Strong uniqueness for both Dirichlet operators and stochastic dynamics to Gibbs measures on a path space with exponential interactions

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Abstract: We prove L^p-uniqueness of Dirichlet operators for Gibbs measures on the path space $C(\mathbb{R}, \mathbb{R}^d)$ associated with exponential type interactions in infinite volume by extending an SPDE approach presented in previous work by the last two named authors. We also give an SPDE characterization of the corresponding dynamics. In particular, we prove existence and uniqueness of a strong solution for the SPDE, though the self-interaction potential is not assumed to be differentiable, hence the drift is possibly discontinuous. As examples, to which our results apply, we mention the stochastic quantization of $P(\phi)_1$-, $\exp(\phi)_1$-, and trigonometric quantum fields in infinite volume. In particular, our results imply essential self-adjointness of the generator of the stochastic dynamics for these models. Finally, as an application of the strong uniqueness result for the SPDE, we prove

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some functional inequalities for diffusion semigroups generated by the above Dirichlet operators. These inequalities are improvements of previous work by the second named author.

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1 Introduction

In recent years, there has been a growing interest in the study of infinite dimensional stochastic dynamics associated with models of Euclidean quantum field theory, hydrodynamics, and statistical mechanics, see, e.g., Liskevich–Röckner [35], Da Prato–Tubaro [19] and Albeverio–Liang–Zegarliński [8], resp. Albeverio–Flandoli–Sinai [2], resp. Albeverio–Kondratiev–Kozitsky–Röckner [7]. Equilibrium states of such dynamics are described by Gibbs measures. The stochastic dynamics corresponding to these states is given by a diffusion semigroup, see, e.g., Albeverio [1]. On some minimal domain of smooth functions, the infinitesimal generator of the semigroup coincides with the Dirichlet operator defined through a classical Dirichlet form of gradient type with a Gibbs measure. From an analytic point of view, it is very important to study L^p-uniqueness of the Dirichlet operator, that is, the question whether or not the Dirichlet operator restricted to the minimal domain has a unique closed extension in the L^p-space of the Gibbs measure under consideration, which generates a C_0-semigroup. As is well known, in the case of $p = 2$, this uniqueness is equivalent to essential self-adjointness. We recall that essential self-adjointness is crucial in applications to quantum mechanics to be sure that solutions of Schrödinger equations are unique. This kind of uniqueness problem on infinite dimensional state spaces has been studied intensively by many authors. In particular, we refer to the recent work [33] by the last two named authors, where essential self-adjointness was proved in the case of $P(\phi)_1$-quantum fields in infinite volume by using an SPDE approach based on Da Prato–Röckner [17]. Besides, in [33] also the relationship between the corresponding dynamics and the $P(\phi)_1$-time evolution, which had been constructed as the strong solution of a parabolic SPDE (2.10) in Iwata [28], is discussed.

The first objective of the present paper is to prove L^p-uniqueness of the Dirichlet operator for all $p \geq 1$, under much weaker conditions on the growth rate of the potential function of the Gibbs measure by a modification of the SPDE approach presented in [33]. Important new examples are $\exp(\phi)_1$-quantum fields in infinite volume in the context of Euclidean quantum field theory. These models were introduced (for the case where $\mathbb{R}$ occurring in (1.1) below is replaced by a 2-dimensional Euclidean space-time $\mathbb{R}^2$ and where $d = 1$) in Høegh-Krohn [26], Albeverio–Høegh-Krohn [6] and further studied e.g., in Simon [47], Fröhlich [22], Albeverio–Gallavotti–Høegh-Krohn [3] and Kusuoka [34].

More precisely, we are concerned with Gibbs measures on an infinite volume path space $C(\mathbb{R},\mathbb{R}^d)$ given by the following formal expression:

$$Z^{-1}\exp\Big\{-\frac{1}{2}\int_{\mathbb{R}}\big((-\Delta_x+m^2)w(x),w(x)\big)_{\mathbb{R}^d}dx \\ -\int_{\mathbb{R}}dx\Big(\int_{\mathbb{R}^d}e^{(w(x),\xi)_{\mathbb{R}^d}}\nu(d\xi)\Big)\Big\}\prod_{x\in\mathbb{R}}dw(x). \tag{1.1}$$

Here Z is a normalizing constant, $m>0$ denotes mass, $\Delta_x := d^2/dx^2$, ν is a bounded positive measure on $\mathbb{R}^d$ with compact support, and $\prod_{x\in\mathbb{R}}dw(x)$ stands for a (heuristic) volume measure on the space of maps from $\mathbb{R}$ into $\mathbb{R}^d$. This has the interpretation of a quantized d-dimensional vector field with an interaction of exponential type in the 1-dimensional space-time $\mathbb{R}$, a model which is known as stochastic quantization of the $\exp(\phi)_1$-quantum field model (with weight measure ν). We should mention that essential self-adjointness of the Dirichlet operators for such $\exp(\phi)_1$-quantum fields was not known yet, although the corresponding stochastic dynamics was constructed by using the Dirichlet form theory in Albeverio–Röckner [12] (see also Hida–Kuo–Potthoff–Streit [25] for an approach based on white noise calculus). Another important new example we handle is the model of trigonometric interactions, defined analogously to (1.1), but with $e^{(w(x),\xi)_{\mathbb{R}^d}}$ replaced by $\cos\{(w(x),\xi)_{\mathbb{R}^d}+\alpha\}, \alpha\in\mathbb{R}$. Such a model was studied (with $\mathbb{R}$ replaced by a 2-dimensional space-time $\mathbb{R}^2$ and assuming $d=1$) e.g., in Albeverio–Høegh-Krohn [5], Fröhlich [22] and Albeverio–Haba–Russo [4]. In the present paper, we, in particular, prove essential self-adjointness of the corresponding Dirichlet operator for all these models. As a consequence, the Dirichlet operator associated with the superposition of polynomial, exponential and trigonometric interactions, is also essentially self-adjoint.

The second objective of the present paper is to discuss a characterization of the stochastic dynamics corresponding to the above Dirichlet operator. Due to general theory, the stochastic dynamics constructed through the Dirichlet form approach solves the parabolic SPDE (2.10) weakly. However, we prove something much better, namely existence and uniqueness of a strong solution. We achieve this by first proving pathwise uniqueness for SPDE (2.10) and then applying the recent work of Ondreját [38] on the Yamada–Watanabe theorem for mild solutions of SPDE. As a consequence, we have the existence of a unique strong solution to SPDE (2.10) by using simple and straightforward arguments which do not rely on any finite volume approximations discussed in [28] in case of polynomial (i.e., smooth) self-interaction.

Here we would like to emphasize that neither of the two uniqueness statements in Theorems 2.7 and 2.8 respectively implies the other (cf. Remark 2.9 below).

The organization of this paper is as follows: In Section 2, we present the framework and state our results. There, we construct Gibbs measures as (1.1) rigorously by using d-dimensional Brownian motion and the ground states of Schrödinger operators on $L^2(\mathbb{R}^d,\mathbb{R})$. After introducing our Dirichlet form and the corresponding Dirichlet operator, we state our main results (Theorems 2.7 and 2.8). Section 3 contains the proofs, in which, we prove our main theorems. In our proof, we regard the Dirichlet operator as a perturbation of the infinite dimensional Ornstein–Uhlenbeck operator by a possibly discontinuous and unbounded drift term. Then we implement a modification of a technique developed in [33] which in turn is based on beautiful results of Da Prato, Tubaro and Priola in [16, 18, 39] for Lipschitz perturbations of the Ornstein–Uhlenbeck operators. (For other works on perturbed infinite dimensional Ornstein–Uhlenbeck operators, see also, e.g., Albeverio–Röckle–Steblovskaya [9] and references therein.) To handle our quite singular drift term, the first thing to do is to check its L^p-integrability. For this purpose, we make use of the asymptotic behavior for the ground state of the Schrödinger operator at infinity which, through the Feynman–Kac formula, has a close connection with the growth rate of the potential function. We introduce an approximation scheme for the potential function by combining the Moreau–Yosida approximation (3.14) with a further regularization (3.18) inspired by [17, 33], and this scheme works efficiently in our proof. To show existence and uniqueness of a strong solution to SPDE (2.10), we firstly identify our diffusion process as a weak solution to an infinite system of SDEs. Secondly, we translate the infinite dimensional SDE into the weak form of SPDE (2.10), and show pathwise uniqueness for it based on Marinelli–Röckner [37]. In Section 4, we discuss some functional inequalities including the logarithmic Sobolev inequality (4.3) as an application of Theorem 2.8, and in Section 5, we give another proof of the logarithmic Sobolev inequality (4.3) by using Lemmas 5.1 and 5.2 on the approximation of the ground state. These lemmas play key roles when we combine some tightness arguments with the previous work to derive inequality (4.3).

2 Framework and Results

We begin by introducing some notation and objects we will be working with. We define a weight function $\rho_r \in C^\infty(\mathbb{R}, \mathbb{R}), r \in \mathbb{R}$, by $\rho_r(x) := e^{r\chi(x)}$, $x \in \mathbb{R}$, where $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ is a positive symmetric convex function satisfying $\chi(x) = |x|$ for $|x| \geq 1$. We fix a positive constant r sufficiently small. In particular, we take $r > 0$ such that $2r^2 < K_1$ if $K_1 > 0$, where the constant K_1 appears in condition **(U1)** below. We set $E := L^2(\mathbb{R}, \mathbb{R}^d; \rho_{-2r}(x)dx)$. This space is a Hilbert space with its inner product defined by

$$(w, \tilde{w})_E := \int_{\mathbb{R}} \big(w(x), \tilde{w}(x)\big)_{\mathbb{R}^d} \rho_{-2r}(x)dx, \quad w, \tilde{w} \in E.$$

Moreover, we set $H := L^2(\mathbb{R}, \mathbb{R}^d)$ and denote by $\|\cdot\|_E$ and $\|\cdot\|_H$ the corresponding norms in E and H, respectively. We regard the dual space E^* of E as $L^2(\mathbb{R}, \mathbb{R}^d; \rho_{2r}(x)dx)$. We endow $C(\mathbb{R}, \mathbb{R}^d)$ with the compact uniform topology and introduce a tempered subspace

$$\mathcal{C} := \{w \in C(\mathbb{R}, \mathbb{R}^d) | \lim_{|x|\to\infty} |w(x)|\rho_{-r}(x) < \infty \text{ for every } r > 0\}.$$

We easily see that the inclusion $\mathcal{C} \subset E \cap C(\mathbb{R}, \mathbb{R}^d)$ is dense with respect to the topology of E. Let $\mathcal{B}$ be the topological σ-field on $C(\mathbb{R}, \mathbb{R}^d)$. For $T_1 < T_2 \in \mathbb{R}$, we define by $\mathcal{B}_{[T_1,T_2]}$ and $\mathcal{B}_{[T_1,T_2],c}$ the sub-σ-fields of $\mathcal{B}$ generated by $\{w(x); T_1 \le x \le T_2\}$ and $\{w(x); x \le T_1, x \ge T_2\}$, respectively. For $T_1, T_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^d$, let $\mathcal{W}^{z_1,z_2}_{[T_1,T_2]}$ be the path space measure of the Brownian bridge such that $w(T_1) = z_1, w(T_2) = z_2$. We sometimes regard this measure as a probability measure on the measurable space $(C(\mathbb{R}, \mathbb{R}^d), \mathcal{B})$ by considering $w(x) = z_1$ for $x \le T_1$ and $w(x) = z_2$ for $x \ge T_2$.

Following Simon [48] and Iwata [27], we now proceed to introduce rigorously the Gibbs measure on $C(\mathbb{R}, \mathbb{R}^d)$. In this paper, we impose the following conditions on the potential function $U \in C(\mathbb{R}^d, \mathbb{R})$:

(U1) There exist a constant $K_1 \in \mathbb{R}$ and a convex function $V : \mathbb{R}^d \to \mathbb{R}$ such that

$$U(z) = \frac{K_1}{2}|z|^2 + V(z), \qquad z \in \mathbb{R}^d.$$

(U2) There exist $K_2 > 0$, $R > 0$ and $\alpha > 0$ such that

$$U(z) \ge K_2|z|^\alpha, \qquad |z| > R.$$

(U3) There exist $K_3, K_4 > 0$ and $0 < \beta < 1 + \frac{\alpha}{2}$ such that

$$|\widetilde{\nabla} U(z)| \le K_3 \exp(K_4|z|^\beta), \qquad z \in \mathbb{R}^d,$$

where $\widetilde{\nabla} U(z) := K_1 z + \partial_0 V(z), z \in \mathbb{R}^d$ and $\partial_0 V$ is the minimal section of the subdifferential ∂V. (The reader is referred to Showalter [46] for the definition of the subdifferential for a convex function and its minimal section. In the case where $U \in C^1(\mathbb{R}^d, \mathbb{R})$, $\widetilde{\nabla} U$ coincides with the usual gradient ∇U.)

Let $H_U := -\frac{1}{2}\Delta_z + U$ be the Schrödinger operator on $L^2(\mathbb{R}^d, \mathbb{R})$, where $\Delta_z := \sum_{i=1}^d \partial^2/\partial z_i^2$ is the d-dimensional Laplacian. Then condition **(U2)** assures that H_U has purely discrete spectrum and a complete set of eigenfunctions (see, e.g., Reed–Simon [40]). We denote by $\lambda_0 (> \min U)$ the minimal eigenvalue and by Ω the corresponding normalized eigenfunction in $L^2(\mathbb{R}^d, \mathbb{R})$. This eigenfunction is called ground state and it can be chosen to be strictly positive. Moreover, it has exponential decay at infinity. To be precise, there exist some positive constants D_1, D_2 such that

$$0 < \Omega(z) \le D_1 \exp\big(- D_2|z| U_{\frac{1}{2}|z|}(z)^{1/2}\big), \quad z \in \mathbb{R}^d, \tag{2.1}$$

where $U_{\frac{1}{2}|z|}(z) := \inf\{U(y) \mid |y - z| \le \frac{1}{2}|z|\}$. See [48, Corollary 25.13] for details.

We are going to define a probability measure μ on $(C(\mathbb{R}, \mathbb{R}^d), \mathcal{B})$. For $T_1 < T_2$, and for all $T_1 \le x_1 < x_2 < \cdots < x_n \le T_2$, $A_1, A_2, \cdots, A_n \in \mathcal{B}(\mathbb{R}^d)$, we define a cylinder set $A \in \mathcal{B}_{[T_1,T_2]}$ by $A := \{w \in C(\mathbb{R}, \mathbb{R}^d) \mid w(x_1) \in A_1, w(x_2) \in A_2, \cdots, w(x_n) \in A_n\}$. Next, we set

$$\begin{aligned}\mu(A) &:= \Big(\Omega, e^{-(x_1-T_1)(H_U-\lambda_0)}\big(\mathbf{1}_{A_1}e^{-(x_2-x_1)(H_U-\lambda_0)}\big(\mathbf{1}_{A_2}\cdots \\ &\qquad e^{-(x_n-x_{n-1})(H_U-\lambda_0)}\big(\mathbf{1}_{A_n}e^{-(T_2-x_n)(H_U-\lambda_0)}\Omega\big)\big)\big)\Big)_{L^2(\mathbb{R}^d,\mathbb{R})} \\ &= e^{\lambda_0(T_2-T_1)}\int_{\mathbb{R}^d}dz_1\Omega(z_1)\int_{\mathbb{R}^d}dz_2\Omega(z_2)p(T_2-T_1,z_1,z_2) \\ &\quad\times\int_{C(\mathbb{R},\mathbb{R}^d)}\mathbf{1}_A(w)\exp\big(-\int_{T_1}^{T_2}U(w(x))dx\big)\mathcal{W}^{z_1,z_2}_{[T_1,T_2]}(dw), \qquad (2.2)\end{aligned}$$

where $p(t,z_1,z_2), t>0, z_1,z_2\in\mathbb{R}^d$, is the transition probability density of standard Brownian motion on $\mathbb{R}^d$, and we used the Feynman–Kac formula for the second line. Then by recalling that $e^{-tH_U}\Omega = e^{-t\lambda_0}\Omega, \|\Omega\|_{L^2(\mathbb{R}^d,\mathbb{R})}=1$ and by the Markov property of the d-dimensional Brownian motion, (2.2) defines a consistent family of probability measures, and hence μ can be extended to a probability measure on $C(\mathbb{R},\mathbb{R}^d)$.

In the same way as [27, Proposition 2.7], we can see that $\mu(\mathcal{C})=1$ and the following DLR-equations hold even if we replace the potential function with polynomial growth by the one satisfying the much weaker condition **(U3)**:

$$\begin{aligned}\mathbb{E}^{\mu}\big[\mathbf{1}_A|\mathcal{B}_{[T_1,T_2],c}\big](\xi) &= Z^{-1}_{[T_1,T_2]}(\xi)\int_A\exp\Big(-\int_{T_1}^{T_2}U(w(x))dx\Big)\mathcal{W}^{\xi(T_1),\xi(T_2)}_{[T_1,T_2]}(dw), \\ &\qquad \mu\text{-a.e. } \xi\in C(\mathbb{R},\mathbb{R}^d), \text{ for all } A\in\mathcal{B}_{[T_1,T_2]}, T_1<T_2, \qquad (2.3)\end{aligned}$$

where $Z_{[T_1,T_2]}(\xi) := \mathbb{E}^{\mathcal{W}^{\xi(T_1),\xi(T_2)}_{[T_1,T_2]}}[\exp(-\int_{T_1}^{T_2}U(w(x))dx)]$ is a normalizing constant. By the continuity of the inclusion map of $\mathcal{C}$ into E, we may regard μ as a probability measure on E by identifying it with its image measure under the inclusion map, and using that, $\mathcal{C}\in\mathcal{B}(E)$ and $\mathcal{B}(E)\cap\mathcal{C}=\mathcal{B}(\mathcal{C})$ by Kuratowski's theorem. The DLR-equations (2.3) imply that the Gibbs measure μ is $C_0^\infty(\mathbb{R},\mathbb{R}^d)$-quasi-invariant, i.e., $\mu(\cdot+k)$ and μ are mutually equivalent, and $\mu(k+dw)=\Lambda(k,w)\mu(dw)$ holds for every $k\in C_0^\infty(\mathbb{R},\mathbb{R}^d)$. In particular by Albeverio–Röckner [10, Proposition 2.7], $\mu(O)>0$ for every open $\emptyset\neq O\subset E$, i.e., the topological support $\mathrm{supp}(\mu)$ is equal to all of E. The Radon-Nikodym density $\Lambda(k,w)$ is represented by

$$\begin{aligned}\Lambda(k,w) &= \exp\Big\{\int_{\mathbb{R}}\Big(U\big(w(x)\big)-U\big(w(x)+k(x)\big) \\ &\qquad -\frac{1}{2}\big|\frac{dk}{dx}(x)\big|^2+(w(x),\Delta_x k(x))_{\mathbb{R}^d}\Big)dx\Big\}. \qquad (2.4)\end{aligned}$$

We give the following examples which satisfy our conditions **(U1)**, **(U2)** and **(U3)**.

Example 2.1 ($P(\phi)_1$-quantum fields) *We consider the case where the potential function U is written as the following potential function on $\mathbb{R}^d$:*

$$U(z)=\sum_{j=0}^{2n}a_j|z|^j, \quad a_{2n}>0, \ n\in\mathbb{N}.$$

Especially, in the case $U(z) = \frac{m^2}{2}|z|^2$, $m > 0$, the corresponding Gibbs measure μ is the Gaussian measure on $\mathcal{C}$ with mean 0 and covariance operator $(-\Delta_x + m^2)^{-1}$. It is just the (space-time) free field of mass m in terms of Euclidean quantum field theory. A double-well potential $U(z) = a(|z|^4 - |z|^2), a > 0$, is also particularly important from the point of view of physics.

Example 2.2 ($\exp(\phi)_1$-quantum fields) *We consider an exponential type potential function $U : \mathbb{R}^d \to \mathbb{R}$ (with weight ν) given by*

$$U(z) = \frac{m^2}{2}|z|^2 + V(z) := \frac{m^2}{2}|z|^2 + \int_{\mathbb{R}^d} e^{(\xi,z)_{\mathbb{R}^d}}\nu(d\xi), \quad z \in \mathbb{R}^d,$$

where ν is a bounded positive measure with $\mathrm{supp}(\nu) \subset \{\xi \in \mathbb{R}^d | \ |\xi| \le L\}$ for some $L > 0$. We note that U is a smooth strictly convex function (i.e., $\nabla^2 U \ge m^2$). Hence we can take $K_1 = m^2$, $K_2 = \frac{m^2}{2}$ and $\alpha = 2$. Moreover,

$$|U(z)| \le \frac{m^2}{2}|z|^2 + \nu(\mathbb{R}^d)e^{L|z|} \le \big(\frac{m^2}{2L^2} + \nu(\mathbb{R}^d)\big)e^{2L|z|}, \quad z \in \mathbb{R}^d,$$

and

$$|\nabla U(z)| \le m^2|z| + \int_{\mathbb{R}^d} |\xi| e^{(\xi,z)_{\mathbb{R}^d}}\nu(d\xi) \le (\frac{m^2}{L} + L\nu(\mathbb{R}^d))e^{L|z|}, \quad z \in \mathbb{R}^d.$$

Thus we can take $\beta = 1$, which satisfies $\beta < 1 + \frac{\alpha}{2}$ in condition **(U3)**.

Remark 2.3 *We discuss a simple example of $\exp(\phi)_1$-quantum fields in the case $d = 1$. This example has been discussed in the 2-dimensional space-time case in [6]. Let δ_a be the Dirac measure at $a \in \mathbb{R}$ and we consider $\nu(d\xi) := \frac{1}{2}\big(\delta_{-a}(d\xi) + \delta_a(d\xi)\big)$, $a > 0$. Then the corresponding potential function is $U(z) = \frac{m^2}{2}z^2 + \cosh(az)$, and (2.1) implies that the Schrödinger operator H_U has a ground state Ω satisfying*

$$0 < \Omega(z) \le D_1 \exp\big(-\frac{D_2}{\sqrt{2}}|z|\,e^{\frac{a}{4}|z|}\big), \quad z \in \mathbb{R}, \tag{2.5}$$

for some $D_1, D_2 > 0$. By the translation invariance of the Gibbs measure μ and (2.5), there exist positive constants M_1 and M_2 such that

$$\begin{aligned} A_T &:= \mu\big(\{w \in C(\mathbb{R},\mathbb{R}) | \ |w(T)| > \frac{4}{a}\log\log T\}\big) \\ &= \int_{|z|>\frac{4}{a}\log\log T} \Omega(z)^2 dz \\ &\le M_1 \exp\big\{-M_2(\log T)(\log\log T)\big\} = M_1 T^{-M_2 \log\log T} \end{aligned} \tag{2.6}$$

for T large enough, and (2.6) implies $\sum_{T=1}^{\infty} A_T < \infty$. Then the first Borel–Cantelli lemma yields

$$\mu\big(\{w \in C(\mathbb{R},\mathbb{R}) | \ \limsup_{T\to\infty} \frac{|w(T)|}{\log\log T} \le \frac{4}{a}\}\big) = 1,$$

and thus μ is supported by a much smaller subset of $C(\mathbb{R},\mathbb{R})$ than $\mathcal{C}$.

Example 2.4 (Trigonometric quantum fields) *We consider a trigonometric type potential function $U : \mathbb{R}^d \to \mathbb{R}$ (with weight ν) given by*

$$U(z) = \frac{m^2}{2}|z|^2 + V(z) := \frac{m^2}{2}|z|^2 + \int_{\mathbb{R}^d} \cos\big\{(\xi, z)_{\mathbb{R}^d} + \alpha\big\}\nu(d\xi), \quad z \in \mathbb{R}^d,$$

where $\alpha \in \mathbb{R}$, $m > 0$, and ν is a bounded signed measure with finite second absolute moment, i.e.,

$$|\nu|(\mathbb{R}^d) < \infty, \quad K(\nu) := \int_{\mathbb{R}^d} |\xi|^2 |\nu|(d\xi) < \infty.$$

This potential function is smooth, and it can be regarded as a bounded perturbation of a quadratic function. Moreover, $\nabla^2 U \geq m^2 - K(\nu)$ and

$$|\nabla U(z)| \leq m^2|z| + K(\nu)^{1/2}|\nu|(\mathbb{R}^d)^{1/2}.$$

This type of potential functions corresponds to quantum field models with "trigonometric interaction" and has been discussed especially in the 2*-dimensional space-time case (see, e.g.,* [5, 22, 25]*).*

Remark 2.5 *Further examples can be obtained by considering $U : \mathbb{R}^d \to \mathbb{R}$ of the form $U(z) = \lambda_1 U_1(z) + \lambda_2 U_2(z) + \lambda_3 U_3(z)$, where $\lambda_i \geq 0$, $i = 1, 2, 3$, and U_1, resp. U_2, resp. U_3, is as given in Example* 2.1*, resp. Example* 2.2*, resp. Example* 2.4*.*

Now we are in a position to introduce the pre-Dirichlet form $(\mathcal{E}, \mathcal{F}\mathcal{C}_b^\infty)$. Let $K \subset E^*$ be a dense linear subspace of E and let $\mathcal{F}\mathcal{C}_b^\infty(K)$ be the space of all smooth cylinder functions on E having the form

$$F(w) = f(\langle w, \varphi_1 \rangle, \ldots, \langle w, \varphi_n \rangle), \quad w \in E,$$

with $n \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$ and $\varphi_1, \ldots, \varphi_n \in K$. Here we set $\langle w, \varphi \rangle := \int_{\mathbb{R}} (w(x), \varphi(x))_{\mathbb{R}^d} dx$ if the integral converges absolutely, and set $\mathcal{F}\mathcal{C}_b^\infty := \mathcal{F}\mathcal{C}_b^\infty(C_0^\infty(\mathbb{R}, \mathbb{R}^d))$ for simplicity. Since we have $\mathrm{supp}(\mu) = E$, two different functions in $\mathcal{F}\mathcal{C}_b^\infty(K)$ represent two different μ-classes. Note that $\mathcal{F}\mathcal{C}_b^\infty$ is dense in $L^p(\mu)$ for all $p \geq 1$. For $F \in \mathcal{F}\mathcal{C}_b^\infty$, we define the H-Fréchet derivative $D_H F : E \to H$ by

$$D_H F(w) := \sum_{j=1}^n \frac{\partial f}{\partial \alpha_j}(\langle w, \varphi_1 \rangle, \ldots, \langle w, \varphi_n \rangle)\varphi_j.$$

Then we consider the pre-Dirichlet form $(\mathcal{E}, \mathcal{F}\mathcal{C}_b^\infty)$ which is given by

$$\mathcal{E}(F, G) = \frac{1}{2} \int_E \big(D_H F(w), D_H G(w)\big)_H \mu(dw), \quad F, G \in \mathcal{F}\mathcal{C}_b^\infty.$$

Proposition 2.6

$$\mathcal{E}(F,G) = -\int_E \mathcal{L}_0 F(w) G(w)\mu(dw), \quad F,G \in \mathcal{FC}_b^\infty, \tag{2.7}$$

where $\mathcal{L}_0 F \in L^p(\mu)$, $p \geq 1$, $F \in \mathcal{FC}_b^\infty$, is given by

$$\begin{aligned}
\mathcal{L}_0 F(w) &= \frac{1}{2}\mathrm{Tr}(D_H^2 F(w)) + \frac{1}{2}\big\langle w, \Delta_x D_H F(w(\cdot))\big\rangle - \frac{1}{2}\big\langle (\widetilde{\nabla} U)(w(\cdot)), D_H F(w)\big\rangle \\
&= \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial\alpha_i\partial\alpha_j}\big(\langle w,\varphi_1\rangle,\ldots,\langle w,\varphi_n\rangle\big)\langle\varphi_i,\varphi_j\rangle \\
&\quad + \frac{1}{2}\sum_{i=1}^n \frac{\partial f}{\partial\alpha_i}\big(\langle w,\varphi_1\rangle,\ldots,\langle w,\varphi_n\rangle\big)\cdot\big\{\langle w,\Delta_x\varphi_i\rangle - \langle(\widetilde{\nabla} U)(w(\cdot)),\varphi_i\rangle\big\}.
\end{aligned}$$

for $F(w) = f(\langle w,\varphi_1\rangle,\ldots,\langle w,\varphi_n\rangle)$.

This proposition means that the operator $\mathcal{L}_0$ is the pre-Dirichlet operator which is associated with the pre-Dirichlet form $(\mathcal{E}, \mathcal{FC}_b^\infty)$. In particular, $(\mathcal{E}, \mathcal{FC}_b^\infty)$ is closable in $L^2(\mu)$. Let us denote by $\mathcal{D}(\mathcal{E})$ the completion of $\mathcal{FC}_b^\infty$ with respect to the $\mathcal{E}_1^{1/2}$-norm. (Here we use standard notations of the theory of Dirichlet forms, see, e.g., [1, 23, 36].) By standard theory (cf. [1, 11, 23, 36]), $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form and the operator $\mathcal{L}_0$ has a self-adjoint extension $(\mathcal{L}_\mu, \mathrm{Dom}(\mathcal{L}_\mu))$, called the Friedrichs extension, corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. The semigroup $\{e^{t\mathcal{L}_\mu}\}_{t\geq 0}$ generated by $(\mathcal{L}_\mu, \mathrm{Dom}(\mathcal{L}_\mu))$ is Markovian, i.e., $0 \leq e^{t\mathcal{L}_\mu}F \leq 1$, μ-a.e. whenever $0 \leq F \leq 1$, μ-a.e. Moreover, since $\{e^{t\mathcal{L}_\mu}\}_{t\geq 0}$ is symmetric on $L^2(\mu)$, the Markovian property implies that

$$\int_E e^{t\mathcal{L}_\mu} F(w)\mu(dw) \leq \int_E F(w)\mu(dw), \quad F \in L^2(\mu),\ F \geq 0,\ \mu\text{-a.e.}$$

Hence $\|e^{t\mathcal{L}_\mu}F\|_{L^1(\mu)} \leq \|F\|_{L^1(\mu)}$ holds for $F \in L^2(\mu)$, and $\{e^{t\mathcal{L}_\mu}\}_{t\geq 0}$ can be extended as a family of C_0-semigroup of contractions in $L^p(\mu)$ for all $p \geq 1$. See e.g., Shigekawa [43, Proposition 2.2] for details.

On the other hand, it is a fundamental question whether the Friedrichs extension is the only closed extension generating a C_0-semigroup on $L^p(\mu), p \geq 1$, which for $p = 2$ is equivalent to the fundamental problem of essential self-adjointness of $\mathcal{L}_0$ in quantum physics (cf. Eberle [21]). Even if $p = 2$, in general there are many lower bounded self-adjoint extensions $\widetilde{\mathcal{L}}$ of $\mathcal{L}_0$ in $L^2(\mu)$ which therefore generate different symmetric strongly continuous semigroups $\{e^{t\widetilde{\mathcal{L}}}\}_{t\geq 0}$. If, however, we have $L^p(\mu)$-uniqueness of $\mathcal{L}_0$ for some $p \geq 2$, there is hence only one semigroup which is strongly continuous and with generator extending $\mathcal{L}_0$. Consequently, in this case, only one such L^p-, hence only one such L^2-dynamics exists, associated with the Gibbs measure μ.

The following theorems are the main results of this paper. For the notions of "quasi-everywhere" and "capacity", we refer to [1, 23, 36].

Theorem 2.7 (1) *The pre-Dirichlet operator* $(\mathcal{L}_0, \mathcal{FC}_b^\infty)$ *is* $L^p(\mu)$*-unique for all* $p \geq 1$*, i.e., there exists exactly one* C_0*-semigroup in* $L^p(\mu)$ *such that its generator extends* $(\mathcal{L}_0, \mathcal{FC}_b^\infty)$*.*
(2) *There exists a diffusion process* $\mathbb{M} := (\Theta, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \{X_t\}_{t\geq 0}, \{\mathbb{P}_w\}_{w\in E})$ *such that the semigroup* $\{P_t\}_{t\geq 0}$ *generated by the unique (self-adjoint) extension of* $(\mathcal{L}_0, \mathcal{FC}_b^\infty)$ *satisfies the following identity for any bounded measurable function* $F : E \to \mathbb{R}$*, and* $t > 0$*:*

$$P_t F(w) = \int_\Theta F(X_t(\omega))\mathbb{P}_w(d\omega), \quad \mu\text{-a.s. } w \in E. \tag{2.8}$$

Moreover, $\mathbb{M}$ *is the unique diffusion process solving the following "componentwise" SDE:*

$$\begin{aligned} \langle X_t, \varphi\rangle &= \langle w, \varphi\rangle + \langle B_t, \varphi\rangle + \frac{1}{2}\int_0^t \big\{\langle X_s, \Delta_x\varphi\rangle - \langle(\widetilde{\nabla}U)(X_s(\cdot)), \varphi\rangle\big\}ds, \\ &\qquad\qquad t > 0,\ \varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^d),\ \mathbb{P}_w\text{-a.s.}, \end{aligned} \tag{2.9}$$

for quasi-every $w \in E$ *and such that its corresponding semigroup given by* (2.8) *consists of locally uniformly bounded (in* t*) operators on* $L^p(\mu), p \geq 1$*, where* $\{B_t\}_{t\geq 0}$ *is an* $\{\mathcal{F}_t\}_{t\geq 0}$*-adapted* H*-cylindrical Brownian motion starting at zero defined on* $(\Theta, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}_w)$ *and* $\widetilde{\nabla}U$ *was defined in condition* **(U3)***.*

Theorem 2.8 *For quasi-every* $w \in E$*, the parabolic SPDE*

$$dX_t(x) = \frac{1}{2}\big\{\Delta_x X_t(x) - (\widetilde{\nabla}U)(X_t(x))\big\}dt + dB_t(x), \quad x \in \mathbb{R},\ t > 0, \tag{2.10}$$

has a unique strong solution $X = \{X_t^w(\cdot)\}_{t\geq 0}$ *living in* $C([0,\infty), E)$*. Namely, there exists a set* $S \subset E$ *with* $\mathrm{Cap}(S) = 0$ *such that for any* H*-cylindrical Brownian motion* $\{B_t\}_{t\geq 0}$ *starting at zero defined on a filtered probability space* $(\Theta, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ *satisfying the usual conditions and an initial datum* $w \in E \setminus S$*, there exists a unique* $\{\mathcal{F}_t\}_{t\geq 0}$*-adapted process* $X = \{X_t^w(\cdot)\}_{t\geq 0}$ *living in* $C([0,\infty), E)$ *satisfying* (2.9)*.*

Remark 2.9 *Obviously, the uniqueness result in Theorem* 2.8 *implies the (thus weaker) uniqueness stated for the diffusion process* $\mathbb{M}$ *in Theorem* 2.7*. However, it does not imply the* $L^p(\mu)$*-uniqueness of the Dirichlet operator. This is obvious, since a priori the latter might have extensions which generate non-Markovian semigroups which thus have no probabilistic interpretation as transition probabilities of a process. Therefore, neither of the two uniqueness results in Theorems* 2.7 *and* 2.8*, i.e.,* $L^p(\mu)$*-uniqueness of the Dirichlet operator and strong uniqueness of the corresponding SPDE respectively, implies the other. We refer to Albeverio–Röckner* [13, Sections 2 and 3] *and see also* [17, Section 8] *for a detailed discussion.*

Remark 2.10 *If the potential function* U *is a* C^1*-function with polynomial growth at infinity, Iwata* [28] *proves that SPDE* (2.10) *has a unique strong solution* $X^w = \{X_t^w(\cdot)\}_{t\geq 0}$

living in $C([0,\infty),\mathcal{C})$ for every initial datum $w\in\mathcal{C}$. On the other hand, in the case of $\exp(\phi)_1$*-quantum fields, since $(\nabla U)(w(\cdot))\notin\mathcal{C}$ for $w\in\mathcal{C}$ in general, we cannot expect to solve SPDE* (2.10) *in $C([0,\infty),\mathcal{C})$ for a given initial datum $w\in\mathcal{C}$. Hence if we replace the state space $\mathcal{C}$ by a much smaller tempered subspace $\mathcal{C}_e$ such that $(\nabla U)(w(\cdot))\in\mathcal{C}_e$ holds for $w\in\mathcal{C}_e$, we might construct a unique strong solution to SPDE* (2.10) *living in $C([0,\infty),\mathcal{C}_e)$ for every initial datum $w\in\mathcal{C}_e$. (A possible candidate for $\mathcal{C}_e$ could be the space of all paths behaving like*

$$|w(x)|\sim\log(\log(\log(\log(\cdots x))))$$

at infinity.) We will discuss this problem in the future.

3 Proof of the Main Results

At the beginning, we give the proof of Proposition 2.6.
Proof of Proposition 2.6: Firstly, we aim to prove that

$$\int_E\Big(\int_{\mathbb{R}}|(\widetilde{\nabla}U)(w(x))|^2\rho_{-2r}(x)dx\Big)^{p/2}\mu(dw)<\infty,\quad p\geq 1. \tag{3.1}$$

By the translation invariance of the Gibbs measure μ, for every $p\geq 2$, it holds that

$$\begin{aligned}
&\int_E\Big(\int_{\mathbb{R}}|(\widetilde{\nabla}U)(w(x))|^2\rho_{-2r}(x)dx\Big)^{p/2}\mu(dw)\\
&\leq\ \int_E\Big\{\Big(\int_{\mathbb{R}}|(\widetilde{\nabla}U)(w(x))|^p\rho_{-2r}(x)dx\Big)\Big(\int_{\mathbb{R}}\rho_{-2r}(x)dx\Big)^{\frac{p-2}{2}}\Big\}\mu(dw)\\
&\leq\ \Big(\frac{1}{r}\Big)^{\frac{p-2}{2}}\int_{\mathbb{R}}\Big(\int_E|(\widetilde{\nabla}U)(w(0))|^p\mu(dw)\Big)\rho_{-2r}(x)dx\\
&\leq\ \Big(\frac{1}{r}\Big)^{p/2}\int_{\mathbb{R}^d}|\widetilde{\nabla}U(z)|^p\Omega(z)^2dz\\
&\leq\ \Big(\frac{K_3^2}{r}\Big)^{p/2}\int_{\mathbb{R}^d}\exp(pK_4|z|^\beta)\Omega(z)^2dz.
\end{aligned}\tag{3.2}$$

On the other hand, condition **(U2)** leads to a lower bound $U_{\frac{1}{2}|z|}(z)\geq\frac{K_2}{2^\alpha}|z|^\alpha$ for $|z|\geq 2R$. Hence we can continue to bound the integral on the right-hand side of (3.2) as follows:

$$\begin{aligned}
&\int_{|z|\geq 2R}\exp(pK_4|z|^\beta)\Omega(z)^2dz\\
&\leq\ D_1^2\int_{|z|\geq 2R}\exp\big\{p(K_4|z|^\beta-D_2|z|U_{\frac{1}{2}|z|}(z)^{1/2})\big\}dz\\
&\leq\ D_1^2\int_{|z|\geq 2R}\exp\Big\{p\Big(K_4|z|^\beta-\frac{D_2K_2^{1/2}}{2^{\alpha/2}}|z|^{1+\frac{\alpha}{2}}\Big)\Big\}dz<\infty,
\end{aligned}\tag{3.3}$$

where we used the estimate (2.1) for the second line and $\beta < 1 + \frac{\alpha}{2}$ for the third line.

Hence by combining (3.2) with (3.3), we see that the left-hand side of (3.2) is finite for all $p \geq 2$. Since μ is a probability measure on E, we have shown that (3.1) holds for all $p \geq 1$. In the same way, we also have

$$\int_E \|w\|_E^p \, \mu(dw) < \infty, \quad p \geq 1. \tag{3.4}$$

Next, we define

$$\beta_\varphi(w) := \langle w, \Delta_x \varphi \rangle - \langle (\widetilde{\nabla} U)(w(\cdot)), \varphi \rangle, \qquad w \in E, \ \varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^d).$$

Then by noting that φ has compact support, one has $\|\Delta_x \varphi\|_{E^*} + \|\varphi\|_{E^*} < \infty$, and (3.1) and (3.4) lead to

$$\begin{aligned} \int_E |\beta_\varphi(w)|^p \mu(dw) \;&\leq\; 2^{p-1}\big(\|\Delta_x \varphi\|_{E^*}^p + \|\varphi\|_{E^*}^p\big) \\ &\quad \times \int_E \Big\{ \|w\|_E^p + \Big(\int_{\mathbb{R}} |(\widetilde{\nabla} U)(w(x))|^2 \rho_{-2r}(x) dx \Big)^{p/2} \Big\} \mu(dw) < \infty. \end{aligned}$$

Thus we have shown that $\mathcal{L}_0 F \in L^p(\mu)$ holds for all $p \geq 1$ and $F \in \mathcal{FC}_b^\infty$. Hence the right-hand side of (2.7) is well-defined and finite, and the quasi-invariance of μ yields (2.7). ∎

Before proceeding to the proofs of our main theorems, we make some preparations. We fix a positive constant $\kappa > 2r^2$, and set

$$G_t w(x) := \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} w(y) dy, \quad t > 0, \ x \in \mathbb{R}.$$

Then by [33, Lemma 3.2], we see that $\{e^{-\kappa t/2} G_t\}_{t \geq 0}$ is a strongly continuous contraction semigroup on E with $\|e^{-\kappa t/2} G_t\|_{L(E,E)} \leq \exp\{-(\frac{\kappa}{2} - r^2)t\}$. Let $A : \mathrm{Dom}(A) \subset E \to E$ be the infinitesimal generator of $\{e^{-\kappa t/2} G_t\}_{t \geq 0}$. We set $e^{tA} := e^{-\kappa t/2} G_t$ throughout this paper. By the Hille–Yosida theorem, $(A, \mathrm{Dom}(A))$ is m-dissipative and it satisfies

$$(Aw, w)_E \leq (r^2 - \frac{\kappa}{2}) \|w\|_E^2, \quad w \in \mathrm{Dom}(A). \tag{3.5}$$

Lemma 3.1 (1) *$C_0^\infty(\mathbb{R}, \mathbb{R}^d)$ is dense in $\mathrm{Dom}(A)$ with respect to the graph norm $\|w\|_A := \|w\|_E + \|Aw\|_E$, $w \in \mathrm{Dom}(A)$, and we have*

$$A\varphi = \frac{1}{2}(\Delta_x - \kappa)\varphi, \quad \varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^d). \tag{3.6}$$

(2) *Let $A^* : \mathrm{Dom}(A^*) \subset E \to E$ denote the adjoint operator of $(A, \mathrm{Dom}(A))$. Then $\mathrm{Dom}(A^*) = \mathrm{Dom}(A)$. Moreover, we have*

$$\begin{aligned} A^* \varphi &= \frac{1}{2} \Delta_x (\rho_{-2r} \cdot \varphi) \rho_{2r} - \frac{\kappa}{2} \varphi \\ &= A\varphi - 2r \frac{d\chi}{dx} \cdot \frac{d\varphi}{dx} + \big\{ 2r^2 \big(\frac{d\chi}{dx} \big)^2 - r \Delta_x \chi \big\} \varphi, \quad \varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^d), \end{aligned} \tag{3.7}$$

and

$$e^{tA^*}w(y) := e^{-\kappa t/2}\rho_{2r}(y)\cdot G_t(\rho_{-2r}\cdot w)(y), \quad t>0,\ y\in\mathbb{R},\ w\in E. \tag{3.8}$$

Proof: (1) By a straightforward computation, we can easily see that $C_0^\infty(\mathbb{R},\mathbb{R}^d)\subset \mathrm{Dom}(A)$ and that (3.6) holds. We introduce

$$\mathcal{C}_\infty^\infty := \bigcap_{k=0}^{\infty}\bigcap_{r>0}\Big\{\varphi\in C^\infty(\mathbb{R},\mathbb{R}^d)\ \big|\ \sup_{x\in\mathbb{R}}\big|\frac{d^k\varphi}{dx^k}(x)\big|\rho_r(x)<\infty\Big\}.$$

Then $C_0^\infty(\mathbb{R},\mathbb{R}^d)\subset\mathcal{C}_\infty^\infty$ and the differential operator A can be naturally extended to the domain $\mathcal{C}_\infty^\infty$ through (3.6). By using the cut-off argument discussed in [33, Lemma 4.7], we can show that $C_0^\infty(\mathbb{R},\mathbb{R}^d)$ is dense in $\mathcal{C}_\infty^\infty$ with respect to the graph norm $\|\cdot\|_A$.

Now, we take a function $\varphi\in\mathcal{C}_\infty^\infty$. Then for every $k\in\mathbb{N}\cup\{0\}$ and $r>0$, we can find a positive constant $C(k,r)$ such that $\big|\frac{d^k\varphi}{dx^k}(x)\big|\le C(k,r)\rho_{-r}(x)$ for all $x\in\mathbb{R}$. Here we recall that

$$\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-y)^2}{2t}}\rho_{-2r}(y)dy\le e^{2r^2t}\rho_{-2r}(x), \quad t>0, x\in\mathbb{R}. \tag{3.9}$$

(cf. e.g., Da Prato–Zabcyzk [20, Lemma 9.44].) Then for every $k\in\mathbb{N}\cup\{0\}$ and $r>0$,

$$\begin{aligned}\big|\frac{d^k}{dx^k}(G_t\varphi)(x)\big|\rho_r(x) &\le \rho_r(x)\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-y)^2}{2t}}\big|\frac{d^k\varphi}{dx^k}(y)\big|dy\\ &\le \rho_r(x)\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-y)^2}{2t}}\big(C(k,2r)\rho_{-2r}(y)\big)dy\\ &\le C(k,2r)\rho_r(x)\big(e^{2r^2t}\rho_{-2r}(x)\big)\\ &\le C(k,2r)e^{2r^2t}<\infty, \quad x\in\mathbb{R},\end{aligned}$$

where we used (3.9) for the third line and $\rho_{-2r}(x)\rho_r(x)=\rho_{-r}(x)\le 1$ for the fourth line. This means that $(e^{-\kappa t/2}G_t)(\mathcal{C}_\infty^\infty)\subset\mathcal{C}_\infty^\infty$ for all $t\ge 0$, and by [21, Theorems 1.2 and 1.3], we see that $\mathcal{C}_\infty^\infty$ is an operator core for A. Hence we have shown that $C_0^\infty(\mathbb{R},\mathbb{R}^d)$ is dense in $\mathrm{Dom}(A)$ with respect to the graph norm $\|\cdot\|_A$.

(2) Since (3.7) and (3.8) follow by straightforward computations, it is sufficient to show the equivalence of the graph norms $\|\varphi\|_A$ and $\|\varphi\|_{A^*}$ for $\varphi\in C_0^\infty(\mathbb{R},\mathbb{R}^d)$.

Using integration by parts, Young's inequality $2ab\le\delta^{-2}a^2+\delta^2b^2$ and that $\|\frac{d\chi}{dx}\|_\infty\le 1$, we obtain

$$\begin{aligned}\|\frac{d\varphi}{dx}\|_E^2 &= -(\varphi,\Delta_x\varphi)_E+2r\int_{\mathbb{R}}\big(\varphi(x),\frac{d\varphi}{dx}(x)\big)_{\mathbb{R}^d}\frac{d\chi}{dx}(x)\rho_{-2r}(x)dx\\ &\le \big(\frac{1}{2\delta^2}\|\varphi\|_E^2+\frac{\delta^2}{2}\|\Delta_x\varphi\|_E^2\big)+r\big(\frac{1}{2r}\|\frac{d\varphi}{dx}\|_E^2+2r\|\varphi\|_E^2\big),\end{aligned}$$

which in turn implies that

$$\|\frac{d\varphi}{dx}\|_E\le\big(2r+\frac{1}{\delta}\big)\|\varphi\|_E+\delta\|\Delta_x\varphi\|_E, \quad \delta>0,\ \varphi\in C_0^\infty(\mathbb{R},\mathbb{R}^d). \tag{3.10}$$

Recalling (3.7), we deduce that

$$
\begin{aligned}
\|A\varphi\|_E &\le \|A^*\varphi\|_E + 2r\|\dot{\varphi}\|_E + (2r^2 + r\|\Delta_x\chi\|_\infty)\|\varphi\|_E \\
&\le \|A^*\varphi\|_E + 2r\big\{\big(2r + \frac{1}{\delta}\big)\|\varphi\|_E + \delta\|\Delta_x\varphi\|_E\big\} + (2r^2 + r\|\Delta_x\chi\|_\infty)\|\varphi\|_E \\
&\le \|A^*\varphi\|_E + 4r\delta\|A\varphi\|_E + \big(6r^2 + \frac{2r}{\delta} + 2r\delta\kappa + r\|\Delta_x\chi\|_\infty\big)\|\varphi\|_E, \qquad (3.11)
\end{aligned}
$$

where we used $\|\frac{d\chi}{dx}\|_\infty \le 1$ again for the first line and (3.10) for the second line.

Now, we choose $\delta := \frac{1}{8r}$. Then (3.11) implies

$$
\|A\varphi\|_E \le 2\|A^*\varphi\|_E + \big(22r^2 + \kappa + r\|\Delta_x\chi\|_\infty\big)\|\varphi\|_E,
$$

and by repeating a similar argument for $A^*\varphi$, we also have

$$
\|A^*\varphi\|_E \le \frac{3}{2}\|A\varphi\|_E + \big(22r^2 + \kappa + r\|\Delta_x\chi\|_\infty\big)\|\varphi\|_E.
$$

This completes the proof. ∎

Proof of Theorem 2.7: (1) Although we mostly follow the argument in [33], which in turn is based on a modification of a technique in [17], we give an outline of the argument for the convenience of the reader. We define $E_U := \{w \in E;\ \|(\widetilde{\nabla} U)(w(\cdot))\|_E < \infty\}$. Then by (3.1), we see that $E_U \in \mathcal{B}(E)$ and $\mu(E_U) = 1$. We define a measurable map $\widetilde{b} : \mathrm{Dom}(\widetilde{b}) \subset E \to E$ with $\mathrm{Dom}(\widetilde{b}) = E_U$ by

$$
\widetilde{b}(w)(\cdot) := -\frac{1}{2}(\partial_0 V)(w(\cdot)) = -\frac{1}{2}\big\{(\widetilde{\nabla} U)(w(\cdot)) - K_1 w(\cdot)\big\}, \quad w \in \mathrm{Dom}(\widetilde{b}). \qquad (3.12)
$$

We note that $\mu(\mathrm{Dom}(\widetilde{b})) = 1$, and since V is convex, $\widetilde{b}$ is dissipative, i.e.,

$$
\big(w_1 - w_2, \widetilde{b}(w_1) - \widetilde{b}(w_2)\big)_E \le 0, \quad w_1, w_2 \in \mathrm{Dom}(\widetilde{b}). \qquad (3.13)
$$

On the other hand, we note that $\widetilde{b}$ is not continuous on E in general. Thus we need to introduce the following regularization scheme. For $\alpha > 0$, we recall the Moreau–Yosida approximation of V which is defined by

$$
V_\alpha(z) := \inf_{y \in \mathbb{R}^d} \big\{\frac{1}{2\alpha}|y - z|^2 + V(y)\big\}, \quad z \in \mathbb{R}^d. \qquad (3.14)
$$

Then $V_\alpha(z) \nearrow V(z)$ for every $z \in \mathbb{R}^d$ as $\alpha \searrow 0$. On the other hand, $\partial_0 V : \mathbb{R}^d \to \mathbb{R}^d$ is maximal dissipative by convexity of V. For $\alpha > 0$, we set $J_\alpha(z) := \big(I_{\mathbb{R}^d} + \alpha\partial_0 V\big)^{-1}(z)$, $z \in \mathbb{R}^d$, and define the Yosida approximation $(\partial_0 V)_\alpha : \mathbb{R}^d \to \mathbb{R}^d$ by

$$
(\partial_0 V)_\alpha(z) := \frac{1}{\alpha}(J_\alpha(z) - z) = (\partial_0 V)(J_\alpha(z)), \quad z \in \mathbb{R}^d.
$$

Then $(\partial_0 V)_\alpha$ is monotone and the following Lipschitz continuity holds:

$$\big|(\partial_0 V)_\alpha(z_1) - (\partial_0 V)_\alpha(z_2)\big| \le \frac{2}{\alpha}|z_1 - z_2|, \quad z_1, z_2 \in \mathbb{R}^d,$$

Furthermore, it is known that $(\partial_0 V)_\alpha(z) = (\partial_0 V_\alpha)(z), z \in \mathbb{R}^d$ (cf. e.g., [46, Proposition 1.8]), and

$$\big|(\partial_0 V)_\alpha(z)\big| \le \big|\partial_0 V(z)\big|, \quad z \in \mathbb{R}^d, \tag{3.15}$$

$$\lim_{\alpha \searrow 0} (\partial_0 V)_\alpha(z) = \partial_0 V(z), \quad z \in \mathbb{R}^d. \tag{3.16}$$

See [46, Theorem 1.1] for details. We define $\widetilde{b}_\alpha : E \to E$ in the same way as $\widetilde{b}$ with $\partial_0 V$ replaced by $(\partial_0 V)_\alpha$. Then $\widetilde{b}_\alpha$ is Lipschitz continuous and dissipative on E. By (3.15) and (3.16), we also have

$$\lim_{\alpha \searrow 0} \widetilde{b}_\alpha(w) = \widetilde{b}(w), \quad w \in \mathrm{Dom}(\widetilde{b}). \tag{3.17}$$

However, since $\widetilde{b}_\alpha$ is not differentiable in general, we need to introduce a further regularization. Let $B : \mathrm{Dom}(B) \subset E \to E$ be a self-adjoint negative definite operator such that B^{-1} is of trace class. For any $\alpha, \beta > 0$, we set

$$\widetilde{b}_{\alpha,\beta}(w) := \int_E e^{\beta B} \widetilde{b}_\alpha\big(e^{\beta B} w + y\big) N_{\frac{1}{2}B^{-1}(e^{2\beta B}-1)}(dy), \quad w \in E, \tag{3.18}$$

where N_Q is the standard centered Gaussian measure with covariance given by a trace class operator Q. Then by applying [20, Theorem 9.19], we prove that $\widetilde{b}_{\alpha,\beta}$ is dissipative, of class C^∞, has bounded derivatives of all orders and

$$\lim_{\beta \searrow 0} \widetilde{b}_{\alpha,\beta}(w) = \widetilde{b}_\alpha(w), \quad \|\widetilde{b}_{\alpha,\beta}(w)\|_E \le C_\alpha(1 + \|w\|_E), \quad w \in E. \tag{3.19}$$

We also define a measurable map $b : \mathrm{Dom}(b) \subset E \to E$ with $\mathrm{Dom}(b) = E_U$ by

$$b(w) := \frac{1}{2}(\kappa - K_1)w + \widetilde{b}(w), \quad w \in \mathrm{Dom}(b), \tag{3.20}$$

and define $b_{\alpha,\beta}$ with $\widetilde{b}_{\alpha,\beta}$ replacing $\widetilde{b}$ in (3.20).

Now, we consider the stochastic evolution equation on E given by

$$\begin{aligned} dX_t &= AX_t dt + b_{\alpha,\beta}(X_t)dt + \sqrt{Q}dW_t \\ &= AX_t dt + \frac{1}{2}(\kappa - K_1)X_t dt + \widetilde{b}_{\alpha,\beta}(X_t)dt + \sqrt{Q}dW_t, \ t \ge 0, \end{aligned} \tag{3.21}$$

where Q is a bounded linear operator on E defined by $Qw := \rho_{-2r} \cdot w, \ w \in E$, and $\{W_t\}_{t\ge 0}$ is an E-cylindrical Brownian motion defined on a fixed filtered probability space

$(\Theta, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Note that Q^{-1} is not bounded on E. This kind of equation is regarded as an abstract formulation of SPDE (2.10) in the sense of [20], i.e., in the mild form. Since each $e^{tA}\sqrt{Q}$ is a Hilbert–Schmidt operator on E and $b_{\alpha,\beta}$ is Lipschitz continuous on E, SPDE (3.21) has a unique mild solution $X = \{X_t^w(\cdot)\}_{t\geq 0}$ living in $C([0,\infty), E)$ for every initial datum $w \in E$. Here we recall that X is a mild solution to SPDE (3.21) with $X_0 = w \in E$ if one has

$$X_t = e^{tA}w + \int_0^t e^{(t-s)A} b_{\alpha,\beta}(X_s)ds + \int_0^t e^{(t-s)A}\sqrt{Q}dW_s, \quad t > 0, \ \mathbb{P}\text{-a.s.} \tag{3.22}$$

By a standard coupling method for SPDEs applied to (3.21), we see that

$$\left\|X_t^w - X_t^{\tilde{w}}\right\|_E \leq e^{\frac{(-K_1+2r^2)t}{2}}\|w-\tilde{w}\|_E, \qquad w, \tilde{w} \in E, \tag{3.23}$$

also holds with probability one. We can then define the transition semigroup corresponding to SPDE (3.21), denoted by $\{P_t^{\alpha,\beta}\}_{t\geq 0}$.

For $F \in \mathcal{FC}_b^\infty$ and $\lambda > (-\frac{K_1}{2} + r^2) \vee 0$, we consider the function

$$\Phi_{\alpha,\beta}(w) := \int_0^\infty e^{-\lambda t} P_t^{\alpha,\beta} F(w)dt, \quad w \in E.$$

Then (3.23) leads us to the estimate

$$\|D\Phi_{\alpha,\beta}(w)\|_E \leq \frac{2}{2\lambda + K_1 - 2r^2}\|DF\|_\infty, \quad w \in E, \tag{3.24}$$

where $DF : E \to E$ is the E-Fréchet derivative of F. We have the relation $D_H F = \sqrt{Q}DF$. By Proposition 2.6, $(\mathcal{L}_0, \mathcal{FC}_b^\infty)$ is dissipative in $L^p(\mu), p \geq 1$, and then it is closable. Let $(\overline{\mathcal{L}}_0, \mathrm{Dom}(\overline{\mathcal{L}}_0))$ denote the closure in $L^p(\mu)$. However, since it is not easy to consider $\overline{\mathcal{L}}_0$ directly, we need to insert a tractable space between $\mathcal{FC}_b^\infty$ and $\mathrm{Dom}(\overline{\mathcal{L}}_0)$. Here we recall some beautiful results on Lipschitz perturbations of Ornstein–Uhlenbeck operators discussed in [16, 18, 39]. By modifying the results in [16, 18, 39] for our use, we deduce that $\Phi_{\alpha,\beta}$ belongs to a "nice" domain $\mathcal{D}(L, C_{b,2}^1(E))$ (see [33] for the precise definition and details) of the Ornstein–Uhlenbeck operator L associated with the SPDE

$$dY_t = AY_t dt + \sqrt{Q}dW_t, \ t \geq 0.$$

Moreover, recalling (3.4), we see that $\overline{\mathcal{L}}_0 F = LF + (b, DF)_E$ for $F \in \mathcal{D}(L, C_{b,2}^1(E))$ and this identity implies the inclusion $\mathcal{D}(L, C_{b,2}^1(E)) \subset \mathrm{Dom}(\overline{\mathcal{L}}_0)$. Hence we have $\Phi_{\alpha,\beta} \in \mathrm{Dom}(\overline{\mathcal{L}}_0) \cap C_b^2(E)$ and moreover $\Phi_{\alpha,\beta}$ satisfies

$$(\lambda - \overline{\mathcal{L}}_0)\Phi_{\alpha,\beta} = F + \big(\widetilde{b}_{\alpha,\beta} - \widetilde{b}, D\Phi_{\alpha,\beta}\big)_E. \tag{3.25}$$

By using (3.24), the right-hand side of (3.25) can be estimated as follows:

$$\begin{aligned} I_{\alpha,\beta} &:= \int_E \left|\big(\widetilde{b}_{\alpha,\beta}(w) - \widetilde{b}(w), D\Phi_{\alpha,\beta}(w)\big)_E\right|^p \mu(dw) \\ &\leq \Big(\frac{2}{2\lambda + K_1 - r^2}\|DF\|_\infty\Big)^p \int_E \|\widetilde{b}_{\alpha,\beta}(w) - \widetilde{b}(w)\|_E^p \mu(dw). \end{aligned} \tag{3.26}$$

Recalling (3.15), (3.16), (3.19) and using Lebesgue's dominated convergence theorem, we conclude that

$$\lim_{\alpha \searrow 0} \lim_{\beta \searrow 0} I_{\alpha,\beta} = \lim_{\alpha \searrow 0} \big(\limsup_{\beta \searrow 0} I_{\alpha,\beta} \big) = 0.$$

From this and (3.25), (3.26), we obtain

$$\lim_{\alpha \searrow 0} \lim_{\beta \searrow 0} (\lambda - \overline{\mathcal{L}}_0)\Phi_{\alpha,\beta} = F \quad \text{in } L^p(\mu).$$

This means that the closure of $\mathrm{Range}(\lambda - \overline{\mathcal{L}}_0)$ contains $\mathcal{F}\mathcal{C}_b^\infty$. Since $\mathcal{F}\mathcal{C}_b^\infty$ is dense in $L^p(\mu)$, $\mathrm{Range}(\lambda - \overline{\mathcal{L}}_0)$ is also dense in $L^p(\mu)$. Then by the Lumer–Phillips theorem, we have that $(\overline{\mathcal{L}}_0, \mathrm{Dom}(\overline{\mathcal{L}}_0))$ generates a C_0-semigroup in $L^p(\mu)$, and this completes the proof of (1).

(2) Since $C_0^\infty(\mathbb{R}, \mathbb{R}^d)$ is dense in E^*, $\mathcal{D}(\mathcal{E})$ coincides with the closure of $\mathcal{F}\mathcal{C}_b^\infty(E^*)$ with respect to the $\mathcal{E}_1^{1/2}$-norm. Thus, we can directly apply the general methods of the theory of Dirichlet forms [1, 36] to prove quasi-regularity of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and the existence of a diffusion process $\mathbb{M}$ properly associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Here, following Röckner [41] and Funaki [24], we introduce scaled Sobolev spaces:

$$H_r^m(\mathbb{R}, \mathbb{R}^d) := \{\varphi |\ \rho_r \varphi \in H^m(\mathbb{R}, \mathbb{R}^d)\}, \quad m \geq 0,\ r \in \mathbb{R},$$

equipped with norms $|\varphi|_{m,r} := \|\rho_r \varphi\|_{H^m(\mathbb{R},\mathbb{R}^d)}$. Note that this norm is equivalent to $\|\varphi\|_{m,r} := \sum_{k=0}^m \|\rho_r\big(\frac{d^k\varphi}{dx^k}\big)\|_{L^2(\mathbb{R},\mathbb{R}^d)}$ in the case $m \in \mathbb{N} \cup \{0\}$. Let $(H_r^m(\mathbb{R}, \mathbb{R}^d))^*$ be the dual space of $H_r^m(\mathbb{R}, \mathbb{R}^d)$. Then we have

$$(H_r^m(\mathbb{R}, \mathbb{R}^d))^* = H_{-r}^{-m}(\mathbb{R}, \mathbb{R}^d) = \{w |\ \rho_{-r} w \in H^{-m}(\mathbb{R}, \mathbb{R}^d)\},$$

and, clearly $H = H_0^0(\mathbb{R}, \mathbb{R}^d)$, $E = H_{-r}^0(\mathbb{R}, \mathbb{R}^d)$. For our later use, we consider a separable Hilbert space $\mathcal{H} := H_{-r}^{-2}(\mathbb{R}, \mathbb{R}^d)$. Since $\mathcal{H}^* = H_r^2(\mathbb{R}, \mathbb{R}^d)$, we have

$$C_0^\infty(\mathbb{R}, \mathbb{R}^d) \subset \mathcal{H}^* \subset E^* \subset H^* \equiv H \subset E \subset \mathcal{H}$$

and the inclusions are dense and continuous.

Let $D := \{\varphi_n\}_{n=1}^\infty \subset C_0^\infty(\mathbb{R}, \mathbb{R}^d)$ be the countable weakly dense $\mathbb{Q}$-linear subspace of $\mathcal{H}^*$ constructed on page 369 of Albeverio–Röckner [12]. Then by [12, Theorem 5.3], for each $n \in \mathbb{N}$, there exists some $S_n \subset E$ with $\mathrm{Cap}(S_n) = 0$ such that the diffusion process $\mathbb{M}$ satisfies

$$\langle X_t, \varphi_n \rangle = \langle w, \varphi_n \rangle + B_t^{(n)} + \frac{1}{2} \int_0^t \beta_{\varphi_n}(X_s) ds, \quad t > 0,\ \mathbb{P}_w\text{-a.s.}, \tag{3.27}$$

for all $w \in E \setminus S_n$, where $\{B_t^{(n)}\}_{t \geq 0}$ is a one-dimensional $\{\mathcal{F}_t\}$-adapted Brownian motion on $(\Theta, \mathcal{F}, \mathbb{P}_w)$ starting at zero multiplied by $\|\varphi_n\|_H$. On the other hand, by recalling (3.1), (3.4) and [12, Lemma 4.2], there exists a set $S_0 \subset E$ with $\mathrm{Cap}(S_0) = 0$ such that

$$\mathbb{P}_w\Big(\int_0^T \big(\|(\tilde{\nabla} U)(X_s(\cdot))\|_E + \|X_s(\cdot)\|_E \big) ds < \infty \text{ for all } T > 0 \Big) = 1 \tag{3.28}$$

for any $w \in E \setminus S_0$. Here we set $S := \cup_{n=0}^{\infty} S_n$. Obviously, $\mathrm{Cap}(S) = 0$. By noting that the embedding map $H \hookrightarrow \mathcal{H}$ is a Hilbert–Schmidt operator (cf. [24, Remark 2.1]), and [12, Remark 6.3], we can apply [12, Lemma 6.1 and Theorem 6.2], which implies that there exists an $\{\mathcal{F}_t\}_{t\geq 0}$-Brownian motion on $(\Theta, \mathcal{F}, \mathbb{P}_w)$ with values in $\mathcal{H}$ starting at zero with covariance $(\cdot,\cdot)_H$ (i.e., an H-cylindrical Brownian motion) under $\mathbb{P}_w$ for every $w \in E \setminus S$ such that

$$\langle B_t, \varphi_n \rangle := {}_{\mathcal{H}}\langle B_t, \varphi_n \rangle_{\mathcal{H}^*} = B_t^{(n)}, \quad n \in \mathbb{N},\ t \geq 0,\ \ \mathbb{P}_w\text{-a.s.}, \quad w \in E \setminus S. \tag{3.29}$$

Since D is dense in $C_0^{\infty}(\mathbb{R}, \mathbb{R}^d)$ with respect to the weak topology of $\mathcal{H}^*$, for every $\varphi \in C_0^{\infty}(\mathbb{R}, \mathbb{R}^d)$, we can take a subsequence $\{\varphi_{n(j)}\}_{j=1}^{\infty} \subset D$ such that $\varphi_{n(j)} \to \varphi$ weakly in $\mathcal{H}^*$ as $j \to \infty$. Furthermore, the Banach–Saks theorem implies that, selecting another subsequence again denoted by $\{\varphi_{n(j)}\}_{j=1}^{\infty}$, the Cesàro mean $\hat{\varphi}_k := \frac{1}{k}\sum_{j=1}^{k} \varphi_{n(j)}, k \in \mathbb{N}$, converges to φ strongly in $\mathcal{H}^*$ as $k \to \infty$. Thus $\|\varphi - \hat{\varphi}_k\|_{E^*} + \|\Delta_x \varphi - \Delta_x \hat{\varphi}_k\|_{E^*} \to 0$ as $k \to \infty$. On the other hand, (3.27) and (3.29) imply

$$\langle X_t, \hat{\varphi}_k \rangle = \langle w, \hat{\varphi}_k \rangle + \langle B_t, \hat{\varphi}_k \rangle + \frac{1}{2}\int_0^t \beta_{\hat{\varphi}_k}(X_s)ds, \quad t > 0,\ \mathbb{P}_w\text{-a.s.}, \tag{3.30}$$

for all $w \in E \setminus S$. Hence due to (3.28) we can take the limit $k \to \infty$ on both sides of (3.30) to obtain SDE (2.9) for all $w \in E \setminus S$. Besides, the uniqueness statement for $\mathbb{M}$ is derived from item (1) (cf. [13, Sections 2 and 3] and also [17, Section 8]). This completes the proof. ∎

Proof of Theorem 2.8: By noting (3.28), the fact that $Q^{-1}(C_0^{\infty}(\mathbb{R}, \mathbb{R}^d)) = C_0^{\infty}(\mathbb{R}, \mathbb{R}^d)$ and Theorem 2.7, we can read (2.9) as

$$\begin{aligned}(X_t, \varphi)_E \ &= \ (w, \varphi)_E + \int_0^t (\sqrt{Q}\varphi, dW_s)_E + \int_0^t \big\{(X_s, A^*\varphi)_E + (b(X_s), \varphi)_E\big\}ds, \\ &\qquad t > 0,\ \varphi \in C_0^{\infty}(\mathbb{R}, \mathbb{R}^d),\ \mathbb{P}_w\text{-a.s.},\ w \in E \setminus S. \end{aligned} \tag{3.31}$$

for all $w \in E \setminus S$, where $\{W_t\}_{t\geq 0}$ is an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted E-cylindrical Brownian motion corresponding to the H-cylindrical Brownian motion $\{B_t\}_{t\geq 0}$ defined on $(\Theta, \mathcal{F}, \mathbb{P}_w)$. (See [33, Remark 3.5] for details.) Furthermore, by recalling Lemma 3.1, we have equation (3.31) for every $\varphi \in \mathrm{Dom}(A^*)$. We also mention that (3.31) is equivalent to the mild-form (3.22) of SPDE (3.21) with $b_{\alpha,\beta}$ replaced by b. We refer to Ondreját [38, Theorem 13] for details.

Now, we prove pathwise uniqueness based on the argument of Marinelli–Röckner [37]. Suppose that $X = X^w$ and $\widetilde{X} = \widetilde{X}^w$ are two weak solutions to SPDE (3.21) defined on the same filtered probability space $(\Theta, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with the same E-cylindrical Brownian motion $\{W_t\}_{t\geq 0}$ and $X_0 = \widetilde{X}_0 = w \in E \setminus S$ such that

$$\int_0^T \|b(X_s)\|_E ds < \infty, \quad \int_0^T \|b(\widetilde{X}_s)\|_E ds < \infty \quad \text{for all } T > 0,\ \mathbb{P}\text{-a.s.} \tag{3.32}$$

We fix $T > 0$ from now on, and set $\Psi_t := X_t - \widetilde{X}_t$. Note that it enjoys an ω-wise equation

$$d\Psi_t = A\Psi_t dt + (b(X_t) - b(\widetilde{X}_t))dt, \quad 0 < t \le T,$$

with the initial datum $\Psi_0 = 0$, again to be understood in the mild form. Since X and $\widetilde{X}$ have continuous paths on E, (3.32) implies that $b(X_\cdot) - b(\widetilde{X}_\cdot) \in L^1([0,T], E)$ and $\sup_{0\le t\le T} \|\Psi_t\|_E < \infty$ hold for $\mathbb{P}$-a.s $\omega \in \Theta$. Let $\{\varphi_n\}_{n=1}^\infty \subset C_0^\infty(\mathbb{R}, \mathbb{R}^d)$ be a CONS of H, and we set $\widetilde{\varphi}_n := \rho_r \varphi_n$ and $e_n := (I + \varepsilon A^*)^{-1}\widetilde{\varphi}_n \in \mathrm{Dom}(A^*)$ for $n \in \mathbb{N}$. We mention that $\{\widetilde{\varphi}_n\}_{n=1}^\infty$ is a CONS of E. Recalling (3.31) and applying Itô's formula, we have

$$\begin{aligned}(e_n, \Psi_t)_E^2 &= 2\int_0^t \Psi_n(s) d\Psi_n(s) \\ &\quad +2\int_0^t (e_n, \Psi_s)_E \big(e_n, b(X_s) - b(\widetilde{X}_s)\big)_E ds \\ &=: 2\big(J_n^1(t) + J_n^2(t)\big), \qquad 0 \le t \le T. \end{aligned} \tag{3.33}$$

For the first term $J_n^1(t)$, Lebesgue's dominated convergence theorem leads us to

$$\begin{aligned}\sum_{n=1}^\infty J_n^1(t) &= \int_0^t \sum_{n=1}^\infty \big((I + \varepsilon A^*)^{-1}\widetilde{\varphi}_n, \Psi_s\big)_E \cdot \big((A(I + \varepsilon A)^{-1})^*\widetilde{\varphi}_n, \Psi_s\big)_E ds \\ &= \int_0^t \sum_{n=1}^\infty \big(\widetilde{\varphi}_n, (I + \varepsilon A)^{-1}\Psi_s\big)_E \cdot \big(\widetilde{\varphi}_n, A(I + \varepsilon A)^{-1}\Psi_s\big)_E ds \\ &= \int_0^t \big((I + \varepsilon A)^{-1}\Psi_s, A(I + \varepsilon A)^{-1}\Psi_s\big)_E ds \\ &\le \big(r^2 - \frac{\kappa}{2}\big) \int_0^t \big\|(I + \varepsilon A)^{-1}\Psi_s\big\|_E^2 ds, \end{aligned} \tag{3.34}$$

where we used $(I+\varepsilon A^*)^{-1} = ((I+\varepsilon A)^{-1})^*$ and the fact that A^* and $(I+\varepsilon A^*)^{-1}$ commute for the second line, and (3.5) for the fourth line.

For the second term $J_n^2(t)$, since we have

$$\begin{aligned}&\int_0^t \big\|b(X_s) - b(\widetilde{X}_s)\big\|_E \|\Psi_s\|_E ds \\ &\quad\le \big(\sup_{0\le t\le T} \|\Psi_s\|_E\big) \int_0^T \big\|b(X_s) - b(\widetilde{X}_s)\big\|_E ds < \infty, \quad 0 \le t \le T, \quad \mathbb{P}\text{-a.s.,}\end{aligned}$$

Lebesgue's dominated convergence theorem also yields

$$\begin{aligned}\sum_{n=1}^\infty J_n^2(t) &= \int_0^t \sum_{n=1}^\infty \big(\widetilde{\varphi}_n, (I + \varepsilon A)^{-1}\Psi_s\big)_E \cdot \big(\widetilde{\varphi}_n, (I + \varepsilon A)^{-1}\big(b(X_s) - b(\widetilde{X}_s)\big)\big)_E ds \\ &= \int_0^t \Big((I + \varepsilon A)^{-1}\Psi_s, (I + \varepsilon A)^{-1}\big(b(X_s) - b(\widetilde{X}_s)\big)\Big)_E ds. \end{aligned} \tag{3.35}$$

Then by putting (3.34) and (3.35) into (3.33), we have

$$\begin{aligned}\|(I+\varepsilon A)^{-1}\Psi_t\|_E^2 &= 2\sum_{n=1}^{\infty}\big(J_n^1(t)+J_n^2(t)\big)\\ &\le (2r^2-\kappa)\int_0^t \big\|(I+\varepsilon A)^{-1}\Psi_s\big\|_E^2 ds\\ &\quad +2\int_0^t \Big((I+\varepsilon A)^{-1}\Psi_s,(I+\varepsilon A)^{-1}\big(b(X_s)-b(\widetilde{X}_s)\big)\Big)_E ds.\end{aligned}$$

Moreover letting $\varepsilon \searrow 0$ on both sides, and recalling the dissipativity (3.13) for $\widetilde{b}$ and (3.20), we obtain that

$$\|\Psi_t\|_E^2 \le (-K_1+2r^2)\int_0^t \|\Psi_s\|_E^2 ds.$$

Hence, we have $\Psi_t = X_t - \widetilde{X}_t = 0$, $0 \le t \le T$, $\mathbb{P}$-almost surely by an application of Gronwall's inequality, which proves the pathwise uniqueness. Then by [38, Theorem 2], a Yamada–Watanabe type argument implies that SPDE (2.10) has a unique strong solution. This completes the proof. ∎

By repeating the same argument as in the above proof, we can easily deduce the following coupling estimates (3.36) and (3.37) which play crucial roles in the next section.

Corollary 3.2 *Let X^w and $X^{\tilde{w}}$ denote the strong solutions of SPDE* (2.10) *with the initial datum $X_0^w = w \in E \setminus S$ and $X_0^{\tilde{w}} = \tilde{w} \in E \setminus S$, respectively. Then*

$$\big\|X_t^w - X_t^{\tilde{w}}\big\|_E \le e^{\frac{(-K_1+2r^2)t}{2}}\|w-\tilde{w}\|_E, \quad t \ge 0,\ \mathbb{P}\text{-}a.s. \tag{3.36}$$

In addition, for every $h \in H \setminus S$, we have

$$\|X_t^{w+h} - X_t^w\|_H \le e^{-\frac{K_1 t}{2}}\|h\|_H, \quad t \ge 0,\ \mathbb{P}\text{-}a.s. \tag{3.37}$$

4 Some Functional Inequalities

In this section, as an application of Theorem 2.8 and Corollary 3.2, we present some functional inequalities for the diffusion semigroup $\{P_t\}_{t\ge0}$ generated by the Dirichlet operator $\mathcal{L}_\mu$. In particular, we prove the gradient estimate for $\{P_t\}_{t\ge0}$ and logarithmic Sobolev inequalities under much weaker conditions on the regularity and the growth rate of the potential function U than in the previous papers [29, 30] (which however already included the $P(\phi)_1$-case).

There are at present some approaches to derive these functional inequalities, and it is well-known that Bakry–Émery's Γ_2-method (cf. Bakry [15]) works efficiently on finite dimensional complete Riemannian manifolds. In contrast to finite dimensions, we face a big difficulty to define the Γ_2-operator when we work in infinite dimensional frameworks, because it is not so easy to check the existence of a suitable core which is not only a ring but also stable under the operations both of the diffusion semigroup and its generator. Hence, we cannot apply this method directly to the infinite dimensional model in the present paper.

On the other hand, we have the coupling estimates (3.36) and (3.37) which are implied by the strong uniqueness of the solution to SPDE (2.10). By making use of them, we can apply the stochastic approach presented in [29, 30].

First, we give the following gradient estimate for $\{P_t\}_{t\geq 0}$.

Proposition 4.1 (Gradient estimate) *For any $F \in \mathcal{D}(\mathcal{E})$, we have the following gradient estimate*

$$\|D(P_tF)(w)\|_H \leq e^{-\frac{K_1 t}{2}} P_t\big(\|DF\|_H\big)(w), \quad \mu\text{-a.e. } w \in E, \ t > 0. \tag{4.1}$$

Proof: The proof is done in the same manner as the proof of [29, Proposition 2.4] together with the coupling estimate (3.37). So we omit it here. ∎

Now, we are in a position to state logarithmic Sobolev inequalities.

Theorem 4.2 (Log-Sobolev inequalities) (1) *For $F \in \mathcal{D}(\mathcal{E})$, we have the following heat kernel log-Sobolev inequality*

$$\begin{aligned} &P_t(F^2 \log F^2)(w) - P_t(F^2)(w) \log P_t(F^2)(w) \\ &\qquad \leq \frac{2(1-e^{-K_1 t})}{K_1} P_t(\|DF\|_H^2)(w), \quad \mu\text{-a.e. } w \in E, \ t > 0. \end{aligned} \tag{4.2}$$

(2) *If $K_1 > 0$, that is, U is strictly convex, then the following log-Sobolev inequality*

$$\int_E F(w)^2 \log\Big(\frac{F(w)^2}{\|F\|_{L^2(\mu)}^2}\Big)\mu(dw) \leq \frac{2}{K_1}\int_E \|D_H F(w)\|_H^2 \mu(dw), \quad F \in \mathcal{D}(\mathcal{E}) \tag{4.3}$$

holds. Consequently, we have the spectral gap estimate $\inf\big(\sigma(-\mathcal{L}_\mu) \setminus \{0\}\big) \geq \frac{K_1}{2}$.

Proof: We first sketch the proof of (1). We refer to [29, 30] for all technical details. We may assume $F \in \mathcal{F}C_b^\infty$, i.e., $F(w) = f(\langle w, \varphi_1\rangle, \ldots, \langle w, \varphi_n\rangle)$, where $\{\varphi_i\}_{i=1}^n \subset C_0^\infty(\mathbb{R}, \mathbb{R}^d)$. Note that P_tF can be extended to a function in $C_b(E)$ by using the coupling estimate (3.36), and the fact that $\mathrm{supp}(\mu) = E$. We fix $\delta > 0$, and introduce a function $G : [0,t] \to L^1(\mu)$ by

$$G(s) := P_{t-s}\big\{(P_s(F^2)+\delta)\log(P_s(F^2)+\delta)\big\}(\cdot), \quad 0 \leq s \leq t.$$

Then G is differentiable with respect to s and

$$\dot{G}(s) = -\frac{1}{2}P_{t-s}\Big\{\frac{\|DP_s(F^2)\|_H^2}{P_s(F^2)+\delta}\Big\}(\cdot), \quad 0 < s < t. \tag{4.4}$$

On the other hand, Proposition 4.1 and Schwarz's inequality imply

$$\|DP_s(F^2)\|_H^2 \le 4e^{-K_1 s}P_s(F^2)\cdot P_s\big(\|DF\|_H^2\big). \tag{4.5}$$

By combining (4.4) with (4.5), we have

$$\dot{G}(s) \ge -2e^{-K_1 s}P_{t-s}\big\{P_s\big(\|DF\|_H^2\big)\big\} = -2e^{-K_1 s}P_t\big(\|DF\|_H^2\big).$$

This imply the heat kernel logarithmic Sobolev inequality (4.2) by first integrating over s from 0 to t and then by letting $\delta \searrow 0$.

Next, we prove (2). By noting that the Gibbs measure μ is the invariant measure for our stochastic dynamics $\mathbb{M}$, we have the following estimate for $w \in E\setminus S$ and $t \ge 0$:

$$\begin{aligned}\big|P_tF(w) - \mathbb{E}^\mu[F]\big| &\le \int_E \mathbb{E}\big[|F(X_t^w) - F(X_t^{\widetilde{w}})|\big]\mu(d\widetilde{w}) \\ &\le \|\nabla f\|_\infty\Big(\sum_{i=1}^n \|\varphi_i\|_{E^*}^2\Big)^{1/2} e^{(\frac{-K_1+2r^2}{2}t)}\int \|w-\widetilde{w}\|_E\ \mu(d\widetilde{w}) \\ &\le \sqrt{2}\|\nabla f\|_\infty\Big(\sum_{i=1}^n \|\varphi_i\|_{E^*}^2\Big)^{1/2} e^{(\frac{-K_1+2r^2}{2}t)}\Big\{\|w\|_E^2 + \int_E \|\widetilde{w}\|_E^2\mu(d\widetilde{w})\Big\}^{1/2},\end{aligned} \tag{4.6}$$

where we used (3.36) for the second line. Since $r>0$ satisfies $2r^2 < K_1$, (4.6) implies the following ergodic property of $\{P_t\}_{t\ge0}$:

$$\lim_{t\to\infty} P_tF(w) = \mathbb{E}^\mu[F], \quad w \in E\setminus S, \tag{4.7}$$

Finally, we have the desired logarithmic Sobolev inequality (4.3) by letting $t\to\infty$ on both sides of (4.2) and using (4.7). This completes the proof of (2). ∎

Remark 4.3 *The logarithmic Sobolev inequality* (4.3) *holds with $K_1 \ge m^2$ in the case of exp(ϕ)$_1$-quantum fields.*

Remark 4.4 *We mention that many other functional inequalities including the dimension free parabolic Harnack inequality (cf.* [29]*) and the Littlewood–Paley–Stein inequality (cf. Kawabi–Miyokawa* [32]*) for our infinite dimensional model can be obtained from the gradient estimate* (4.1)*. In particular, it is a fundamental and important problem in harmonic analysis and potential theory to ask for boundedness of the Riesz transform $R_\alpha(\mathcal{L}_p) := D_H(\alpha - \mathcal{L}_p)^{-1/2}$ on $L^p(\mu)$ for all $p>1$ and some $\alpha>0$, where $\mathcal{L}_p$ is the extension of $(\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty)$ in $L^p(\mu)$, because boundedness of $R_\alpha(\mathcal{L}_p)$ yields the Meyer equivalence of first order Sobolev norms. In* [44]*, Shigekawa studied this problem in a general framework assuming the intertwining property of the diffusion semigroup $\{P_t\}_{t\ge0}$ and another semigroup $\{\overrightarrow{P}_t\}_{t\ge0}$ acting on vector-valued functions. We note that essential self-adjointness of $(\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty)$ as obtained in Theorem* 2.7 *plays a crucial role to prove this property for our model. (See e.g., Shigekawa* [45] *and Kawabi* [31]*.) We will discuss boundedness of the Riesz transform by making use of the Littlewood–Paley–Stein inequality and this intertwining property in a forthcoming paper.*

5 Appendix: Another Approach to the Log-Sobolev Inequality (4.3)

In this section, we give another approach to the log-Sobolev inequality (4.3). First, we prepare the following lemma taken from Arai–Hirokawa [14, Lemma 4.9]:

Lemma 5.1 *Let $\{T_n\}_{n=1}^{\infty}$ and T be self-adjoint operators on a Hilbert space $\mathcal{H}$ having a common core $\mathcal{D}$ such that, for all $\psi \in \mathcal{D}$, $T_n\psi \to T\psi$ as $n \to \infty$. Let ψ_n be a normalized eigenvectors of T_n with eigenvalue E_n : $T_n\psi_n = E_n\psi_n$. Assume that $E := \lim_{n\to\infty} E_n$ exists and that the weak limit $w\text{-}\lim_{n\to\infty}\psi_n = \psi$ also exists and one has $\psi \neq 0$. Then ψ is an eigenvector of T with eigenvalue E. In particular, if ψ_n is a ground state of T_n, then ψ is a ground state of T.*

Lemma 5.2 *Let $U_N(z) := \frac{K_1}{2}|z|^2 + V_{1/N}(z), N = 1, 2, \ldots,$ be potential functions, where $V_{1/N}$ is the Moreau–Yosida approximation of V. We consider the Schrödinger operator $H_{U_N} = -\frac{1}{2}\Delta_z + U_N$ on $L^2(\mathbb{R}^d, \mathbb{R})$, and denote by $(\lambda_0)_N$ and Ω_N the minimal eigenvalue and the (normalized) ground state of H_{U_N}, respectively. Then the following properties hold under the assumption $K_1 > 0$:*
(1) $(\lambda_0)_N \nearrow \lambda_0$ as $N \to \infty$.
(2) There exists a sub-sequence $\{N(k)\}_{k=1}^{\infty}$ of $N \to \infty$ such that $\|\Omega_{N(k)} - \Omega\|_{L^2(\mathbb{R}^d,\mathbb{R})} \to 0$ as $k \to \infty$.

Proof. (1) Since $U_N \nearrow U$ as $N \to \infty$, we have $(\lambda_0)_1 \leq (\lambda_0)_2 \leq \cdots \leq \lambda_0$. Moreover, recalling (2.1) and taking into account the estimate $(U_N)_{\frac{1}{2}|z|}(z) \geq \frac{K_1}{8}|z|^2, z \in \mathbb{R}^d$ for every $N \in \mathbb{N}$, we have the following uniform pointwise upper bound for $\{\Omega_N\}_{N=1}^{\infty}$:

$$0 < \Omega_N(z) \leq D_1 \exp(-\frac{D_2 K_1^{1/2}}{2\sqrt{2}}|z|^2), \quad z \in \mathbb{R}^d. \tag{5.1}$$

On the other hand, the variational characterization of the minimal eigenvalue and the ground state implies

$$\begin{aligned} (\lambda_0)_N &= \big(\Omega_N, H_{U_N}\Omega_N\big)_{L^2(\mathbb{R}^d,\mathbb{R})} \\ &= \big(\Omega_N, H_U\Omega_N\big)_{L^2(\mathbb{R}^d,\mathbb{R})} - \big(\Omega_N, (V - V_N)\Omega_N\big)_{L^2(\mathbb{R}^d,\mathbb{R})} \\ &\geq \lambda_0 - \big(\Omega_N, (V - V_N)\Omega_N\big)_{L^2(\mathbb{R}^d,\mathbb{R})}, \end{aligned} \tag{5.2}$$

and by Lebesgue's monotone convergence theorem, we also have

$$\begin{aligned} 0 &\leq \lim_{N\to\infty} \big(\Omega_N, (V - V_N)\Omega_N\big)_{L^2(\mathbb{R}^d,\mathbb{R})} \\ &\leq D_1^2 \lim_{N\to\infty} \int_{\mathbb{R}^d} (V(z) - V_N(z)) \exp(-\frac{D_2 K_1^{1/2}}{\sqrt{2}}|z|^2)dz \\ &= D_1^2 \int_{\mathbb{R}^d} \lim_{N\to\infty} (V(z) - V_N(z)) \exp(-\frac{D_2 K_1^{1/2}}{\sqrt{2}}|z|^2)dz = 0, \end{aligned} \tag{5.3}$$

where we used (5.1) for the second line.

Hence by combining (5.2) with (5.3), we have $\lim_{N\to\infty}(\lambda_0)_N \geq \lambda_0$, which completes the proof of (1).

(2) We take $C_0^\infty(\mathbb{R}^d,\mathbb{R})$ as a common core of the Schrödinger operators $\{H_{U_N}\}_{N=1}^\infty$ and H_U (cf. [40, Theorem X.28]), and by Lebesgue's monotone convergence theorem, we can easily see that $H_{U_N}\psi \to H_U\psi$ for all $\psi \in C_0^\infty(\mathbb{R}^d,\mathbb{R})$ as $N\to\infty$. Since $\|\Omega_N\|_{L^2(\mathbb{R}^d,\mathbb{R})}=1$ for all $N\in\mathbb{N}$, there exist a sub-sequence $\{N(k)\nearrow\infty\}$ and a function $\psi\in L^2(\mathbb{R}^d,\mathbb{R})$ such that $\Omega_{N(k)}\to\psi$ weakly as $k\to\infty$. On the other hand, by [48, Theorem 25.16], there exist some positive constants D_3, D_4 independent of N such that

$$\Omega_N(z) \geq D_3 \exp\big(-D_4|z|U_N^{(\infty)}(z)^{1/2}\big), \quad z\in\mathbb{R}^d, \tag{5.4}$$

where $U_N^{(\infty)}(z) := \inf\{U_N(y) \mid |y|\leq 3|z|\}$. Recalling condition **(U3)**, we see that

$$\begin{aligned} U_N^{(\infty)}(z) &\leq \inf\{U(y) \mid |y|\leq 3|z|\} \\ &\leq |U(0)| + 3K_3|z|\exp(3^\beta K_4|z|^\beta), \quad z\in\mathbb{R}^d. \end{aligned} \tag{5.5}$$

Then combining (5.4) with (5.5), we deduce that

$$\begin{aligned} \Omega_N(z) &\geq D_3\exp\Big\{-D_4|z|\sqrt{|U(0)|+3K_3|z|\exp(3^\beta K_4|z|^\beta)}\Big\} \\ &=: \Psi(z), \quad z\in\mathbb{R}^d, \end{aligned} \tag{5.6}$$

and hence the uniform pointwise lower estimate (5.5) implies that

$$\lim_{k\to\infty}(\Omega_{N(k)},\Psi)_{L^2(\mathbb{R}^d,\mathbb{R})} \geq \|\Psi\|^2_{L^2(\mathbb{R}^d,\mathbb{R})} > 0$$

and we now see that $\psi\neq 0$ holds.

Now by item (1) and Lemma 5.1, it follows that ψ is a ground state of H_U. However, since we already know the uniqueness of the ground state of H_U, $\Omega_{N(k)}\to\psi=\Omega$ weakly as $k\to\infty$. Moreover since

$$\lim_{k\to\infty}\|\Omega_{N(k)}\|_{L^2(\mathbb{R}^d,\mathbb{R})} = \|\Omega\|_{L^2(\mathbb{R}^d,\mathbb{R})} = 1,$$

we conclude that $\lim_{k\to\infty}\|\Omega_{N(k)}-\Omega\|_{L^2(\mathbb{R}^d,\mathbb{R})}=0$. ∎

Proof of the Log-Sobolev Inequality (4.3): By the same procedure as in Section 2, we can construct a Gibbs measure μ_N with $\mu_N(\mathcal{C})=1$ if we replace U by U_N. As we have seen in the proof of Theorem 2.7, $\widetilde{\nabla}U_{1/N}(z) = K_1 z + \partial_0(V_{1/N})(z), z\in\mathbb{R}^d$, is Lipschitz continuous. Thus, we can apply [30, Theorem 1.2], and we see that the following logarithmic Sobolev inequality holds for each μ_N:

$$\int_E F(w)^2\log\Big(\frac{F(w)^2}{\|F\|^2_{L^2(\mu_N)}}\Big)\mu_N(dw) \leq \frac{2}{K_1}\int_E \|D_H F(w)\|_H^2\mu_N(dw), \quad F\in\mathcal{FC}_b^\infty. \tag{5.7}$$

Next, we aim to prove the tightness of the family of probability measures $\{\mu_N\}_{N=1}^{\infty}$ on $\mathcal{C}$. Due to [28, Lemma 5.4], it suffices to verify the following two conditions:

(i) There exists a constant $\gamma > 0$ such that $\sup_{N\in\mathbb{N}} \int_{\mathcal{C}} |w(0)|^{\gamma} \mu_N(dw) < \infty$.

(ii) For each $r > 0$, there exist constants $p, q, M > 0$ independent of N such that $\int_{\mathcal{C}} |w(x_1) - w(x_2)|^p \mu_N(dw) \le M|x_1 - x_2|^{2+q} \rho_r(x_1)$ for $x_1, x_2 \in \mathbb{R}$ with $|x_1 - x_2| \le 1$.

By combining the translation invariance of μ_N with estimate (5.1), we have

$$\int_{\mathcal{C}} |w(0)|^2 \mu_N(dw) = \int_{\mathbb{R}^d} |z|^2 \Omega_N(z)^2 dz \le D_1^2 \int_{\mathbb{R}^d} |z|^2 \exp(-\frac{D_2 K_1^{1/2}}{\sqrt{2}} |z|^2) dz.$$

Hence we have shown that condition (i) holds with $\gamma = 2$. Besides, in a similar way to [27], we see that

$$\begin{aligned}
&\int_{\mathcal{C}} |w(x_1) - w(x_2)|^{2m} \mu_N(dw) \\
&\qquad \le \exp\left\{\big((\lambda_0)_N - \inf_{z\in\mathbb{R}^d} U_N(z)\big)|x_1 - x_2|\right\}\big(\sup_{z\in\mathbb{R}^d} \Omega_N(z)\big) \\
&\qquad\quad \times \int_{\mathbb{R}^d} \Omega_N(z) dz \cdot (2m-1)!! \cdot |x_1 - x_2|^m \\
&\qquad \le \exp\left\{\big(\lambda_0 - \inf_{z\in\mathbb{R}^d} U_1(z)\big)|x_1 - x_2|\right\} D_1^2 \Big(\frac{\sqrt{2}\pi}{D_2 K_1^{1/2}}\Big)^{d/2} (2m-1)!! \cdot |x_1 - x_2|^m
\end{aligned}$$

for every $m \in \mathbb{N}$, where $(2m-1)!! := \prod_{k=1}^{m} (2k-1)$ and we used Lemma 5.2 and (5.1) for the third line. Hence we can find a positive constant C independent of N such that

$$\int_{\mathcal{C}} |w(x_1) - w(x_2)|^{2m} \mu_N(dw) \le C|x_1 - x_2|^m, \quad \text{for } x_1, x_2 \in \mathbb{R} \text{ with } |x_1 - x_2| \le 1,$$

and hence we have proven condition (ii).

Thus we can find a sub-sequence $\{N(j) \nearrow \infty\}$ such that $\mu_{N(j)}$ converges to some probability measure μ_* weakly on $\mathcal{C}$. On the other hand, by virtue of the Feynman–Kac formula, we have

$$\lim_{N\to\infty} \|e^{-tH_{U_N}}\psi - e^{-tH_U}\psi\|_{L^2(\mathbb{R}^d,\mathbb{R})} = 0, \quad \psi \in L^2(\mathbb{R}^d, \mathbb{R}). \tag{5.8}$$

Then by putting (5.8) and Lemma 5.2 into (2.2), we see that there exists a sub-sequence $\{N(k) \nearrow \infty\}$ of $\{N(j)\}$ such that $\lim_{k\to\infty} \mu_{N(k)}(A) = \mu(A)$ for each cylinder set $A \in \mathcal{B}_{[T_1,T_2]}, T_1 < T_2$. Hence we obtain $\mu_* = \mu$.

Finally, since $F \in \mathcal{FC}_b^{\infty}$ can be regarded as an element of $C_b(\mathcal{C})$ in a natural way, we can take the limit $k \to \infty$ on both sides of (5.7). This implies the desired inequality (4.3).

∎

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