1}$ we have

$$\begin{aligned} & \left(U_{n,n+1} \left(\prod_l \mathbf{H}(u_l) \right)_{n+1} \right) (\lambda) \\ &= \sum_{i=1}^d p_{\theta, z, z'}^{\uparrow}(\lambda, \lambda \cup \square_i) \prod_l \mathbf{H}(u_l, \lambda \cup \square_i), \quad \lambda \in \mathbb{Y}_n \end{aligned}$$

Substituting the explicit expression (5.3) for the up probabilities and using Lemma 6.4 we rewrite this equality as

$$\begin{aligned} & \left((zz' + \theta n) U_{n,n+1} \left(\prod_l \mathbf{H}(u_l) \right)_{n+1} \right) (\lambda) \\ &= \left\{ \sum_{i=1}^d (z + x_i)(z' + x_i) \prod_l \frac{(u_l - x_i)(u_l - x_i + \theta - 1)}{(u_l - x_i - 1)(u_l - x_i + \theta)} \cdot \pi_i^\uparrow(X; Y) \right\} \\ & \quad \times \prod_l \mathbf{H}(u_l; \lambda), \quad \lambda \in \mathbb{Y}_n. \quad (7.3) \end{aligned}$$

Here, as usual, $(X; Y)$ are the Kerov interlacing coordinates of λ and $\pi_i^\uparrow(X; Y)$ denote the numbers π_i^\uparrow defined in (5.2).

Likewise, by the definition of $D_{n+1,n}$,

$$\begin{aligned} & \left(D_{n+1,n} \left(\prod_l \mathbf{H}(u_l) \right)_n \right) (\lambda) \\ &= \sum_{j=1}^{d-1} p^\downarrow(\lambda, \lambda \setminus \square_j) \prod_l \mathbf{H}(u_l, \lambda \setminus \square_j), \quad \lambda \in \mathbb{Y}_{n+1} \end{aligned}$$

Substituting the explicit expression for the down probabilities (see (4.6) and (4.5)) and using Lemma 6.10 we rewrite this as

$$\begin{aligned} & \left(\theta(n+1) D_{n+1,n} \left(\prod_l \mathbf{H}(u_l) \right)_n \right) (\lambda) \\ &= \left\{ \sum_{j=1}^{d-1} \prod_l \frac{(u_l - y_j + 1)(u_l - y_j - \theta)}{(u_l - y_j)(u_l - y_j - \theta + 1)} \cdot \pi_j^\downarrow(X; Y) \right\} \\ & \quad \times \prod_l \mathbf{H}(u_l; \lambda), \quad \lambda \in \mathbb{Y}_{n+1}. \quad (7.4) \end{aligned}$$

Here $\pi_j^\downarrow(X; Y)$ are the numbers π_j^\downarrow defined in (4.4).

It is convenient to introduce a special notation for the expressions in the curly brackets that appear in (7.3) and (7.4):

$$F^\uparrow(u_1, u_2, \dots; \lambda) = \sum_{i=1}^d (z + x_i)(z' + x_i) \prod_l \frac{(u_l - x_i)(u_l - x_i + \theta - 1)}{(u_l - x_i - 1)(u_l - x_i + \theta)} \cdot \pi_i^\uparrow(X; Y) \quad (7.5)$$

$$F^\downarrow(u_1, u_2, \dots; \lambda) = \sum_{j=1}^{d-1} \prod_l \frac{(u_l - y_j + 1)(u_l - y_j - \theta)}{(u_l - y_j)(u_l - y_j - \theta + 1)} \cdot \pi_j^\downarrow(X; Y) \quad (7.6)$$

Lemma 7.2. *As functions in λ , both $F^\uparrow(u_1, u_2, \dots; \lambda)$ and $F^\downarrow(u_1, u_2, \dots; \lambda)$ are elements of the algebra \mathbb{A}_θ . More precisely, the both expressions can be viewed as elements of $\mathbb{A}_\theta[[u_1^{-1}, u_2^{-1}, \dots]]$.*

Proof. Observe that the i th product in (7.5) and the j th product in (7.6) can be viewed as elements of $\mathbb{R}[x_i][[u_1^{-1}, u_2^{-1}, \dots]]$ and $\mathbb{R}[y_j][[u_1^{-1}, u_2^{-1}, \dots]]$, respectively, and then apply Lemma 6.11. \square

Formulas (7.3) and (7.4) combined with Lemma 7.2 show that the operators $(zz' + \theta n)U_{n, n+1}$ and $\theta(n+1)D_{n+1, n}$ are indeed induced by certain operators U and D acting in \mathbb{A}_θ , and the transformation of the generating series $\mathbf{H}(u_1)\mathbf{H}(u_2)\dots$ under the action of these two operators looks as follows (it is convenient to omit the argument λ in the formulas below):

$$\begin{aligned} U(\mathbf{H}(u_1)\mathbf{H}(u_2)\dots) &= F^\uparrow(u_1, u_2, \dots)\mathbf{H}(u_1)\mathbf{H}(u_2)\dots \\ D(\mathbf{H}(u_1)\mathbf{H}(u_2)\dots) &= F^\downarrow(u_1, u_2, \dots)\mathbf{H}(u_1)\mathbf{H}(u_2)\dots \end{aligned} \quad (7.7)$$

These nice formulas contain in a compressed form all the information about the action of U and D on the basis elements \mathbf{h}_ρ . Our next step is to extract from (7.7) some explicit expressions for $U\mathbf{h}_\rho$ and $D\mathbf{h}_\rho$ using (7.2) and Lemma 6.11.

7.3. Action of D and U in the basis $\{\mathbf{h}_\rho\}$. We need to introduce some notation. Expand the products (6.7) and (6.11) about $u = \infty$:

$$\begin{aligned} \frac{(u-x)(u-x+\theta-1)}{(u-x-1)(u-x+\theta)} &= \sum_{s=0}^{\infty} a_s(x)u^{-s}, & a_s &\in \mathbb{R}[x], \\ \frac{(u-y+1)(u-y-\theta)}{(u-y)(u-y-\theta+1)} &= \sum_{s=0}^{\infty} b_s(y)u^{-s}, & b_s &\in \mathbb{R}[y]. \end{aligned} \quad (7.8)$$

Lemma 7.3. *We have*

$$a_0(x) = b_0(y) \equiv 1, \quad a_1(x) = b_1(y) \equiv 0,$$

and $a_s(x)$ and $b_s(y)$ are polynomials of degree $s - 2$ for $s \geq 2$. More precisely, the two top degree terms of these polynomials are as follows

$$\begin{aligned} a_s(x) &= (s-1)\theta x^{s-2} + \frac{(s-1)(s-2)}{2} \theta(1-\theta)x^{s-3} + \dots, & s \geq 2 \\ b_s(y) &= -(s-1)\theta y^{s-2} + \frac{(s-1)(s-2)}{2} \theta(1-\theta)y^{s-3} + \dots, & s \geq 2. \end{aligned}$$

s

Proof. Setting $v = u^{-1}$ we get

$$\begin{aligned} \frac{(u-x)(u-x+\theta-1)}{(u-x-1)(u-x+\theta)} &= 1 + \frac{\theta v}{1+\theta} \left(\frac{1}{1-(x+1)v} - \frac{1}{1-(x-\theta)v} \right) \\ &= 1 + \frac{\theta}{1+\theta} \sum_{s \geq 1} v^s ((x+1)^{s-1} - (x-\theta)^{s-1}) \\ &= 1 + \frac{\theta}{1+\theta} \sum_{s \geq 2} v^s \left((s-1)(1+\theta)x^{s-2} + \frac{(s-1)(s-2)}{2} (1-\theta^2)x^{s-3} + \dots \right) \\ &= 1 + \sum_{s \geq 2} v^s \left((s-1)\theta x^{s-2} + \frac{(s-1)(s-2)}{2} \theta(1-\theta)x^{s-3} + \dots \right), \end{aligned}$$

which proves the claim concerning the first expansion. For the second expansion the computation is analogous:

$$\begin{aligned} \frac{(u-y+1)(u-y-\theta)}{(u-y)(u-y-\theta+1)} &= 1 - \frac{\theta v}{1-\theta} \left(\frac{1}{1-yv} - \frac{1}{1-(y+\theta-1)v} \right) \\ &= 1 - \frac{\theta v}{1-\theta} \sum_{s \geq 1} v^{s-1} (y^{s-1} - (y+\theta-1)^{s-1}) \\ &= 1 - \frac{\theta}{1-\theta} \sum_{s \geq 2} v^s \left((s-1)(1-\theta)y^{s-2} - \frac{(s-1)(s-2)}{2} (1-\theta)^2 y^{s-3} + \dots \right) \\ &= 1 + \sum_{s \geq 2} v^s \left(-(s-1)\theta y^{s-2} + \frac{(s-1)(s-2)}{2} \theta(1-\theta)y^{s-3} + \dots \right). \end{aligned}$$

□

For a partition $\sigma = (\sigma_1, \sigma_2, \dots)$ we set

$$a_\sigma(x) = \prod_i a_{\sigma_i}(x), \quad b_\sigma(y) = \prod_i b_{\sigma_i}(y). \quad (7.9)$$

Note that these polynomials vanish if σ has a part equal to 1, because $a_1(x)$ and $b_1(x)$ are identically equal to 0.

Observe that (7.8) and (7.9) imply

$$\begin{aligned} \prod_l \frac{(u_l - x)(u_l - x + \theta - 1)}{(u_l - x - 1)(u_l - x + \theta)} &= \sum_{\sigma} a_{\sigma}(x) m_{\sigma}(u_1^{-1}, u_2^{-1}, \dots) \\ \prod_l \frac{(u_l - y + 1)(u_l - y - \theta)}{(u_l - y)(u_l - y - \theta + 1)} &= \sum_{\sigma} b_{\sigma}(y) m_{\sigma}(u_1^{-1}, u_2^{-1}, \dots). \end{aligned} \quad (7.10)$$

Next, introduce linear maps

$$\begin{aligned} f &\rightarrow \langle f \rangle^{\uparrow}, & \mathbb{R}[x] &\rightarrow \mathbb{A}_{\theta}, \\ g &\rightarrow \langle g \rangle^{\downarrow}, & \mathbb{R}[y] &\rightarrow \mathbb{A}_{\theta}, \end{aligned}$$

by setting

$$\langle x^m \rangle^{\uparrow} = \mathbf{h}_m, \quad \langle y^m \rangle^{\downarrow} = \hat{\mathbf{e}}_{m+2}, \quad m = 0, 1, 2, \dots, \quad \mathbf{h}_0 := 1. \quad (7.11)$$

This definition is inspired by Lemma 6.11.

Finally, let $c_{\sigma\tau}^{\rho}$ be the structure constants of the algebra Λ in the basis of monomial symmetric functions:

$$m_{\sigma} m_{\tau} = \sum_{\rho} c_{\sigma\tau}^{\rho} m_{\rho}.$$

Note that $c_{\sigma\tau}^{\rho}$ vanishes unless $|\rho| = |\sigma| + |\tau|$.

Now we are in a position to compute $U\mathbf{h}_{\rho}$ and $D\mathbf{h}_{\rho}$:

Lemma 7.4. *With the notation introduced above we have*

$$U\mathbf{h}_{\rho} = \sum_{\sigma, \tau: |\sigma|+|\tau|=|\rho|} c_{\sigma\tau}^{\rho} \langle (z+x)(z'+x)a_{\sigma}(x) \rangle^{\uparrow} \mathbf{h}_{\tau}, \quad (7.12)$$

$$D\mathbf{h}_{\rho} = \sum_{\sigma, \tau: |\sigma|+|\tau|=|\rho|} c_{\sigma\tau}^{\rho} \langle b_{\sigma}(y) \rangle^{\downarrow} \mathbf{h}_{\tau}. \quad (7.13)$$

Proof. Write

$$F^{\uparrow}(u_1, u_2, \dots) = \sum_{\sigma} F_{\sigma}^{\uparrow} m_{\sigma}(u_1^{-1}, u_2^{-1}, \dots), \quad F_{\sigma}^{\uparrow} \in \mathbb{A}_{\theta},$$

$$F^{\downarrow}(u_1, u_2, \dots) = \sum_{\sigma} F_{\sigma}^{\downarrow} m_{\sigma}(u_1^{-1}, u_2^{-1}, \dots), \quad F_{\sigma}^{\downarrow} \in \mathbb{A}_{\theta}.$$

From (7.2) and (7.7) we get

$$\sum_{\rho} m_{\rho}(u_1^{-1}, u_2^{-1}, \dots) U\mathbf{h}_{\rho} = \left(\sum_{\sigma} F_{\sigma}^{\uparrow} m_{\sigma}(u_1^{-1}, u_2^{-1}, \dots) \right) \left(\sum_{\tau} m_{\tau}(u_1^{-1}, u_2^{-1}, \dots) \mathbf{h}_{\tau} \right),$$

which implies

$$U\mathbf{h}_{\rho} = \sum_{\sigma, \tau: |\sigma|+|\tau|=|\rho|} c_{\sigma\tau}^{\rho} F_{\sigma}^{\uparrow} \mathbf{h}_{\tau}.$$

Likewise,

$$D\mathbf{h}_\rho = \sum_{\sigma, \tau: |\sigma|+|\tau|=\rho} c_{\sigma\tau}^\rho F_\sigma^\downarrow \mathbf{h}_\tau.$$

It remains to prove that

$$F_\sigma^\uparrow = \langle (z+x)(z'+x)a_\sigma(x) \rangle^\uparrow, \quad F_\sigma^\downarrow = \langle b_\sigma(y) \rangle^\downarrow,$$

but this directly follows from (7.5), (7.6), and (7.10).

Finally, we note that F_σ^\uparrow and F_σ^\downarrow vanish if σ has a part equal to 1, because in this case $a_\sigma(x) \equiv 0$ and $b_\sigma(y) \equiv 0$. This agrees with the remark made just below (7.2). \square

7.4. Top degree terms of D : proof of claim (i) of Theorem 7.1. The existence of the operator $D : \mathbb{A}_\theta \rightarrow \mathbb{A}_\theta$ satisfying (7.1) has been established above (see (7.7)), and its uniqueness is obvious.

By virtue of (7.13) we can write

$$D = \sum_{\sigma} D_\sigma, \quad D_\sigma \mathbf{h}_\rho = \sum_{\tau: |\tau|=\rho-|\sigma|} \langle b_\sigma(y) \rangle^\downarrow c_{\sigma\tau}^\rho \mathbf{h}_\tau. \quad (7.14)$$

Lemma 7.5. *Let $\sigma \neq \emptyset$. Then*

$$\deg D_\sigma \leq \max_{\rho, \tau} (\ell(\rho) - \ell(\tau) - 2\ell(\sigma) + 1), \quad (7.15)$$

where the maximum is taken over all pairs (ρ, τ) such that $c_{\sigma\tau}^\rho \neq 0$.

Furthermore, a more rough but simpler estimate is

$$\deg D_\sigma \leq -\ell(\sigma) + 1. \quad (7.16)$$

Proof. We have

$$\begin{aligned} \deg D_\sigma &\leq \max_{\rho, \tau} (\deg \langle b_\sigma(y) \rangle^\downarrow + \deg \mathbf{h}_\tau - \deg \mathbf{h}_\rho) \\ &= \max_{\rho, \tau} (\deg \langle b_\sigma(y) \rangle^\downarrow + |\tau| - \ell(\tau) - |\rho| + \ell(\rho)) \\ &= \max_{\rho, \tau} (\deg \langle b_\sigma(y) \rangle^\downarrow - |\sigma| - \ell(\tau) + \ell(\rho)). \end{aligned}$$

Here the first line holds by the very definition of D_σ , the second line holds because

$$\deg \mathbf{h}_\tau = |\tau| - \ell(\tau), \quad \deg \mathbf{h}_\rho = |\rho| - \ell(\rho)$$

for any τ and ρ , and the third line holds because $c_{\sigma\tau}^\rho \neq 0$ implies $|\rho| = |\sigma| + |\tau|$.

Let us write down $\langle b_\sigma(y) \rangle^\downarrow$ in more detail. Set $\sigma = (\sigma_1, \dots, \sigma_{\ell(\sigma)})$. Here $\ell(\sigma) \geq 1$ because $\sigma \neq \emptyset$ by the assumption. We may assume that σ does not

have parts equal to 1; otherwise $D_\sigma = 0$ because $b_1(y) \equiv 0$. Thus, $\sigma_i \geq 2$ for all i and we have

$$\begin{aligned} \langle b_\sigma(y) \rangle^\downarrow &= \left\langle \prod_{i=1}^{\ell(\sigma)} b_{\sigma_i}(y) \right\rangle^\downarrow \\ &= \left\langle \prod_{i=1}^{\ell(\sigma)} \left(-(\sigma_i - 1)\theta y^{\sigma_i - 2} + \frac{(\sigma_i - 1)(\sigma_i - 2)}{2} \theta(1 - \theta)y^{\sigma_i - 3} + \dots \right) \right\rangle^\downarrow, \end{aligned}$$

where we have used Lemma 7.3.

The expression inside the brackets has degree $|\sigma| - 2\ell(\sigma)$ in y . Consequently, the top degree term of $\langle b_\sigma(y) \rangle^\downarrow$ is equal, within a nonzero scalar factor, to $\hat{\mathbf{e}}_{|\sigma| - 2\ell(\sigma) + 2}$, and the degree of this element is $|\sigma| - 2\ell(\sigma) + 1$.

Therefore,

$$\deg D_\sigma \leq \max_{\rho, \tau} (|\sigma| - 2\ell(\sigma) - |\sigma| - \ell(\tau) + \ell(\rho) + 1) = \max_{\rho, \tau} (\ell(\rho) - \ell(\tau) - 2\ell(\sigma) + 1),$$

which is (7.15).

To deduce (7.16) we observe that $c_{\sigma\tau}^\rho \neq 0$ implies $\ell(\rho) \leq \ell(\sigma) + \ell(\tau)$. \square

Corollary 7.6. *If $\ell(\sigma) \geq 3$ then $\deg D_\sigma \leq -2$.*

Proof. Indeed, this immediately follows from (7.16). \square

By Corollary 7.6, to prove claim (i) of Theorem 7.1 it suffices to examine the contribution of the operators D_σ with $\ell(\sigma) = 0$ (that is, $\sigma = \emptyset$), $\ell(\sigma) = 1$, and $\ell(\sigma) = 2$. We do this in the three lemmas below.

Lemma 7.7 (Contribution from $\sigma = \emptyset$). $D_\emptyset = \mathbf{h}_2$.

Proof. Indeed, if $\sigma = \emptyset$ then τ has to be equal to ρ , and then $c_{\sigma\tau}^\rho = 1$. On the other hand, $\langle b_\sigma(y) \rangle^\downarrow$ reduces to $\langle 1 \rangle^\downarrow = \hat{\mathbf{e}}_2$.

Next, $\hat{\mathbf{e}}_2 = -\mathbf{e}_2$ (see Lemma 6.12) and the identity $h_2 + e_2 = h_1^2$ in Λ implies the identity $-\mathbf{e}_2 = \mathbf{h}_2 + \mathbf{h}_1^2$ in \mathbb{A}_θ . Since $\mathbf{h}_1 = 0$, we conclude from (7.14) that D_\emptyset is the operator of multiplication by \mathbf{h}_2 . \square

Lemma 7.8 (Contribution from σ 's with $\ell(\sigma) = 2$).

$$\sum_{\sigma: \ell(\sigma)=2} D_\sigma = \frac{1}{2}\theta^2 \sum_{r,s \geq 2} (r-1)(s-1)\mathbf{h}_{r+s-2} \frac{\partial^2}{\partial \mathbf{h}_r \partial \mathbf{h}_s} + \text{terms of degree } \leq -2.$$

Proof. Let $\ell(\sigma) = 2$, so that $\sigma = (\sigma_1 \geq \sigma_2 > 0)$. Recall that we may assume $\sigma_2 \geq 2$ (otherwise $D_\sigma = 0$). Below ρ and τ are the same as in Lemma 7.5. In particular, $\ell(\rho) \leq \ell(\tau) + 2$. If $\ell(\rho) < \ell(\tau) + 2$, then the argument of Lemma 7.5 says that the corresponding contribution to D_σ has degree ≤ -2 . Thus, we may take into account only those (ρ, σ) for which $\ell(\rho) = \ell(\tau) + 2$. This

means $\rho = \sigma \cup \tau$, that is, the nonzero parts of ρ are the disjoint union of those in σ and τ . In other words, for some $i < j \leq \ell(\rho)$

$$\sigma_1 = \rho_i, \quad \sigma_2 = \rho_j, \quad \tau = \{\rho_1, \dots, \rho_{\ell(\rho)}\} \setminus \{\rho_i, \rho_j\}.$$

In this case $c_{\sigma\tau}^\rho = 1$

Furthermore, the argument in Lemma 7.5 also shows that in $\langle b_\sigma(y) \rangle^\downarrow$, only the top degree term is relevant. This top degree term is

$$\langle (-\sigma_1 - 1)\theta y^{\sigma_1 - 2}(-\sigma_2 - 1)\theta y^{\sigma_2 - 2} \rangle^\downarrow = \theta^2(\sigma_1 - 1)(\sigma_2 - 1)\hat{\mathbf{e}}_{\sigma_1 + \sigma_2 - 2}.$$

It follows (see (7.12)) that

$$\left(\sum_{\sigma: \ell(\sigma)=2} D_\sigma \right) \mathbf{h}_\rho = \theta^2 \sum_{1 \leq i < j \leq \ell(\rho)} (\rho_i - 1)(\rho_j - 1)\hat{\mathbf{e}}_{\rho_i + \rho_j - 2} \mathbf{h}_{\rho \setminus \{\rho_i, \rho_j\}} + \text{negligible terms.}$$

Therefore,

$$\begin{aligned} \sum_{\sigma: \ell(\sigma)=2} D_\sigma &= \theta^2 \sum_{r_1 > r_2 \geq 2} (r_1 - 1)(r_2 - 1)\hat{\mathbf{e}}_{r_1 + r_2 - 2} \frac{\partial^2}{\partial \mathbf{h}_{r_1} \partial \mathbf{h}_{r_2}} \\ &\quad + \frac{1}{2}\theta^2 \sum_{r \geq 2} (r - 1)^2 \hat{\mathbf{e}}_{2r - 2} \frac{\partial^2}{\partial \mathbf{h}_r^2} + \text{terms of degree } \leq -2. \end{aligned}$$

Observe that

$$\hat{\mathbf{e}}_r = \mathbf{h}_r + \text{lower degree terms}, \quad r \geq 2.$$

Indeed, recall that $\hat{\mathbf{e}}_r = (-1)^{r-1} \mathbf{e}_r$ (Lemma 6.12). In the algebra Λ , one has

$$\begin{aligned} (-1)^{r-1} \mathbf{e}_r &= h_r - (h_1 h_{r-1} + h_2 h_{r-2} + \dots + h_{r-1} h_1) \\ &\quad + \text{linear combination of triple, etc., products of } h_1, h_2, \dots \end{aligned} \quad (7.17)$$

Projecting to \mathbb{A}_θ we get

$$\begin{aligned} \hat{\mathbf{e}}_r &= \mathbf{h}_r - (\mathbf{h}_2 \mathbf{h}_{r-2} + \dots + \mathbf{h}_{r-2} \mathbf{h}_2) \\ &\quad + \text{linear combination of triple, etc., products of } \mathbf{h}_2, \mathbf{h}_3, \dots, \end{aligned} \quad (7.18)$$

because $\mathbf{h}_1 = 0$. In (7.17), all terms are homogeneous elements of Λ of one and the same degree r . However, in (7.18) the only terms of highest degree (with respect to the filtration of \mathbb{A}_θ) are $\hat{\mathbf{e}}_r$ and \mathbf{h}_r . Consequently, replacing $\hat{\mathbf{e}}_r$ by \mathbf{h}_r affects only negligible terms.

Thus, we get

$$\begin{aligned} \sum_{\sigma: \ell(\sigma)=2} D_\sigma &= \theta^2 \sum_{r_1 > r_2 \geq 2} (r_1 - 1)(r_2 - 1) \mathbf{h}_{r_1+r_2-2} \frac{\partial^2}{\partial \mathbf{h}_{r_1} \mathbf{h}_{r_2}} \\ &\quad + \frac{1}{2} \theta^2 \sum_{r \geq 2} (r - 1)^2 \mathbf{h}_{2r-2} \frac{\partial^2}{\partial \mathbf{h}_r^2} + \text{terms of degree } \leq -2, \end{aligned}$$

which is equivalent to the desired expression. \square

Lemma 7.9 (Contribution from σ 's with $\ell(\sigma) = 1$).

$$\begin{aligned} \sum_{\sigma: \ell(\sigma)=1} D_\sigma &= -\theta \sum_{r \geq 2} (r - 1) \mathbf{h}_r \frac{\partial}{\partial \mathbf{h}_r} \\ &\quad + \frac{1}{2} \theta (1 - \theta) \sum_{r \geq 3} (r - 1)(r - 2) \mathbf{h}_{r-1} \frac{\partial}{\partial \mathbf{h}_r} \\ &\quad + \frac{1}{2} \theta \sum_{r, s \geq 2} (r + s) \mathbf{h}_r \mathbf{h}_s \frac{\partial}{\partial \mathbf{h}_{r+s}} \\ &\quad + \text{terms of degree } \leq -2. \end{aligned}$$

Proof. Let $\ell(\sigma) = 1$, so that $\sigma = (s)$ with $s \geq 2$. Below ρ and τ are the same as in Lemma 7.5. Two cases are possible: $\ell(\tau) = \ell(\rho) - 1$ and $\ell(\tau) = \ell(\rho)$. Let us examine them separately.

Assume $\ell(\tau) = \ell(\rho) - 1$. This means that, for some $i = 1, \dots, \ell(\rho)$, we have $\sigma = (\rho_i)$ and $\tau = \rho \setminus \{\rho_i\}$. Note that then $c_{\sigma\tau}^\rho = 1$. We argue as in the proof of Lemma 7.8, the only difference is that we have to take into account not only the top degree term in $\langle b_\sigma(y) \rangle^\downarrow$ but also the next term. Thus, applying Lemma 7.3, we write

$$\langle b_\sigma(y) \rangle^\downarrow = \langle b_s(y) \rangle^\downarrow = -(s - 1) \theta \hat{\mathbf{e}}_s + \frac{(s - 1)(s - 2)}{2} \theta (1 - \theta) \hat{\mathbf{e}}_{s-1} + \dots$$

According to (7.14), this gives rise to the terms

$$-\theta \sum_{r \geq 2} (r - 1) \hat{\mathbf{e}}_r \frac{\partial}{\partial \mathbf{h}_r} + \frac{1}{2} \theta (1 - \theta) \sum_{r \geq 3} (r - 1)(r - 2) \hat{\mathbf{e}}_{r-1} \frac{\partial}{\partial \mathbf{h}_r}.$$

Now, assume $\ell(\tau) = \ell(\rho)$. This means that τ is obtained from ρ by subtracting s from one of the parts ρ_i of ρ ; moreover, this part ρ_i should be $\geq s + 2$. Note that $c_{\sigma\tau}^\rho$ is just equal to the multiplicity of that part in ρ . Note also that only the top degree term in $\langle b_s(y) \rangle^\downarrow$ has a relevant contribution. This gives rise to the terms

$$-\theta \sum_{r \geq 4, 2 \leq s \leq r-2} (s - 1) \hat{\mathbf{e}}_s \mathbf{h}_{r-s} \frac{\partial}{\partial \mathbf{h}_r}.$$

Next, the above two expressions involve $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_{r-1}$, and $\hat{\mathbf{e}}_s$, which we have to express in terms of \mathbf{h}_i 's. This should be done as follows:

$$\hat{\mathbf{e}}_r = \mathbf{h}_r - (\mathbf{h}_2 \mathbf{h}_{r-2} + \cdots + \mathbf{h}_{r-2} \mathbf{h}_2) + \cdots, \quad \hat{\mathbf{e}}_{r-1} = \mathbf{h}_{r-1} + \cdots, \quad \hat{\mathbf{e}}_s = \mathbf{h}_s + \cdots,$$

where the rest terms denoted by dots contribute only to terms of degree ≤ -2 in D . Collecting all the terms together and slightly changing the notation of indices we get

$$\begin{aligned} & -\theta \sum_{r \geq 2} (r-1) \mathbf{h}_r \frac{\partial}{\partial \mathbf{h}_r} + \theta \sum_{r \geq 2, s \geq 2} (r+s-1) \mathbf{h}_r \mathbf{h}_s \frac{\partial}{\partial \mathbf{h}_{r+s}} \\ & + \frac{1}{2} \theta (1-\theta) \sum_{r \geq 3} (r-1)(r-2) \mathbf{h}_{r-1} \frac{\partial}{\partial \mathbf{h}_r} - \theta \sum_{r \geq 2, s \geq 2} (s-1) \mathbf{h}_r \mathbf{h}_s \frac{\partial}{\partial \mathbf{h}_{r+s}}. \end{aligned}$$

Now, putting together the second and fourth sums, there is a simplification, which finally leads to the desired expression. \square

The expressions obtained in Lemmas 7.7, 7.8, and 7.9 give together the result stated in claim (i) of Theorem 7.1.

7.5. Top degree terms of U : proof of claim (ii) of Theorem 7.1. The strategy of the proof is the same as in the preceding subsection. However, we have to slightly modify our arguments because of the following circumstances:

- Formula (7.12), as compared to formula (7.13), contains the additional factors $(z+x)(z'+x)$.

- As seen from (7.11), there is a subtle difference in the behavior of the degree of $\langle x^m \rangle^\uparrow = \mathbf{h}_m$ and the degree of $\langle y^m \rangle^\downarrow = \hat{\mathbf{e}}_{m+2}$. For the latter quantity we have a “regular” expression $\deg \langle y^m \rangle^\downarrow = m+1$, valid for all $m \geq 0$, while a similar expression for the former quantity, $\deg \langle x^m \rangle^\uparrow = m-1$, holds for $m \geq 1$ but fails for $m=0$.

In accordance with (7.12), it is convenient to decompose U as follows

$$U = \sum_{\sigma} (U_{\sigma}^0 + U_{\sigma}^1 + U_{\sigma}^2),$$

where

$$U_{\sigma}^0 \mathbf{h}_{\rho} = z z' \sum_{\tau} \langle a_{\sigma}(x) \rangle^\uparrow c_{\sigma\tau}^{\rho} \mathbf{h}_{\tau} \quad (7.19)$$

$$U_{\sigma}^1 \mathbf{h}_{\rho} = (z+z') \sum_{\tau} \langle a_{\sigma}(x)x \rangle^\uparrow c_{\sigma\tau}^{\rho} \mathbf{h}_{\tau} \quad (7.20)$$

$$U_{\sigma}^2 \mathbf{h}_{\rho} = \sum_{\tau} \langle a_{\sigma}(x)x^2 \rangle^\uparrow c_{\sigma\tau}^{\rho} \mathbf{h}_{\tau}. \quad (7.21)$$

Lemma 7.10 (Compare to Lemma 7.5). *Let $\sigma \neq \emptyset$. Then the following estimate for $\deg U_\sigma^1$ and $\deg U_\sigma^2$ holds*

$$\deg U_\sigma^p \leq \max_{\rho, \tau} (\ell(\rho) - \ell(\tau) - 2\ell(\sigma) - 1 + p), \quad p = 1, 2,$$

where the maximum is taken over all pairs (ρ, τ) such that $c_{\sigma\tau}^\rho \neq 0$.

Furthermore, a more rough but simpler estimate is

$$\deg U_\sigma^p \leq -\ell(\sigma) - 1 + p, \quad p = 1, 2.$$

Notice that the case of U_σ^0 requires a special investigation.

Proof. The argument is completely similar to that in Lemma 7.5. We have

$$\begin{aligned} \deg U_\sigma^p &\leq \max_{\rho, \tau} (\deg \langle a_\sigma(x)x^p \rangle^\dagger + \deg \mathbf{h}_\tau - \deg \mathbf{h}_\rho) \\ &= \max_{\rho, \tau} (\deg \langle a_\sigma(x)x^p \rangle^\dagger + |\tau| - \ell(\tau) - |\rho| + \ell(\rho)) \\ &= \max_{\rho, \tau} (\deg \langle a_\sigma(x)x^p \rangle^\dagger - |\sigma| - \ell(\tau) + \ell(\rho)) \end{aligned}$$

Since $p > 0$ by the assumption, the polynomial $a_\sigma(x)x^p$ has degree > 0 even if the polynomial $a_\sigma(x)$ is a constant. Therefore,

$$\deg \langle a_\sigma(x)x^p \rangle^\dagger = |\sigma| - 2\ell(\sigma) + p - 1,$$

which gives the first estimate. Then the second estimate follows from the inequality $\ell(\rho) \leq \ell(\sigma) + \ell(\tau)$. □

Corollary 7.11 (Compare to Corollary 7.6). *We have:*

- (i) $\deg U_\sigma^1 \leq -2$ if $\ell(\sigma) \geq 2$;
- (ii) $\deg U_\sigma^2 \leq -2$ if $\ell(\sigma) \geq 3$;

Proof. Indeed, this follows at once from the second estimate in Lemma 7.10. □

We will examine the cases $p = 0$, $p = 1$, and $p = 2$ separately.

Lemma 7.12. (Contribution from U_σ^0 's)

$$\sum_{\sigma} U_\sigma^0 = zz' + \theta zz' \frac{\partial}{\partial \mathbf{h}_2} + \text{terms of degree } \leq -2.$$

Proof. The contribution of U_\emptyset^0 is the constant term zz' : this is shown by the same argument as in Lemma 7.7.

Assume $\sigma \neq \emptyset$. Then $\sigma = (\sigma_1, \dots, \sigma_{\ell(\sigma)})$ with $\ell(\sigma) \geq 1$. Recall that all σ_i are ≥ 2 . Arguing as in Lemma 7.10 we get

$$\begin{aligned} \deg U_\sigma^0 &\leq \max_{\rho, \tau} (\deg \langle a_\sigma(x)x^p \rangle^\dagger + |\tau| - \ell(\tau) - |\rho| + \ell(\rho)) \\ &\leq \max_{\rho, \tau} (\deg \langle a_\sigma(x) \rangle^\dagger - |\sigma| + \ell(\sigma)), \quad \text{because } -\ell(\tau) + \ell(\rho) \leq \ell(\sigma). \end{aligned}$$

Here ρ and τ are the same as in Lemma 7.10.

In the “regular case”, when the polynomial $a_\sigma(x)$ has degree > 0 , we can apply the formula $\deg\langle a_\sigma(x) \rangle^\dagger = |\sigma| - 2\ell(\sigma) - 1$, which implies

$$\deg U_\sigma^0 \leq -\ell(\sigma) - 1 \leq -2.$$

The “irregular case” occurs when $\sigma_1 = \cdots = \sigma_{\ell(\sigma)} = 2$ (see Lemma 7.3). Then $\deg\langle a_\sigma(x) \rangle^\dagger = 0$ and we get a weaker inequality

$$\deg U_\sigma^0 \leq -\ell(\sigma).$$

If $\ell(\sigma) \geq 2$, this is enough to conclude $\deg U_\sigma^0 \leq -2$.

Finally, examine the case $\sigma = (2)$. There are two possibilities: $\ell(\tau) = \ell(\rho) - 1$ and $\ell(\tau) = \ell(\rho)$. In the latter case the estimate can be refined because then $-\ell(\tau) + \ell(\rho) = 0$ is strictly smaller than $\ell(\sigma) = 1$, which again implies $\deg U_\sigma^0 \leq -2$.

Thus, the only substantial contribution arises when $\sigma = (2)$ and $\tau = \rho \setminus \{(2)\}$. Taking into account (7.19) and Lemma 7.3, this gives rise to the term $\theta z z' \partial / \partial \mathbf{h}_2$. \square

Lemma 7.13 (Contribution from U_σ^1 's).

$$\sum_{\sigma} U_{\sigma}^1 = \theta(z + z') \sum_{r \geq 3} (r-1) \mathbf{h}_{r-1} \frac{\partial}{\partial \mathbf{h}_r} + \text{terms of degree } \leq -2.$$

Proof. Observe that $U_{\emptyset}^1 = 0$ because $\langle a_{\emptyset}(x)x \rangle^\dagger = \mathbf{h}_1 = 0$. By Corollary 7.11, it suffices to examine the case $\ell(\sigma) = 1$, that is, $\sigma = (s)$ with $s \geq 2$. We have

$$\langle a_s(x)x \rangle^\dagger = (s-1)\theta \mathbf{h}_{s-1} + \dots,$$

where the rest terms are negligible. Furthermore, if $\ell(\tau) = \ell(\rho)$ then the estimate of Corollary 7.11 can be refined, which implies that the contribution is negligible. Thus, we may assume $\ell(\tau) = \ell(\rho) - 1$, that is, $\tau = \rho \setminus \{(s)\}$. In accordance with (7.20), this produces the desired expression. Notice also that the restriction $r \geq 3$ arises because $\mathbf{h}_{r-1} = 0$ for $r = 2$. \square

It remains to compute $\sum_{\sigma} U_{\sigma}^2$. Here our arguments are strictly parallel to those of the preceding subsection, because, due to the extra factor x^2 , the element $\langle a_\sigma(x)x^2 \rangle^\dagger$ has the same degree as the element $\langle b_\sigma(x) \rangle^\dagger$, which we examined in the preceding subsection.

In the next three lemmas we rely on (7.21).

Lemma 7.14. (Contribution from U_{\emptyset}^2) $U_{\emptyset}^2 = \mathbf{h}_2$.

Proof. The same argument as in Lemma 7.7. The situation is even simpler because we do not need to convert $\hat{\mathbf{e}}_2$ to \mathbf{h}_2 . \square

Lemma 7.15 (Contribution from U_σ^2 's with $\ell(\sigma) = 2$).

$$\sum_{\sigma: \ell(\sigma)=2} U_\sigma^2 = \frac{1}{2}\theta^2 \sum_{r,s \geq 2} (r-1)(s-1) \mathbf{h}_{r+s-2} \frac{\partial^2}{\partial \mathbf{h}_r \partial \mathbf{h}_s} + \text{terms of degree } \leq -2$$

Proof. The argument is exactly the same as in the proof of Lemma 7.8. Instead of Lemma 7.5 we refer to its analog, Lemma 7.10. Again, we do not need to convert $\hat{\mathbf{e}}_r$ to \mathbf{h}_r , which slightly shortens the proof. \square

Lemma 7.16 (Contribution from U_σ^2 's with $\ell(\sigma) = 1$).

$$\begin{aligned} \sum_{\sigma: \ell(\sigma)=1} U_\sigma^2 &= \theta \sum_{r \geq 2} (r-1) \mathbf{h}_r \frac{\partial}{\partial \mathbf{h}_r} \\ &+ \frac{1}{2}\theta(1-\theta) \sum_{r \geq 3} (r-1)(r-2) \mathbf{h}_{r-1} \frac{\partial}{\partial \mathbf{h}_r} \\ &+ \frac{1}{2}\theta \sum_{r,s \geq 2} (r+s-2) \mathbf{h}_r \mathbf{h}_s \frac{\partial}{\partial \mathbf{h}_{r+s}} + \text{terms of degree } \leq -2. \end{aligned}$$

Proof. We argue as in the proof of Lemma 7.9. \square

Lemmas 7.14, 7.15, and 7.16 together give

$$\begin{aligned} U^2 &= \mathbf{h}_2 + \frac{1}{2}\theta^2 \sum_{r,s \geq 2} (r-1)(s-1) \mathbf{h}_{r+s-2} \frac{\partial^2}{\partial \mathbf{h}_r \partial \mathbf{h}_s} \\ &+ \theta \sum_{r \geq 2} (r-1) \mathbf{h}_r \frac{\partial}{\partial \mathbf{h}_r} \\ &+ \frac{1}{2}\theta(1-\theta) \sum_{r \geq 3} (r-1)(r-2) \mathbf{h}_{r-1} \frac{\partial}{\partial \mathbf{h}_r} \\ &+ \frac{1}{2}\theta \sum_{r,s \geq 2} (r+s-2) \mathbf{h}_r \mathbf{h}_s \frac{\partial}{\partial \mathbf{h}_{r+s}} \\ &+ \text{terms of degree } \leq -2. \end{aligned}$$

Adding this with the expressions obtained in Lemmas 7.12 and 7.13 we finally get the result indicated in Claim (ii) of Theorem 7.1.

This completes the proof of Theorem 7.1.

8. COMPUTATION OF $T_n - 1$

As in Section 7, here we are dealing with a fixed triple (θ, z, z') of parameters. As usual, we assume $\theta > 0$. As for the couple (z, z') , we now assume that it belongs to the principal or complementary series (Definition 5.4), so that $p_{\theta, z, z'}^\dagger$ is a true system of transition probabilities.

Definition 8.1 (The operator T_n). Recall that in Definition 2.3 we have introduced the up-down Markov chains associated with arbitrary systems p^\uparrow and p^\downarrow of up and down transition probabilities. Now we take the concrete systems $p^\uparrow = p_{\theta, z, z'}^\uparrow$, $p^\downarrow = p_\theta^\downarrow$ and denote by T_n the transition operator of the corresponding up-down chain of level n , $n = 1, 2, \dots$. We regard T_n as an operator in the space $\text{Fun}(\mathbb{Y}_n)$ of functions on \mathbb{Y}_n :

$$T_n F(\lambda) = \sum_{\nu, \kappa} p_{\theta, z, z'}^\uparrow(\lambda, \nu) p_\theta^\downarrow(\nu, \kappa) F(\kappa), \quad F \in \text{Fun}(\mathbb{Y}_n),$$

summed over all $\nu \in \mathbb{Y}_{n+1}$ and $\kappa \in \mathbb{Y}_n$ such that $\lambda \nearrow \nu \searrow \kappa$.

In the notation introduced in the beginning of Section 7,

$$T_n = U_{n, n+1} D_{n+1, n}.$$

Set

$$\varepsilon_n = \frac{1}{(\theta^{-1} z z' + n)(n+1)}, \quad n = 1, 2, \dots \quad (8.1)$$

Equivalently,

$$\varepsilon_n^{-1} = \frac{(z z' + \theta n)(\theta(n+1))}{\theta^2}.$$

Clearly, $\varepsilon_n \sim n^{-2}$ (in Theorem 9.6 below we simply take $\varepsilon_n = n^{-2}$). Recall that given a function $F \in \mathbb{A}_\theta$, we denote by $F_n \in \text{Fun}(\mathbb{Y}_n)$ the restriction of F to the finite subset \mathbb{Y}_n .

Theorem 8.2. *There exists a unique operator $\tilde{B} : \mathbb{A}_\theta \rightarrow \mathbb{A}_\theta$ such that for any $F \in \mathbb{A}_\theta$ and each $n = 1, 2, \dots$*

$$\varepsilon_n^{-1} (T_n - 1) F_n = (\tilde{B} F)_n.$$

The operator \tilde{B} has degree 0 and its top degree component looks as follows

$$\begin{aligned} \tilde{B} = & \sum_{r, s \geq 3} (r-1)(s-1) (\mathbf{h}_2 \mathbf{h}_{r+s-2} - \mathbf{h}_r \mathbf{h}_s) \frac{\partial^2}{\partial \mathbf{h}_r \partial \mathbf{h}_s} \\ & + \sum_{r \geq 3} [(\theta^{-1} - 1)(r-1)(r-2) \mathbf{h}_2 \mathbf{h}_{r-1} + \theta^{-1}(z+z')(r-1) \mathbf{h}_2 \mathbf{h}_{r-1} \\ & \quad - (r-1)(r-2) \mathbf{h}_r - \theta^{-1} z z' (r-1) \mathbf{h}_r] \frac{\partial}{\partial \mathbf{h}_r} \\ & + \theta^{-1} \sum_{r, s \geq 2} (r+s-1) \mathbf{h}_2 \mathbf{h}_r \mathbf{h}_s \frac{\partial}{\partial \mathbf{h}_{r+s}} + \text{terms of degree } < 0. \quad (8.2) \end{aligned}$$

Proof. The uniqueness claim is obvious. The existence of \tilde{B} follows from Theorem 7.1. Indeed, by virtue of (6.12) and Proposition 5.1, $\mathbf{h}_2(\lambda) = \theta n$. Using

this and expressing $U_{n,n+1}$ and $D_{n+1,n}$ through U and D , as indicated in Theorem 7.1, we get

$$\begin{aligned} (T_n - 1)F_n &= U_{n,n+1}D_{n+1,n}F_n - F_n = \frac{(UDF)_n - (zz' + \theta n)(\theta(n+1))F_n}{(zz' + \theta n)(\theta(n+1))} \\ &= \frac{((UD - (\mathbf{h}_2 + zz')(\mathbf{h}_2 + \theta))F)_n}{(zz' + \theta n)(\theta(n+1))}. \end{aligned}$$

Therefore,

$$\tilde{B} = \theta^{-2}(UD - (\mathbf{h}_2 + zz')(\mathbf{h}_2 + \theta)). \quad (8.3)$$

To compute $UD - (\mathbf{h}_2 + zz')(\mathbf{h}_2 + \theta)$, within negligible terms, we write

$$U = \mathbf{h}_2 + U_0 + U_{-1} + \dots, \quad D = \mathbf{h}_2 + D_0 + D_{-1} + \dots,$$

where U_0 and D_0 are the terms of degree 0 and U_{-1} and D_{-1} are the terms of degree -1 .

Note that \mathbf{h}_2 (more precisely, the operator of multiplication by \mathbf{h}_2) has degree 1. Let us check that the operator $UD - (\mathbf{h}_2 + zz')(\mathbf{h}_2 + \theta)$, which could have degree 2, is actually of degree 0, due to cancelation of the terms of degree 2 and 1.

Indeed, the degree 2 terms in $UD - (\mathbf{h}_2 + zz')(\mathbf{h}_2 + \theta)$ are

$$\mathbf{h}_2^2 - \mathbf{h}_2^2 = 0.$$

The degree 1 terms in $UD - (\mathbf{h}_2 + zz')(\mathbf{h}_2 + \theta)$ are

$$\mathbf{h}_2 D_0 + U_0 \mathbf{h}_2 - (zz' + \theta) \mathbf{h}_2.$$

Substitute here the explicit expressions for D_0 and U_0 taken from Theorem 7.1,

$$\begin{aligned} D_0 &= -\theta \sum_{r \geq 2} (r-1) \mathbf{h}_r \frac{\partial}{\partial \mathbf{h}_r} \\ U_0 &= zz' + \theta \sum_{r \geq 2} (r-1) \mathbf{h}_r \frac{\partial}{\partial \mathbf{h}_r}. \end{aligned}$$

Since

$$U_0 \mathbf{h}_2 = \mathbf{h}_2 U_0 + [U_0, \mathbf{h}_2] = \mathbf{h}_2 U_0 + \theta \mathbf{h}_2,$$

all the terms of degree 1 are indeed cancelled out.

Now, let us examine the degree 0 terms in $UD - (\mathbf{h}_2 + zz')(\mathbf{h}_2 + \theta)$. These are

$$\mathbf{h}_2 D_{-1} + U_{-1} \mathbf{h}_2 + U_0 D_0 - \theta zz'.$$

To compute $\mathbf{h}_2 D_{-1} + U_{-1} \mathbf{h}_2$ we substitute the explicit expressions for D_{-1} and U_{-1} taken from Theorem 7.1,

$$\begin{aligned} D_{-1} &= \frac{1}{2} \theta^2 \sum_{r,s \geq 2} (r-1)(s-1) \mathbf{h}_{r+s-2} \frac{\partial^2}{\partial \mathbf{h}_r \partial \mathbf{h}_s} \\ &\quad + \frac{1}{2} \theta (1-\theta) \sum_{r \geq 3} (r-1)(r-2) \mathbf{h}_{r-1} \frac{\partial}{\partial \mathbf{h}_r} \\ &\quad + \frac{1}{2} \theta \sum_{r,s \geq 2} (r+s) \mathbf{h}_r \mathbf{h}_s \frac{\partial}{\partial \mathbf{h}_{r+s}} \end{aligned}$$

and

$$\begin{aligned} U_{-1} &= \theta z z' \frac{\partial}{\partial \mathbf{h}_2} + \theta (z + z') \sum_{r \geq 3} (r-1) \mathbf{h}_{r-1} \frac{\partial}{\partial \mathbf{h}_r} \\ &\quad + \frac{1}{2} \theta^2 \sum_{r,s \geq 2} (r-1)(s-1) \mathbf{h}_{r+s-2} \frac{\partial^2}{\partial \mathbf{h}_r \partial \mathbf{h}_s} \\ &\quad + \frac{1}{2} \theta (1-\theta) \sum_{r \geq 3} (r-1)(r-2) \mathbf{h}_{r-1} \frac{\partial}{\partial \mathbf{h}_r} \\ &\quad + \frac{1}{2} \theta \sum_{r,s \geq 2} (r+s-2) \mathbf{h}_r \mathbf{h}_s \frac{\partial}{\partial \mathbf{h}_{r+s}}. \end{aligned}$$

Note that

$$[U_{-1}, \mathbf{h}_2] = \theta z z' + \theta^2 \sum_{r \geq 2} (r-1) \mathbf{h}_r \frac{\partial}{\partial \mathbf{h}_r}.$$

Therefore,

$$\begin{aligned}
\mathbf{h}_2 D_{-1} + U_{-1} \mathbf{h}_2 &= \mathbf{h}_2 D_{-1} + \mathbf{h}_2 U_{-1} + [U_{-1}, \mathbf{h}_2] \\
&= \theta^2 \sum_{r,s \geq 2} (r-1)(s-1) \mathbf{h}_2 \mathbf{h}_{r+s-2} \frac{\partial^2}{\partial \mathbf{h}_r \partial \mathbf{h}_s} \\
&\quad + \theta z z' \mathbf{h}_2 \frac{\partial}{\partial \mathbf{h}_2} \\
&\quad + \theta(z+z') \sum_{r \geq 3} (r-1) \mathbf{h}_2 \mathbf{h}_{r-1} \frac{\partial}{\partial \mathbf{h}_r} \\
&\quad + \theta(1-\theta) \sum_{r \geq 3} (r-1)(r-2) \mathbf{h}_2 \mathbf{h}_{r-1} \frac{\partial}{\partial \mathbf{h}_r} \\
&\quad + \theta \sum_{r,s \geq 2} (r+s-1) \mathbf{h}_2 \mathbf{h}_r \mathbf{h}_s \frac{\partial}{\partial \mathbf{h}_{r+s}} \\
&\quad + \theta^2 \sum_{r \geq 2} (r-1) \mathbf{h}_r \frac{\partial}{\partial \mathbf{h}_r} + \theta z z'.
\end{aligned}$$

Next, using the above expressions for U_0 and D_0 we get

$$\begin{aligned}
U_0 D_0 - \theta z z' &= -\theta^2 \left(\sum_{r \geq 2} (r-1) \mathbf{h}_r \frac{\partial}{\partial \mathbf{h}_r} \right)^2 - \theta z z' \sum_{r \geq 2} (r-1) \mathbf{h}_r \frac{\partial}{\partial \mathbf{h}_r} - \theta z z' \\
&= -\theta^2 \sum_{r,s \geq 2} (r-1)(s-1) \mathbf{h}_r \mathbf{h}_s \frac{\partial^2}{\partial \mathbf{h}_r \partial \mathbf{h}_s} - \theta^2 \sum_{r \geq 2} (r-1)^2 \mathbf{h}_r \frac{\partial}{\partial \mathbf{h}_r} \\
&\quad - \theta z z' \sum_{r \geq 2} (r-1) \mathbf{h}_r \frac{\partial}{\partial \mathbf{h}_r} - \theta z z'.
\end{aligned}$$

Adding together the above two expressions we see that the terms involving $\partial/\partial \mathbf{h}_2$ cancel out, which plays the crucial role in Corollary 8.7 below. Then we divide by θ^2 in accordance with (8.3) and finally get (8.2). \square

From the definition of the filtration in \mathbb{A}_θ (Definition 6.2) we see that there is a natural isomorphism between the associated graded algebra

$$\text{gr } \mathbb{A}_\theta = \bigoplus_{m=0}^{\infty} (\mathbb{A}_\theta^{(m)} / \mathbb{A}_\theta^{(m-1)})$$

and the algebra $\Lambda = \mathbb{R}[p_1, p_2, \dots]$ of symmetric functions: the top degree terms of the generators $p_m^* \in \mathbb{A}_\theta$ are identified with the homogeneous generators $p_m \in \Lambda$. This isomorphism $\text{gr } \mathbb{A}_\theta \simeq \Lambda$ should not be confused with the covering homomorphism $\Lambda \rightarrow \mathbb{A}_\theta$.

Definition 8.3 (The operator B). Observe that the operator $\tilde{B} : \mathbb{A}_\theta \rightarrow \mathbb{A}_\theta$ (Theorem 8.2) has degree 0. Therefore, it gives rise to an operator in the associated graded algebra. Using the identification $\text{gr } \mathbb{A}_\theta = \Lambda$, we denote the latter operator as $B : \Lambda \rightarrow \Lambda$. Note that B is homogeneous of degree 0

Theorem 8.4. *The operator $B : \Lambda \rightarrow \Lambda$ just defined has the following form*

$$\begin{aligned} B = & \sum_{k,l \geq 2} kl(p_1 p_{k+l-1} - p_k p_l) \frac{\partial^2}{\partial p_k \partial p_l} \\ & + \sum_{k \geq 2} [(1-\theta)k(k-1)p_1 p_{k-1} + (z+z')k p_1 p_{k-1} - k(k-1)p_k - k\theta^{-1} z z' p_k] \frac{\partial}{\partial p_k} \\ & + \theta \sum_{k,l \geq 1} (k+l+1) p_1 p_k p_l \frac{\partial}{\partial p_{k+l+1}}. \end{aligned} \quad (8.4)$$

Before proving the theorem let us state a lemma.

Lemma 8.5. *Consider the algebra of polynomials in countably many generators $\mathbb{R}[x_1, x_2, \dots]$ with the filtration determined by setting $\deg x_k = k$, and let y_1, y_2, \dots be another sequence of elements of the same algebra such that*

$$x_k = y_k + \text{terms of degree } < k, \quad k = 1, 2, \dots,$$

so that $\{y_1, y_2, \dots\}$ is also a system of generators.

Then we have

$$\frac{\partial}{\partial x_k} = \frac{\partial}{\partial y_k} + \text{terms of degree } < -k, \quad k = 1, 2, \dots$$

Proof of Lemma 8.5. Indeed,

$$\frac{\partial}{\partial x_k} = \sum_{l \geq 1} \frac{\partial y_l}{\partial x_k} \frac{\partial}{\partial y_l}.$$

Since

$$y_l = x_l + R_l, \quad \text{where } \deg R_l < l,$$

we have

$$\frac{\partial}{\partial x_k} = \sum_{l \geq k} \frac{\partial y_l}{\partial x_k} \frac{\partial}{\partial y_l} = \frac{\partial}{\partial y_k} + \sum_{l > k} \frac{\partial R_l}{\partial x_k} \frac{\partial}{\partial y_l}.$$

The degree of the l th summand in the last sum is strictly less than $(l-k) - l = -k$, so that all these summands are negligible. \square

Proof of Theorem 8.4. By virtue of (6.3) and (6.5), and because of $\mathbf{h}_1 = \mathbf{p}_1 = 0$,

$$1 + \sum_{k \geq 1} \mathbf{h}_{k+1} u^{-k-1} = \exp \left(\sum_{k \geq 1} \frac{\mathbf{p}_{k+1}}{k+1} u^{-k-1} \right).$$

It follows that

$$\mathbf{h}_{k+1} = \frac{\mathbf{P}^{k+1}}{k+1} + \text{terms of degree } < k, \quad k = 1, 2, \dots$$

Next, by (6.8)

$$\frac{\mathbf{P}^{k+1}}{k+1} = \theta p_k^* + \text{terms of degree } < k, \quad k = 1, 2, \dots$$

Therefore,

$$\mathbf{h}_{k+1} = \theta p_k^* + \text{terms of degree } < k, \quad k = 1, 2, \dots \quad (8.5)$$

It follows that we may apply Lemma 8.5 to the generators $x_k = \mathbf{h}_{k+1}$ and $y_k = \theta p_k^*$, $k = 1, 2, \dots$. This gives us

$$\frac{\partial}{\partial \mathbf{h}_{k+1}} = \frac{1}{\theta} \frac{\partial}{\partial p_k^*} + \text{terms of degree } < -k, \quad k = 1, 2, \dots \quad (8.6)$$

Substituting (8.5) and (8.6) into (8.2) we get a similar expression for the operator \tilde{B} in terms of generators p_1^*, p_2^*, \dots and the corresponding partial derivatives.

Finally, it is readily seen that to get B it suffices to replace p_k^* with p_k . This leads to (8.4). \square

Definition 8.6 (The quotient algebra Λ°). Consider the principal ideal $(p_1 - 1)\Lambda$ in the algebra Λ generated by the element $p_1 - 1$ and let $\Lambda^\circ = \Lambda / (p_1 - 1)\Lambda$ denote the corresponding quotient algebra. Because the ideal is not homogeneous, there is no natural graduation in Λ° , but Λ° inherits the filtration of Λ . Given $\varphi \in \Lambda$, we denote by φ° the image of φ under the canonical projection $\Lambda \rightarrow \Lambda^\circ$. Due to the natural isomorphism of Λ° with the polynomial algebra $\mathbb{R}[p_2^\circ, p_3^\circ, \dots]$ we may introduce in Λ° the differential operators $\partial / \partial p_k^\circ$, $k \geq 2$.

Corollary 8.7. *The operator $B : \Lambda \rightarrow \Lambda$ introduced in Definition 8.3 and computed in Theorem 8.4 preserves the principal ideal $(p_1 - 1)\Lambda \subset \Lambda$ and hence gives rise to an operator $A : \Lambda^\circ \rightarrow \Lambda^\circ$. We have*

$$\begin{aligned} A = & \sum_{k,l \geq 2} kl(p_{k+l-1}^\circ - p_k^\circ p_l^\circ) \frac{\partial^2}{\partial p_k^\circ \partial p_l^\circ} \\ & + \sum_{k \geq 2} [(1-\theta)k(k-1)p_{k-1}^\circ + (z+z')kp_{k-1}^\circ - k(k-1)p_k^\circ - k\theta^{-1}zz'p_k^\circ] \frac{\partial}{\partial p_k^\circ} \\ & + \theta \sum_{k,l \geq 1} (k+l+1)p_k^\circ p_l^\circ \frac{\partial}{\partial p_{k+l+1}^\circ} \quad (8.7) \end{aligned}$$

with the understanding that $p_1^\circ = 1$.

Proof. This immediately follows from Theorem 8.4 because the expression (8.4) for the operator B does not contain $\partial / \partial p_1$. \square

9. CONSTRUCTION OF MARKOV PROCESSES

9.1. An operator semigroup approximation theorem. We start with the statement of a well-known general result on approximation of continuous contraction semigroups by discrete ones. Our basic reference is the book [EK2] by Ethier and Kurtz, where one can also find references to original papers.

Assume we are given real Banach spaces L, L_1, L_2, \dots together with bounded linear operators $\pi_n : L \rightarrow L_n, n = 1, 2, \dots$, such that $\sup_n \|\pi_n\| < \infty$.

Definition 9.1 (Convergence of vectors in varying Banach spaces). Let $f \in L$ and $f_n \in L_n, n = 1, 2, \dots$. Write $f_n \rightarrow f$ if

$$\lim_{n \rightarrow \infty} \|f_n - \pi_n f\| = 0.$$

In particular, if $f_n = \pi_n f$ then $f_n \rightarrow f$ for trivial reasons. Clearly, if $f_n \rightarrow f$ and $g_n \rightarrow g$ then $f_n + g_n \rightarrow f + g$. Generally speaking, it may happen that $f_n \rightarrow f$ and $f_n \rightarrow g$ with $f \neq g$. However, such an unpleasant situation can be excluded by imposing an extra assumption like (9.4) below. In the concrete situation we will be dealing with, (9.4) is satisfied but this is not required for Theorem 9.3 below.

Definition 9.2 (Approximation of operator semigroups). Let $L, \{L_n\}$, and $\{\pi_n\}$ be as above; $\{T(t)\}_{t \geq 0}$ be a strongly continuous contraction semigroup in L ; T_n be a contraction in $L_n, n = 1, 2, \dots$; $\{\varepsilon_n\}$ be a sequence of numbers such that $\varepsilon_n > 0$ and $\varepsilon_n \rightarrow 0$. Each contraction T_n generates a discrete semigroup, $\{T_n^m\}_{m=0,1,\dots}$, in L_n .

Let us say that these discrete semigroups *approximate*, as $n \rightarrow \infty$, the continuous semigroup $\{T(t)\}$ if for any $f \in L$

$$T_n^{\lceil t\varepsilon_n^{-1} \rceil} \pi_n f \rightarrow T(t)f$$

uniformly on arbitrarily large bounded intervals $0 \leq t \leq t_0$. That is,

$$\|T_n^{\lceil t\varepsilon_n^{-1} \rceil} \pi_n f - \pi_n T(t)f\| \rightarrow 0$$

uniformly on $t \in [0, t_0]$.

Let us emphasize that the definition substantially depends on the sequence $\{\varepsilon_n\}$ which determines the time scaling.

Theorem 9.3. Consider the same data as in Definition 9.2, and set

$$A_n = \varepsilon_n^{-1}(T_n - 1), \quad n = 1, 2, \dots \quad (9.1)$$

Let $\mathcal{F} \subset L$ be a dense subspace and $A : \mathcal{F} \rightarrow L$ be a closable operator such that its closure \bar{A} coincides with the generator of the semigroup $\{T(t)\}$. Next, let us assume that the operators A_n converge to the operator A in the following “extended” sense: For any $f \in \mathcal{F}$ there exists a sequence $\{f_n \in L_n\}$ such that

$$f_n \rightarrow f \quad \text{and} \quad A_n f_n \rightarrow Af. \quad (9.2)$$

Then the semigroups $\{T_n^m\}$ approximate, as $n \rightarrow \infty$, the semigroup $\{T(t)\}$ in the sense of Definition 9.2.

Proof. This is exactly the implication (c) \Rightarrow (a) in [EK2, Chapter 1, Theorem 7.5]. \square

The notion of “extended convergence” (9.2) turns out to be well adapted to the application we need. In the context of the paper [BO8], where we were concerned with the case $\theta = 1$, we could manage with a weaker version of the theorem, based on a stronger assumption: For any $f \in \mathcal{F}$, $A_n \pi_n f \rightarrow Af$. However, in the case of general $\theta > 0$ such a weaker version seems to be insufficient.

9.2. The Thoma simplex Ω and the embeddings $\mathbb{Y}_n \hookrightarrow \Omega$. We return to our concrete situation. As L_n we take the finite-dimensional vector space $\text{Fun}(\mathbb{Y}_n)$ with the supremum norm, and as T_n we take the Markov transition operator introduced in Definition 8.1. Clearly, T_n is a contraction. To define the Banach space L and the operators $\pi_n : L \rightarrow L_n$ we need a preparation.

Let $[0, 1]^\infty$ (the infinite-dimensional cube) be the direct product of countably many copies of the closed unit interval $[0, 1]$. We equip $[0, 1]^\infty$ with the product topology; then we get a compact topological space.

Recall (see Subsection 1.1) that the Thoma simplex Ω is the subset of couples $(\alpha; \beta) \in [0, 1]^\infty \times [0, 1]^\infty$ satisfying the conditions

$$\alpha_1 \geq \alpha_2 \geq \dots \geq 0, \quad \beta_1 \geq \beta_2 \geq \dots \geq 0, \quad \sum_i \alpha_i + \sum_j \beta_j \leq 1.$$

Clearly, Ω is closed in the topology of $[0, 1]^\infty \times [0, 1]^\infty$ and hence is a compact topological space.

We set $L = C(\Omega)$, the Banach space of real-valued continuous functions on Ω with the supremum norm.

For any n , we define an embedding $\iota_{\theta, n} : \mathbb{Y}_n \rightarrow \Omega$ in the following way. Given $\lambda \in \mathbb{Y}_n$, we divide the shape of λ in the quarter (r, s) plane into two parts, \mathfrak{A} and \mathfrak{B} :

$$\mathfrak{A} = \{(r, s) \in \text{shape}(\lambda) \mid s \geq \theta r\}, \quad \mathfrak{B} = \{(r, s) \in \text{shape}(\lambda) \mid s \leq \theta r\}.$$

Let a_i denote the area of the intersection of \mathfrak{A} with the i th row of boxes, $i = 1, 2, \dots$. Likewise, let b_j be the area of the intersection of \mathfrak{B} with the j th column of boxes, $j = 1, 2, \dots$. The sequences a_1, a_2, \dots and b_1, b_2, \dots are nonincreasing, have finitely many nonzero terms, and $\sum a_i + \sum b_j = n$. Now, we set

$$\iota_{\theta, n}(\lambda) = (\alpha; \beta) := \left(\frac{a_1}{n}, \frac{a_2}{n}, \dots; \frac{b_1}{n}, \frac{b_2}{n}, \dots \right) \in \Omega. \quad (9.3)$$

It is easy to check that $\iota_{\theta,n}$ is indeed an embedding. Using it we define the operator $\pi_n : L \rightarrow L_n$, that is, $\pi_n : C(\Omega) \rightarrow \text{Fun}(\mathbb{Y}_n)$, by setting

$$(\pi_n f)(\lambda) = f(\iota_{\theta,n}(\lambda)), \quad f \in C(\Omega), \quad \lambda \in \mathbb{Y}_n.$$

Clearly, $\|\pi_n\| \leq 1$. Moreover, the operators π_n possess the following property: For any $f \in L = C(\Omega)$,

$$\|f\| = \lim_{n \rightarrow \infty} \|\pi_n f\|. \quad (9.4)$$

Indeed, this follows from the obvious fact that any open subset in Ω has a nonempty intersection with $\iota_{\theta,n}(\mathbb{Y}_n)$ for all n large enough.

9.3. Thoma measures and moment coordinates. The content of this subsection is parallel to that of [BO8, Subsection 4.4] where we considered the particular case $\theta = 1$.

Recall that to any point $(\alpha; \beta) \in \Omega$ we have assigned a probability measure $\nu_{\alpha;\beta}$ on the closed interval $[-\theta, 1]$, see (1.4). The measure $\nu_{\alpha;\beta}$ is called the *Thoma measure* corresponding to $(\alpha; \beta)$. Recall also that the moments $q_k = q_k(\alpha; \beta)$ of $\nu_{\alpha;\beta}$ are given by formula (1.3) and note that the 0th moment is always equal to 1. As was already said in the Introduction, we call q_1, q_2, \dots the *moment coordinates* of the point $(\alpha; \beta) \in \Omega$. Observe that they are continuous functions on Ω . Indeed, since α_i 's decrease, the condition $\sum \alpha_i \leq 1$ implies $\alpha_i \leq i^{-1}$ for any $i = 1, 2, \dots$, whence $\alpha_i^{k+1} \leq i^{-k-1}$. Similarly, $\beta_i^{k+1} \leq i^{-k-1}$. It follows that the both series in (1.3) are uniformly convergent on Ω , which implies their continuity as functions on Ω .¹⁸

Let $\mathfrak{M}([-\theta, 1])$ denote the space of probability Borel measures on $[-\theta, 1]$ equipped with the weak topology. Since this topology is determined by convergence of moments, the assignment $(\alpha; \beta) \mapsto \nu_{\alpha;\beta}$ determines a homeomorphism of the Thoma simplex on a compact subset of $\mathfrak{M}([-\theta, 1])$.

Note also that the moment coordinates are algebraically independent as functions on Ω . Indeed, this holds even we restrict them on the subset with all β_i 's equal to 0. It follows that the algebra of polynomials $\mathbb{R}[q_1, q_2, \dots]$ can be viewed as a subalgebra of the Banach algebra $C(\Omega)$. Since this subalgebra contains 1 and separates points, it is dense in $C(\Omega)$.

Using the correspondence

$$p_k^\circ \longleftrightarrow q_{k-1}, \quad k = 2, 3, \dots,$$

we may identify the algebras Λ° and $\mathbb{R}[q_1, q_2, \dots]$, which makes it possible to realize Λ° as a dense subalgebra of $C(\Omega)$. In what follow we will often identify elements $f \in \Lambda^\circ$ and the corresponding continuous functions $f(\alpha; \beta)$ on Ω .

¹⁸This argument substantially relies on the fact that $k+1 \geq 2$. Note that the function $(\alpha; \beta) \mapsto \sum \alpha_i + \sum \beta_i$ is not continuous on Ω .

This also enables us to assign to any element $\varphi \in \Lambda$ a continuous function on Ω ; according to our convention, this is simply $\varphi^\circ(\alpha; \beta)$. In equivalent terms, the morphism $\varphi \mapsto \varphi^\circ(\cdot)$ is determined by setting

$$p_1^\circ(\alpha; \beta) \equiv 1, \quad p_k^\circ(\alpha; \beta) = \sum_{i=1}^{\infty} \alpha_i^k + (-\theta)^{k-1} \sum_{i=1}^{\infty} \beta_i^k, \quad k = 2, 3, \dots,$$

which is precisely the definition given in [KOO].

We take $\Lambda^\circ = \mathbb{R}[p_2^\circ, p_3^\circ, \dots] = \mathbb{R}[q_1, q_2, \dots]$ as the dense subspace $\mathcal{F} \subset C(\Omega)$ which has been mentioned in Theorem 1.1 and in Theorem 9.3.

9.4. Boundary z -measures on Ω . Fix a couple (z, z') from the principal or complementary series and consider the system $\{M_{\theta, z, z'}^{(n)}\}$ of z -measures that we have defined in Section 5. Recall that in Subsection 9.2 we have defined the embeddings $\iota_{\theta, n} : \mathbb{Y}_n \hookrightarrow \Omega$ (formula (9.3)).

Theorem 9.4. *As $n \rightarrow \infty$, the pushforward of the z -measure $M_{\theta, z, z'}^{(n)}$ under $\iota_{\theta, n}$ weakly converges to a probability measure $M_{\theta, z, z'}$ on Ω .*

Proof. See [KOO, Section 8, proof of Theorem B]. \square

We call the limit measure $M_{\theta, z, z'}$ on Ω the *boundary z -measure*. This agrees with the definitions given in Section 2 after the statement of Theorem 2.2. Indeed, as shown in [KOO], the topological space $\Omega(p^\perp)$ corresponding to the system $p^\perp = p_\theta^\perp$ can be identified with the Thoma simplex Ω and $M_{\theta, z, z'}$ is just the measure M that appears in (2.3) when $\{M^{(n)}\} = M_{\theta, z, z'}^{(n)}$.

Note that the kernel $\mathcal{K}(\lambda, \omega)$ that appears in formula (2.3) has the following form:

$$\mathcal{K}(\lambda, \omega) = \dim_\theta \lambda \cdot (\mathcal{P}_\lambda^{(\theta)})^\circ(\alpha; \beta), \quad \lambda \in \mathbb{Y}, \quad \omega = (\alpha; \beta) \in \Omega,$$

where $\dim_\theta \lambda$ is a certain “ θ -version” of the conventional dimension function $\dim \lambda$ (see [KOO, §6]) and $\mathcal{P}_\lambda^{(\theta)} \in \Lambda$ is the Jack symmetric function with parameter θ and index λ .

9.5. Asymptotics of θ -regular functions. Introduce a notation: if $F \in \mathbb{A}_\theta$ is an element of degree $\leq m$, that is, $F \in \mathbb{A}_\theta^{(m)}$ (see Definition 6.2), then $[F]_m$ will denote its highest homogeneous term, which is an element of the quotient space $\mathbb{A}_\theta^{(m)} / \mathbb{A}_\theta^{(m-1)}$ identified with the m th homogeneous component of the graded algebra Λ .

Theorem 9.5. *Let $F \in \mathbb{A}_\theta$ be an element of degree $\leq m$ and $\varphi = [F]_m \in \Lambda$, as defined above. There exists a constant $C > 0$ depending only on F , such that for any $n = 1, 2, \dots$ and any $\lambda \in \mathbb{Y}_n$ the following estimate holds*

$$\left| \frac{F(\lambda)}{n^m} - \varphi^\circ(\iota_{\theta, n}(\lambda)) \right| \leq \frac{C}{\sqrt{n}}.$$

Proof. This result was proved in [KOO, Theorem 8.1], only the definition of embeddings $\mathbb{Y}_n \hookrightarrow \Omega$ employed in [KOO] slightly differs from that given in Subsection 9.2 above. However, the difference between the two definitions is unessential. This is seen from the proof given in [KOO] and the following observation: If $\lambda \in \mathbb{Y}_n$, $(\alpha; \beta) = \iota_{\theta, n}(\lambda)$, and $(\alpha'; \beta')$ is the image of λ according to the definition of [KOO], then

$$|\alpha_i - \alpha'_i| \leq \frac{1}{n}, \quad |\beta_i - \beta'_i| \leq \frac{1}{n}, \quad i = 1, 2, \dots,$$

and the number of nonzero coordinates is of order \sqrt{n} . Alternatively, the reader may simply take the definition of [KOO]. The reason to modify that definition is purely aesthetic: it is slightly asymmetric with respect to transposition of rows and columns of a diagram. \square

9.6. The main results. Now we are in a position to state and prove the main results of the present paper. For the reader's convenience let us recall the basic data and definitions:

- We fix the three basic parameters θ, z, z' , where $\theta > 0$ and the couple (z, z') belongs to the principal or complementary series (Proposition 5.3 and Definition 5.4).
- Ω is the Thoma simplex; it is a compact topological space (Subsection 9.2).
- $C(\Omega)$ is the Banach algebra of continuous real-valued functions on Ω with supremum norm.
- Λ° is the quotient of the algebra Λ of symmetric functions modulo the principal ideal $(p_1 - 1)\Lambda$ (Definition 8.6); Λ° is embedded into $C(\Omega)$ as a dense subalgebra (Subsection 9.3).
- $A : \Lambda^\circ \rightarrow \Lambda^\circ$ is the operator defined in Corollary 8.7, formula (8.7); we regard A as a densely defined operator in the Banach space $C(\Omega)$ (note that under the identification $p_{k+1}^\circ = q_k$, (8.7) coincides with (1.5)).
- $T_n : \text{Fun}(\mathbb{Y}_n) \rightarrow \text{Fun}(\mathbb{Y}_n)$, $n = 1, 2, \dots$, are the Markov chain transition operators introduced in Definition 8.1.

Theorem 9.6. (i) *The operator A is closable and its closure \bar{A} generates a strongly continuous contraction semigroup $\{T(t)\}_{t \geq 0}$ in $C(\Omega)$.*

(ii) *As $n \rightarrow \infty$, the discrete semigroups in the finite-dimensional spaces $\text{Fun}(\mathbb{Y}_n)$ generated by the contractions T_n approximate the semigroup $\{T(t)\}_{t \geq 0}$ in the sense of Definition 9.2 with the following choice of the time scaling factors: $\varepsilon_n = n^{-2}$.*

(iii) *The semigroup $\{T(t)\}_{t \geq 0}$ is a conservative Markov semigroup.*

Recall that the last property means that each operator $T(t)$ preserves the constant function 1 and maps into itself the cone of nonnegative functions in $C(\Omega)$.

Proof. Step 1. As in (9.1), set $A_n = \varepsilon_n^{-1}(T_n - 1)$. Let us prove that the operators A_n approximate the operator A in the “extended” sense, as explained in Theorem 9.3.

First of all, it is more convenient to re-define the factors ε_n according to (8.1). Since the new factors are asymptotically equivalent to n^{-2} , this does not affect the result.

Given $f \in \Lambda^\circ$, fix a natural number m so large that $\deg f \leq m$. By the very definition of Λ° and the filtration therein, there exists a homogeneous element $\varphi \in \Lambda$ of degree m such that $\varphi^\circ = f$. Next, choose an arbitrary element $F \in \mathbb{A}_\theta^{(m)}$ such that $[F]_m = \varphi$, see Subsection 9.4 for the notation. Finally, set

$$f_n = \frac{F_n}{n^m}, \quad n = 1, 2, \dots$$

Here, in accordance with the notation of Subsection 7.1, F_n stands for the restriction of F to the subset $\mathbb{Y}_n \subset \mathbb{Y}$, so that $F_n \in \text{Fun}(\mathbb{Y}_n)$.

By virtue of Theorem 9.5, $\|f_n - \pi_n f\| \leq C/\sqrt{n}$, so that $f_n \rightarrow f$ in the sense of Definition 9.1.

Further, set $G = \tilde{B}F$, where the operator $\tilde{B} : \mathbb{A}_\theta \rightarrow \mathbb{A}_\theta$ has been introduced in Theorem 8.2, and also set

$$g_n = \frac{G_n}{n^m}, \quad n = 1, 2, \dots$$

By virtue of Theorem 8.2, $G \in \mathbb{A}_\theta^{(m)}$ and $[G]_m = B[F]_m = B\varphi$, where the operator $B : \Lambda \rightarrow \Lambda$ has been introduced in Definition 8.3. Applying again Theorem 9.5 we get $g_n \rightarrow g$, where $g := (B\varphi)^\circ$.

On the other hand, Corollary 8.7 says that $(B\varphi)^\circ = A\varphi^\circ = Af$. Therefore, $A_n f \rightarrow Af$. Thus, we have proved the required “extended” convergence $A_n \rightarrow A$.

Step 2. Let us prove that A is a dissipative operator, that is, for any $s > 0$ and any $f \in \Lambda^\circ \subset C(\Omega)$ we have the inequality $\|(s - A)f\| \geq s\|f\|$.

Indeed, according to the result of step 1, there exist $f_n \in \text{Fun}(\mathbb{Y}_n)$ such that $f_n \rightarrow f$ and $A_n f_n \rightarrow Af$. Since the operators T_n are contractions, the operators A_n are dissipative. Consequently, $\|(s - A_n)f_n\| \geq s\|f_n\|$.

On the other hand, because of (9.4), in our situation, convergence of vectors (in the sense of Definition 9.1) implies convergence of their norms. Therefore, we may pass to the limit in the above inequality for f_n and get the desired inequality for f .

Step 3. As is seen from (8.7), the operator A does not raise degree in the sense of the canonical filtration of the algebra Λ° (this is also obvious because B preserves the graduation in Λ). Let $\Lambda^{\circ(m)} \subset \Lambda^\circ$ stand for the subspace of elements of degree $\leq m$. Each such subspace has finite dimension and is invariant under A . Because A is dissipative (step 2), the operator $s - A$ maps

$\Lambda^{\circ(m)}$ onto itself for any $s > 0$ and any m . Since the subspaces $\Lambda^{\circ(m)}$ form an ascending chain and their union is the whole space Λ° , we conclude that $s - A$ maps Λ° onto itself. The combination of this property and the dissipativity property entails that A is closable and its closure \bar{A} serves as the generator of a strongly continuous contraction semigroup $\{T(t)\}_{t \geq 0}$ in $C(\Omega)$: this is a version of the Hille–Yosida theorem, see, e. g., Theorem 2.12 in [EK2]. Thus, we have checked claim (i) of the theorem.

Step 4. Now claim (ii) immediately follows from Theorem 9.3. Indeed, we have just established the existence of the semigroup $\{T(t)\}$, and the validity of the hypothesis of Theorem 9.3 has been verified on step 1.

Step 5. Let us check claim (iii). Since the constant term of the differential operator A vanishes, we have $A1 = 0$, which implies $T(t)1 = 1$ for all $t \geq 0$. Therefore, the semigroup is conservative.

Let us to prove that if $f \in C(\Omega)$ is nonnegative then so is $T(t)f$.

Obviously, $\pi_n f$ is a nonnegative function for any n . Since T_n is the transition operator of a Markov chain, $T_n^m \pi_n f$ is a nonnegative function on \mathbb{Y}_n for any natural number m . In particular, $T^{[t\varepsilon_n^{-1}]} \pi_n f \geq 0$.

On the other hand, because claim (ii) has already been established, we know that $\delta_n := \|T_n^{[t\varepsilon_n^{-1}]} \pi_n f - \pi_n T(t)f\| \rightarrow 0$ as $n \rightarrow \infty$ (see Definition 9.2). Therefore, $\pi_n T(t)f \geq -\delta_n$ on \mathbb{Y}_n . In other words, $T(t)f \geq -\delta_n$ on the subset $\iota_{\theta,n}(\mathbb{Y}_n) \subset \Omega$. As pointed out in the very end of Subsection 9.2, this finite subset becomes more and more dense in Ω as $n \rightarrow \infty$. Since $\delta_n \rightarrow 0$ and the function $T(t)f$ is continuous, it is nonnegative on the whole Ω .

This concludes the proof. \square

By a well-known general result (see [EK2, Chapter 4, Theorem 2.7]), the Markov semigroup $\{T(t)\}$ constructed in Theorem 9.6 gives rise to a strong Markov process in Ω with càdlàg sample paths. Let us denote this process by $\omega_{\theta,z,z'}(t)$. Actually, due to the knowledge of the explicit form of the pre-generator A (formula (8.7)) one can get a stronger result:

Theorem 9.7. *The Markov process $\omega_{\theta,z,z'}(t)$ has continuous sample paths.*

Proof. The argument is exactly the same as in the case $\theta = 1$, see [BO8, Theorem 8.1]. One shows that any smooth cylinder function in the moment coordinates $q_1 = p_2^\circ, q_2 = p_3^\circ, \dots$ enters the domain of the generator \bar{A} ([BO8, Corollary 7.4]). Using this, one can verify the Dynkin–Kinney condition. \square

The next results are related to the boundary z -measure $M_{\theta,z,z'}$ defined in Subsection 9.4. Below the angular brackets $\langle \cdot, \cdot \rangle$ denote the pairing between functions and measures. Consider the inner product in $C(\Omega)$ determined by

$$(f, g) = \langle fg, M_{\theta,z,z'} \rangle. \quad (9.5)$$

Lemma 9.8. (i) *If $f \in C(\Omega)$, $f_n \in \text{Fun}(\mathbb{Y}_n)$, and $f_n \rightarrow f$ in the sense of Definition 9.1, then*

$$\langle f_n, M_{\theta, z, z'}^{(n)} \rangle \rightarrow \langle f, M_{\theta, z, z'} \rangle.$$

(ii) *If $f_n \rightarrow f$ and $g_n \rightarrow g$, where $f, g \in C(\Omega)$ and $f_n, g_n \in \text{Fun}(\mathbb{Y}_n)$, then $f_n g_n \rightarrow fg$.*

Proof. (i) Set $\widetilde{M}_{\theta, z, z'}^{(n)} = \iota_{\theta, n}(M_{\theta, z, z'}^{(n)})$; this is a probability measure on Ω . By Theorem 9.4, the measures $\widetilde{M}_{\theta, z, z'}^{(n)}$ weakly converge to the measure $M_{\theta, z, z'}$ as $n \rightarrow 0$. Therefore,

$$\langle f, \widetilde{M}_{\theta, z, z'}^{(n)} \rangle \rightarrow \langle f, M_{\theta, z, z'} \rangle.$$

On the other hand,

$$\langle f, \widetilde{M}_{\theta, z, z'}^{(n)} \rangle = \langle \pi_n f, M_{\theta, z, z'}^{(n)} \rangle.$$

Further, the assumption $f_n \rightarrow f$ just means $\|f_n - \pi_n f\| \rightarrow 0$, so that

$$\langle f_n, M_{\theta, z, z'}^{(n)} \rangle - \langle \pi_n f, M_{\theta, z, z'}^{(n)} \rangle \rightarrow 0.$$

This proves the claim.

(ii) We know that $\|f_n - \pi_n f\| \rightarrow 0$ and $\|g_n - \pi_n g\| \rightarrow 0$, and we have to check that

$$\|f_n g_n - \pi_n(fg)\| \rightarrow 0.$$

Since $\pi_n(fg) = (\pi_n f)(\pi_n g)$ and all the functions under consideration are uniformly bounded, this is obvious. \square

Theorem 9.9. (i) *The pre-generator $A : \Lambda^\circ \rightarrow \Lambda^\circ$ is symmetric with respect to the inner product (9.5) in the space Λ° and can be diagonalized in an appropriate orthogonal basis.*

(ii) *The spectrum of A is $\{0\} \cup \{-\sigma_m : m = 2, 3, \dots\}$ where*

$$\sigma_m = m(m-1 + \theta^{-1} z z'), \quad m = 2, 3, \dots,$$

*the eigenvalue 0 is simple, and the multiplicity of $-\sigma_m$ equals the number of partitions of m without parts equal to 1.*¹⁹

Note that this result also describes the spectrum of the closure of A in the Hilbert space $L^2(\Omega, M_{\theta, z, z'})$.

Proof. (i) We have to prove that for any $f, g \in \Lambda^\circ$

$$\langle (Af)g - f(Ag), M_{\theta, z, z'} \rangle = 0. \quad (9.6)$$

As shown on step 1 of the proof of Theorem 9.6, there exist sequences $\{f_n \in \text{Fun}(\mathbb{Y}_n)\}$ and $\{g_n \in \text{Fun}(\mathbb{Y}_n)\}$ such that

$$f_n \rightarrow f, \quad A_n f_n \rightarrow Af, \quad g_n \rightarrow g, \quad A_n g_n \rightarrow Ag \quad (9.7)$$

¹⁹Denoting by $\mathfrak{p}(m)$ the number of *all* partitions of m , the multiplicity in question can be written as $\mathfrak{p}(m) - \mathfrak{p}(m-1)$.

in the sense of Definition 9.1.

On the other hand, observe that the transition operator $T_n : \text{Fun}(\mathbb{Y}_n) \rightarrow \text{Fun}(\mathbb{Y}_n)$ is symmetric with respect to the inner product in $\text{Fun}(\mathbb{Y}_n)$ given by the formula similar to (9.5) but with $M_{\theta,z,z'}^{(n)}$ instead of $M_{\theta,z,z'}$. Indeed, this follows from the fact that $M_{\theta,z,z'}^{(n)}$ is the symmetrizing measure (Proposition 2.5). Therefore, A_n is also symmetric and hence

$$\langle (A_n f_n)g_n - f_n(A_n g_n), M_{\theta,z,z'}^{(n)} \rangle = 0. \quad (9.8)$$

Now we apply Lemma 9.8. By virtue of its claim (ii), (9.7) implies

$$(A_n f_n)g_n - f_n(A_n g_n) \rightarrow (A f)g - f(A g),$$

and then claim (i) makes it possible to pass to the limit in (9.8), which gives (9.6).

The existence of an orthogonal eigenbasis for A follows from the fact that A preserves each of the finite-dimensional subspaces $\Lambda^{\circ(m)}$ forming the filtration of Λ° .

(ii) Set

$$E = \sum_{k \geq 2} k p_k^{\circ} \frac{\partial}{\partial p_k^{\circ}}.$$

As seen from (8.7), A can be represented as the sum of the operator

$$- E(E - 1 + \theta^{-1} z z') \quad (9.9)$$

and a rest term which has degree ≤ -1 . The operator (9.9) is diagonalized in the (non-orthogonal) basis formed by 1 and the monomials in the generators $p_2^{\circ}, p_3^{\circ}, \dots$, and has precisely the spectrum indicated in the statement of the proposition. The rest term, obviously, does not affect the spectrum. \square

Theorem 9.10. (i) *The Markov process $\omega_{\theta,z,z'}(t)$ has the boundary measure $M_{\theta,z,z'}$ as a unique stationary distribution.*

(ii) *It is also a symmetrizing measure.*

(iii) *The process is ergodic in the sense that for any $f \in C(\Omega)$,*

$$\lim_{t \rightarrow +\infty} \|T(t)f - \langle f, M_{\theta,z,z'} \rangle 1\| = 0,$$

where 1 is the constant function equal to one.

Proof. This result relies on Theorem 9.9: the argument is exactly the same as in the proof of Theorem 8.3 from [BO8]. \square

Remark 9.11. Recall the notation $\mathcal{F} = \Lambda^{\circ}$ for the domain of the pre-generator A of the Markov process $\omega_{\theta,z,z'}(t)$. The pre-Dirichlet form on $\mathcal{F} \times \mathcal{F}$ corresponding to the pre-generator can be written in the following way

$$- \int_{\Omega} (A F)(\alpha; \beta) G(\alpha; \beta) M_{\theta,z,z'}(d\alpha d\beta) = \int_{\Omega} \Gamma(F, G)(\alpha; \beta) M_{\theta,z,z'}(d\alpha d\beta),$$

where the $F, G \in \mathcal{F}$ and $\Gamma(\cdot, \cdot)$ (the ‘‘square field operator’’) is a symmetric bilinear map $\mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ which *does not depend* on the parameters θ, z, z' :

$$\Gamma(F, G) = \sum_{i,j=1}^{\infty} (i+1)(j+1)(q_{i+j} - q_i q_j) \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial q_j} = \sum_{k,l=2}^{\infty} kl(p_{k+l-1}^{\circ} - p_k^{\circ} p_l^{\circ}) \frac{\partial F}{\partial p_k^{\circ}} \frac{\partial G}{\partial p_l^{\circ}}.$$

The proof is exactly the same as in [BO8, Theorem 8.4]; it relies on the fact that the coefficients of the second derivatives in (1.5) or (8.7) do not depend on the parameters.

Remark 9.12. Here is a complement to the remark made in Subsection 1.2 about the possibility to degenerate the pre-generator A given by formula (1.5) (or, equivalently, by (8.7)) to the Ethier–Kurtz–Schmuland operator (1.7) in the limit regime (1.6). Recently Petrov [Pe1] found a two-parameter generalization of the Ethier–Kurtz diffusion associated to Pitman’s two-parameter generalization $P(\alpha, \tau)$ of the Poisson–Dirichlet distribution (here $\alpha \in [0, 1]$ is the additional parameter and τ should be strictly greater than $-\alpha$).²⁰ The corresponding two-parameter extension of the operator (1.7) has the form

$$\sum_{i,j \geq 1} (i+1)(j+1)(q_{i+j} - q_i q_j) \frac{\partial^2}{\partial q_i \partial q_j} + \sum_{i \geq 1} (i+1)[(i-\alpha)q_{i-1} - (i+\tau)q_i] \frac{\partial}{\partial q_i}, \quad (9.10)$$

see [Pe1, formula (16)]. Now, observe that this more general operator can also be obtained by degeneration from (1.5): to achieve this we have to impose the following conditions on the asymptotics of our triple (θ, z, z') :

$$\theta \rightarrow 0, \quad zz' \rightarrow 0, \quad \theta^{-1}zz' \rightarrow \tau, \quad z + z' \rightarrow -\alpha. \quad (9.11)$$

Moreover, one can show that in this limit regime, the down transition probabilities $p_{\theta}^{\downarrow}(\lambda, \mu)$, the up transition probabilities $p_{\theta, z, z'}^{\uparrow}(\lambda, \nu)$, and the weights $M_{\theta, z, z'}^{(n)}(\lambda)$ converge to the respective quantities considered in [Pe1]. However, for $\alpha \neq 0$, the limit regime (9.11) is *incompatible* with the restrictions on (z, z') that ensure positivity of the up transition probabilities and the z -measures (see [BO5, Proposition 2.3]). That is, if we wish to perform the limit (9.11) with a nonzero α , then we inevitably have to admit those (z, z') ’s for which the pre-limit quantities $p_{\theta, z, z'}^{\uparrow}(\lambda, \nu)$ and $M_{\theta, z, z'}^{(n)}(\lambda)$ can take negative or even complex values, which makes the limit transition purely formal.

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²⁰I recall that I am writing $PD(\alpha, \tau)$ instead of the conventional $PD(\alpha, \theta)$ to avoid a conflict of notation.

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