SPDE in Hilbert Space with Locally Monotone Coefficients *

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Abstract

The aim of this paper is to extend the usual framework of SPDE with monotone coefficients to include a large class of cases with merely locally monotone coefficients. This new framework is conceptually not more involved than the classical one, but includes many more fundamental examples not included previously. Thus our main result can be applied to various types of SPDEs such as stochastic reaction-diffusion equations, stochastic Burgers type equation, stochastic 2-D Navier-Stokes equation, stochastic p-Laplace equation and stochastic porous media equation with non-monotone perturbations.

AMS Subject Classification: 60H15, 37L30, 34D45
Keywords: Stochastic evolution equation; locally monotone; Navier-Stokes equation; Burgers type equation; variational approach.

1 Introduction

Let

$$V \subset H \equiv H^* \subset V^*$$

be a Gelfand triple, *i.e.* $(H, \langle \cdot, \cdot \rangle_H)$ is a separable Hilbert space and identified with its dual space by the Riesz isomorphism, V is a reflexive Banach space such that it is continuously and densely embedded into H. If ${}_{V^*}\langle \cdot, \cdot \rangle_V$ denotes the dualization between V and its dual space V^*, then it follows that

$${}_{V^*}\langle u, v \rangle_V = \langle u, v \rangle_H, \ u \in H, v \in V.$$

*Supported in part by the SFB-701, the BiBoS-Research Center and NNSFC(10721091).
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Let $\{W_t\}_{t\geq 0}$ be a cylindrical Wiener process on a separable Hilbert space U w.r.t a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and $(L_2(U;H), \|\cdot\|_2)$ denotes the space of all Hilbert-Schmidt operators from U to H. We consider the following stochastic evolution equation

$$\mathrm{d}X_t = A(t, X_t)\mathrm{d}t + B(t, X_t)\mathrm{d}W_t, \tag{1.1}$$

where for some fixed time T

$$A : [0,T] \times V \times \Omega \to V^*; \quad B : [0,T] \times V \times \Omega \to L_2(U;H)$$

are progressively measurable, *i.e.* for every $t \in [0,T]$, these maps restricted to $[0,t] \times V \times \Omega$ are $\mathcal{B}([0,t]) \otimes \mathcal{B}(V) \otimes \mathcal{F}_t$-measurable (where $\mathcal{B}$ denotes the corresponding Borel σ-algebra).

It is well known that (1.1) has a unique solution if A, B satisfy the classical monotone and coercivity conditions (cf. [19, 29]), which we recall in the Appendix below. The theory of monotone operators starts from substantial work of Minty [26] and Browder [8] for PDE. We refer to [39, 36] for a detailed exposition and references. In recent years, this variational approach has been also used intensively for analyzing SPDE driven by infinite dimensional Wiener process. Unlike the semigroup approach (cf. [12]), it is not necessary to have a linear operator in the drift part which has to generate a semigroup. Hence the variational approach can be used to investigate nonlinear SPDE which are not necessarily of semilinear type. For general results on the existence and uniqueness of solutions to SPDE we refer to [28, 19, 14, 30, 38]. Within this framework many different types of properties have already been established, *e.g.* see [10, 23, 31, 35] for the small noise large deviation principle, [15, 16] for discretization approximation schemes to the solutions of SPDE, [37, 20, 21, 22] for the dimension-free Harnack inequality and resulting ergodicity, compactness and contractivity properties of the associated transition semigroups, and [24, 5, 13] for the invariance of subspaces and existence of random attractors for corresponding random dynamical systems. As one typical example of SPDE in this framework, the stochastic porous media equation has been extensively studied in [1, 2, 3, 4, 11, 34].

The main aim of this paper is to provide a more general framework for the variational approach, being conceptually not more complicated than the classical one (cf. [19]), but including a large number of new applications as e.g. fundamental SPDE as the stochastic 2-D Navier-Stokes equation and stochastic Burgers type equation. The main changes consist of localizing the monotonicity condition and relaxing the growth condition. This new framework is, in addition, more stable with respect to perturbations. We refer to Section 3 below for details. In particular, we can simplify the related approach to the stochastic 2-D Navier-Stokes equation in the nice paper [25], which inspired us a lot to start this work. However, our approach also easily covers the case of arbitrary multiplicative noise, whereas in [25] only additive noise was considered. It is also straight forward to extend our new framework to more noise terms, e.g. Levy noise (cf. [14] for the classical case). This and further new applications will be the subject of future work.

Let us now state the precise conditions on the coefficients of (1.1):

Suppose there exist constants $\alpha > 1$, $\beta \geq 0$, $\theta > 0$, K and a positive adapted process $f \in L^1([0,T] \times \Omega; \mathrm{d}t \times \mathbb{P})$ such that the following conditions hold for all $v, v_1, v_2 \in V$ and $(t, \omega) \in [0,T] \times \Omega$.

$(H1)$ (Hemicontinuity) The map $s \mapsto {}_{V^*}\langle A(t, v_1 + sv_2), v\rangle_V$ is continuous on $\mathbb{R}$.

$(H2)$ (Local Monotonicity)

$$2{}_{V^*}\langle A(t, v_1) - A(t, v_2), v_1 - v_2\rangle_V + \|B(t, v_1) - B(t, v_2)\|_2^2 \leq (K + \rho(v_2)) \|v_1 - v_2\|_H^2,$$

where $\rho : V \to [0, +\infty)$ is a measurable function and locally bounded in V.

$(H3)$ (Coercivity)

$$2{}_{V^*}\langle A(t, v), v\rangle_V + \|B(t, v)\|_2^2 + \theta\|v\|_V^\alpha \leq f_t + K\|v\|_H^2.$$

$(H4)$ (Growth)

$$\|A(t, v)\|_{V^*}^{\frac{\alpha}{\alpha-1}} \leq (f_t + K\|v\|_V^\alpha)(1 + \|v\|_H^\beta).$$

Remark 1.1. (1) $(H2)$ is essentially weaker than the standard monotonicity $(A2)$ (i.e. $\rho \equiv 0$). One typical form of $(H2)$ in applications is

$$\rho(v) = C\|v\|^\gamma,$$

where $\|\cdot\|$ is some norm on V and C, γ are some constants.

One typical example is the stochastic 2-D Navier-Stokes equation on a bounded or unbounded domain, which satisfies $(H2)$ but does not satisfy $(A2)$ (see Section 3). In fact, if $A(t, v) = \nu P_H \Delta v - P_H [(v \cdot \nabla)v]$, we have

$$2{}_{V^*}\langle A(t, v_1) - A(t, v_2), v_1 - v_2\rangle_V \leq -\frac{\nu}{2}\|v_1 - v_2\|_V^2 + \left(\nu + \frac{16}{\nu^3}\|v_2\|_{L^4}^4\right) \|v_1 - v_2\|_H^2.$$

(2) $(H4)$ is also weaker than the standard growth condition $(A4)$ (see the Appendix) assumed in the literature (cf. [19, 39, 29]). The advantage of $(H4)$ is, *e.g.*, to include many semilinear type equations with nonlinear perturbation terms. For example, if we consider a reaction-diffusion type equation, *i.e.* $A(u) = \Delta u + F(u)$, then for verifying $(H3)$ we have $\alpha = 2$. Hence $(A4)$ would imply that F has at most linear growth. However, we can allow F to have some polynomial growth by using the weaker condition $(H4)$ here. We refer to Section 3 for more details.

Definition 1.1. (Solution of SEE) A continuous H-valued $(\mathcal{F}_t)$-adapted process $\{X_t\}_{t\in[0,T]}$ is called a solution of (1.1), if for its $\mathrm{d}t \otimes \mathbb{P}$-equivalent class $\bar{X}$ we have

$$\bar{X} \in L^\alpha([0, T] \times \Omega, \mathrm{d}t \otimes \mathbb{P}; V) \cap L^2([0, T] \times \Omega, \mathrm{d}t \otimes \mathbb{P}; H)$$

and $\mathbb{P} - a.s.$,

$$X_t = X_0 + \int_0^t A(s, \bar{X}_s)\mathrm{d}s + \int_0^t B(s, \bar{X}_s)\mathrm{d}W_s, \ t \in [0, T].$$

Now we can state the main result.

Theorem 1.1. *Suppose* $(H1)-(H4)$ *hold for* $f \in L^{p/2}([0,T]\times\Omega; \mathrm{d}t\times\mathbb{P})$ *with some* $p \geq \beta+2$, *and there exists a constant* C *such that*

$$\begin{aligned} &\|B(t,v)\|_2^2 \leq C(f_t + \|v\|_H^2),\ t \in [0,T], v \in V; \\ &\rho(v) \leq C(1+\|v\|_V^\alpha)(1+\|v\|_H^\beta),\ v \in V. \end{aligned} \tag{1.2}$$

Then for any $X_0 \in L^p(\Omega \to H; \mathcal{F}_0; \mathbb{P})$ (1.1) *has a unique solution* $\{X_t\}_{t\in[0,T]}$ *and satisfies*

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|X_t\|_H^p + \int_0^T \|X_t\|_V^\alpha \mathrm{d}t\right) < \infty.$$

Moreover, if $A(t,\cdot)(\omega), B(t,\cdot)(\omega)$ *are independent of* $t \in [0,T]$ *and* $\omega \in \Omega$, *then the solution* $\{X_t\}_{t\in[0,T]}$ *of (1.1) is a Markov process.*

2 Proof of the main theorem

The first step of the proof is mainly based on the Galerkin approximation. Let

$$\{e_1, e_2, \cdots\} \subset V$$

be an orthonormal basis of H and let $H_n := span\{e_1, \cdots, e_n\}$ such that $span\{e_1, e_2, \cdots\}$ is dense in V. Let $P_n : V^* \to H_n$ be defined by

$$P_n y := \sum_{i=1}^n {}_{V^*}\langle y, e_i\rangle_V e_i,\ y \in V^*.$$

Obviously, $P_n|_H$ is just the orthogonal projection onto H_n in H and we have

$${}_{V^*}\langle P_n A(t,u), v\rangle_V = \langle P_n A(t,u), v\rangle_H = {}_{V^*}\langle A(t,u), v\rangle_V,\ u \in V, v \in H_n.$$

Let $\{g_1, g_2, \cdots\}$ be an orthonormal basis of U and

$$W_t^{(n)} := \sum_{i=1}^n \langle W_t, g_i\rangle_U g_i = \tilde{P}_n W_t,$$

where $\tilde{P}_n$ is the orthogonal projection onto $span\{g_1, \cdots, g_n\}$ in U.

Then for each finite $n \in \mathbb{N}$ we consider the following stochastic equation on H_n

$$\mathrm{d}X_t^{(n)} = P_n A(t, X_t^{(n)})\mathrm{d}t + P_n B(t, X_t^{(n)})\mathrm{d}W_t^{(n)},\ X_0^{(n)} = P_n X_0. \tag{2.1}$$

By the classical result for the solvability of SDE in finite dimensional space [19, 29] we know that (2.1) has a unique strong solution.

In order to construct the solution of (1.1), we need some a priori estimates for $X^{(n)}$. For convenience we use following notations:

$$\begin{aligned} &K = L^\alpha([0,T]\times\Omega \to V; \mathrm{d}t\times\mathbb{P}); \\ &K^* = L^{\frac{\alpha}{\alpha-1}}([0,T]\times\Omega \to V^*; \mathrm{d}t\times\mathbb{P}); \\ &J = L^2([0,T]\times\Omega \to L_2(U;H); \mathrm{d}t\times\mathbb{P}). \end{aligned}$$

Lemma 2.1. *Under the assumptions in Theorem 1.1, there exists $C > 0$ such that for all $n \in \mathbb{N}$*

$$\|X^{(n)}\|_K + \sup_{t\in[0,T]} \mathbb{E}\|X_t^{(n)}\|_H^2 \leq C.$$

Proof. The conclusion follows from $(H3)$ by using the same argument as in [29, Lemma 4.2.9]. Hence we omit the details here. □

Lemma 2.2. *Under the assumptions in Theorem 1.1, there exists $C > 0$ such that for all $n \in \mathbb{N}$ we have*

$$\mathbb{E}\sup_{t\in[0,T]} \|X_t^{(n)}\|_H^p + \mathbb{E}\int_0^T \|X_t^{(n)}\|_H^{p-2}\|X_t^{(n)}\|_V^\alpha \mathrm{d}t \leq C\left(\mathbb{E}\|X_0\|_H^p + \mathbb{E}\int_0^T f_t^{p/2}\mathrm{d}t\right). \tag{2.2}$$

In particular, there exists $C > 0$ such that for all $n \in \mathbb{N}$

$$\|A(\cdot, X^{(n)})\|_{K^*} \leq C.$$

Proof. By Itô's formula, Young's inequality and (1.2) we have

$$\begin{aligned}
\|X_t^{(n)}\|_H^p =& \|X_0^{(n)}\|_H^p + p(p-2)\int_0^t \|X_s^{(n)}\|_H^{p-4}\|(P_nB(s,X_s^{(n)})\tilde{P}_n)^* X_s^{(n)}\|_H^2 \mathrm{d}s \\
&+ \frac{p}{2}\int_0^t \|X_s^{(n)}\|_H^{p-2}\left(2_{V^*}\langle A(s,X_s^{(n)},X_s^{(n)}\rangle_V + \|P_nB(s,X_s^{(n)})\tilde{P}_n\|_2^2\right)\mathrm{d}s \\
&+ p\int_0^t \|X_s^{(n)}\|_H^{p-2}\langle X_s^{(n)}, P_nB(s,X_s^{(n)})\mathrm{d}W_s^{(n)}\rangle_H \\
\leq& \|X_0\|_H^p - \frac{p\theta}{2}\int_0^t \|X_s^{(n)}\|_H^{p-2}\|X_s^{(n)}\|_V^\alpha \mathrm{d}s \\
&+ C\int_0^t \left(\|X_s^{(n)}\|_H^p + f_s\cdot\|X_s^{(n)}\|_H^{p-2}\right)\mathrm{d}s \\
&+ p\int_0^t \|X_s^{(n)}\|_H^{p-2}\langle X_s^{(n)}, P_nB(s,X_s^{(n)})\mathrm{d}W_s^{(n)}\rangle_H \\
\leq& \|X_0\|_H^p - \frac{p\theta}{2}\int_0^t \|X_s^{(n)}\|_H^{p-2}\|X_s^{(n)}\|_V^\alpha \mathrm{d}s \\
&+ C\int_0^t \left(\|X_s^{(n)}\|_H^p + f_s^{p/2}\right)\mathrm{d}s \\
&+ p\int_0^t \|X_s^{(n)}\|_H^{p-2}\langle X_s^{(n)}, P_nB(s,X_s^{(n)})\mathrm{d}W_s^{(n)}\rangle_H, \quad t\in[0,T],
\end{aligned} \tag{2.3}$$

where C is a generic constant (independent of n) and may change from line to line.

For any given n we define the stopping time

$$\tau_R^{(n)} = \inf\{t\in[0,T] : \|X_t^{(n)}\|_H > R\}\wedge T,\ R > 0.$$

Here we take $\inf \emptyset = \infty$. It's obvious that

$$\lim_{R\to\infty} \tau_R^{(n)} = T,\ \mathbb{P}-a.s.,\ n \in \mathbb{N}.$$

Then by the Burkholder-Davis-Gundy inequality we have

$$\begin{aligned}
&\mathbb{E} \sup_{r\in[0,t]} \left| \int_0^r \|X_s^{(n)}\|_H^{p-2} \langle X_s^{(n)}, P_n B(s, X_s^{(n)}) \mathrm{d}W_s^{(n)} \rangle_H \right| \\
\leq & 3\mathbb{E} \left(\int_0^t \|X_s^{(n)}\|_H^{2p-2} \|B(s, X_s^{(n)})\|_2^2 \mathrm{d}s \right)^{1/2} \\
\leq & 3\mathbb{E} \left(\sup_{s\in[0,t]} \|X_s^{(n))}\|_H^{2p-2} \cdot C \int_0^t \left(\|X_s^{(n))}\|_H^2 + f_s \right) \mathrm{d}s \right)^{1/2} \qquad (2.4) \\
\leq & 3\mathbb{E} \left[\varepsilon \sup_{s\in[0,t]} \|X_s^{(n)}\|_H^p + C_\varepsilon \left(\int_0^t (\|X_s^{(n)}\|_H^2 + f_s) \mathrm{d}s \right)^{p/2} \right] \\
\leq & 3\varepsilon \mathbb{E} \sup_{s\in[0,t]} \|X_s^{(n)}\|_H^p + 3 \cdot (2T)^{p/2-1} C_\varepsilon \mathbb{E} \int_0^t \left(\|X_s^{(n)}\|_H^p + f_s^{p/2} \right) \mathrm{d}s,\ t \in [0, \tau_R^{(n)}],
\end{aligned}$$

where $\varepsilon > 0$ is a small constant and C_ε comes from Young's inequality.

Then by (2.3), (2.4) and Gronwall's lemma we have

$$\mathbb{E} \sup_{t\in[0,\tau_R^{(n)}]} \|X_t^{(n)}\|_H^p + \mathbb{E} \int_0^{\tau_R^{(n)}} \|X_s^{(n)}\|_H^{p-2} \|X_s^{(n)}\|_V^\alpha \mathrm{d}s \leq C \left(\mathbb{E}\|X_0\|_H^p + \mathbb{E} \int_0^T f_s^{p/2} \mathrm{d}s \right),\ n \geq 1,$$

where C is a constant independent of n.

For $R \to \infty$, (2.2) follows from the monotone convergence theorem.

Moreover, by $(H4)$ and $p \geq \beta + 2$ we have

$$\|A(\cdot, X^{(n)})\|_{K^*} \leq C,\ n \geq 1,$$

where C is a constant independent of n. □

Proof of Theorem 1.1. (1) Existence: By Lemma 2.1 and 2.2 there exists a subsequence $n_k \to \infty$ such that

(i) $X^{(n_k)} \to \bar{X}$ weakly in K and weakly star in $L^p(\Omega; L^\infty([0,T]; H))$.

(ii) $Y^{(n_k)} := A(\cdot, X^{(n_k)}) \to Y$ weakly in K^*.

(iii) $Z^{(n_k)} := P_{n_k} B(\cdot, X^{(n_k)}) \to Z$ weakly in J and hence

$$\int_0^\cdot P_{n_k} B(s, X_s^{(n_k)}) \mathrm{d}W_s^{(n_k)} \to \int_0^\cdot Z_s \mathrm{d}W_s$$

weakly in $L^\infty([0,T], \mathrm{d}t; L^2(\Omega, \mathbb{P}; H))$.

Now we define

$$X_t := X_0 + \int_0^t Y_s \mathrm{d}s + \int_0^t Z_s \mathrm{d}W_s,\ t \in [0,T], \tag{2.5}$$

it is easy to show that $X = \bar{X}$ $\mathrm{d}t \otimes \mathbb{P}$-a.e.

Then by [29, Theorem 4.2.5] we know that X is an H-valued continuous $(\mathcal{F}_t)$-adapted process and

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|X_t\|_H^p + \int_0^T \|X_t\|_V^\alpha \mathrm{d}t\right) < \infty.$$

Therefore, it remains to verify that

$$A(\cdot, \bar{X}) = Y,\ B(\cdot, \bar{X}) = Z\ \mathrm{d}t \otimes \mathbb{P} - a.e.$$

Define

$$\mathcal{M} = \left\{\phi : \phi \text{ is } V\text{-valued } (\mathcal{F}_t)\text{-adapted process such that } \mathbb{E}\int_0^T \rho(\phi_s) ds < \infty\right\}.$$

For $\phi \in K \cap \mathcal{M} \cap L^p(\Omega; L^\infty([0,T]; H))$,

$$\begin{aligned}
&\mathbb{E}\left(e^{-\int_0^t (K+\rho(\phi_s))\mathrm{d}s} \|X_t^{(n_k)}\|_H^2\right) - \mathbb{E}\left(\|X_0^{(n_k)}\|_H^2\right) \qquad (2.6)\\
=&\mathbb{E}\Bigg[\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\mathrm{d}r}\Bigg(2_{V^*}\langle A(s, X_s^{(n_k)}), X_s^{(n_k)}\rangle_V + \|P_{n_k} B(s, X_s^{(n_k)})\tilde{P}_{n_k}\|_2^2\\
&- (K+\rho(\phi_s))\|X_s^{(n_k)}\|_H^2\Bigg)\mathrm{d}s\Bigg] \qquad (2.7)\\
\le&\mathbb{E}\Bigg[\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\mathrm{d}r}\Bigg(2_{V^*}\langle A(s, X_s^{(n_k)}), X_s^{(n_k)}\rangle_V + \|B(s, X_s^{(n_k)})\|_2^2\\
&- (K+\rho(\phi_s))\|X_s^{(n_k)}\|_H^2\Bigg)\mathrm{d}s\Bigg] \qquad (2.8)\\
=&\mathbb{E}\Bigg[\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\mathrm{d}r}\Bigg(2_{V^*}\langle A(s, X_s^{(n_k)}) - A(s,\phi_s), X_s^{(n_k)} - \phi_s\rangle_V\\
&+ \|B(s, X_s^{(n_k)}) - B(s,\phi_s)\|_2^2 - (K+\rho(\phi_s))\|X_s^{(n_k)} - \phi_s\|_H^2\Bigg)\mathrm{d}s\Bigg]\\
&+ \mathbb{E}\Bigg[\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\mathrm{d}r}\Bigg(2_{V^*}\langle A(s, X_s^{(n_k)}) - A(s,\phi_s), \phi_s\rangle_V + 2_{V^*}\langle A(s,\phi_s), X_s^{(n_k)}\rangle_V\\
&- \|B(s,\phi_s)\|_2^2 + 2\langle B(s, X_s^{(n_k)}), B(s,\phi_s)\rangle_{L_2(U,H)}\\
&- 2(K+\rho(\phi_s))\langle X_s^{(n_k)}, \phi_s\rangle_H + (K+\rho(\phi_s))\|\phi_s\|_H^2\Bigg)\mathrm{d}s\Bigg].
\end{aligned}$$

Let $k \to \infty$, by $(H2)$ and the lower semicontinuity (cf. e.g. [29, (4.2.27)] for details) we have for every nonnegative $\psi \in L^\infty([0,T];\mathrm{d}t)$,

$$
\begin{aligned}
(2.9)\qquad & \mathbb{E}\left[\int_0^T \psi_t \left(e^{-\int_0^t (K+\rho(\phi_s))\mathrm{d}s}\|X_t\|_H^2 - \|X_0\|_H^2\right)\right] \\
\leq & \liminf_{k\to\infty} \mathbb{E}\left[\int_0^T \psi_t \left(e^{-\int_0^t (K+\rho(\phi_s))\mathrm{d}s}\|X_t^{(n_k)}\|_H^2 - \|X_0^{(n_k)}\|_H^2\right)\right] \\
\leq & \mathbb{E}\Bigg[\int_0^T \psi_t \Bigg(\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\mathrm{d}r}\Bigg(2_{V^*}\langle Y_s - A(s,\phi_s),\phi_s\rangle_V \\
& + 2_{V^*}\langle A(s,\phi_s),\bar{X}_s\rangle_V - \|B(s,\phi_s)\|_2^2 + 2\langle Z_s, B(s,\phi_s)\rangle_{L_2(U,H)} \\
& - 2(K+\rho(\phi_s))\langle X_s,\phi_s\rangle_H + (K+\rho(\phi_s))\|\phi_s\|_H^2\Bigg)\mathrm{d}s\Bigg)\mathrm{d}t\Bigg].
\end{aligned}
$$

By Itô's formula we have for $\phi \in K \cap \mathcal{M} \cap L^p(\Omega; L^\infty([0,T];H))$,

$$
\begin{aligned}
(2.10)\qquad & \mathbb{E}\left(e^{-\int_0^t (K+\rho(\phi_s))\mathrm{d}s}\|X_t\|_H^2\right) - \mathbb{E}\left(\|X_0\|_H^2\right) \\
= & \mathbb{E}\left[\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\mathrm{d}r}\left(2_{V^*}\langle Y_s,\bar{X}_s\rangle_V + \|Z_s\|_2^2 - (K+\rho(\phi_s))\|X_s\|_H^2\right)\mathrm{d}s\right].
\end{aligned}
$$

By inserting (2.10) into (2.9) we obtain

$$
\begin{aligned}
(2.11)\qquad 0 \geq & \mathbb{E}\Bigg[\int_0^T \psi_t \Bigg(\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\mathrm{d}r}(2_{V^*}\langle Y_s - A(s,\phi_s),\bar{X}_s - \phi_s\rangle_V \\
& + \|B(s,\phi_s) - Z_s\|_2^2 - (K+\rho(\phi_s))\|X_s - \phi_s\|_H^2)\mathrm{d}s\Bigg)\mathrm{d}t\Bigg].
\end{aligned}
$$

Note that (1.2), Lemma 2.1 and 2.2 imply that

$$
\bar{X} \in K \cap \mathcal{M} \cap L^p(\Omega; L^\infty([0,T];H)).
$$

By taking $\phi = \bar{X}$ we obtain that $Z = B(\cdot,\bar{X})$. Next, we first take $\phi = \bar{X} - \varepsilon\tilde{\phi}v$ for $\tilde{\phi} \in L^\infty([0,T]\times\Omega;\mathrm{d}t\otimes\mathbb{P};\mathbb{R})$ and $v \in V$, then we divide by ε and let $\varepsilon \to 0$ to derive that

$$
(2.12)\qquad 0 \geq \mathbb{E}\Bigg[\int_0^T \psi_t \Bigg(\int_0^t e^{-\int_0^s (K+\rho(\bar{X}_r))\mathrm{d}r}\tilde{\phi}_s {}_{V^*}\langle Y_s - A(s,\bar{X}_s), v\rangle_V \mathrm{d}s\Bigg)\mathrm{d}t\Bigg].
$$

By the arbitrariness of ψ and $\tilde{\phi}$, we conclude that $Y = A(\cdot,\bar{X})$.

Hence $\bar{X}$ is a solution of (1.1).

(2) Uniqueness: Suppose X_t, Y_t are the solutions of (1.1) with initial conditions X_0, Y_0 respectively, i.e.

$$
(2.13)\qquad \begin{aligned}
X_t &= X_0 + \int_0^t A(s,X_s)\mathrm{d}s + \int_0^t B(s,X_s)\mathrm{d}W_s,\ t\in[0,T]; \\
Y_t &= Y_0 + \int_0^t A(s,Y_s)\mathrm{d}s + \int_0^t B(s,Y_s)\mathrm{d}W_s,\ t\in[0,T].
\end{aligned}
$$

Then by the product rule, Itô's formula and $(H2)$ we have

$$\begin{aligned}
&e^{-\int_0^t (K+\rho(Y_s))\mathrm{d}s}\|X_t - Y_t\|_H^2 \\
\leq &\|X_0 - Y_0\|_H^2 + 2\int_0^t e^{-\int_0^s (K+\rho(Y_r))\mathrm{d}r}\langle X_s - Y_s, B(s,X_s)\mathrm{d}W_s - B(s,Y_s)\mathrm{d}W_s\rangle_H,\ t\in[0,T].
\end{aligned}$$

By a standard localization argument we have

$$\mathbb{E}\left[e^{-\int_0^t (K+\rho(Y_s))\mathrm{d}s}\|X_t - Y_t\|_H^2\right] \leq \mathbb{E}\|X_0 - Y_0\|_H^2,\ t\in[0,T].$$

If $X_0 = Y_0, \mathbb{P}-a.s.$, then

$$\mathbb{E}\left[e^{-\int_0^t (K+\rho(Y_s))\mathrm{d}s}\|X_t - Y_t\|_H^2\right] = 0,\ t\in[0,T].$$

Since (1.2) and Lemma 2.1, 2.2 imply that

$$\int_0^t (K+\rho(Y_s))\mathrm{d}s < \infty,\ \mathbb{P}-a.s.,\ t\in[0,T],$$

we have

$$X_t = Y_t,\ \mathbb{P}-a.s.,\ t\in[0,T].$$

Therefore, the pathwise uniqueness follows from the path continuity of X, Y in H.

(3) Markov property: the proof of Markov property is standard, we refer to [29, Proposition 4.3.5] or [19, Theorem II.2.4]. □

3 Application to examples

Obviously, the main result can be applied to stochastic evolution equations with monotone coefficients (cf. [29] for the stochastic porous medium equation and stochastic p-Laplace equation) and non-monotone perturbations (*e.g.* some locally Lipschitz perturbation) in the drift. Below we present some examples where the coefficients are only locally monotone, hence the classical result of monotone operators can not be applied.

In this section we use the notation D_i to denote the spatial derivative $\frac{\partial}{\partial x_i}$, $\Lambda \subseteq \mathbb{R}^d$ is an open bounded domain with smooth boundary. For the standard Sobolev space $W_0^{1,p}(\Lambda)$ $(p \geq 2)$ we always use the following (equivalent) Sobolev norm:

$$\|u\|_{1,p} := \left(\int_\Lambda |\nabla u(x)|^p dx\right)^{1/p}.$$

For simplicity we only consider examples where the coefficients are time independent, but one can easily adapt those examples to the time dependent case.

Lemma 3.1. *Consider the Gelfand triple*

$$V := W_0^{1,2}(\Lambda) \subseteq H := L^2(\Lambda) \subseteq W^{-1,2}(\Lambda)$$

and the operator

$$A(u) = \Delta u + \sum_{i=1}^{d} f_i(u) D_i u,$$

where f_i $(i = 1, \cdots, d)$ are bounded Lipschitz functions on $\mathbb{R}$.

(1) *If $d < 3$, then there exists a constant K such that*

$$_{V^*}\langle A(u) - A(v), u - v\rangle_V \leq -\|u - v\|_V^2 + \left(K + K\|v\|_V^2\right)\|u - v\|_H^2,\ u, v \in V.$$

(2) *If $d = 3$, then there exists a constant K such that*

$$_{V^*}\langle A(u) - A(v), u - v\rangle_V \leq -\|u - v\|_V^2 + \left(K + K\|v\|_V^4\right)\|u - v\|_H^2,\ u, v \in V.$$

(3) *If f_i are independent of u for $i = 1, \cdots, d$, i.e.*

$$A(u) = \Delta u + \sum_{i=1}^{d} f_i \cdot D_i u,$$

then for any $d \geq 1$ we have

$$_{V^*}\langle A(u) - A(v), u - v\rangle_V \leq -\|u - v\|_V^2 + K\|u - v\|_H^2,\ u, v \in V.$$

Proof. (1) Since all f_i are bounded and Lipschitz, we have

$$\begin{aligned}
&_{V^*}\langle A(u) - A(v), u - v\rangle_V \\
&= -\|u - v\|_V^2 + \sum_{i=1}^{d} \int_\Lambda \left(f_i(u) D_i u - f_i(v) D_i v\right)(u - v)\, dx \\
&= -\|u - v\|_V^2 + \sum_{i=1}^{d} \int_\Lambda \left(f_i(u)(D_i u - D_i v) + D_i v(f_i(u) - f_i(v))\right)(u - v)\, dx \\
&\leq -\|u - v\|_V^2 + \sum_{i=1}^{d} \Bigg[\left(\int_\Lambda (D_i u - D_i v)^2 dx\right)^{1/2} \left(\int_\Lambda f_i^2(u)(u - v)^2 dx\right)^{1/2} \\
&\quad + \left(\int_\Lambda (D_i v)^2 dx\right)^{1/2} \left(\int_\Lambda \left(f_i(u) - f_i(v)\right)^2 (u - v)^2 dx\right)^{1/2} \Bigg] \\
&\leq -\|u - v\|_V^2 + K\|u - v\|_V \left(\int_\Lambda (u - v)^2 dx\right)^{1/2} + K\|v\|_V \left(\int_\Lambda (u - v)^4 dx\right)^{1/2} \\
&\leq -\frac{3}{4}\|u - v\|_V^2 + K\|u - v\|_H^2 + K\|v\|_V \|u - v\|_{L^4}^2,\ u, v \in V,
\end{aligned} \tag{3.1}$$

where K is a generic constant that may change from line to line.

For $d<3$, we have the following well known estimate on $\mathbb{R}^2$ (see [25, Lemma 2.1])

$$\|u\|_{L^4}^4 \leq 2\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2,\ u\in W_0^{1,2}(\Lambda). \tag{3.2}$$

Hence combining with (3.1) we have

$$_{V^*}\langle A(u)-A(v),u-v\rangle_V \leq -\frac{1}{2}\|u-v\|_V^2+\left(K+K\|v\|_V^2\right)\|u-v\|_H^2,\ u,v\in V.$$

(2) For $d=3$ we use the following estimate (cf.[25])

$$\|u\|_{L^4}^4 \leq 4\|u\|_{L^2}\|\nabla u\|_{L^2}^3,\ u\in W_0^{1,2}(\Lambda), \tag{3.3}$$

then the second assertion can be derived similarly from (3.1) and Young's inequality.

(3) This assertion can be easily derived as (3.1). □

Example 3.2. *(Semilinear stochastic equations)*
Let Λ be an open bounded domain in $\mathbb{R}^d$ with smooth boundary. We consider the following triple

$$V:=W_0^{1,2}(\Lambda)\subseteq H:=L^2(\Lambda)\subseteq (W_0^{1,2}(\Lambda))^*$$

and the semilinear stochastic equation

$$\mathrm{d}X_t=\left(\Delta X_t+\sum_{i=1}^d f_i(X_t)D_iX_t+g(X_t)\right)\mathrm{d}t+B(X_t)\mathrm{d}W_t, \tag{3.4}$$

where W_t is a Wiener process on $L^2(\Lambda)$ and f_i,g,B satisfy the following conditions:

(i) f_i are bounded Lipschitz functions on $\mathbb{R}$ for $i=1,\cdots,d$;

(ii) g is a continuous function on $\mathbb{R}$ such that

$$\begin{aligned}|g(x)|&\leq C(|x|^r+1),\ x\in\mathbb{R};\\ (g(x)-g(y))(x-y)&\leq C(1+|y|^s)(x-y)^2,\ x,y\in\mathbb{R}.\end{aligned} \tag{3.5}$$

where C,r,s are some positive constants.

(iii) $B:V\to L_2(L^2(\Lambda))$ is Lipschitz.

Then we have the following result:

(1) If $d=1,r=3,s=2$, then for any $X_0\in L^6(\Omega,\mathcal{F}_0,\mathbb{P};H)$, (3.4) has a unique solution $\{X_t\}_{t\in[0,T]}$ and this solution satisfies

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|X_t\|_H^6+\int_0^T\|X_t\|_V^2\mathrm{d}t\right)<\infty. \tag{3.6}$$

(2) If $d=2,r=\frac{7}{3},s=2$, then for any $X_0\in L^6(\Omega,\mathcal{F}_0,\mathbb{P};H)$, (3.4) has a unique solution $\{X_t\}_{t\in[0,T]}$ and this solution satisfies (3.6).

(3) If $d=3,r=\frac{7}{3},s=\frac{4}{3}$, $f_i,i=1,\cdots,d$ are bounded measurable functions and independent of u, then for any $X_0\in L^6(\Omega,\mathcal{F}_0,\mathbb{P};H)$, (3.4) has a unique solution $\{X_t\}_{t\in[0,T]}$ and this solution satisfies (3.6).

Proof. (1) We define the operator

$$A(u) = \Delta u + \sum_{i=1}^{d} f_i(u) D_i u + g(u),\ u \in V.$$

The hemicontinuity $(H1)$ follows easily from the continuity of f and g.

Note that (3.5) and (3.2) implies

$$\begin{aligned} {}_{V^*}\langle g(u) - g(v), u - v\rangle_V \leq & C\left(1 + \|v\|_{L^{2s}}^s\right)\|u - v\|_{L^4}^2 \\ \leq & \frac{1}{4}\|u - v\|_V^2 + C\left(1 + \|v\|_{L^{2s}}^{2s}\right)\|u - v\|_H^2,\ u, v \in V. \end{aligned} \tag{3.7}$$

Therefore, by Lemma 3.1 we have for $d < 3$

$$2_{V^*}\langle A(u) - A(v), u - v\rangle_V \leq -\frac{1}{2}\|u - v\|_V^2 + C\left(1 + \|v\|_V^2 + \|v\|_{L^{2s}}^{2s}\right)\|u - v\|_H^2,\ u, v \in V,$$

i.e. $(H2)$, $(H3)$ hold with $\rho(v) = \|v\|_V^2 + \|v\|_{L^{2s}}^{2s}$ and $\alpha = 2$.

For $d = 1, r = 3$, by the Sobolev embedding theorem we have

$$\|g(u)\|_{V^*} \leq C\left(1 + \|u\|_{L^3}^3\right) \leq C\left(1 + \|u\|_V\|u\|_H^2\right),\ u \in V.$$

Then it is easy to show that

$$\|A(u)\|_{V^*} \leq C\left(1 + \|u\|_V + \|u\|_V\|u\|_H^2\right),\ u \in V.$$

Hence $(H4)$ holds with $\beta = 4$.

Therefore, all assertions follow from Theorem 1.1 by taking $p = 6$.

(2) For $d = 2, 3$ we have

$$\|g(u)\|_{V^*} \leq C\left(1 + \|u\|_{L^{6r/5}}^r\right),\ u \in V.$$

For $r = \frac{7}{3}$, by the interpolation theorem we have

$$\|u\|_{L^{6r/5}} \leq \|u\|_{L^2}^{4/7}\|u\|_{L^6}^{3/7},\ u \in W_0^{1,2}(\Lambda) \subseteq L^6(\Lambda).$$

Then

$$\|g(u)\|_{V^*} \leq C\left(1 + \|u\|_{L^{6r/5}}^r\right) \leq C\left(1 + \|u\|_H^{4/3}\|u\|_V\right),\ u \in V. \tag{3.8}$$

Hence $(H4)$ holds for $d = 2, 3$ with $\beta = 8/3$.

Therefore, for $d = 2$, all assertions follow from Theorem 1.1 by taking $p = 6$ (in fact, $p = 14/3$ is enough).

(3) If $d = 3$ and $f_i, i = 1, 2, 3$ are bounded measurable functions and independent of u, then by Lemma 3.1 and (3.3) we have

$$2_{V^*}\langle A(u) - A(v), u - v\rangle_V \leq -\frac{1}{2}\|u - v\|_V^2 + K\left(1 + \|v\|_{L^{2s}}^{4s}\right)\|u - v\|_H^2,\ u, v \in V.$$

Hence $(H2)$, $(H3)$ hold with $\rho(v) = \|v\|_{L^{2s}}^{4s}$ and $\alpha = 2$.

Since $s = \frac{4}{3}$, by the interpolation inequality we have

$$\|u\|_{L^{2s}} \leq \|u\|_{L^2}^{5/8} \|u\|_{L^6}^{3/8}, \ u \in V.$$

Therefore,

$$\|u\|_{L^{2s}}^{4s} \leq C\|u\|_H^{10/3} \|u\|_V^2, \ u \in V,$$

i.e. (1.2) holds with $\beta = 10/3$.

Hence combining with (3.8) we can take $p = 6$ (in fact, $p \geq 16/3$ is enough).

Then all assertions follow from Theorem 1.1. □

Remark 3.1. One obvious generalization is that one can replace Δ in (3.4) by the p-Laplace operator $\mathbf{div}(|\nabla u|^{p-2}\nabla u)$ or the more general quasi-linear differential operator

$$\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D_\alpha A_\alpha(Du),$$

where $Du = (D_\beta u)_{|\beta| \leq m}$. Under certain assumptions (cf.[39, Proposition 30.10]) this operator satisfies the monotonicity and coercivity condition. Then, according to Theorem 1.1, we can obtain the existence and uniqueness of solutions to this type of quasi-linear SPDE with non-monotone perturbations (*e.g.* some locally Lipschitz lower order terms).

Now we apply Theorem 1.1 to the stochastic 2-D Navier-Stokes equation.

Let Λ be a bounded domain in $\mathbb{R}^2$ with smooth boundary. Define

$$V = \left\{ v \in W_0^{1,2}(\Lambda, \mathbb{R}^2) : \nabla \cdot v = 0 \ a.e. \text{ in } \Lambda \right\}, \ \|v\|_V := \left(\int_\Lambda |\nabla v|^2 dx \right)^{1/2},$$

and H is the closure of V in the following norm

$$\|v\|_H := \left(\int_\Lambda |v|^2 dx \right)^{1/2}.$$

The linear operator P_H (Helmhotz-Hodge projection) and A (Stokes operator with viscosity constant ν) are defined by

$$P_H : L^2(\Lambda, \mathbb{R}^2) \to H \quad \text{orthogonal projection;}$$

$$A : W^{2,2}(\Lambda, \mathbb{R}^2) \cap V \to H, \ Au = \nu P_H \Delta u.$$

It is well known then the Navier-Stokes equation can be reformulated as follows:

$$u' = Au + F(u) + f, \ u(0) = u_0 \in H, \tag{3.9}$$

where $f \in L^2(0, T; V^*)$ denotes some external force and

$$F : \mathcal{D}_F \subset H \times V \to H, \ F(u, v) = -P_H \left[(u \cdot \nabla) v \right], F(u) = F(u, u).$$

It is standard that using the Gelfand triple

$$V \subseteq H \equiv H^* \subseteq V^*,$$

we see that the following mappings

$$A : V \to V^*,\ F : V \times V \to V^*$$

are well defined. In particular, we have

$$_{V^*}\langle F(u,v), w\rangle_V = -_{V^*}\langle F(u,w), v\rangle_V,\ _{V^*}\langle F(u,v), v\rangle_V = 0,\ u,v,w \in V.$$

Now we consider the stochastic 2-D Navier-Stokes equation

$$\mathrm{d}X_t = (AX_t + F(X_t) + f_t)\,\mathrm{d}t + B(X_t)\mathrm{d}W_t, \tag{3.10}$$

where W_t is a Wiener process on H.

Example 3.3. *(Stochastic 2-D Navier-Stokes equation) Suppose that $X_0 \in L^4(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ and $B : V \to L_2(H)$ satisfies*

$$\|B(v_1) - B(v_2)\|_2^2 \le K\left(1 + \|v_2\|^4_{L^4(\Lambda;\mathbb{R}^2)}\right)\|v_1 - v_2\|_H^2,\ v_1, v_2 \in V,$$

where K is some constant. Then (3.10) has a unique solution $\{X_t\}_{t\in[0,T]}$ and this solution satisfies

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|X_t\|_H^4 + \int_0^T \|X_t\|_V^2 \mathrm{d}t\right) < \infty.$$

Proof. The hemicontinuity $(H1)$ is obvious since F is a bilinear map.

Note that $_{V^*}\langle F(v), v\rangle_V = 0$, it is also easy to show $(H3)$ with $\alpha = 2$:

$$\begin{aligned} _{V^*}\langle Av + F(v) + f_t, v\rangle_V &\le -\nu\|v\|_V^2 + \|f_t\|_{V^*}\|v\|_V \le -\frac{\nu}{2}\|v\|_V^2 + C\|f_t\|^2_{V^*},\ v \in V, \\ \|B(v)\|_2^2 &\le 2K\|v\|_H^2 + 2\|B(0)\|_2^2,\ v \in V. \end{aligned}$$

Recall the following estimates (cf. e.g.[25, Lemma 2.1, 2.2])

$$\begin{aligned} &|_{V^*}\langle F(w), v\rangle_V| \le 2\|w\|_{L^4(\Lambda;\mathbb{R}^2)}\|v\|_V; \\ &|_{V^*}\langle F(w), v\rangle_V| \le 2\|w\|_V^{3/2}\|w\|_H^{1/2}\|v\|_{L^4(\Lambda;\mathbb{R}^2)}, v, w \in V. \end{aligned} \tag{3.11}$$

Then we have

$$\begin{aligned} _{V^*}\langle F(u) - F(v), u - v\rangle_V &= -_{V^*}\langle F(u, u-v), v\rangle_V + _{V^*}\langle F(v, u-v), v\rangle_V \\ &= -_{V^*}\langle F(u-v), v\rangle_V \\ &\le 2\|u-v\|_V^{3/2}\|u-v\|_H^{1/2}\|v\|_{L^4(\Lambda;\mathbb{R}^2)} \\ &\le \frac{\nu}{2}\|u-v\|_V^2 + \frac{32}{\nu^3}\|v\|^4_{L^4(\Lambda;\mathbb{R}^2)}\|u-v\|_H^2,\ u, v \in V. \end{aligned} \tag{3.12}$$

Hence we have the local monotonicity $(H2)$ with $\rho(v) = \|v\|^4_{L^4(\Lambda;\mathbb{R}^2)}$:

$$_{V^*}\langle Au + F(u) - Av - F(v), u - v\rangle_V \leq -\frac{\nu}{2}\|u - v\|^2_V + \frac{32}{\nu^3}\|v\|^4_{L^4(\Lambda;\mathbb{R}^2)}\|u - v\|^2_H.$$

(3.11) and (3.2) imply that $(H4)$ and (1.2) hold with $\beta = 2$.

Therefore, the existence and uniqueness of solutions to (3.10) follow from Theorem 1.1 by taking $p = 4$. □

Remark 3.2. (1) If the noise in (3.10) is additive type, then the existence and uniqueness of solutions to (3.10) has been established in [25]. Here we can conclude the same result for (3.10) with general multiplicative noise by a direct application of our main result.

(2) For the 3-D Navier-Stokes equation, we recall the following well known estimate (cf. e.g. [25, (2.5)])

$$\|\psi\|^4_{L^4} \leq 4\|\psi\|_{L^2}\|\nabla\psi\|^3_{L^2},\ \psi \in W^{1,2}_0(\Lambda;\mathbb{R}^3).$$

Then one can show that

$$\begin{aligned}
_{V^*}\langle F(u) - F(v), u - v\rangle_V &= -_{V^*}\langle F(u - v), v\rangle_V \\
&\leq 2\|u - v\|^{7/4}_V\|u - v\|^{1/4}_H\|v\|_{L^4(\Lambda;\mathbb{R}^3)} \\
&\leq \frac{\nu}{2}\|u - v\|^2_V + \frac{2^{12}}{\nu^7}\|v\|^8_{L^4(\Lambda;\mathbb{R}^3)}\|u - v\|^2_H,\ u, v \in V.
\end{aligned}$$

Hence we have the following local monotonicity $(H2)$:

$$_{V^*}\langle Au + F(u) - Av - F(v), u - v\rangle_V \leq -\frac{\nu}{2}\|u - v\|^2_V + \frac{2^{12}}{\nu^7}\|v\|^8_{L^4(\Lambda;\mathbb{R}^3)}\|u - v\|^2_H.$$

Another form of local monotonicity can be derived similarly:

$$\begin{aligned}
_{V^*}\langle F(u) - F(v), u - v\rangle_V &= -_{V^*}\langle F(u - v), v\rangle_V \\
&\leq 2\|u - v\|^{3/2}_V\|u - v\|^{1/2}_H\|v\|_{L^6(\Lambda;\mathbb{R}^3)} \\
&\leq \frac{\nu}{2}\|u - v\|^2_V + \frac{32}{\nu^3}\|v\|^4_{L^6(\Lambda;\mathbb{R}^3)}\|u - v\|^2_H,\ u, v \in V.
\end{aligned}$$

(3) Concerning the growth condition, we have in the 3-D case that

$$\|F(u)\|_{V^*} \leq 2\|u\|^2_{L^4(\Lambda;\mathbb{R}^3)} \leq 4\|u\|^{1/2}_H\|u\|^{3/2}_V,\ u \in V.$$

Unfortunately, this is not enough to verify $(H4)$ in Theorem 1.1.

(4) One should note that the only role of $(H4)$ is to assure that $\|A(\cdot, X^{(n)})\|_{K^*}$ is uniformly bounded for all n (see Lemma 2.2). Therefore, one can replace $(H4)$ by some weaker growth condition once we can derive some stronger a priori estimate for $X^{(n)}$ in the Galerkin approximation (e.g. as in [33]). One good example of a further generalization of our main result is that we can apply Theorem 1.1 (with a revised version of $(H4)$) to derive the existence

and uniqueness of solutions to the following stochastic tamed 3-D Navier-Stokes equation with smooth enough initial condition:

$$\mathrm{d}X_t = \big(AX_t + F(X_t) + f_t - P_H g_N(|X_t|^2)X_t\big)\,\mathrm{d}t + B(X_t)\mathrm{d}W_t,$$

where the taming function $g_N : \mathbb{R}_+ \to \mathbb{R}_+$ is smooth and satisfies for some $N > 0$

$$\begin{cases} g_N(r) = 0, \text{ if } r \le N, \\ g_N(r) = (r-N)/\nu, \text{ if } r \ge N+1, \\ g_N'(r) \ge 0, \text{ if } r \ge 0. \end{cases}$$

We refer to [32, 33] for more details on the stochastic tamed 3-D Navier-Stokes equation.

4 Appendix: The classical monotone and coercive conditions

For the existence and uniqueness of the solution to (1.1) we recall the following classical monotone and coercive conditions on A and B.

Suppose there exist constants $\alpha > 1$, $\theta > 0$, K and a positive adapted process $f \in L^1([0,T]\times\Omega; \mathrm{d}t\times\mathbb{P})$ such that the following conditions hold for all $v, v_1, v_2 \in V$ and $(t,\omega) \in [0,T]\times\Omega$.

$(A1)$ (Hemicontinuity) The map $s \mapsto {}_{V^*}\langle A(t, v_1 + sv_2), v\rangle_V$ is continuous on $\mathbb{R}$.

$(A2)$ (Monotonicity)

$$2\,{}_{V^*}\langle A(t,v_1) - A(t,v_2), v_1 - v_2\rangle_V + \|B(t,v_1) - B(t,v_2)\|_2^2 \le K\|v_1 - v_2\|_H^2.$$

$(A3)$ (Coercivity)

$$2\,{}_{V^*}\langle A(t,v), v\rangle_V + \|B(t,v)\|_2^2 + \theta\|v\|_V^\alpha \le f_t + K\|v\|_H^2.$$

$(A4)$ (Boundedness)

$$\|A(t,v)\|_{V^*} \le f_t^{(\alpha-1)/\alpha} + K\|v\|_V^{\alpha-1}.$$

Theorem 4.1. ([19] Theorems II.2.1, II.2.2) *Suppose $(A1)-(A4)$ hold, then for any $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ (1.1) has a unique solution $\{X_t\}_{t\in[0,T]}$ and this solution satisfies*

$$\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_H^2 < \infty.$$

Moreover, we have the following Itô formula

$$\begin{aligned}\|X_t\|_H^2 =& \|X_0\|_H^2 + \int_0^t \big(2\,{}_{V^*}\langle A(s,X_s), X_s\rangle_V + \|B(s,X_s)\|_2^2\big)\,\mathrm{d}s \\ &+2\int_0^t \langle X_s, B(s,X_s)\mathrm{d}W_s\rangle_H,\ t\in[0,T],\ \mathbb{P}-a.s.\end{aligned}$$

References

[1] V. Barbu, G. Da Prato and M. Röckner, *Existence of strong solutions for stochastic porous media equation under general monotonicity conditions*, Ann. Probab. **37** (2009), 428–452.

[2] V. Barbu, G. Da Prato and M. Röckner, *Existence and uniqueness of nonnegative solutions to the stochastic porous media equation*, Indiana Univ. Math. J. **57** (2008), no. 1, 187–212.

[3] V. Barbu, G. Da Prato and M. Röckner, *Some results on stochastic porous media equations*, Bollettino U.M.I. **9** (2008), 1–15.

[4] V. Barbu, G. Da Prato and M. Röckner, *Stochastic porous media equation and self-organized criticality*, Commun. Math. Phys. **285** (2009), 901–923.

[5] W. Beyn, B. Gess, P. Lescot and M. Röckner, *The global random attractor for a class of stochastic porous media equations*, Preprint.

[6] H. Brézis, *Équations et inéquations non linéaires dans les espaces vectoriels en dualité* , Ann. Inst. Fourier **18** (1968), 115–175.

[7] F. E. Browder, *Nonlinear elliptic boundary value problems*, Bull. Amer. Math. Soc. **69** (1963), 862–874.

[8] F. E. Browder, *Non-linear equations of Evolution*, Ann. Math. **80** (1964), 485–523.

[9] F. E. Browder, *Pseudo-monotone operators and nonlinear elliptic boundary value problems on unbounded domains*, Proc. Natl. Acad. Sci. USA **74** (1977), 2659–2661.

[10] P.L. Chow, *Large deviation problem for some parabolic Itô equations*, Comm. Pure Appl. Math. **45** (1992), 97–120.

[11] G. Da Prato, M. Röckner, B.L. Rozovskii and F.-Y. Wang, *Strong solutions to stochastic generalized porous media equations: existence, uniqueness and ergodicity*, Comm. Part. Diff. Equa. **31** (2006), 277–291.

[12] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions,* Encyclopedia of Mathematics and its Applications, Cambridge University Press. 1992.

[13] B. Gess, W. Liu and M. Röckner, *The existence of random attractor for stochastic evolution equations with general additive noise*, Preprint.

[14] I. Gyöngy, *On stochastic equations with respect to semimartingale III*, Stochastics **7** (1982), 231–254.

[15] I. Gyöngy and A. Millet, *Rate of convergence of implicit approximations for stochastic evolution equations*, Interdiscip. Math. Sci., vol. 2, pp. 281–310, World Sci. Publ., Hackensack, 2007.

[16] I. Gyöngy and A. Millet, *Rate of convergence of space time approximations for stochastic evolution equations*, Pot. Anal. **30** (2009), 29–64.

[17] J. Leray and J. L. Lions, *Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder*, Bull. Soc. Math. France **93** (1965), 97–107.

[18] J. L. Lions, *Some methods of solving nonlinear boundary value problems*, Mir, Moscow, 1972, Russian translation. 163–194.

[19] N.V. Krylov and B.L. Rozovskii, *Stochastic evolution equations*, Translated from Itogi Naukii Tekhniki, Seriya Sovremennye Problemy Matematiki **14** (1979), 71–146.

[20] W. Liu and F.-Y. Wang, *Harnack inequality and strong Feller property for stochastic fast diffusion equations*, J. Math. Anal. Appl. **342** (2008), 651–662.

[21] W. Liu, *On the stochastic p-Laplace equation,* J. Math. Anal. Appl. **360** (2009), 737–751.

[22] W. Liu, *Harnack inequality and applications for stochastic evolution equations with monotone drifts*, J. Evol. Equat. **9**(2009), 747–770.

[23] W. Liu, *Large deviations for stochastic evolution equations with small multiplicative noise*, Appl. Math. Optim. **61** (2010), 27–56.

[24] W. Liu, *Invariance of subspaces under the solution flow of SPDE,* Infin. Dimens. Anal. Quantum Probab. Relat. Top. **13** (2010), 87–98.

[25] J.-L. Menaldi and S.S. Sritharan, *Stochastic 2-D Navier-Stokes equation*, Appl. Math. Optim. 46(2002), 31–53.

[26] G.J. Minty, *Monotone (non-linear) operators in Hilbert space*, Duke. Math. J. **29** (1962), 341–346.

[27] G.J. Minty, *On a monotonicity method for the solution of non-linear equations in Banach space*, Proc. Nat. Acad. Sci. USA **50** (1963), 1038–1041.

[28] E. Pardoux, *Equations aux dérivées partielles stochastiques non linéaires monotones*, Ph.D. thesis, Université Paris XI, 1975.

[29] C. Prévôt and M. Röckner, *A concise course on stochastic partial differential equations*, Lecture Notes in Mathematics, vol. 1905, Springer, 2007.

[30] J. Ren, M. Röckner and F.-Y. Wang, *Stochastic generalized porous media and fast diffusion equations*, J. Diff. Equa. **238** (2007), 118–152.

[31] J. Ren and X. Zhang, *Fredlin-Wentzell's large deviation principle for stochastic evolution equations*, J. Funct. Anal. **254** (2008), 3148–3172.

[32] M. Röckner and X. Zhang, *Stochastic tamed 3D Navier-Stokes equations: existence, uniqueness and small time large deviation principles*, Probab. Theory Related Fields **145** (2009), no. 1-2, 211–267.

[33] M. Röckner and T. Zhang, *Stochastic tamed 3D Navier-Stokes equations: existence, uniqueness and small time large deviation principles*, BiBoS-Preprint 10-03-339.

[34] M. Röckner and F.-Y. Wang, *Non-monotone stochastic porous media equation*, J. Diff. Equa. **245** (2008), 3898–3935.

[35] M. Röckner, F.-Y. Wang and L. Wu, *Large deviations for stochastic generalized porous media equations*, Stoc. Proc. Appl. **116** (2006), 1677–1689.

[36] R.E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, Mathematical Surveys and Monographs 49, American Mathematical Society, Providence, 1997.

[37] F.-Y. Wang, *Harnack inequality and applications for stochastic generalized porous media equations*, Ann. Probab. **35** (2007), 1333–1350.

[38] X. Zhang, *On stochastic evolution equations with non-Lipschitz coefficients*, Stoc. Dyna. **9** (2009), 549–595.

[39] E. Zeidler, *Nonlinear Functional Analysis and its Applications II/B: Nonlinear Monotone operators*, Springer-Verlag, New York, 1990.