BV functions in a Gelfand triple and the stochastic reflection problem on a convex set of a Hilbert space *
Fonctions BV dans triplet de Gelfand et le probleme de reflexion sur un ensemble convexe d'un espace de Hilbert

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Abstract

In this note we introduce BV functions in a Gelfand triple, which is an extension of BV functions in [1], by using Dirichlet form theory. By this definition, we can consider the stochastic reflection problem associated with a self-adjoint operator A and a cylindrical Wiener process on a convex set Γ. We prove the existence and uniqueness of a stong solution of this problem when Γ is a regular convex set. The result is also extended to the non-symmetric case. Finally, we extend our results to the case when $\Gamma = K_\alpha$, where $K_\alpha = \{f \in L^2(0,1)|f \geq -\alpha\}, \alpha \geq 0$

Résumé.

Dans ce papier, on introduit des fonctions BV dans un triplet de Gelfand qui est une extension de fonctions BV dans [1] en utilizant la forme de Dirichlet. Par cette définition, on peut considérer le problème de réflexion stochastique associé a un opérateur auto-adjoint A et un processus de Wiener cylindrique sur un ensemble convexe Γ. Nous démontrons l'existence et l'unicité d'une solution forte de ce probleme si Γ et un ensemble convexe régulier. Le résultat est aussi étendu au cas non-symétrique. Finalement, nous utilisons les fonctions BV dans le cas $\Gamma = K_\alpha$, où $K_\alpha = \{f \in L^2(0,1)|f \geq -\alpha\}$, $\alpha \geq 0$.

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1. Dirichlet form and BV functions——Given a real separable Hilbert space H(with scalar product $\langle \cdot, \cdot \rangle$ and norm denoted by $|\cdot|$), assume that:

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Hypothesis 1.1. $A : D(A) \subset H \to H$ is a linear self-adjoint operator on H such that $\langle Ax, x\rangle \geq \delta|x|^2, \forall x \in D(A)$ for some $\delta > 0$. Moreover, A^{-1} is of trace class. $\{e_j\}$ is an orthonormal basis in H consisting of eigen-functions for A, that is, $Ae_j = \alpha_j e_j, j \in \mathbb{N}$, where $\alpha_j \geq \delta$.

In the following $D\varphi : H \to H$ is the Frêchet-derivative of a function $\varphi : H \to \mathbb{R}$. By $C_b^1(H)$ we shall denote the set of all bounded differentiable functions with continuous and bounded derivatives. For $K \subset H$, the space $C_b^1(K)$ is defined as the space of restrictions of all functions in $C_b^1(H)$ to the subset K. μ will denote the Gaussian measure in H with mean 0 and covariance operator $Q := \frac{1}{2}A^{-1}$. For $\rho \in L_+^1(H,\mu)$, we consider $\mathcal{E}^\rho(u,v) = \frac{1}{2}\int_H \langle Du, Dv\rangle \rho(z)\mu(dz), u, v \in C_b^1(F)$, where $F = Supp[\rho \cdot \mu]$ and $L_+^1(H,\mu)$ denotes the set of all non-negative elements in $L^1(H,\mu)$. Let $QR(H)$ be the set of all functions $\rho \in L_+^1(H,\mu)$ such that $(\mathcal{E}^\rho, C_b^1(F))$ is closable on $L^2(F;\rho\cdot\mu)$. Its closure is denoted by $(\mathcal{E}^\rho, \mathcal{F}^\rho)$.

Theorem 1.2. Let $\rho \in QR(H)$. Then $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ is a quasi-regular local Dirichlet form on $L^2(F;\rho\cdot\mu)$ in the sense of [6, IV Definition 3.1].

By virtue of Theorem 1.2 and [6], there exists a diffusion process $M^\rho = (X_t, P_z)$ on F associated with the Dirichlet form $(\mathcal{E}^\rho, \mathcal{F}^\rho)$. M^ρ will be called distorted OU process on F. Since constant functions are in $\mathcal{F}^\rho$ and $\mathcal{E}^\rho(1,1) = 0$, M^ρ is recurrent and conservative. Let $A_{1/2}(x) := \int_0^x (\log(1+s))^{1/2}ds, x \geq 0$ and let ψ be its complementary function, namely, $\psi(y) := \int_0^y (A'_{1/2})^{-1}(t)dt = \int_0^y (\exp(t^2)-1)dt$. Define $L(\log L)^{1/2} := \{f | A_{1/2}(|f|) \in L^1\}, L^\psi := \{g|\psi(c|g|) \in L^1 \text{ for some } c > 0\}$ (cf.[7]). Let $c_j, j \in \mathbb{N}$, be a sequence in $[1,\infty)$. Define $H_1 := \{x \in H | \sum_{j=1}^\infty \langle x, e_j\rangle^2 c_j^2 < \infty\}$, equipped with the inner product $\langle x, y\rangle_{H_1} := \sum_{j=1}^\infty c_j^2 \langle x, e_j\rangle\langle y, e_j\rangle$. Then clearly $(H_1, \langle,\rangle_{H_1})$ is a Hilbert space such that $H_1 \subset H$ continuously and densely. Identifying H with its dual we obtain the continuous and dense embeddings $H_1 \subset H(\equiv H^*) \subset H_1^*$. It follows that ${}_{H_1}\langle z, v\rangle_{H_1^*} = \langle z, v\rangle_H \forall z \in H_1, v \in H$ and that (H_1, H, H_1^*) is a Gelfand triple. We also introduce a family of H-valued function on H by

$$(C_b^1)_{D(A)\cap H_1} = \{G : G(z) = \sum_{j=1}^m g_j(z)l^j, g_j \in C_b^1(H), l^j \in D(A) \cap H_1\}$$

Denote by D^* the adjoint of $D : C_b^1(H) \subset L^2(H,\mu) \to L^2(H,\mu;H)$. For $\rho \in L(\log L)^{1/2}(H,\mu)$, we put $V(\rho) := \sup_{G\in(C_b^1)_{D(A)\cap H_1}, \|G\|_{H_1}\leq 1} \int_H D^*G(z)\rho(z)\mu(dz)$. A function ρ on H is called a BV function in the Gelfand triple (H_1, H, H_1^*)(denoted $\rho \in BV(H, H_1)$ in notation), if $\rho \in L(\log L)^{1/2}(H,\mu)$ and $V(\rho)$ is finite. When $H_1 = H = H_1^*$, this coincides with the definition of BV functions defined in [1] and clearly $BV(H,H) \subset BV(H,H_1)$. This definition is a modification of BV function in abstract Wiener space introduced in [3] and [4].

Theorem 1.3. (i) Suppose $\rho \in BV(H,H_1) \cap L_+^1(H,\mu)$, then there exist a positive finite measure $\|d\rho\|$ on H and a Borel-measurable map $\sigma_\rho : H \to H_1^*$ such that $\|\sigma_\rho(z)\|_{H_1^*} = 1$ $\|d\rho\| - a.e, V(\rho) = \|d\rho\|(H)$,

$$\int_H D^*G(z)\rho(z)\mu(dz) = \int_H {}_{H_1}\langle G(z), \sigma_\rho(z)\rangle_{H_1^*} \|d\rho\|(dz), \forall G \in (C_b^1)_{D(A)\cap H_1}. \tag{1.1}$$

Further, if $\rho \in QR(H)$, $\|d\rho\|$ is $\mathcal{E}^\rho$-smooth, also, σ_ρ and $\|d\rho\|$ are uniquely determined.
(ii) Conversely, if Eq.(1.1) holds for $\rho \in L(\log L)^{1/2}(H,\mu)$ and for some positive finite measure $\|d\rho\|$ and a map σ_ρ with the stated properties, then $\rho \in BV(H,H_1)$ and $V(\rho) = \|d\rho\|(H)$.

Theorem 1.4 Let $\rho \in QR(H) \cap BV(H,H_1)$ and consider the measure $\|d\rho\|$ and σ_ρ from Theorem 1.3(i). Then there is an $\mathcal{E}^\rho$-exceptional set $S \subset F$ such that $\forall z \in F\backslash S$, under P_z there exists an $\mathcal{M}_t$- cylindrical Wiener process W^z, such that the sample paths of the associated distorted

OU-process M^ρ on F satisfy the following: for $l \in D(A) \cap H_1$

$$\langle l, X_t - X_0 \rangle = \int_0^t \langle l, dW_s^z \rangle + \frac{1}{2}\int_0^t {}_{H_1}\langle l, \sigma_\rho(X_s)\rangle_{H_1^*} dL_s^{\|d\rho\|} - \int_0^t \langle Al, X_s \rangle ds \ \forall t \geq 0 \ P_z\text{-a.s.}$$

Here $L_t^{\|d\rho\|}$ is the real valued PCAF associated with $\|d\rho\|$ by the Revuz correspondence.

2. Reflected OU process——Consider the situation when $\rho = I_\Gamma$, the indicator of a set.

Remark 2.1 We emphasize that if Γ is a convex closed set in H, then for each $z, l \in H$ the set $\{s \in \mathbb{R} | z + sl \in \Gamma\}$ is a closed interval in $\mathbb{R}$, whose indicator function hence trivially has the Hamza property. Hence, in particular, $I_\Gamma \in QR(H)$.

2.1 Reflected OU processes on regular convex set——Denote the corresponding objects $\sigma_\rho, \|dI_\Gamma\|$ in Theorem 1.3(i) by $-\mathbf{n}_\Gamma, \|\partial\Gamma\|$, respectively.

Hypothesis 2.1.1 There exists a convex C^∞ function $g : H \to R$ with $g(0) = 0, g'(0) = 0$, and $D^2 g$ strictly positively definite, that is,$\langle D^2 g(x)h, h\rangle \geq \gamma |h|^2, \forall h \in H$ where $\gamma > 0$, such that

$$\Gamma = \{x \in H : g(x) \leq 1\}, \partial\Gamma = \{x \in H : g(x) = 1\}$$

Moreover, we also suppose that $D^2 g$ is bounded on Γ. Finally, we also suppose that g and all its derivatives grow at infinity at most polynomially.

By using [2, Lemma 2.1], we have (1.1) for $\rho = I_\Gamma$ with $H = H_1$. By the continuity property of surface measure given in [5], we have the following two theorems.

Theorem 2.1.2 Assume Hypothesis 2.1.1. Then $I_\Gamma \in BV(H, H) \cap QR(H)$.

Theorem 2.1.3 Assume Hypothesis 2.1.1. Then there exists an $\mathcal{E}^\rho$-exceptional set $S \subset F$ such that $\forall z \in F \backslash S$, under P_z there exists an $\mathcal{M}_t$- cylindrical Wiener process W^z, such that the sample paths of the associated reflected OU-process M^ρ on F with $\rho = I_\Gamma$ satisfy the following: for $l \in D(A) \cap H_1$

$$\langle l, X_t - X_0 \rangle = \int_0^t \langle l, dW_s^z \rangle - \frac{1}{2}\int_0^t \langle l, \mathbf{n}_\Gamma(X_s) dL_s^{\|\partial\Gamma\|}\rangle - \int_0^t \langle Al, X_s \rangle ds \ \forall t \geq 0 P_z - a.e.$$

where $\mathbf{n}_\Gamma := \frac{Dg}{|Dg|}$ is the exterior normal to Γ, satisfying $\langle \mathbf{n}_\Gamma(x), x - y\rangle \geq 0$, for any $y \in \Gamma, x \in \partial\Gamma$ and $\|\partial\Gamma\| = \mu_{\partial\Gamma}$, where $\mu_{\partial\Gamma}$ is the surface measure induced by μ (c.f [2], [5]).

Let Γ satisfy Hypothesis 2.1.1 and A satisfy Hypothesis 1.1. Consider the following stochastic differential inclusion in the Hilbert space H,

$$\begin{cases} dX(t) + (AX(t) + N_\Gamma(X(t)))dt \ni dW(t), \\ X(0) = x \end{cases} \tag{2.1}$$

where $W(t)$ is a cylindrical Wiener process in H on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and $N_\Gamma(x)$ is the normal cone to Γ at x.

Definition 2.1.4 A pair of continuous $H \times R$ valued and $\mathcal{F}_t$-adapted processes $(X(t), L(t)), t \in [0, T]$, is called a solution of (2.1) if the following conditions hold:
(i) $X(t) \in \Gamma, P - a.s.$ for all $t \in [0, T]$,
(ii) L is an increasing process with the property $\int_0^t I_{\partial\Gamma}(X_s(\omega)) dL_s(\omega) = L_t(\omega), t \geq 0$ and we have for any $l \in D(A)$, $\langle l, X_t(\omega) - x\rangle = \langle l, W_t(\omega) - \int_0^t \mathbf{n}_\Gamma(X_s(\omega)) dL_s(\omega)\rangle - \langle Al, \int_0^t X_s(\omega) ds\rangle$ where $\mathbf{n}_\Gamma$ is the exterior normal to Γ, satisfying $\langle \mathbf{n}_\Gamma(x), x - y\rangle \geq 0, \forall y \in \Gamma, x \in \partial\Gamma$.

Theorem 2.1.5 If Γ satisfies Hypothesis 2.1.1, then there exists M, $I_\Gamma \cdot \mu(M) = 1$, such that for every $x \in M$, (2.1) has a pathwise unique continuous strong solution in the sense of Definition 2.1.4, such that $X(t) \in M$ for all $t \geq 0$ P_x-a.s.

Remark 2.1.6 We can extend all these results to non-symmetric Dirichlet forms obtained by first order perturbation of the above Dirichlet form.

2.2 Reflection OU processes on a class of convex sets——Now we consider the case when $H = L^2(0,1), \rho = I_{K_\alpha}$, where $K_\alpha = \{f \in H | f \geq -\alpha\}, \alpha \geq 0$ and $A = -\frac{1}{2}\frac{d^2}{dr^2}$ with Dirichlet boundary condition on [0,1]. Take $c_j = (j\pi)^{\frac{1}{2}+\varepsilon}$ if $\alpha > 0$, $c_j = (j\pi)^\beta$ if $\alpha = 0$, where $\varepsilon \in (0, \frac{3}{2}]$ and $\beta \in (\frac{3}{2}, 2]$ respectively. By using [8, (1) (2), Theorem 5], we can prove the following theorem.

Theorem 2.2.1 $I_{K_\alpha} \in BV(H, H_1) \cap QR(H)$.

Remark 2.2.2 It has been proved by Guan Qingyang that I_{K_α} is not in $BV(H,H)$. Since we have Theorem 2.2.1, we denote the corresponding objects σ_ρ, $\|dI_{K_\alpha}\|$ in Theorem 1.3 (i) by $n_\alpha, |\sigma_\alpha|$, respectively.

Theorem 2.2.3 Let $\rho = I_{K_\alpha}$. Then there is an $\mathcal{E}^\rho$-exceptional set $S \subset F$ such that $\forall z \in F \backslash S$, under P_z there exists an $\mathcal{M}_t$- cylindrical Wiener process W^z, such that the sample paths of the associated distorted OU-process M^ρ on F satisfy the following: for $l \in D(A) \cap H_1$

$$\langle l, X_t - X_0 \rangle = \int_0^t \langle l, dW_s \rangle + \frac{1}{2}\int_0^t {}_{H_1}\langle l, n_\alpha(X_s)\rangle_{H_1^*} dL_s^{|\sigma_\alpha|} - \int_0^t \langle Al, X_s \rangle ds \ P_z - a.e.$$

Here, $L_t^{|\sigma_\alpha|}(\omega)$ is a real valued PCAF associated with $|\sigma_\alpha|$ by the Revuz correspondence, satisfying $I_{\{X_s+\alpha\neq 0\}} dL_s^{|\sigma_\alpha|} = 0$, and for every $z \in F$, $P_z[X_t \in C_0[0,1]$ for a.e. $t \in [0,\infty)] = 1$

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