Stochastic 3D tamed Navier Stokes equations: existence, uniqueness and small time large deviation principles

Michael Röckner and Tusheng Zhang

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Department of Mathematics, University of Bielefeld, Box 100131,D-33501, Bielefeld, Germany
School of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, England, U.K. Email: tzhang@maths.man.ac.uk

Abstract

In this paper, we first give a new proof for the existence and uniqueness of strong solutions to stochastic 3D tamed Navier Stokes equations in case of Dirichlet boundary conditions for the Stokes-Laplacian. Then we prove a small time large deviation principle for the solutions.

Key words: Stochastic 3D tamed Navier-Stokes equations, strong solutions, large deviations, exponential equivalence, energy identity.

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1 Introduction

It is well known that the stochastic Navier-Stokes equation with Dirichlet boundary condition describes the time evolution of an incompressible fluid and is given by

$$\begin{cases} du - \nu\triangle u\ dt + (u\cdot\nabla)u\ dt + \nabla p\ dt = gdt + \sigma(t,u)dW(t), \\ \hfill (div\ u)(t,x) = 0,\ t>0, \\ \hfill u(0,x) = u_0(x). \end{cases}$$

While the stochastic 2D Navier-Stokes equation has been studied extensively in the literature, there exist serious obstacles to tackle stochastic 3D Navier-Stokes equations. One of them is the lack of uniquenes. Existence of martingale solutions and stationary solutions of the stochastic 3D Navier-Stokes equation was proved by Flandoli and

Gatarek in [FG] and later by Mikulevicius and Rozovskii in [MR] under more general conditions. Existence of Markov selections was proved in [FR], [DO] and [GRZ]. Recently, the following stochastic 3D tamed Navier-Stokes equations was proposed in [RZ1](see also [RZ2] for the deterministic case)

$$\begin{aligned} du(t) &= -Au(t)dt - B(u(t))dt - \mathcal{P}g_N(|u|^2(t))u(t)dt + \sum_{k=1}^{\infty} \sigma_k(u(t))dW_k(t) \\ u(0) &= u_0 \in H^1, \end{aligned} \tag{1.1}$$

where g_N is a smooth function from R_+ to R_+ being nonzero only for large arguments, see the next section for the precise definitions of g_N and the coefficients. The motivation to study (1.1) originates from the deterministic case, i.e., when the noise is zero. In that case (cf [RZ2]) a bounded strong solution of the classical $3D$ Navier-Stokes equation coincides with the solution of (1.1) (with $\sigma_k = 0$, $\forall k$) for large enough N. Existence and uniqueness of strong solutions (in the probabilistic sense), Feller properties and invariant measures were obtained in [RZ1]. However, since the underlying domain in [RZ1] was all of R^3 or the torus , the existence of a strong solution was obtained indirectly via the Yamada-Watanabe Theorem by proving the existence of martingale solutions and pathwise uniqueness.

The purpose of this paper is two-fold. The first is in case of a bounded underlying domain and taking Dirichlet boundary conditions to prove the existence of a strong solution of the stochastic 3D tamed Navier-Stokes equation directly, based on Galerkin's approximation and on a kind of local monotonicity of the coefficients. The second part is to prove a small time large deviation principle (LDP) for the stochastic 3D tamed Navier-Stokes equations on $C([0,1];H^1)$.

Though our interest here is in small time LDP, let us briefly mention that the small noise LDP for stochastic partial differential equations (SPDEs) has been studied by many people. For example, for SPDE with monotone coefficients under very general conditions this LDP has been proved in [L], strongly generalizing a corresponding former result by P.L. Chow (1992). In 2004 a small noise LDP for stochastic reaction diffusion equations with nonlinear reaction term was established by Cerrai and Röckner in [CR] generalizing an early result by R. Sowers from (1992) in [S]. For stochastic Burgers'-type SPDEs this was achieved by Cardon-Weber (1999) in [CW]. A uniform LDP for parabolic SPDEs was proved by Chenal and Millet (1997) in [CM2]. In [RS], Rovira and Sanz-Sole (1996) proved an LDP for a class of nonlinear hyperbolic SPDEs.

A small time large deviation principle for stochastic parabolic equations was obtained by one of authors in [Z]. For the general theory of large deviations, the reader is referred to the monograph [DZ]. Because of the different nature of nonlinearities for different types of equations, the large deviations for SPDE have to be dealt with on a case by case basis.

For small time asymptotics of diffusion processes in finite and infinite dimensions we refer the reader to [V], [HR] respectively.

The small noise large deviation of the stochastic 2D Navier-Stokes equations was established in [CM1] correcting an error/gap in [S.S] and the large deviation of occupation measures was considered in [G]. The small time large deviation principle for the 2D stochastic Navier-Stokes equation was treated in [XZ] and the small noise large deviation for the 3D tamed stochastic Navier-Stokes equation in [RZZ].

To obtain the small time large deviation principle for the stochastic 3D tamed Navier-Stokes equation, as one expects, the main difficulty lies in dealing with the nonlinear term $B(u) = \mathcal{P}\big((u \cdot \nabla)u\big)$ and the unbounded term $Au = -\nu \triangle u$. To control $B(u)$, the main idea is to show that the probability that the solution stays outside an energy ball is exponentially small so that we can restrict the solution to a sufficiently large energy ball. Our approach is close to that of [XZ]. However, the treatment of the nonlinear terms is different from that in [XZ] because of the well known difference between the 2D and 3D- case for Navier Stokes equations.

2 Notations

Let $u(x) = (u^1(x), u^2(x), u^3(x))$ be a vector valued function on a bounded domain $D \subset R^3$. The following notations will be used.

$$|u|^2 := \sum_{i=1}^{3} |u^i|^2, \qquad \partial_i u^j := \frac{\partial u^j}{\partial x_i},$$

$$\nabla u^j := (\partial_1 u^j, \partial_2 u^j, \partial_3 u^j), \quad \Delta u^j := \sum_{i=1}^{3} \partial_i^2 u^j,$$

$$\partial_i u := (\partial_i u^1, \partial_i u^2, \partial_i u^3),$$

$$(\lambda I - \Delta)^{\frac{m}{2}} u := ((\lambda I - \Delta)^{\frac{m}{2}} u^1, (\lambda I - \Delta)^{\frac{m}{2}} u^2, (\lambda I - \Delta)^{\frac{m}{2}} u^3), \lambda, m \geq 0$$

$$div(u) := \sum_{i=1}^{3} \partial_i u^i, \qquad (u \cdot \nabla)u := \sum_{i=1}^{3} u^i \partial_i u.$$

Let $C_0^\infty(D; R^3)$ denote the set of all smooth functions from D to R^3 with compact supports. For $p \geq 1$, let $L^p(D; R^3)$ be the vector valued L^p-space in which the norm is denoted by $|| \cdot ||_{L^p}$. For an non-negative integer $m \geq 0$, let $W_0^{m,2}$ be the usual Sobolev space on D with values in R^3, i.e., the closure of $C_0^\infty(D; R^3)$ with respect to the norm:

$$||u||^2_{W_0^{m,2}} = \int_D |(I - \Delta)^{\frac{m}{2}} u|^2 dx.$$

Recall the following Gagliardo-Nirenberg interpolation inequality. If

$$\frac{1}{q} = \frac{1}{2} - \frac{m\alpha}{3}, \quad 0 \le \alpha \le 1,$$

then for any $u \in W_0^{m,2}$

$$||u||_{L^q} \le C_{m,q} ||u||^{\alpha}_{W_0^{m,2}} ||u||^{1-\alpha}_{L^2}. \tag{2.1}$$

Set

$$H^m := \{u \in W_0^{m,2} : div(u) = 0\}.$$

The norm of $W_0^{m,2}$ restricted to H^m will be denoted by $|| \cdot ||_{H^m}$. Remark that H^0 is a closed linear subspace of the Hilbert space $L^2(D; R^3)$. Let $\mathcal{P}$ be the orthogonal projection from $L^2(D; R^3)$ to H^0. It is well known that $\mathcal{P}$ commutes with the derivative operators.

For $u, v \in L^2(D; R^3)$ set

$$B(u, v) := \mathcal{P}((u \cdot \nabla)v), \qquad Au = -\nu \Delta u.$$

If $u = v$, we write $B(u) = B(u, u)$. Let V be defined by

$$V := \{u : u \in C_0^\infty(D; R^3), div(u) = 0\}.$$

Throughout this paper, $g_N(\cdot)$ will denote a fixed smooth function from R_+ to R_+ such that for some $N > 0$,

$$\begin{cases} g_N(r) = 0, & \text{if } r \le N, \\ g_N(r) = \frac{(r-N)}{\nu}, & \text{if } r \ge N+1, \\ 0 \le g_N'(r) \le C, & r \ge 0. \end{cases} \tag{2.2}$$

3 Existence and uniqueness

For simplicity we take $\nu = 1$. Let $(W_k(t), k \ge 1)$ be a sequence of independent $\mathcal{F}_t$-Brownian motions defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Consider the stochastic 3D tamed Navier-Stokes equation:

$$\begin{aligned} du(t) &= -Au(t)dt - B(u(t))dt - \mathcal{P}\big(g_N(|u|^2)u(t)\big)dt + \sum_{k=1}^{\infty} \sigma_k(u(t))dW_k(t) \\ u(0) &= u_0 \in H^1. \end{aligned} \tag{3.1}$$

Here $\sigma_k(\cdot), k \ge 1$, is a sequence of mappings from H^1 (H^0) into $H^1(H^0)$. Consider the following hypotheses.

(H.1) .

$$\sum_{k=1}^{\infty} ||\sigma_k(u)||^2_{H^0} < \infty, \text{for} \quad u \in H^0.$$

(H.2) .

$$\sum_{k=1}^{\infty} ||\sigma_k(u)||_{H^1}^2 < \infty, \text{for} \quad u \in H^1.$$

(H.3) .

$$\sum_{k=1}^{\infty} ||\sigma_k(u) - \sigma_k(v)||_{H^0}^2 \le c(||u - v||_{H^0}^2)$$

(H.4) .

$$\sum_{k=1}^{\infty} ||\sigma_k(u) - \sigma_k(v)||_{H^1}^2 \le c(||u - v||_{H^1}^2).$$

(H.1), (H.2) imply that for every $u \in H^1(H^0 \quad resp.)$ the linear map $\sigma(u) := (\sigma_k(u))_{k\in N} : l_2 \to H^1(H^0 \quad resp.)$ defined by

$$\sigma(u)h := \sum_{k=1}^{\infty} \sigma_k(u)h_k, h = (h_k)_{k\in N} \in l_2,$$

is in $L_2(l_2, H^1(H^0 \quad resp.))$ (=Hilbert-Schmidt operators from l_2 to $H^1(H^0 \quad resp.)$ and (H.3), (H.4) imply that $u : | \to \sigma(u)$ is Lipschitz. For simplicity, in this section we write

$$F(u) := -Au - B(u) - \mathcal{P}\big(g_N(|u|^2)u\big).$$

The following inequality can be found in [H]:

$$\sup_x |u(x)|^2 \le C||\Delta u||_{H^0} \cdot ||\nabla u||_{H^0} \qquad (3.1)'.$$

Theorem 3.1 *Assume (H.1) − (H.4) hold and* $u_0 \in L^2(\Omega, \mathcal{F}_0; H^1)$. *Then there exists a unique solution to the stochastic 3D-tamed Navier-Stokes equation (3.1) that satisfies the following energy inequality:*

$$E\bigg(\sup_{0\le t\le T} ||u(t)||_{H^1}^2\bigg) + \int_0^T E[||u(t)||_{H^2}^2]dt < \infty. \qquad (3.2)$$

Proof. The uniqueness can be proved as in [RZ1]. Therefore, we only prove the existence. We will use Galerkin approximation combined with a kind of local monotonicity of the 3D-tamed equation. We will do this in two steps.

Step 1. Assume $u_0 \in L^6(\Omega, \mathcal{F}_0; H^1)$.

Let $\{e_i, i \ge 1\} \subset H^2$ be a fixed orthonormal basis of H^0 consisting of eigenvectors of Δ, so that it is also orthogonal in H^1. Since D is bounded, such an orthonormal basis exists. Denote by Π_n the orthogonal projection from H^0 onto the finite dimensional space $H_n := span(e_1, e_2, ..., e_n)$:

$$\Pi_n v := \sum_{i=1}^{n} < v, e_i >_{H^0} e_i.$$

Then Π_n is also the orthogonal projection onto H_n in H^1. Consider the following finite dimensional stochastic differential equation in H_n

$$\begin{cases} du_n(t) & = [\Pi_n F(u_n(t)]dt + \sum_{k=1}^{\infty} \Pi_n \sigma_k(u(t))dW_k(t), \\ u_n(0) & = \Pi_n u_0. \end{cases} \tag{3.3}$$

By Lemma 2.4 in [RZ1] and (H.1), we have for $u \in H_n$

$$\begin{aligned} \langle \Pi_n F(u), u \rangle &\leq C_N ||u||_{H^0}^2, \\ \sum_{k=1}^{\infty} ||\Pi_n \sigma_k(u)||_{H^0}^2 &\leq C(1 + ||u||_{H^0}^2). \end{aligned} \tag{3.4}$$

It follows from [K] that equation (3.3) admits a unique, continuous adapted solution $u_n(t), t \geq 0$. Now we will give a uniform energy estimate for the family $\{u_n, n \geq 1\}$. Recall the following estimates ($\nu = 1$) for $u \in H^2$ from the proof of Lemma 2.3 in [RZ1]:

$$- < Au, u >_{H^1} = -||u||_{H^2}^2 + ||\nabla u||_{L^2}^2 + ||u||_{L^2}^2 \tag{3.5}$$

$$- < B(u), u >_{H^1} \leq \frac{1}{2}||u||_{H^2}^2 + \frac{1}{2}|||u| \cdot |\nabla u|||_{L^2}^2 \tag{3.6}$$

$$- < g_N(|u|^2)u, u >_{H^1} \leq -|||u| \cdot |\nabla u|||_{L^2}^2 + (CN)||\nabla u||_{L^2}^2. \tag{3.7}$$

By (3.5)–(3.7) and Itô's formula , we have

$$\begin{aligned} ||u_n(t)||_{H^1}^2 \;=\;& ||u_n(0)||_{H^1}^2 - 2\int_0^t < u_n(s), Au_n(s) >_{H^1} ds - 2\int_0^t < u_n(s), B(u_n(s)) >_{H^1} ds \\ & +2\sum_{k=1}^{\infty}\int_0^t < u_n(s), \sigma_k(u_n(s) >_{H^1} dW_k(s) + \sum_{k=1}^{\infty}\int_0^t |\sigma_k(u_n(s))|_{H^1}^2 ds \\ & - \int_0^t < u_n(s), \mathcal{P}(g_N(|u_n|^2)u_n(s)) >_{H^1} ds \\ \leq\;& ||u_0||_{H^1}^2 - \int_0^t ||u_n(s)||_{H^2}^2 ds - \int_0^t |||u_n(s)| \cdot |\nabla u_n(s)|||_{L^2}^2 ds \\ & +C_N \int_0^t (1 + ||u_n(s)||_{H^1}^2) ds + 2\sum_{k=1}^{\infty}\int_0^t < u_n(s), \sigma_k(u_n(s) >_{H^1} dW_k(s). \end{aligned} \tag{3.8}$$

Taking expectation,

$$\begin{aligned}E[||u_n(t)||_{H^1}^2] &\le E[||u_0||_{H^1}^2] - \int_0^t E[||u_n(s)||_{H^2}^2]ds - \int_0^t E[|||u_n(s)| \cdot |\nabla u_n(s)|||_{L^2}^2]ds \\ &\quad + C\int_0^t (1 + E[||u_n(s)||_{H^1}^2)])ds, \qquad (3.9)\end{aligned}$$

Gronwall's inequality yields

$$\begin{aligned}&\sup_{0\le t\le T} E[||u_n(t)||_{H^1}^2] + \int_0^T E[||u_n(s)||_{H^2}^2]ds + \int_0^T E[|||u_n(s)| \cdot |\nabla u_n(s)|||_{L^2}^2]ds \\ &\le C_N(1 + E[||u_0||_{H^1}^2]). \qquad (3.10)\end{aligned}$$

Using (3.10), (3.8), and applying Burkholder's inequality to the martingale

$$M_t = 2\sum_{k=1}^{\infty} \int_0^t < u_n(s), \sigma_k(u_n(s) >_{H^1} dW_k(s),$$

we can further strengthen (3.10) to

$$\begin{aligned}&E\left(\sup_{0\le t\le T} ||u_n(t)||_{H^1}^2 \right) + \int_0^T E[||u_n(s)||_{H^2}^2]ds + \int_0^t E[|||u_n(s)| \cdot |\nabla u_n(s)|||_{L^2}^2]ds \\ &\le C_N(1 + E[||u_0||_{H^1}^2]), \qquad (3.11)\end{aligned}$$

for all $n \ge 1$. Next we show

$$\sup_n \int_0^T E[||u_n(t)||_{H^1}^6]dt < \infty. \qquad (3.12)$$

To this end, we apply Ito's formula to function $f(x) = x^3$ and the real-valued process $Y(t) = ||u_n(t)||_{H^1}^2$ to get

$$
\begin{aligned}
& ||u_n(t)||_{H^1}^6 \\
= \;& ||u_n(0)||_{H^1}^6 + 6\int_0^t ||u_n(s)||_{H^1}^4 < u_n(s), F(u_n)(s) >_{H^1} ds \\
& +6\sum_{k=1}^{\infty}\int_0^t ||u_n(s)||_{H^1}^4 < u_n(s), \Pi_n\sigma_k(u_n(s) >_{H^1} dW_k(s) \\
& +3\sum_{k=1}^{\infty}\int_0^t ||u_n(s)||_{H^1}^4 |\Pi_n\sigma_k(u_n(s))|_{H^1}^2 ds \\
& +12\sum_{k=1}^{\infty}\int_0^t ||u_n(s)||_{H^1}^2 < u_n(s), \Pi_n\sigma_k(u_n(s) >_{H^1}^2 ds \\
\leq \;& ||u_0||_{H^1}^6 - 3\int_0^t ||u_n(s)||_{H^1}^4 ||u_n(s)||_{H^2}^2 ds \\
& -3\int_0^t ||u_n(s)||_{H^1}^4 |||u_n(s)| \cdot |\nabla u_n(s)|||_{L^2}^2 ds \\
& +C_N\int_0^t ||u_n(s)||_{H^1}^6 ds \\
& +6\sum_{k=1}^{\infty}\int_0^t ||u_n(s)||_{H^1}^4 < u_n(s), \Pi_n\sigma_k(u_n(s) >_{H^1} dW_k(s). \qquad (3.13)
\end{aligned}
$$

Now (3.13), a standard stopping argument and an application of Gronwall's lemma after taking expectation yields (3.12). As a consequence, by $(3.1)'$ and Sobolev imbedding we get that

$$
\begin{aligned}
& \sup_n \int_0^T E[||\Pi_n F(u_n(t))||_{L^2}^2] dt \\
\leq \;& C\sup_n \int_0^T (E[||u_n(t)||_{H^1}^6] + E[||u_n(t)||_{H^2}^2]) dt < \infty. \qquad (3.14)
\end{aligned}
$$

Now the inequalities (3.11), (3.14) imply that there exist a subsequence of processes, still denoted by $(u_n, n \geq 1)$, and a process

$$\tilde{u} \in L^2(\Omega_T, H^2) \cap L^2(\Omega, L^\infty([0,T], H^1)),$$

$F \in L^2(\Omega_T, H^0)$ and $\tilde{\sigma} := (\tilde{\sigma}_k)_{k\in N} \in L_2(l_2, H^1)$ for which the following hold:

(i) $u_n \to \tilde{u}$ weakly in $L^2(\Omega_T, H^2)$, hence weakly in $L^2(\Omega_T, H^1)$.

(ii) $u_n \to \tilde{u}$ in $L^2(\Omega, L^\infty([0,T], H^1))$ with respect to the weak star topology,

(iii) $\Pi_n F(u_n) \to F$ weakly in $L^2(\Omega_T, H^0)$,

(iv) $\Pi_n \sigma(u_n) \to \tilde{\sigma}$ weakly in $L^2(\Omega_T, L_2(l_2, H^1))$,

(v) $u_n \to \tilde{u}$ weakly also in $L^6(\Omega_T, H^1)$,
where $\Omega_T = [0,T] \times \Omega$.

Now following the same arguments as in the proof of Theorem 4.2.4 in [PR] (see also the proof of Theorem 3.1 in [CM1]) we can show that, for $0 \le t \le T$, if we define

$$u(t) := u_0 + \int_0^t F(s)ds + \sum_{k=1}^{\infty} \int_0^t \tilde{\sigma}_k(s)dW_k(s), \tag{3.15}$$

then $u = \tilde{u}, \quad dt \times P - a.e.$ below. We note that by [[L1], Corollary 1.14 and Theorem 4.36] and since H^2 is continuously embedded into the domain of $I - \Delta$ on H^0 with Dirichlet boundary conditions, it immediately follows that u has continuous paths in H^1. To complete the proof of the theorem, we need to show that $F(s) = F(\tilde{u}(s)) = F(u(s))$ and $\tilde{\sigma}_k(s) = \sigma_k(\tilde{u}(s)) = \sigma_k(u(s))$ a.e. on Ω_T. To establish these relations, we will use the same idea as in [S.S] which in turn is a modification of an argument in [KR]. But, first we will need several estimates. Let $u_1, u_2 \in H^2 \subset H^1$. We have

$$- < A(u_1 - u_2), u_1 - u_2 >_{H^0} = -||u_1 - u_2||^2_{H^1} + ||u_1 - u_2||^2_{H^0}. \tag{3.16}$$

Using the property $< B(w,v), v >_{H^0} = 0$, we see that

$$\begin{aligned}
& - < B(u_1,u_1) - B(u_2,u_2), u_1 - u_2 >_{H^0} = < B(u_1 - u_2, u_1 - u_2), u_2 >_{H^0} \\
\le & \int_{R^3} |((u_1 - u_2) \cdot \nabla)(u_1 - u_2)||u_2|(x)dx \\
\le & C \sup_x |u_2|(x)(||u_1 - u_2||_{H^1}||u_1 - u_2||_{H^0}) \\
\le & \frac{1}{2}||u_1 - u_2||^2_{H^1} + C \sup_x |u_2|^2(x)||u_1 - u_2||^2_{H^0} \\
\le & \frac{1}{2}||u_1 - u_2||^2_{H^1} + C||u_2||_{H^1}||u_2||_{H^2}||u_1 - u_2||^2_{H^0}.
\end{aligned} \tag{3.17}$$

As $g_N \ge 0$, we have

$$\begin{aligned}
& - < g_N(|u_1|^2)u_1 - g_N(|u_2|^2)u_2, u_1 - u_2 >_{H^0} \\
= & - < g_N(|u_2|^2)u_2 - g_N(|u_1|^2)u_1, u_2 - u_1 >_{H^0} \\
= & - < g_N(|u_2|^2)(u_2 - u_1), u_2 - u_1 >_{H^0} + < (g_N(|u_1|^2) - g_N(|u_2|^2))u_1, u_2 - u_1 >_{H^0} \\
\le & < (g_N(|u_1|^2) - g_N(|u_2|^2))u_1, u_2 - u_1 >_{H^0} .
\end{aligned} \tag{3.18}$$

Because $0 \leq g'_N(r) \leq 2$ it follows that

$$\begin{aligned}
& < (g_N(|u_1|^2) - g_N(|u_2|^2))u_1, u_2 - u_1 >_{H^0} \\
= & \int_{\{|u_1| \geq |u_2|\}} ((g_N(|u_1|^2) - g_N(|u_2|^2))[u_1 \cdot u_2 - |u_1|^2] dx \\
& + \int_{\{|u_1| < |u_2|\}} ((g_N(|u_1|^2) - g_N(|u_2|^2))u_1 \cdot (u_2 - u_1) dx \\
\leq & \int_{\{|u_1| < |u_2|\}} ((g_N(|u_1|^2) - g_N(|u_2|^2))u_1 \cdot (u_2 - u_1) dx \\
\leq & C \int_{\{|u_1| < |u_2|\}} \big| |u_1|^2 - |u_2|^2 \big| \cdot |u_1| \cdot |u_2 - u_1| dx \\
\leq & C \int_{\{|u_1| < |u_2|\}} (|u_1| + |u_2|)|u_1| \cdot |u_2 - u_1|^2 dx \\
\leq & 2C \int_{\{|u_1| < |u_2|\}} |u_2|^2(x) |u_2 - u_1|^2(x) dx \leq 2C \sup_x |u_2|^2 \int_{R^3} |u_2 - u_1|^2 dx \\
\leq & C ||u_2||_{H^1} ||u_2||_{H^2} ||u_1 - u_2||^2_{H^0}. \qquad (3.19)
\end{aligned}$$

Putting (3.16)-(3.19) together we obtain that for all $u_1, u_2 \in H^2$

$$\begin{aligned}
& < F(u_1) - F(u_2), u_1 - u_2 >_{H^0} \\
\leq & -\frac{1}{2} ||u_1 - u_2||^2_{H^1} + C(||u_2||_{H^1} ||u_2||_{H^2} + 1) ||u_1 - u_2||^2_{H^0}. \qquad (3.20)
\end{aligned}$$

Fix an integer K. Take $v \in L^2(\Omega_T, H_K)$, where H_K is the linear span of $e_1, e_2, ...e_K$. By Ito's formula, writing $u = u - v + v$, we have

$$\begin{aligned}
& E[||u(t)||^2_{H^0} e^{-r(t)}] - E[||u_0||^2_{H^0}] \\
= & E[\int_0^t 2e^{-r(s)} < F(s), u(s) >_{H^0} ds] + E[\int_0^t e^{-r(s)} \sum_{k=1}^{\infty} ||\tilde{\sigma}_k(s)||^2_{H^0} ds] \\
& -E[\int_0^t e^{-r(s)} r'(s) ||u(s)||^2_{H^0} ds] \\
= & E[\int_0^t 2e^{-r(s)} < F(s), u(s) >_{H^0} ds] + E[\int_0^t e^{-r(s)} \sum_{k=1}^{\infty} ||\tilde{\sigma}_k(s)||^2_{H^0} ds] \\
& -E[\int_0^t e^{-r(s)} r'(s) \{ ||u(s) - v(s)||^2_{H^0} + 2 < u(s) - v(s), v(s) >_{H^0} + ||v(s)||^2_{H^0} \} ds], \\
& \qquad (3.21)
\end{aligned}$$

where $r(t)$ is a non-negative stochastic process which is absolutely continuous and to be chosen later. A similar expression also holds for $E[||u_n(t)||^2_{H^0} e^{-r(t)}] - E[||u_0||^2_{H^0}]$.

For any nonnegative $\psi \in L^\infty([0,T],R)$, the weak convergence implies that

$$
\begin{aligned}
&\int_0^T \psi(t)dtE[||u(t)||^2_{H^0}e^{-r(t)}] - E[||u_0||^2_{H^0}] = \int_0^T \psi(t)dtE[||\tilde{u}(t)||^2_{H^0}e^{-r(t)}] - E[||u_0||^2_{H^0}] \\
\leq\ & \liminf_{n\to\infty} \int_0^T \psi(t)dtE[||u_n(t)||^2_{H^0}e^{-r(t)}] - E[||u_0||^2_{H^0}]\}. \qquad (3.22)
\end{aligned}
$$

By substituting the corresponding expressions, (3.22) becomes

$$
\begin{aligned}
&\int_0^T \psi(t)dt\Big\{E[\int_0^t 2e^{-r(s)} < F(s), u(s) >_{H^0} ds] + E[\int_0^t e^{-r(s)} \sum_{k=1}^\infty ||\tilde{\sigma}_k(s)||^2_{H^0} ds] \\
&-E[\int_0^t e^{-r(s)} r'(s)\{||u(s)-v(s)||^2_{H^0} + 2 < u(s)-v(s), v(s) >_{H^0}\}ds]\Big\} \\
\leq\ & \liminf_{n\to\infty} \int_0^T \psi(t)dt\Big\{E[\int_0^t 2e^{-r(s)} < F(u_n(s)), u_n(s) >_{H^0} ds] \\
&+E[\int_0^t e^{-r(s)} \sum_{k=1}^\infty ||\Pi_n\sigma_k(u_n(s))||^2_{H^0} ds] \\
&-E[\int_0^t e^{-r(s)} r'(s)\{||u_n(s)-v(s)||^2_{H^0} + 2 < u_n(s)-v(s), v(s) >_{H^0}\}ds]\Big\} \\
:=\ & \liminf_{n\to\infty} Z_n, \qquad (3.23)
\end{aligned}
$$

where $Z_n = Z_n^1 + Z_n^2 + Z_n^3$ with

$$
\begin{aligned}
Z_n^1 = & \int_0^T \psi(t)dt\Big\{E[\int_0^t e^{-r(s)}\{-r'(s)||u_n(s)-v(s)||^2_{H^0} \\
&+2 < F(u_n(s)) - F(v(s)), u_n(s)-v(s) >_{H^0} \\
&+\sum_{k=1}^\infty ||\Pi_n\sigma_k(u_n(s)) - \Pi_n\sigma_k(v(s))||^2_{H^0}\}ds]\Big\}, \qquad (3.24)
\end{aligned}
$$

$$
\begin{aligned}
Z_n^2 = & \int_0^T \psi(t)dt\Big\{E[\int_0^t e^{-r(s)}\{-2r'(s) < u_n(s)-v(s), v(s) >_{H^0} \\
&+2 < F(u_n(s)), v(s) >_{H^0} +2 < F(v(s)), u_n(s) >_{H^0} \\
&-2 < F(v(s)), v(s) >_{H^0} +2\sum_{k=1}^\infty < \Pi_n\sigma_k(u_n(s)), \sigma_k(v(s)) >_{H^0}\}ds]\Big\}, \qquad (3.25)
\end{aligned}
$$

$$
\begin{aligned}
Z_n^3 = & \int_0^T \psi(t)dt\Big\{E[\int_0^t e^{-r(s)}\{2\sum_{k=1}^\infty < \Pi_n\sigma_k(u_n(s)), \Pi_n\sigma_k(v(s)) - \sigma_k(v(s)) >_{H^0} \\
&-\sum_{k=1}^\infty ||\Pi_n\sigma_k(v(s))||^2_{H^0}\}ds]\Big\}. \qquad (3.26)
\end{aligned}
$$

Set $r'(s) = c + C(||v(s)||_{H^1}||v(s)||_{H^2} + 1)$. In view of (3.20) and (H.3) we see that $Z_n^1 \leq 0$. By the weak convergence, it is clear that $Z_n^2 \to Z^2$, where

$$\begin{aligned} Z^2 &= \int_0^T \psi(t)dt \Big\{ E[\int_0^t e^{-r(s)}\{-2r'(s) < u(s) - v(s), v(s) >_{H^0} \\ &\quad +2 < F(s), v(s) >_{H^0} +2 < F(v(s)), u(s) >_{H^0} \\ &\quad -2 < F(v(s)), v(s) >_{H^0} +2\sum_{k=1}^{\infty} < \tilde{\sigma}_k(s), \sigma_k(v(s)) >_{H^0}\}ds] \Big\} \end{aligned} \tag{3.27}$$

Also

$$Z_n^3 \to Z^3 := -\int_0^T \psi(t)dt \sum_{k=1}^{\infty} E[\int_0^t ||\sigma_k(v(s))||^2_{H^0}\}ds] \tag{3.28}$$

Combining (3.23)-(3.28), after some cancelations it turns out that

$$\begin{aligned} &\int_0^T \psi(t)dt \Big\{ E[\int_0^t e^{-r(s)}\{-r'(s)||u(s) - v(s)||^2_{H^0} + 2 < F(s) - F(v(s)), u(s) - v(s) >_{H^0} \\ &+ \sum_{k=1}^{\infty} ||\tilde{\sigma}_k(s) - \sigma_k(v(s))||^2_{H^0}\}ds] \Big\} \leq 0 \end{aligned} \tag{3.29}$$

As K is arbitrary, by approximation it is seen that (3.29) holds true for every $v \in L^2(\Omega_T, H^2)$. In particular, take $v(s) = u(s)$ in (3.29) to obtain $\tilde{\sigma}_k(s) = \sigma_k(u(s))$ for every $k \geq 1$. For $\lambda \in [-1, 1]$, $\tilde{v} \in L^\infty(\Omega_T, H^2)$, set $v_\lambda(s) = u(s) - \lambda\tilde{v}(s)$. Replace v by v_λ in (3.29) to get

$$E[\int_0^T e^{-r_\lambda(s)}\{-\lambda^2 r'_\lambda(s)||\tilde{v}(s)||^2_{H^0} + 2\lambda < F(s) - F(v_\lambda(s)), \tilde{v}(s) >_{H^0}\}ds] \leq 0, \tag{3.30}$$

where $r_\lambda(s)$ is defined as $r(s)$ with v replaced by v_λ. Dividing (3.30) by λ we obtain

$$E[\int_0^T e^{-r_\lambda(s)}\{-\lambda r'_\lambda(s)||\tilde{v}(s)||^2_{H^0} + 2 < F(s) - F(v_\lambda(s)), \tilde{v}(s) >_{H^0}\}ds] \leq 0 \tag{3.31}$$

for $\lambda > 0$, and

$$E[\int_0^T e^{-r_\lambda(s)}\{-\lambda r'_\lambda(s)||\tilde{v}(s)||^2_{H^0} + 2 < F(s) - F(v_\lambda(s)), \tilde{v}(s) >_{H^0}\}ds] \geq 0 \tag{3.32}$$

for $\lambda < 0$.

Note that by (3.20)

$$\begin{aligned} &| < F(u(s)) - F(v_\lambda(s)), \tilde{v}(s) >_{H^0} | \\ \leq\ & \frac{|\lambda|}{2}||\tilde{v}(s)||^2_{H^1} + |\lambda|C'_1(||\tilde{v}(s)||^2_{H^0}||u(s)||_{H^1}||u(s)||_{H^2}) + |\lambda|C_1||\tilde{v}(s)||^2_{H^0}. \end{aligned} \tag{3.33}$$

Hence by (vi) the dominated convergence theorem yields

$$\begin{aligned}
&\lim_{\lambda\to 0} E[\int_0^T e^{-r_\lambda(s)} < F(s) - F(v_\lambda(s)), \tilde{v}(s) >_{H^0} ds] \\
= \; & E[\int_0^T e^{-r_0(s)} < F(s) - F(u(s)), \tilde{v}(s) >_{H^0} ds] \qquad (3.34)
\end{aligned}$$

Let $\lambda \to 0^+$ in (3.31) and $\lambda \to 0^-$ in (3.32) to obtain

$$E[\int_0^T e^{-r_0(s)} < F(s) - F(u(s)), \tilde{v}(s) >_{H^0} ds] = 0$$

As $\tilde{v}$ is arbitrary, we conclude that $F(s) = F(u(s))$ a.e. om Ω_T. Hence,

$$u(t) = u_0 - \int_0^t Au(s)ds - \int_0^t B(u(s))ds - \int_0^t g_N(|u|^2)u(s)ds + \sum_{k=1}^{\infty} \int_0^t \sigma_k(u(s))dW_k(s) \qquad (3.35)$$

Step 2. General case: $E||u_0||_{H^1}^2] < \infty$.

Take any sequence $Y_n(0) \in L^6(\Omega, \mathcal{F}_0; H^1)$ that satisfies $E[||Y_n(0) - u_0||_{H^2}^2] \to 0$. Let $Y_n(t), t \geq 0$ be the solution of the following equation:

$$\begin{aligned}
dY_n(t) &= -AY_n(t)dt - B(Y_n(t))dt - g_N(|Y_n|^2)Y_n(t)dt + \sum_{k=1}^{\infty} \sigma_k(Y_n(t))dW_k(t) \\
Y_n(0) &= Y_n(0) \in H^1.
\end{aligned}$$

The existence of Y_n is guaranteed by step 1. Moreover, as the proof of (3.11) we can show that

$$\begin{aligned}
&\sup_n \left\{ E\big(\sup_{0\leq t\leq T} ||Y_n(t)||_{H^1}^2 \big) + \int_0^T E[||Y_n(t)||_{H^2}^2]dt \right\} \\
\leq \; & C \sup_n (E[||Y_n(0)||_{H^1}^2]) < \infty \qquad (3.36)
\end{aligned}$$

This implies that there exist a subsequence (still use the same notation) of $\{Y_n, n \geq 1\}$ and a process $Y \in L^2(\Omega, L^\infty([0,T], H^1)) \cap L^2(\Omega_T, H^2)$ for which the following hold:
(i) $Y_n \to Y$ weakly in $L^2(\Omega_T, H^2)$,
(ii) $Y_n \to Y$ in $L^2(\Omega, L^\infty([0,T], H^1))$ equipped with the weak star topology.

Next we show that Y_n also converges to Y in probability in $L^\infty([0,T], H^0)$. For $R > 0$, define the stopping time

$$\tau_R^n := \inf\{t \in [0,\infty) : ||Y_n(t)||_{H^1} > R\}.$$

τ_R^n is really a stopping time since Y_n is continuous in H^1. Then it follows from (3.36) that there exists a constant M , independent of n, R, so that

$$P(\tau_R^n \le T) \le P(\sup_{0\le t\le T} ||Y_n(t)||_{H^1} > R) \le \frac{M}{R^2} \tag{3.37}$$

When R is fixed, as in the proof of Theorem 3.7 in [RZ1], we find that

$$E\left[\sup_{0\le t\le T} ||Y_n(t\wedge \tau_R^n \wedge \tau_R^m) - Y_m(t\wedge \tau_R^n \wedge \tau_R^m)||_{H^0}^2\right] \le C_{R,T} E[||Y_n(0) - Y_m(0)||_{H^0}^2] \tag{3.38}$$

For $\eta > 0$ and any $R > 0$, we have

$$\begin{aligned} & P(\sup_{0\le t\le T} ||Y_n(t) - Y_m(t)||_{H^0} > \eta) \\ \le\ & P(\tau_R^n \le T) + P(\tau_R^m \le T) \\ & + P(\sup_{0\le t\le T} ||Y_n(t\wedge \tau_R^n \wedge \tau_R^m) - Y_m(t\wedge \tau_R^n \wedge \tau_R^m)||_{H^0} > \eta) \end{aligned} \tag{3.39}$$

Given an arbitrarily small constant $\delta > 0$. In view of (3.37), one can choose R such that $P(\tau_R^n \le T) \le \frac{\delta}{4}$ and $P(\tau_R^m \le T) \le \frac{\delta}{4}$. For such R, by (3.38) there exists N_0 such that for $m, n \ge N_0$,

$$P(\sup_{0\le t\le T} ||Y_n(t\wedge \tau_R^n \wedge \tau_R^m) - Y_m(t\wedge \tau_R^n \wedge \tau_R^m)||_{H^0} > \eta) \le \frac{\delta}{4}$$

Therefore,

$$P(\sup_{0\le t\le T} ||Y_n(t) - Y_m(t)||_{H^0} > \eta) \le \delta$$

Thus

$$\lim_{n,m\to\infty} P(\sup_{0\le t\le T} ||Y_n(t) - Y_m(t)||_{H^0} > \eta) = 0 \tag{3.40}$$

This proves that Y_n converges to Y in probability in $L^\infty([0,T], H^0)$. Finally we want to show Y solves the equation (3.1). To this end, it suffices to prove that for $v \in V$,

$$\begin{aligned} & < Y(t), v >_{H^0} \\ =\ & < u_0, v >_{H^0} - \int_0^t < AY(s), v >_{H^0} ds - \int_0^t < B(Y(s)), v >_{H^0} ds \\ & - \int_0^t < g_N(|Y|^2)Y(s), v >_{H^0} ds + \sum_{k=1}^\infty \int_0^t < \sigma_k(Y(s)), v >_{H^0} dW_k(s) \end{aligned} \tag{3.41}$$

But for every $n \ge 1$, we know that

$$\begin{aligned} & < Y_n(t), v >_{H^0} \\ =\ & < Y_n(0), v >_{H^0} - \int_0^t < AY_n(s), v >_{H^0} ds - \int_0^t < B(Y_n(s)), v >_{H^0} ds \\ & - \int_0^t < g_N(|Y_n|^2)Y_n(s), v >_{H^0} ds + \sum_{k=1}^\infty \int_0^t < \sigma_k(Y_n(s)), v >_{H^0} dW_k(s) \end{aligned} \tag{3.42}$$

Note that

$$-\int_0^t < B(Y_n(s)), v >_{H^0} ds = \int_0^t < Y_n^*(s) \cdot Y_n(s), \nabla v >_{H^0} ds$$

Letting $n \to \infty$, thanks to the convergence in probablility and also the weak convergence, by dominated convergence theorem we see that each term in (3.42) tends to the corresponding term in (3.41). Hence the proof is complete. ■

4 Statement of the large deviation principle

Consider again the stochastic 3D tamed Navier-Stokes equation:

$$\begin{aligned} du(t) &= -Au(t)dt - B(u(t))dt - \mathcal{P}g_N(|u|^2)u(t)dt + \sum_{k=1}^{\infty} \sigma_k(u(t))dW_k(t) \\ u(0) &= u_0 \in H^1. \end{aligned}$$

Here $\sigma_k(\cdot), k \geq 1$ is a sequence of mappings from H^1 (H^2) into H^1 (H^2). Consider the following hypotheses.

(A.1) .

$$\sum_{k=1}^{\infty} ||\sigma_k(u)||_{H^2}^2 \leq c(1 + ||u||_{H^2}^2)$$

(A.2) .

$$\sum_{k=1}^{\infty} ||\sigma_k(u)||_{H^1}^2 \leq c(1 + ||u||_{H^1}^2)$$

(A.3) .

$$\sum_{k=1}^{\infty} ||\sigma_k(u) - \sigma_k(v)||_{H^1}^2 \leq c(||u - v||_{H^1}^2)$$

(A.4) .

$$\sum_{k=1}^{\infty} ||\sigma_k(u) - \sigma_k(v)||_{H^2}^2 \leq c(||u - v||_{H^2}^2)$$

Consider the small time process $u(\varepsilon t)$. By the scaling property of the Brownian motion, $u(\varepsilon \cdot)$ coincides in law with the solution of the following stochastic 3D tamed Navier-Stokes equation:

$$\begin{aligned} u^\varepsilon(t) &= u_0 - \varepsilon \int_0^t Au^\varepsilon(s)ds - \varepsilon \int_0^t B(u^\varepsilon(s))ds - \varepsilon \int_0^t \mathcal{P}(g_N(|u|^2)u)ds \\ &\quad + \sum_{k=1}^{\infty} \sqrt{\varepsilon} \int_0^t \sigma_k(u^\varepsilon(s))dW_k(s). \end{aligned} \tag{4.1}$$

We know that the stochastic tamed NSE (4.1) has a unique strong solution $u^\varepsilon \in L^2(\Omega; C([0,T]; H^1)) \cap L^2(\Omega \times [0,T]; H^2)$. Set

$$\begin{aligned} \mathcal{H} &= \{h = (h_1, h_2, ..., h_k, ...); \quad h(\cdot) : [0,T] \to l^2 \quad \text{such that} \\ & \qquad h \quad \text{is absolutely continuous and} \quad \sum_{k=1}^{\infty} \int_0^T \dot{h}_k(t)^2 dt < \infty\} \end{aligned}$$

For $h \in \mathcal{H}$, let $u^h(t)$ denote the solution of the following deterministic equation:

$$\begin{aligned} du^h(t) &= \sum_{k=1}^{\infty} \sigma_k(u^h(t))\dot{h}_k(t)dt \\ u^h(0) &= u_0. \end{aligned} \tag{4.2}$$

For $h(t) = \sum_{k=1}^{\infty} h_k(t)e_k \in \mathcal{H}$, define

$$I(h) = \frac{1}{2}\sum_{k=1}^{\infty} \int_0^T (\dot{h}_k(t))^2 dt.$$

For $f \in C([0,T]; H^1)$, define

$$\mathcal{L}_f = \{h \in \mathcal{H} : \quad f(\cdot) = u^h(\cdot)\}.$$

Define

$$R(f) = \begin{cases} \inf_{h\in\mathcal{L}_f} I(h) & \text{if} \quad \mathcal{L}_f \neq \emptyset, \\ +\infty & \text{if} \quad \mathcal{L}_f = \emptyset. \end{cases}$$

Theorem 4.1 *Assume (A.1)-(A.4). Let μ_ε be the law of u^ε on the space $C([0,T]; H^1)$. Then $\{\mu_\varepsilon, \varepsilon > 0\}$ satisfies a large deviation principle with rate function $R(f)$, i.e.,*

(1) for every closed subset $C \subset C([0,T]; H^1)$,

$$\limsup_{\varepsilon\to 0} \varepsilon^2 log\mu_\varepsilon(C) \leq -inf_{f\in C} R(f), \tag{4.3}$$

(2) for every open subset $G \subset C([0,T]; H^1)$,

$$\liminf_{\varepsilon\to 0} \varepsilon^2 log\mu_\varepsilon(G) \geq -inf_{f\in G} R(f). \tag{4.4}$$

5 Proof of Theorem 4.1

This section is devoted to the proof Theorem 4.1, which will be split into a number of lemmas. Let $v^\varepsilon(\cdot)$ be the solution of the stochastic differential equation:

$$v^\varepsilon(t) = u(0) + \sqrt{\varepsilon}\sum_{k=1}^{\infty}\int_0^t \sigma_k(v^\varepsilon(s))dW_k(s), \tag{5.1}$$

and ν^ε be the law of $v^\varepsilon(\cdot)$ on the $C([0,T];H^1)$. Then by [DZ], we know that ν^ε satisfies a large deviation principle with rate function $R(\cdot)$. Our task is to show that the two families of probability measures μ^ε and ν^ε are exponentially equivalent, that is, for any $\delta > 0$,

$$\lim_{\varepsilon\to 0}\varepsilon \log P\big(\sup_{0\le t\le T}|u^\varepsilon(t) - v^\varepsilon(t)|^2 > \delta\big) = -\infty. \tag{5.2}$$

Then Theorem 4.1 follows from the fact (see e.g. [DZ]) that if one of the two exponentially equivalent families satisfies a large deviation principle, so does the other.

We begin with the following lemma which provides an estimate of the probability that the solution of (4.1) leaves an energy ball. It will play a crucial role in the rest of the paper.

Lemma 5.1

$$\lim_{M\to\infty}\sup_{0<\varepsilon\le 1}\varepsilon\log P\big(|u^\varepsilon|_{H^1}^{H^2}(T) > M\big) = -\infty, \tag{5.3}$$

where $|u^\varepsilon|_{H^1}^{H^2}(T) := \sup_{0\le t\le T}||u^\varepsilon(t)||_{H^1}^2 + \varepsilon\int_0^T ||u^\varepsilon(t)||_{H^2}^2 dt$.

Proof: By Itô's formula, we have

$$\begin{aligned}
||u^\varepsilon(t)||_{H^1}^2 &= ||u_0||_{H^1}^2 - 2\varepsilon\int_0^t < u^\varepsilon(s), Au^\varepsilon(s) > ds - 2\varepsilon\int_0^t < u^\varepsilon(s), B(u^\varepsilon(s)) >_{H^1} ds \\
&\quad + 2\sqrt{\varepsilon}\sum_{k=1}^{\infty}\int_0^t < u^\varepsilon(s), \sigma_k(u^\varepsilon(s) >_{H^1} dW_k(s) + \varepsilon\sum_{k=1}^{\infty}\int_0^t ||\sigma_k(u^\varepsilon(s))||_{H^1}^2 ds \\
&\quad - 2\varepsilon\int_0^t < u^\varepsilon(s), \mathcal{P}(g_N(|u^\varepsilon|^2)u^\varepsilon) >_{H^1} ds \\
&:= ||u_0||_{H^1}^2 + I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t).
\end{aligned} \tag{5.4}$$

We now estimate each of the terms. First, we have

$$\begin{aligned}
I_1(t) &= 2\varepsilon\int_0^t < \Delta u^\varepsilon(s), (I-\Delta)u^\varepsilon(s) >_{L^2} ds \\
&= -2\varepsilon\int_0^t ||u^\varepsilon(s)||_{H^2}^2 ds + 2\varepsilon\int_0^t ||\nabla u^\varepsilon(s)||_{L^2}^2 ds + 2\varepsilon\int_0^t ||u^\varepsilon(s)||_{L^2}^2 ds.
\end{aligned} \tag{5.5}$$

In view of (3.6),

$$\begin{aligned}|I_2(t)| &\le \varepsilon\int_0^t ||(I-\Delta)u^\varepsilon(s)||_{L^2}^2 ds + \varepsilon\int_0^t ||(u\cdot\nabla)u^\varepsilon(s)||_{L^2}^2 ds\\ &\le \varepsilon\int_0^t ||u^\varepsilon(s)||_{H^2}^2 ds + \varepsilon\int_0^t ||(u\cdot\nabla)u^\varepsilon(s)||_{L^2}^2 ds\\ &\le \varepsilon\int_0^t ||u^\varepsilon(s)||_{H^2}^2 ds + \varepsilon\int_0^t |||u|\cdot|\nabla u^\varepsilon(s)|||_{L^2}^2 ds,\end{aligned}\tag{5.6}$$

By (A.2),

$$|I_4(t)| \le \varepsilon\cdot L\int_0^t (1+||u^\varepsilon(s)||_{H^1}^2)ds.\tag{5.7}$$

As $g_N(r)\ge r-C_N$ for some constant C_N, we have

$$\begin{aligned}& I_5(t)\\ =\ & -2\varepsilon\int_0^t\int_D |\nabla u^\varepsilon(s)|^2 g_N(|u^\varepsilon(s)|^2)dxds - \varepsilon\int_0^t\int_D g_N'(|u^\varepsilon(s)|^2)|\nabla|u^\varepsilon|^2|^2 dxds\\ & -\varepsilon\int_0^t\int_D |u^\varepsilon(s)|^2 g_N(|u^\varepsilon(s)|^2)dxds\\ \le\ & -2\varepsilon\int_0^t |||u|\cdot|\nabla u^\varepsilon(s)|||_{L^2}^2 ds + C_N\varepsilon\int_0^t ||\nabla u^\varepsilon(s)||_{L^2}^2 ds.\end{aligned}\tag{5.8}$$

Substituting (5.5), (5.6), (5.7) and (5.8) into (5.4) we obtain

$$\begin{aligned}|u^\varepsilon(t)|^2 + \varepsilon\int_0^t ||u^\varepsilon||_{L^2}^2 ds &\le \big(||u_0||_{H^1}^2 + \varepsilon LT\big) + C_L\varepsilon\int_0^t ||u^\varepsilon(s)||_{H^1}^2 ds\\ &\quad +2\sqrt{\varepsilon}|\sum_{k=1}^\infty\int_0^t < u^\varepsilon(s),\sigma_k(u^\varepsilon(s) >_{H^1} dW_k(s)|.\end{aligned}$$

Therefore,

$$\begin{aligned}|u^\varepsilon|_{H^1}^{H^2}(t) &\le 2\big(||u_0||_{H^1}^2 + \varepsilon LT\big) + C_L\varepsilon\int_0^t |u^\varepsilon|_{H^1}^{H^2}(s)ds\\ &\quad +4\sqrt{\varepsilon}\sup_{0\le t\le T}|\sum_{k=1}^\infty\int_0^t < u^\varepsilon(s),\sigma_k(u^\varepsilon(s) >_{H^1} dW_k(s)|.\end{aligned}$$

Hence, for $p\ge 2$, we have,

$$\begin{aligned}&\big(E(|u^\varepsilon|_{H^1}^{H^2}(t))^p\big)^{\frac{1}{p}}\\ \le\ & 2\big(||u_0||_{H^1}^2+\varepsilon LT\big) + C_L\varepsilon\big(E(\int_0^t |u^\varepsilon|_{H^1}^{H^2}(s)ds)^p\big)^{\frac{1}{p}}\\ +\ & 4\sqrt{\varepsilon}\big(E(\sup_{0\le s\le t}|\sum_{k=1}^\infty\int_0^s < u^\varepsilon(r),\sigma_k(u^\varepsilon(r) >_{H^1} dW_k(r)|^p\big)^{\frac{1}{p}}.\end{aligned}\tag{5.9}$$

To estimate the stochastic integral term, we will use the following remarkable result from [D1], [BY] which says that there exists a universal constant c such that, for any $p \geq 2$ and for any continuous martingale (M_t) with $M_0 = 0$, one has

$$||M_t^*||_p \leq cp^{\frac{1}{2}}|| < M >_t^{\frac{1}{2}} ||_p, \tag{5.10}$$

where $M_t^* = \sup_{0\leq s\leq t} |M_s|$ and $||\cdot||_p$ stands for the L^p-norm. We emphasize that what we need is the precise factor $p^{\frac{1}{2}}$ on the right.

Thus,

$$\begin{aligned}
& 4\sqrt{\varepsilon}\big(E(\sup_{0\leq s\leq t}|\sum_{k=1}^{\infty}\int_0^s < u^\varepsilon(r), \sigma_k(u^\varepsilon(r) >_{H^1} dW_k(r)|^p\big)^{\frac{1}{p}} \\
\leq\ & 4c\sqrt{p\varepsilon}\big(E(\int_0^t \sum_{k=1}^{\infty} < u^\varepsilon(s), \sigma_k(u^\varepsilon(s) >_{H^1}^2 ds)^{\frac{p}{2}}\big)^{\frac{1}{p}} \\
\leq\ & 4c\sqrt{p\varepsilon}\big(E(\int_0^t ||u^\varepsilon(s)||_{H^1}^2(1 + ||u^\varepsilon(s)||_{H^1}^2)ds)^{\frac{p}{2}}\big)^{\frac{1}{p}} \\
\leq\ & 4c\sqrt{p\varepsilon}\big[\big(E(\int_0^t (1 + ||u^\varepsilon(s)||_{H^1}^2)^2 ds)^{\frac{p}{2}}\big)^{\frac{2}{p}}\big]^{\frac{1}{2}} \\
\leq\ & 4c\sqrt{p\varepsilon}\big[\big(E(\int_0^t (1 + ||u^\varepsilon(s)||_{H^1}^4)ds)^{\frac{p}{2}}\big)^{\frac{2}{p}}\big]^{\frac{1}{2}} \\
\leq\ & 4c\sqrt{p\varepsilon}\big[\int_0^t 1 + \big(E||u^\varepsilon(s)||_{H^1}^{2p}\big)^{\frac{2}{p}} ds\big]^{\frac{1}{2}},
\end{aligned} \tag{5.11}$$

where (A.2) has been used. On the other hand,

$$2\varepsilon L\big(E(\int_0^t |u^\varepsilon|_{H^1}^{H^2}(s)ds)^p\big)^{\frac{1}{p}} \leq 2\varepsilon L\int_0^t (E|u^\varepsilon|_{H^1}^{H^2}(T))^p)^{\frac{1}{p}} ds. \tag{5.12}$$

Combining (5.9), (5.11) and (5.12), we arrive at

$$\begin{aligned}
& \big(E(|u^\varepsilon|_{H^1}^{H^2}(t))^p\big)^{\frac{2}{p}} \\
\leq\ & 8\big(||u_0||_{H^1}^2 + \varepsilon LT\big)^2 + 8\varepsilon^2 L^2 T\int_0^t \big(E(|u^\varepsilon|_{H^1}^{H^2}(s))^p\big)^{\frac{2}{p}} ds \\
& +32c^2 p\varepsilon T + 32c^2 p\varepsilon\int_0^t \big(E(|u^\varepsilon|_{H^1}^{H^2}(s))^p\big)^{\frac{2}{p}} ds.
\end{aligned} \tag{5.13}$$

Applying the Gronwall inequality, we obtain

$$\begin{aligned}
& \big(E(|u^\varepsilon|_{H^1}^{H^2}(T))^p\big)^{\frac{2}{p}} \\
\leq\ & \big[8\big(||u_0||_{H^1}^2 + \varepsilon LT\big)^2 + 32c^2 p\varepsilon T\big]\cdot exp(8\varepsilon^2 L^2 T + 32c^2 p\varepsilon T).
\end{aligned} \tag{5.14}$$

Since $P(|u^\varepsilon|_{H^1}^{H^2}(T) > M) \leq M^{-p}E(|u^\varepsilon|_{H^1}^{H^2}(T))^p$, take $p = \frac{1}{\varepsilon}$ in (5.14) to get

$$\begin{aligned} & \varepsilon \log P(|u^\varepsilon|_{H^1}^{H^2}(T) > M) \\ \leq & -\log M + \log(E(|u^\varepsilon|_{H^1}^{H^2}(T))^p)^{\frac{1}{p}} \\ \leq & -\log M + \log \sqrt{\left[8\left(||u_0||_{H^1}^2 + \varepsilon LT\right)^2 + 32c^2T\right]} + 4\varepsilon^2L^2T + 16c^2T. \end{aligned}$$

Therefore,

$$\begin{aligned} & \sup_{0<\varepsilon\leq 1} \varepsilon \log P(|u^\varepsilon|_{H^1}^{H^2}(T) > M) \\ \leq & -\log M + \log \sqrt{\left[8\left(||u_0||_{H^1}^2 + LT\right)^2 + 32c^2\right]} + 16c^2 + 4L^2T. \end{aligned}$$

Letting $M \to \infty$ on both side of the above inequality, we complete the proof. ■

Since H^2 is dense in H^1, there exists a sequence $\{u_n(0)\}_{n=1}^\infty \subset H^2$ such that

$$\lim_{n\to+\infty} ||u_n(0) - u_0||_{H^1} = 0.$$

Let $u_n^\varepsilon(\cdot)$ be the solution of (4.1) with initial value $u_n(0)$. From the proof of Lemma 3.1, it is easily seen that the following is also true.

$$\lim_{M\to+\infty} \sup_n \sup_{0<\varepsilon\leq 1} \varepsilon \log P((|u_n^\varepsilon|_{H^1}^{H^2}(T))^2 > M) = -\infty. \tag{5.15}$$

Let $v_n^\varepsilon(\cdot)$ be the solution of (5.1) with the initial value $u_n(0)$. We have the following result whose proof is very similar to (but simpler than) that of Lemma.

Lemma 5.2

$$\lim_{M\to\infty} \sup_n \sup_{0<\varepsilon\leq 1} \varepsilon \log P(\sup_{0\leq t\leq 1} ||v_n^\varepsilon(t)||_{H^1}^2 > M) = -\infty.$$

Moreover, for any fixed $n \in \mathbb{Z}^+$,

$$\lim_{M\to\infty} \sup_{0<\varepsilon\leq 1} \varepsilon \log P(\sup_{0\leq t\leq 1} ||v_n^\varepsilon(t)||_{H^2}^2 > M) = -\infty.$$

The following estimates will be used frequently in the sequel. By Hölder's inequality and Sobolev imbedding, we have

$$\begin{aligned}
||B(u)-B(v)||_{L^2}^2 &= \int_{R^3} |(u\cdot\nabla)u-(v\cdot\nabla)v|^2(x)dx \\
&\leq 2\int_{R^3}\sum_{i=1}^3 (u^i-v^i)^2\sum_{k=1}^3\sum_{i=1}^3(\partial_i u^k)^2dx \\
&\quad +2\int_{R^3}\sum_{i=1}^3 (v^i)^2\sum_{k=1}^3\sum_{i=1}^3(\partial_i u^k-\partial_i v^k)^2dx \\
&\leq 2\sup_x\sum_{i=1}^3(u^i(x)-v^i(x))^2||u||_{H^1}^2+2\sup_x\sum_{i=1}^3(v^i(x))^2||u-v||_{H^1}^2 \\
&\leq 2C||u-v||_{H^2}||u-v||_{H^1}||u||_{H^1}^2+2C||v||_{H^2}||v||_{H^1}||u-v||_{H^1}^2 \quad (5.16)
\end{aligned}$$

and

$$\begin{aligned}
& ||g_N(|u|^2)u-g_N(|v|^2)v||_{L^2} \\
\leq\ & ||u-v||_{L^6}(||u||_{L^6}^2+||v||_{L^6}^2) \\
\leq\ & C||u-v||_{H^1}(||u||_{H^1}^2+||v||_{H^1}^2). \quad (5.17)
\end{aligned}$$

Lemma 5.3 *For any* $\delta>0$,

$$\lim_{n\to+\infty}\sup_{0<\varepsilon\leq 1}\varepsilon\log P(\sup_{0\leq t\leq 1}||u^\varepsilon(t)-u_n^\varepsilon(t)||_{H^1}^2>\delta)=-\infty. \quad (5.18)$$

Proof: As an equation in L^2, we have

$$\begin{aligned}
u^\varepsilon(t)-u_n^\varepsilon(t) &= u_0-u_n(0)-\varepsilon\int_0^t A(u^\varepsilon(s)-u_n^\varepsilon(s))ds-\varepsilon\int_0^t(B(u^\varepsilon(s))-B(u_n^\varepsilon(s)))ds \\
&+ \sqrt{\varepsilon}\sum_{k=1}^\infty\int_0^t(\sigma_k(u^\varepsilon(s))-\sigma_k(u_n^\varepsilon(s))dW_k(s) \\
&- \varepsilon\int_0^t(g_N(|u^\varepsilon(s)|^2)u^\varepsilon(s)-g_N(|u_n^\varepsilon(s)|^2)u_n^\varepsilon(s))ds. \quad (5.19)
\end{aligned}$$

For $M>0$, define stopping times

$$t_{\varepsilon,M}=\inf\{t:\varepsilon\int_0^t||u^\varepsilon(r)||_{H^2}^2dr>M,\ \text{or}\ |u^\varepsilon(t)|_{H^1}^2>M\}.$$

$$t_{\varepsilon,M}^n=\inf\{t:\varepsilon\int_0^t||u_n^\varepsilon(r)||_{H^2}^2dr>M,\ \text{or}\ |u_n^\varepsilon(t)|_{H^1}^2>M\}.$$

Put $\tau_{\varepsilon,M} = t_{\varepsilon,M} \wedge t^n_{\varepsilon,M}$. By Itô's formula, we have

$$
\begin{aligned}
& ||u^\varepsilon(t) - u^\varepsilon_n(t)||^2_{H^1} \\
= \quad & ||u_0 - u_n(0)||^2_{H^1} - 2\varepsilon \int_0^t < A(u^\varepsilon(s) - u^\varepsilon_n(s)), u^\varepsilon(s) - u^\varepsilon_n(s) >_{H^1} ds \\
& -2\varepsilon \int_0^t < B(u^\varepsilon(s)) - B(u^\varepsilon_n(s)), u^\varepsilon(s) - u^\varepsilon_n(s) >_{H^1} ds \\
& -2\varepsilon \int_0^t < g_N(|u^\varepsilon(s)|^2) u^\varepsilon(s) - g_N(|u^\varepsilon_n(s)|^2) u^\varepsilon_n(s), u^\varepsilon(s) - u^\varepsilon_n(s) >_{H^1} ds \\
& +\varepsilon \int_0^t \sum_{k=1}^{\infty} ||\sigma_k(u^\varepsilon(s)) - \sigma_k(u^\varepsilon_n(s))||^2_{H^1} ds \\
& +2\sqrt{\varepsilon} \sum_{k=1}^{\infty} \int_0^t < \sigma_k(u^\varepsilon(s)) - \sigma_k(u^\varepsilon_n(s)), u^\varepsilon(s) - u^\varepsilon_n(s) >_{H^1} dW_k(s) \\
:= \quad & ||u_0 - u_n(0)||^2_{H^1} + J_{n,1}(t) + J_{n,2}(t) + J_{n,3}(t) + J_{n,4}(t) + J_{n,5}(t). \qquad (5.20)
\end{aligned}
$$

We will bound each of the terms on the right.

$$
\begin{aligned}
J_{n,1}t) \quad = \quad & -2\varepsilon \int_0^t ||u^\varepsilon(s) - u^\varepsilon_n(s)||^2_{H^2} ds + 2\varepsilon \int_0^t ||\nabla u^\varepsilon(s) - \nabla u^\varepsilon_n(s)||^2_{L^2} ds \\
& +2\varepsilon \int_0^t ||u^\varepsilon(s) - u^\varepsilon_n(s)||^2_{L^2} ds \\
\leq \quad & -2\varepsilon \int_0^t ||u^\varepsilon(s) - u^\varepsilon_n(s)||^2_{H^2} ds + 4\varepsilon \int_0^t ||u^\varepsilon(s) - u^\varepsilon_n(s)||^2_{H^1} ds. \qquad (5.21)
\end{aligned}
$$

In view of (5.16) and by Young's inequality,

$$
\begin{aligned}
J_{n,2}(t) \quad \leq \quad & 2\varepsilon \int_0^t ||B(u^\varepsilon(s)) - B(u^\varepsilon_n(s))||_{L^2} ||u^\varepsilon(s) - u^\varepsilon_n(s)||_{H^2} ds \\
\leq \quad & 4\varepsilon \int_0^t ||u^\varepsilon(s) - u^\varepsilon_n(s)||^{\frac{3}{2}}_{H^2} ||u^\varepsilon(s) - u^\varepsilon_n(s)||^{\frac{1}{2}}_{H^1} ||u^\varepsilon(s)||_{H^1} ds \\
& +4\varepsilon \int_0^t ||u^\varepsilon(s) - u^\varepsilon_n(s)||_{H^2} ||u^\varepsilon(s) - u^\varepsilon_n(s)||_{H^1} ||u^\varepsilon_n(s)||^{\frac{1}{2}}_{H^1} ||u^\varepsilon_n(s)||^{\frac{1}{2}}_{H^2} ds \\
\leq \quad & \frac{1}{2}\varepsilon \int_0^t ||u^\varepsilon(s) - u^\varepsilon_n(s)||^2_{H^2} ds + C\varepsilon \int_0^t ||u^\varepsilon(s) - u^\varepsilon_n(s)||^2_{H^1} ||u^\varepsilon(s)||^4_{H^1} ds \\
& +C\varepsilon \int_0^t ||u^\varepsilon(s) - u^\varepsilon_n(s)||^2_{H^1} ||u^\varepsilon_n(s)||_{H^1} ||u^\varepsilon_n(s)||_{H^2}. \qquad (5.22)
\end{aligned}
$$

(5.17) yields

$$
\begin{aligned}
J_{n,3}(t) &\leq 2\varepsilon\int_0^t ||g_N(|u^\varepsilon(s)|^2)u^\varepsilon(s) - g_N(|u_n^\varepsilon(s)|^2)u_n^\varepsilon(s)||_{L^2}||u^\varepsilon(s) - u_n^\varepsilon(s)||_{H^2}ds \\
&\leq 4\varepsilon\int_0^t ||u^\varepsilon(s) - u_n^\varepsilon(s)||_{L^6}||u^\varepsilon(s) - u_n^\varepsilon(s)||_{H^2}(||u^\varepsilon(s)||_{L^6}^2 + ||u_n^\varepsilon(s)||_{L^6}^2)ds \\
&\leq 4\varepsilon\int_0^t ||u^\varepsilon(s) - u_n^\varepsilon(s)||_{H^1}||u^\varepsilon(s) - u_n^\varepsilon(s)||_{H^2}(||u^\varepsilon(s)||_{H^1}^2 + ||u_n^\varepsilon(s)||_{H^1}^2)ds \\
&\leq \frac{1}{4}\varepsilon\int_0^t ||u^\varepsilon(s) - u_n^\varepsilon(s)||_{H^2}^2ds \\
&\quad +C\varepsilon\int_0^t ||u^\varepsilon(s) - u_n^\varepsilon(s)||_{H^1}^2(||u^\varepsilon(s)||_{H^1}^2 + ||u_n^\varepsilon(s)||_{H^1}^2)^2ds. \qquad (5.23)
\end{aligned}
$$

Using(A.3), we obtain

$$
J_{n,4}(t) \leq C\varepsilon\int_0^t ||u^\varepsilon(s) - u_n^\varepsilon||_{H^1}^2ds \qquad (5.24)
$$

We substitute the above estimates into (5.20) to obtain

$$
\begin{aligned}
&||u^\varepsilon(t) - u_n^\varepsilon(t)||_{H^1}^2 \\
\leq\; &C\varepsilon\int_0^t ||u^\varepsilon(s) - u_n^\varepsilon(s)||_{H^1}^2\big[1 + ||u_n^\varepsilon(s)||_{H^1}^4 + ||u^\varepsilon(s)||_{H^1}^4 \\
&\quad +||u^\varepsilon(s)||_{H^1}^8 + ||u_n^\varepsilon(s)||_{H^1}^2 + ||u_n^\varepsilon(s)||_{H^2}^2\big]ds \\
&+||u_0 - u_n(0)||_{H^1}^2 + |M_t^\varepsilon|, \qquad (5.25)
\end{aligned}
$$

where

$$
M_t^\varepsilon = \sqrt{\varepsilon}\sum_{k=1}^\infty \int_0^t < u^\varepsilon(s) - u_n^\varepsilon(s), \sigma_k(u^\varepsilon(s)) - \sigma_k(u_n^\varepsilon(s)) >_{H^1} dW_k(s) \qquad (5.26)
$$

We apply Gronwall's inequality, (5.25) and the definition of $\tau_{\varepsilon,M}$ to get

$$
\begin{aligned}
&\sup_{0\leq s\leq t}(||u^\varepsilon(s\wedge\tau_{\varepsilon,M}) - u_n^\varepsilon(s\wedge\tau_{\varepsilon,M})||_{H^1}^2) \\
\leq\; &\big(||u_0 - u_n(0)||_{H^1}^2 + \sup_{0\leq s\leq t}|M_{s\wedge\tau_{\varepsilon,M}}^\varepsilon|\big) \\
&\times exp\{C\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}}(1 + ||u^\varepsilon(s)||_{H^1}^2 + ||u^\varepsilon(s)||_{H^1}^4 + ||u_n^\varepsilon(s)||_{H^1}^2 + ||u_n^\varepsilon(s)||_{H^1}^4 + ||u_n^\varepsilon(s)||_{H^2}^2)ds\} \\
\leq\; &\big(||u_0 - u_n(0)||_{H^1}^2 + \sup_{0\leq s\leq t}|M_{s\wedge\tau_{\varepsilon,M}}^\varepsilon|\big)exp\{C\varepsilon(T + 2M^2T + 2M^4T + M)\}. \qquad (5.27)
\end{aligned}
$$

Set $C^\varepsilon_M = C\varepsilon(T + 2M^2T + 2M^4T + M)$. By virtue of the Martingale Inequality (5.10) it follows from (5.27) that

$$\begin{aligned}
&(E[\sup_{0\le s\le t}(||u^\varepsilon(s\wedge\tau_{\varepsilon,M}) - u^\varepsilon_n(s\wedge\tau_{\varepsilon,M})||^{2p}_{H^1})])^{\frac{1}{p}}\\
\le\ & exp(C^\varepsilon_M)||u_0 - u_n(0)||^2_{H^1} + cexp(C^\varepsilon_M)\sqrt{p\varepsilon}\big(E(\int_0^t\sum_{k=1}^\infty < u^\varepsilon(s\wedge\tau_{\varepsilon,M})\\
&\quad -u^\varepsilon_n(s\wedge\tau_{\varepsilon,M}), \sigma_k(u^\varepsilon(s\wedge\tau_{\varepsilon,M}) - \sigma_k(u^\varepsilon_n(s\wedge\tau_{\varepsilon,M}) >^2_{H^1} ds)^{\frac{p}{2}}\big)^{\frac{1}{p}}\\
\le\ & exp(C^\varepsilon_M)||u_0 - u_n(0)||^2_{H^1}\\
&+cCexp(C^\varepsilon_M)\sqrt{p\varepsilon}\big(E(\int_0^t ||u^\varepsilon(s\wedge\tau_{\varepsilon,M}) - u^\varepsilon_n(s\wedge\tau_{\varepsilon,M})||^4_{H^1})ds)^{\frac{p}{2}}\big)^{\frac{1}{p}}\\
\le\ & exp(C^\varepsilon_M)||u_0 - u_n(0)||^2_{H^1} + cCexp(C^\varepsilon_M)\sqrt{p\varepsilon}\big[\big(E(\int_0^t ||u^\varepsilon(s\wedge\tau_{\varepsilon,M}) - u^\varepsilon_n(s\wedge\tau_{\varepsilon,M})||^4_{H^1}ds)^{\frac{p}{2}}\big)^{\frac{2}{p}}\big]^{\frac{1}{2}}\\
\le\ & exp(C^\varepsilon_M)||u_0 - u_n(0)||^2_{H^1} + cCexp(C^\varepsilon_M)\sqrt{p\varepsilon}\big[\int_0^t \big(E||u^\varepsilon(s\wedge\tau_{\varepsilon,M}) - u^\varepsilon_n(s\wedge\tau_{\varepsilon,M})||^{2p}_{H^1}\big)^{\frac{2}{p}}ds\big]^{\frac{1}{2}}
\end{aligned} \tag{5.28}$$

Hence,

$$\begin{aligned}
&(E[\sup_{0\le s\le t}(||u^\varepsilon(s\wedge\tau_{\varepsilon,M}) - u^\varepsilon_n(s\wedge\tau_{\varepsilon,M})||^{2p}_{H^1})])^{\frac{2}{p}}\\
\le\ & cCexp(2C^\varepsilon_M)p\varepsilon\big[\int_0^t \big(E||u^\varepsilon(s\wedge\tau_{\varepsilon,M}) - u^\varepsilon_n(s\wedge\tau_{\varepsilon,M})||^{2p}_{H^1}\big)^{\frac{2}{p}}ds\big]\\
&+2exp(2C^\varepsilon_M)||u_0 - u_n(0)||^4_{H^1}.
\end{aligned} \tag{5.29}$$

By Gronwall's inequality, this yields

$$\begin{aligned}
&(E[\sup_{0\le s\le T}(||u^\varepsilon(s\wedge\tau_{\varepsilon,M}) - u^\varepsilon_n(s\wedge\tau_{\varepsilon,M})||^{2p}_{H^1})])^{\frac{2}{p}}\\
\le\ & 2exp(2C^\varepsilon_M)||u_0 - u_n(0)||^4_{H^1}exp\{cCexp(2C^\varepsilon_M)p\varepsilon T\}.
\end{aligned} \tag{5.30}$$

Choose $p = \frac{2}{\varepsilon}$ to obtain

$$\begin{aligned}
&\sup_{0<\varepsilon\le 1}\varepsilon\log P(\sup_{0\le t\le T}||u^\varepsilon(t\wedge\tau_{\varepsilon,M}) - u^\varepsilon_n(t\wedge\tau_{\varepsilon,M})||^2_{H^1} > \delta)\\
\le\ & \sup_{0\le\varepsilon\le 1}\varepsilon\log\frac{E[\sup_{0\le t\le T}||u^\varepsilon(t\wedge\tau_{\varepsilon,M}) - u^\varepsilon_n(t\wedge\tau_{\varepsilon,M})||^{2p}_{H^1}]}{\delta^p}\\
\le\ & log2 + 2C^1_M + 2cCexp(2C^1_M) + 4\log||u_0 - u_n(0)||_{H^1}\\
\to\ & -\infty,\quad \text{as } n\to+\infty.
\end{aligned} \tag{5.31}$$

For any given $R > 0$, by Lemma 5.1, and (5.15) there exists a constant M such that for any $\varepsilon\in(0,1]$ and any $n\ge 1$ the following inequalities hold,

$$P((|u^\varepsilon|^{H^2}_{H^1}(T))^2 > M) \le e^{-\frac{R}{\varepsilon}}, \tag{5.32}$$

$$P((|u_n^\varepsilon|_{H^1}^{H^2}(T))^2 > M) \leq e^{-\frac{R}{\varepsilon}}. \tag{5.33}$$

For such M, by (5.31), there exits a positive integer N, such that for any $n \geq N$,

$$\begin{aligned}
&\sup_{0<\varepsilon\leq T} \varepsilon \log P(\sup_{0\leq t\leq 1} ||u^\varepsilon(t) - u_n^\varepsilon(t)||_{H^1}^2 > \delta, |u^\varepsilon|_{H^1}^{H^2}(T))^2 \leq M, |u_n^\varepsilon|_{H^1}^{H^2}(T))^2 \leq M) \\
\leq &\sup_{0<\varepsilon\leq 1} \varepsilon \log P(\sup_{0\leq t\leq T} ||u^\varepsilon(t\wedge\tau_{\varepsilon,M}) - u_n^\varepsilon(t\wedge\tau_{\varepsilon,M})||_{H^1}^2 > \delta) \leq -R. \qquad (5.34)
\end{aligned}$$

Putting (5.32) and (5.34) together, one sees that there exists a positive integer N, such that for any $n \geq N, \varepsilon \in (0,1]$

$$P(\sup_{0\leq t\leq T} ||u^\varepsilon(t) - u_n^\varepsilon(t)||_{H^1}^2 > \delta) \leq 3e^{-\frac{R}{\varepsilon}}. \tag{5.35}$$

Since R is arbitrary, the lemma follows. ■

The next Lemma can be proved similarly as Lemma 5.3.

Lemma 5.4 *For any $\delta > 0$,*

$$\lim_{n\to+\infty} \sup_{0<\varepsilon\leq 1} \varepsilon \log P(\sup_{0\leq t\leq 1} ||v^\varepsilon(t) - v_n^\varepsilon(t)||_{H^1}^2 > \delta) = -\infty. \tag{5.36}$$

The following result says that for a fixed integer n, the two families $\{u_n^\varepsilon, \varepsilon > 0\}$ $\{v_n^\varepsilon, \varepsilon > 0\}$ are exponentially equivalent.

Lemma 5.5 *For any fixed positive integer n, any $\delta > 0$,*

$$\lim_{\varepsilon\to 0} \varepsilon \log P(\sup_{0\leq t\leq T} ||u_n^\varepsilon(t) - v_n^\varepsilon(t)||_{H^1}^2 > \delta) = -\infty. \tag{5.37}$$

Proof: As the integer n is fixed, for simplicity, we drop the index n everywhere in the proof. Observe

$$\begin{aligned}
&u^\varepsilon(t) - v^\varepsilon(t) \\
= &-\varepsilon\int_0^t A(u^\varepsilon(s) - v^\varepsilon(s))ds - \varepsilon\int_0^t Av^\varepsilon(s)ds - \varepsilon\int_0^t B(u^\varepsilon(s))ds \\
+ &\sqrt{\varepsilon}\sum_{k=1}^\infty \int_0^t (\sigma_k(u^\varepsilon(s)) - \sigma_k(v^\varepsilon(s))dW_k(s) - \varepsilon\int_0^t g_N(|u^\varepsilon(s)|^2)u^\varepsilon(s)ds. \qquad (5.38)
\end{aligned}$$

For any $M > 0$, define stopping times

$$t_{\varepsilon,M} = \inf\{t : \varepsilon\int_0^t ||u^\varepsilon(r)||_{H^2}^2 dr > M, \text{ or } ||u^\varepsilon(t)||_{H^1}^2 > M\}.$$

$$s_{\varepsilon,M} = \inf\{t : ||v^\varepsilon(t)||_{H^2} > M\}.$$

Put $\tau_{\varepsilon,M}=t_{\varepsilon,M}\wedge s^n_{\varepsilon,M}$. By Itô's formula, we have

$$\begin{aligned}
&||u^\varepsilon(t)-u^\varepsilon_n(t)||^2_{H^1}\\
=&\ -2\varepsilon\int_0^t <A(u^\varepsilon(s)-v^\varepsilon(s)),u^\varepsilon(s)-v^\varepsilon(s)>_{H^1}ds-2\varepsilon\int_0^t<Av^\varepsilon(s),u^\varepsilon(s)-v^\varepsilon(s)>_{H^1}ds\\
&-2\varepsilon\int_0^t<B(u^\varepsilon(s)),u^\varepsilon(s)-v^\varepsilon(s)>_{H^1}ds\\
&-2\varepsilon\int_0^t<g_N(|u^\varepsilon(s)|^2)u^\varepsilon(s),u^\varepsilon(s)-v^\varepsilon(s)>_{H^1}ds\\
&+\varepsilon\int_0^t\sum_{k=1}^\infty||\sigma_k(u^\varepsilon(s))-\sigma_k(v^\varepsilon(s))||^2_{H^1}ds\\
&+2\sqrt{\varepsilon}\sum_{k=1}^\infty\int_0^t<\sigma_k(u^\varepsilon(s))-\sigma_k(v^\varepsilon(s)),u^\varepsilon(s)-u^\varepsilon_n(s)>_{H^1}dW_k(s)\\
:=&\ I_{n,1}(t)+I_{n,2}(t)+I_{n,3}(t)+I_{n,4}(t)+I_{n,5}(t)+I_{n,6}(t). \qquad (5.39)
\end{aligned}$$

For the first term, we have

$$\begin{aligned}
I_{n,1}t) &= -2\varepsilon\int_0^t||u^\varepsilon(s)-v^\varepsilon(s)||^2_{H^2}ds+2\varepsilon\int_0^t||\nabla u^\varepsilon(s)-\nabla v^\varepsilon(s)||^2_{L^2}ds\\
&\quad+2\varepsilon\int_0^t||u^\varepsilon(s)-v^\varepsilon(s)||^2_{L^2}ds\\
&\le -2\varepsilon\int_0^t||u^\varepsilon(s)-v^\varepsilon(s)||^2_{H^2}ds+4\varepsilon\int_0^t||u^\varepsilon(s)-v^\varepsilon(s)||^2_{H^1}ds. \qquad (5.40)
\end{aligned}$$

For the second term, it holds that

$$\begin{aligned}
I_{n,2}(t) &\le 2\varepsilon\int_0^t||u^\varepsilon(s)-v^\varepsilon(s)||_{H^2}||Av^\varepsilon(s)||_{L^2}ds\\
&\le \frac{1}{2}\varepsilon\int_0^t||u^\varepsilon(s)-u^\varepsilon_n(s)||^2_{H^2}ds+2\varepsilon\int_0^t||v^\varepsilon(s)||^2_{H^2}ds. \qquad (5.41)
\end{aligned}$$

Apply the Sobolev imbedding to the non-linear term $B(\cdot)$,

$$\begin{aligned}
I_{n,3}(t) &\le 2\varepsilon\int_0^t||B(u^\varepsilon(s))||_{L^2}||u^\varepsilon(s)-u^\varepsilon_n(s)||_{H^2}ds\\
&\le \frac{1}{2}\varepsilon\int_0^t||u^\varepsilon(s)-v^\varepsilon(s)||^2_{H^2}ds+C\varepsilon\int_0^t||u^\varepsilon(s)||^3_{H^1}||u^\varepsilon(s)||_{H^2}ds. \qquad (5.42)
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_{n,4}(t) &\le \frac{1}{2}\varepsilon\int_0^t||u^\varepsilon(s)-v^\varepsilon(s)||^2_{H^2}ds+2\varepsilon\int_0^t||g_N(|u^\varepsilon|^2(s))u^\varepsilon(s)||^2_{L^2}\\
&\le \frac{1}{2}\varepsilon\int_0^t||u^\varepsilon(s)-v^\varepsilon(s)||^2_{H^2}ds+C\varepsilon\int_0^t||u^\varepsilon(s)||^6_{H^1}. \qquad (5.43)
\end{aligned}$$

Taking into account (A.3),

$$I_{n,5}(t) \le C\varepsilon \int_0^t ||u^\varepsilon(s) - v^\varepsilon||^2_{H^1} ds. \tag{5.44}$$

Substituting the above estimates into (5.39) we obtain

$$\begin{aligned}
& \sup_{0\le s\le t} ||u^\varepsilon(s\wedge\tau_{\varepsilon,M}) - v^\varepsilon(s\wedge\tau_{\varepsilon,M})||^2_{H^1} \\
\le\ & C\varepsilon \int_0^{t\wedge\tau_{\varepsilon,M}} (||u^\varepsilon(s)||^6_{H^1} + ||u^\varepsilon(s)||^3_{H^1}||u^\varepsilon(s)||_{H^2} + ||v^\varepsilon(s)||^2_{H^2}] ds \\
& + C\varepsilon \int_0^t ||u^\varepsilon(s\wedge\tau_{\varepsilon,M}) - v^\varepsilon(s\wedge\tau_{\varepsilon,M})||^2_{H^1} ds + \sup_{0\le s\le t} |I_{n,6}(s\wedge\tau_{\varepsilon,M})| \\
\le\ & C\varepsilon(M^6T + M^3(T+M) + M^2T) \\
& + C\varepsilon \int_0^t ||u^\varepsilon(s\wedge\tau_{\varepsilon,M}) - v^\varepsilon(s\wedge\tau_{\varepsilon,M})||^2_{H^1} ds + \sup_{0\le s\le t} |I_{n,6}(s\wedge\tau_{\varepsilon,M})|.
\end{aligned} \tag{5.45}$$

Similar to the proof of (5.29), using the Martingale inequality, it follows from (5.45) that

$$\begin{aligned}
& (E[\sup_{0\le s\le t}(||u^\varepsilon(s\wedge\tau_{\varepsilon,M}) - u_n^\varepsilon(s\wedge\tau_{\varepsilon,M})||^{2p}_{H^1})])^{\frac{2}{p}} \\
\le\ & c\varepsilon^2(M^6T + M^3(T+M) + M^2T)^2 + C_T\varepsilon^2 \int_0^t (E[\sup_{0\le r\le s}(||u^\varepsilon(r\wedge\tau_{\varepsilon,M}) - u_n^\varepsilon(r\wedge\tau_{\varepsilon,M})||^{2p}_{H^1})])^{\frac{2}{p}} ds \\
+\ & Cp\varepsilon \int_0^t (E[\sup_{0\le r\le s}(||u^\varepsilon(r\wedge\tau_{\varepsilon,M}) - u_n^\varepsilon(r\wedge\tau_{\varepsilon,M})||^{2p}_{H^1})])^{\frac{2}{p}} ds.
\end{aligned} \tag{5.46}$$

By Gronwall's inequality, this yields

$$\begin{aligned}
& E[\sup_{0\le s\le T}(||u^\varepsilon(s\wedge\tau_{\varepsilon,M}) - u_n^\varepsilon(s\wedge\tau_{\varepsilon,M})||^{2p}_{H^1})] \\
\le\ & C^{\frac{p}{2}}\varepsilon^p(M^6T + M^3(T+M) + M^2T)^p exp(\frac{p}{2}C_T\varepsilon^2 + \frac{p^2}{2}\varepsilon).
\end{aligned} \tag{5.47}$$

Choose $p = \frac{2}{\varepsilon}$ to get

$$\begin{aligned}
& \varepsilon \log P(\sup_{0\le t\le T} ||u^\varepsilon(t\wedge\tau_{\varepsilon,M}) - v^\varepsilon(t\wedge\tau_{\varepsilon,M})||^2_{H^1} > \delta) \\
\le\ & \varepsilon \log \frac{E[\sup_{0\le t\le T} ||u^\varepsilon(t\wedge\tau_{\varepsilon,M}) - v^\varepsilon(t\wedge\tau_{\varepsilon,M})||^{2p}_{H^1}]}{\delta^p} \\
\le\ & logC_T + C_T\varepsilon^2 + C_T + 2log\big(\varepsilon(M^6T + M^3(T+M) + M^2T)\big) \\
\to\ & -\infty, \quad \text{as} \quad \varepsilon \to 0.
\end{aligned} \tag{5.48}$$

For any given $R > 0$, by Lemma 5.1 , and Lemma 5.2 there exists a constant M such that for any $\varepsilon \in (0,1]$ the following inequalities hold,

$$P((|u^\varepsilon|_{H^1}^{H^2}(T))^2 > M) \leq e^{-\frac{R}{\varepsilon}}, \tag{5.49}$$

$$P(\sup_{0\leq t\leq T} ||v^\varepsilon||_{H^2}^2 > M) \leq e^{-\frac{R}{\varepsilon}}. \tag{5.50}$$

For such M, (5.48) implies that there exits a positive number ε_0, such that for $\varepsilon \leq \varepsilon_0$,

$$\begin{aligned} & \varepsilon \log P(\sup_{0\leq t\leq 1} ||u^\varepsilon(t) - v^\varepsilon(t)||_{H^1}^2 > \delta, |u^\varepsilon|_{H^1}^{H^2}(T))^2 \leq M, \sup_{0\leq t\leq T} ||^\varepsilon||_{H^2}^2 \leq M) \\ \leq \; & \varepsilon \log P(\sup_{0\leq t\leq T} ||u^\varepsilon(t\wedge\tau_{\varepsilon,M}) - v^\varepsilon(t\wedge\tau_{\varepsilon,M})||_{H^1}^2 > \delta) \leq -R. \end{aligned} \tag{5.51}$$

Combining (5.49) and (5.51) together, one can find a positive number ε_0, such that for $\varepsilon \leq \varepsilon_0$

$$P(\sup_{0\leq t\leq T} ||u^\varepsilon(t) - v^\varepsilon(t)||_{H^1}^2 > \delta) \leq 3e^{-\frac{R}{\varepsilon}}. \tag{5.52}$$

Since R was arbitrary, the lemma follows. ■

Now we are in the position to complete the proof of Theorem 4.1, that is, the proof of (5.2). By Lemma 5.3 and Lemma 5.4, we have for any $R > 0$ that there exists a N_0 satisfying

$$P(\sup_{0\leq t\leq 1} ||u^\varepsilon(t) - u^\varepsilon_{N_0}(t)||_{H^1}^2 > \delta) \leq e^{-\frac{R}{\varepsilon}} \quad \text{for any } \varepsilon \in (0,1], \tag{5.53}$$

and

$$P(\sup_{0\leq t\leq 1} ||v^\varepsilon(t) - v^\varepsilon_{N_0}(t)||_{H^1}^2 > \delta) \leq e^{-\frac{R}{\varepsilon}} \quad \text{for any } \varepsilon \in (0,1]. \tag{5.54}$$

In view of Lemma 5.5, for such N_0, there exists ε_0 such that for any $\varepsilon \in (0,\varepsilon_0]$

$$P(\sup_{0\leq t\leq 1} ||u^\varepsilon_{N_0}(t) - v^\varepsilon_{N_0}(t)||_{H^1} > \delta) \leq e^{-\frac{R}{\varepsilon}}. \tag{5.55}$$

Thus, for any $\varepsilon \in (0,\varepsilon_0]$,

$$P(\sup_{0\leq t\leq 1} ||u^\varepsilon(t) - v^\varepsilon(t)||_{H^1} > \delta) \leq 3e^{-\frac{R}{\varepsilon}}. \tag{5.56}$$

Since R is arbitrary, we conclude that

$$\lim_{\varepsilon\to 0} \varepsilon \log P(\sup_{0\leq t\leq 1} ||u^\varepsilon(t) - v^\varepsilon(t)||_{H^1}^2 > \delta) = -\infty. \quad ■$$

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