Log-Harnack Inequality for Stochastic Differential Equations in Hilbert Spaces and its Consequences*

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Abstract

A logarithmic type Harnack inequality is established for the semigroup of solutions to a stochastic differential equation in Hilbert spaces with non-additive noise. As applications, the strong Feller property as well as the entropy-cost inequality for the semigroup are derived with respect to the corresponding distance (cost function).

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1 Introduction

Under a curvature condition the second named author established the following type dimension-free Harnack inequality for diffusion semigroups on a Riemannian manifold M ([14]):

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$$(P_t f)^\alpha(x) \le (P_t f^\alpha)(x) \mathrm{e}^{c(t)\rho(x,y)^2}, \quad f \ge 0, t > 0,\ \alpha > 1,\ x, y \in M,$$

where $c(t) > 0$ is explicitly determined by α and the curvature lower bound. This inequality has been efficiently applied to the study of functional inequalities for the associated Dirichlet form, the hyper-/super-/ultracontractivity properties of the semigroup, strong Feller property as well as estimates on the heat kernel of the semigroup (cf. [8, 13, 15, 6] and references therein). To establish this inequality for diffusions with curvature unbounded below, a coupling method is developed in [1]. This method works also for infinite dimensional SPDE provided the noise is additive and non-degenrate, see e.g. [16, 9, 10, 11, 6] for Harnack inequalities for several different classes of SPDE. The aim of this paper is to extend the study to stochastic differential equations with non-additive noises.

Let us start from the following Itô stochastic differential equation on $\mathbb{R}^n$:

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t,$$

where $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n$ are continuous, B_t is the Brownian motion in $\mathbb{R}^n$. In this case the solution is a diffusion process with corresponding generator (Kolmogorov operator)

$$L = \sum_{i,j=1}^n a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^n b_i(x)\partial_i,$$

where $(a_{ij})_{1\le i,j\le n} = \frac{1}{2}\sigma^*\sigma$. If a_{ij} and b_i are regular enough such that Bakry-Emery's Γ_2 condition (see [2])

$$\Gamma_2(f,f) := \frac{1}{2}L\langle a\nabla f, \nabla f\rangle - \langle a\nabla f, \nabla Lf\rangle \ge -K\langle a\nabla f, \nabla f\rangle \tag{1.1}$$

holds for all smooth f and some constant K, then the curvature condition used in [14] holds for the Riemannian metric $\langle u, v\rangle_a := \langle a^{-1}u, v\rangle, u, v \in \mathbb{R}^n$, induced by the diffusion coefficient. Thus, one derives the desired Harnack inequality for the associated diffusion semigroup. Theoretically one may use this argument to establish the Harnack inequality for non-constant a also in infinite dimensions. For $n \to \infty$ condition (1.1) is, however, too complicated to verify or does not hold. This is the main reason why all existing results in this direction for infinite dimensional SDE are merely proved for additive noise (i.e. for the constant diffusion case).

In this paper we shall analyse the following log-Harnack inequality allowing the diffusion to be non-constant:

$$P_t \log f(x) \le \log P_t f(y) + \frac{K\rho_a(x,y)^2}{2(1-\mathrm{e}^{-2Kt})}, \quad t > 0, x, y \in \mathbb{R}^n, f > 0,$$

where ρ_a is the distance induced by the metric $\langle\cdot,\cdot\rangle_a$. This inequality was first presented in the proof of [3, Lemma 4.2] under the Γ_2 condition (1.1) for P_tf in place of f, which is crucial for the proof of the HWI inequality [3, Theorem 4.3]. We will see in Section 2 that this type of inequality can be derived by using a standard dissipative type condition which is explicit and dimension free. Combining this observation with an approximation argument, we are able to establish the inequality for infinite dimensional diffusions on Hilbert spaces, and furthermore derive the strong Feller property of the semigroup and entropy inequalities for the heat kernel.

We will work with the following semi-linear stochastic differential equation on a separable Hilbert space $(\mathbb{H},\langle\cdot,\cdot\rangle,\|\cdot\|)$ (cf. [7]):

$$\mathrm{d}X_t = \big(AX_t + F(X_t)\big)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \tag{1.2}$$

where W_t is a cylindrical Brownian motion on $\mathbb{H}$ on some filtered probability space $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t))$. F is a Lipschtiz continuous function on $\mathbb{H}$, and $\sigma(x)=\tilde\sigma_1(x)+\tilde\sigma_0$ for a linear operator $\tilde\sigma_0$ and a Hilbert-Schmidt operator-valued function $\tilde\sigma_1$ such that $\sigma^*\sigma\ge\tilde\sigma_0^2$. We shall assume:

$(H1)$ A is a self-adjoint operator on $\mathbb{H}$ generating a contractive compact semigroup T_t. In this case $-A$ has discrete spectrum $0\le\lambda_1\le\lambda_2\le\cdots$ with corresponding eigenbasis $\{e_i\}_{i\ge1}$ of $\mathbb{H}$. Let $\mathbb{H}_n=\mathrm{span}\{e_1,\cdots,e_n\}, n\ge1$.

$(H2)$ $\tilde\sigma_0e_i=q_ie_i$ for a sequence $\{q_i>0:\ i\ge1\}$ such that $\sigma^*\sigma\ge\tilde\sigma_0^2$ and $\sum_{i=1}^\infty\frac{q_i^2}{1+\lambda_i}<\infty$.

$(H3)$ $\tilde\sigma_1:=\sigma-\tilde\sigma_0$ is Hilbert-Schmidt and there exists a constant $C>0$ such that

$$\|F(x)-F(y)\|+\|\tilde\sigma_1(x)-\tilde\sigma_1(y)\|_{HS}\le C\|x-y\|,\quad x,y\in\mathbb{H}.$$

$(H4)$ There exists a constant $K\in\mathbb{R}$ such that

$$2\langle F(x)-F(y),\tilde\sigma_0^{-2}(x-y)\rangle+\|\tilde\sigma_0^{-1}(\tilde\sigma_1(x)-\tilde\sigma_1(y))\|_{HS}^2\le K\|\tilde\sigma_0^{-1}(x-y)\|^2$$

holds for all $x,y\in\mathbb{H}$ with $x-y\in\cup_{n=1}^\infty\mathbb{H}_n$.

We note that $(H3)$ implies $(H4)$ in case $\tilde\sigma_0$ and $\tilde\sigma_0^{-1}$ are both bounded. Obviously, $(H1)$–$(H3)$ imply the existence and the uniqueness of the mild solution to (1.2), that is, for any $x\in\mathbb{H}$ there exists a unique $\mathbb{H}$ valued adapted process X_t, which is continuous in $L^2(\Omega,\mathbb{P})$, such that (cf. [7])

$$X_t=T_tx+\int_0^tT_{t-s}F(X_s)\mathrm{d}s+\int_0^tT_{t-s}\sigma(X_s)\mathrm{d}W_s.$$

Let P_t be the associated Markov semigroup, i.e.

$$P_t f(x) = \mathbb{E} f(X_t), \quad f \in \mathscr{B}_b(\mathbb{H}),$$

where $\mathscr{B}_b(\mathbb{H})$ is the set of all bounded measurable functions on $\mathbb{H}$. In this paper we shall establish a log-Harnack inequality for P_t by using $(H4)$ in place of the Γ_2 condition.

Theorem 1.1. *If* $(H1)$*–* $(H4)$ *hold then for any strictly positive* $f \in \mathscr{B}_b(\mathbb{H})$,

$$P_t \log f(x) \le \log P_t f(y) + \frac{K\|\tilde{\sigma}_0^{-1}(x-y)\|^2}{2(1-\mathrm{e}^{-Kt})}, \quad t > 0, x, y \in \mathbb{H},$$

where $\|\tilde{\sigma}_0^{-1}x\|^2 := \sum_{i=1}^\infty q_i^{-2}\langle x, e_i\rangle^2 \in [0, \infty]$.

As applications of Theorem 1.1, we have the following results on the strong Feller property, heat kernel inequality and entropy-cost inequality. To state these results, let us introduce some notions. Let

$$\mathbb{H}_0 = \{x \in \mathbb{H} : \|x\|_0 := \|\tilde{\sigma}_0^{-1}x\| < \infty\}.$$

We call P_t $\mathbb{H}_0$-strong Feller if for any $f \in \mathscr{B}_b(\mathbb{H})$,

$$\lim_{\|y-x\|_0 \to 0} P_t f(y) = P_t f(x), \quad x \in \mathbb{H}. \tag{1.3}$$

When $\tilde{\sigma}_0^{-1}$ is bounded then $\mathbb{H}_0 = \mathbb{H}$ and $\mathbb{H}_0$-strong Feller, implies $\mathbb{H}$-strong Feller. Next, let P_t be $\mathbb{H}_0$-strong Feller and let μ be a probability measure on $\mathbb{H}$ such that for some C, $\alpha > 0$,

$$\int P_t f \mathrm{d}\mu \le Ce^{\alpha t} \int f \mathrm{d}\mu \quad \text{for all } f \in \mathscr{B}_b(\mathbb{H}),\ f \ge 0, \tag{1.4}$$

(which holds e.g. if μ is P_t-invariant). Such measures always exist. Take e.g. for $x_0 \in \mathbb{H}$, $\mu(dy) := \int_0^\infty e^{-s}P_s(x_0, dy)ds$. Then μ satisfies (1.4) with $\alpha = 1 = C$. Suppose that μ is fully supported on $\mathbb{H}_0$, i.e. $\mu(U) > 0$ for every nonempty $\|\cdot\|_0$-open set $U \subset \mathbb{H}_0$. Then it is easy to see that for every $x \in \mathbb{H}_0$, $P_t(x, dy)$ has a transition density $p_t(x, y)$ with respect to μ.

Remark. Obviously $(\mathbb{H}_0, \|\cdot\|_0)$ is separable. Hence there exists μ as in (1.4) fully supported on $\mathbb{H}_0$. Indeed, take a countable $\|\cdot\|_0$-dense subset $\{x_n | n \in \mathbb{N}\}$ of $\mathbb{H}_0$. Then

$$\mu(dy) := \sum_{n=1}^\infty \frac{1}{2^n} \int_0^\infty e^{-s}P_s(x_n, dy)\mathrm{d}s$$

is a probability measure on $\mathbb{H}$, satisfying (1.4). Furthermore, if $U \subset \mathbb{H}_0$ is $\|\cdot\|_0$-open such that $\mu(U) = 0$. Then for $\varphi(x) := \inf\{\|x - y\|_0 :\ y \in U^c\}$ we have

$$\int \varphi \mathrm{d}\mu = 0.$$

Hence by a diagonal argument we can find a zero sequence $(t_k)_{k\in\mathbb{N}}$ such that $P_{t_k}\varphi(x_n)=0$ for all $k,n\in\mathbb{N}$. Taking $k\to\infty$ we obtain $\varphi(x_n)=0$ for all $n\in\mathbb{N}$. But if $U\neq\emptyset$, then $x_{n_0}\in U$ for some $n_0\in\mathbb{N}$, so $\varphi(x_{n_0})>0$. This contradiction shows that $U=\emptyset$.

Finally, for two probability measures μ_1,μ_2 on $\mathbb{H}$, let $W_0(\mu_1,\mu_2)$ be the L^2-Wasserstein distance or L^2-transportation cost between them with respect to the cost function $(x,y)\mapsto \|x-y\|_0$. More precisely, with $\mathscr{C}(\mu_1,\mu_2)$ denoting the set of all couplings of μ_1 and μ_2, we have

$$W_0(\mu_1,\mu_2)^2=\inf_{\pi\in\mathscr{C}(\mu_1,\mu_2)}\int_{\mathbb{H}\times\mathbb{H}}\|(\tilde\sigma_0^{-1}(x-y)\|^2\pi(\mathrm{d}x,\mathrm{d}y).$$

Corollary 1.2. *Let* (*H*1)– (*H*4) *hold. Then:*

(1) *For any $t>0$, P_t is $\mathbb{H}_0$-strong Feller. Let μ be P_t-subinvariant* (*i.e.,* (1.4) *holds with $C=1,\ \alpha=0$*)*. Then* (1.3) *holds for all μ-exponentially integrable functions f.*

(2) *Let μ be as in* (1.4) *above, fully supported on $\mathbb{H}_0$. Then for every $x\in\mathbb{H}_0$, $P_t(x,dy)$ has a transition density $p_t(x,y)$ satisfying the following entropy inequality*

$$\int_{\mathbb{H}}p_t(x,z)\log p_t(x,z)\mu(\mathrm{d}z)\le\log C+\alpha t-\log\int_{\mathbb{H}}\exp\Big[-\frac{K\|x-y\|_0^2}{2(1-\mathrm{e}^{-Kt})}\Big]\mu(\mathrm{d}y),\quad t>0,x\in\mathbb{H}.$$

(3) *Let μ be P_t-subinvariant. Then the following entropy-cost inequality holds for the adjoint operator P_t^* of P_t in $L^2(\mu)$:*

$$\mu((P_t^*f)\log P_t^*f)\le\frac{K}{2(1-\mathrm{e}^{-Kt})}W_0(f\mu,\mu)^2,\quad t>0,f\ge0,\mu(f)=1.$$

In Section 2 we shall prove the log-Harnack inequality for diffusion semigroups on $\mathbb{R}^n$ and then we extend this to an infinite dimensional setting in Section 3 by finite-dimensional approximations. Finally, Corollary 1.2 will be proved in Section 4.

2 Log-Harnack inequality on $\mathbb{R}^n$

Consider the following SDE on $\mathbb{R}^n$:

$$\mathrm{d}X_t=b(X_t)\mathrm{d}t+\sigma(X_t)\mathrm{d}B_t,\tag{2.1}$$

where B_t is Brownian motion on $\mathbb{R}^n$, $b:\mathbb{R}^n\to\mathbb{R}^n$ and $\sigma:\mathbb{R}^n\to\mathbb{R}^n\otimes\mathbb{R}^n$ are locally Lipschitzian and of at most linear growth. Hence the equation has a unique strong solution, which is non-explosive. Let $\tilde\sigma_0$ be a (strictly) positive definite symmetric matrix such that $\sigma^*\sigma\ge\tilde\sigma_0^2$. Assume that

(2.2) $\|(\tilde\sigma_0^{-1}(\sigma(x)-\sigma(y)\|_{HS}^2+2\langle\tilde\sigma_0^{-1}(b(x)-b(y)),\tilde\sigma_0^{-1}(x-y)\rangle \le K\|\tilde\sigma_0^{-1}(x-y)\|^2,\quad x,y\in\mathbb{R}^n$

holds for some constant $K\in\mathbb{R}$.

Theorem 2.1. *Assume* (2.2) *and that the solution to* (2.1) *is non-explosive. Then the associated Markov semigroup P_t satisfies*

$$P_t\log f(x)\le \log P_tf(y)+\frac{K\|\tilde\sigma_0^{-1}(x-y)\|^2}{2(1-\mathrm{e}^{-Kt})},\quad t>0, x,y\in\mathbb{R}^n$$

for all $f\in\mathscr{B}_b(\mathbb{R}^n)$, $f\ge 0$.

By a standard approximation argument we can assume that $f\in C_b^\infty(\mathbb{R}^n)$. Furthermore, we can approximate b,σ by smooth b_n,σ_n such that the corresponding semigroups converge pointwise on $f\in C_b^\infty(\mathbb{R}^n)$ and such that $P_tC_b^2\subset C_b^2$ for all $t>0$.

To prove the log-Harnack inequality, we need the following gradient estimate on P_t.

Lemma 2.2. *Under the assumptions of Theorem* 2.1 *we have*

$$\|\tilde\sigma_0\nabla P_tf\|^2(x)\le \mathrm{e}^{Kt}P_t\|\tilde\sigma_0\nabla f\|^2(x),\quad f\in C_b^1(\mathbb{R}^n), x\in\mathbb{R}^n.$$

Proof. For $x,y\in\mathbb{R}^n$, let X_t and Y_t be the solutions to (2.1) with $X_0=x$ and $Y_0=y$ respectively. By Itô's formula and (2.2) we obtain

$$\begin{aligned}&\mathrm{d}\|\tilde\sigma_0^{-1}(X_t-Y_t)\|^2=2\langle\tilde\sigma_0^{-1}(X_t-Y_t),\tilde\sigma_0^{-1}(\sigma(X_t)-\sigma(Y_t))\mathrm{d}B_t\rangle\\&+\big\{\|(\tilde\sigma_0^{-1}(\sigma(X_t)-\sigma(Y_t))\|_{HS}^2+2\langle\tilde\sigma_0^{-1}(b(X_t)-b(Y_t)),\tilde\sigma_0^{-1}(X_t-Y_t)\rangle\big\}\mathrm{d}t\\&\le 2\langle\tilde\sigma_0^{-1}(X_t-Y_t),\tilde\sigma_0^{-1}(\sigma(X_t)-\sigma(Y_t))\mathrm{d}B_t\rangle+K\|\tilde\sigma_0^{-1}(X_t-Y_t)\|^2\mathrm{d}t.\end{aligned}$$

Since the solution to (2.1) is non-explosive, this implies

$$\mathbb{E}\|\tilde\sigma_0^{-1}(X_t-Y_t)\|^2\le \mathrm{e}^{Kt}\|\tilde\sigma_0^{-1}(x-y)\|^2.$$

Therefore,

$$\begin{aligned}\|\tilde\sigma_0\nabla P_tf\|^2(x)&=\limsup_{y\to x}\frac{|P_tf(y)-P_tf(x)|^2}{\|\tilde\sigma_0^{-1}(x-y)\|^2}=\limsup_{y\to x}\Big(\frac{\mathbb{E}(f(Y_t)-f(X_t))}{\|\tilde\sigma_0^{-1}(x-y)\|}\Big)^2\\&\le\limsup_{y\to x}\Big(\mathbb{E}\frac{|f(Y_t)-f(X_t)|^2}{\|\tilde\sigma_0^{-1}(Y_t-X_t)\|^2}\Big)\frac{\mathbb{E}\|\tilde\sigma_0^{-1}(X_t-Y_t)\|^2}{\|\tilde\sigma_0^{-1}(x-y)\|^2}\\&\le \mathrm{e}^{Kt}\mathbb{E}\|\tilde\sigma_0\nabla f\|^2(X_t).\end{aligned}$$

This implies the desired gradient estimate. □

Proof of Theorem 2.1. We may assume $f \geq 1$. For fixed $x \in \mathbb{R}^n$, let $X_0 = x$. By Itô's formula we have

$$\begin{aligned} \mathrm{d}\log P_{t-s}f(X_s) &= \langle \nabla \log P_{t-s}f(X_s), \sigma(X_s)\mathrm{d}B_s\rangle + L\log P_{t-s}f(X_s)\mathrm{d}s - \frac{LP_{t-s}f}{P_{t-s}f}(X_s)\mathrm{d}s \\ &= \langle \nabla \log P_{t-s}f(X_s), \sigma(X_s)\mathrm{d}B_s\rangle - \frac{1}{2}\|\sigma\nabla\log P_{t-s}f\|^2(X_s)\mathrm{d}s. \end{aligned}$$

Letting

$$\tau_k = \inf\{t \geq 0 : \ \|X_t\| \geq k\}, \quad k \geq 1,$$

we obtain

$$\mathbb{E}\log P_{t-s\wedge\tau_k}f(X_{s\wedge\tau_k}) - P_tf(x) = -\frac{1}{2}\mathbb{E}\int_0^{s\wedge\tau_k} \|\sigma\nabla\log P_{t-r}f\|^2(X_r)\mathrm{d}r.$$

Since the process is non-explosive, we have $\tau_k \to \infty$. Thus, due to the dominated convergence theorem, as $k \to \infty$ the left-hand side goes to $P_s\log P_{t-s}f(x) - \log P_tf(x)$, while by the monotone convergence theorem, the right-hand side goes to $-\frac{1}{2}\int_0^s P_r\|\sigma\nabla\log P_{t-r}f\|^2(x)\mathrm{d}r$. So, $\int_0^t P_r\|\sigma\nabla\log P_{t-r}f\|^2(x)\mathrm{d}r < \infty$ and

$$P_s\log P_{t-s}f(x) - \log P_tf(x) = -\frac{1}{2}\int_0^s P_r\|\sigma\nabla\log P_{t-r}f\|^2(x)\mathrm{d}r, \quad s \in [0,t]. \tag{2.3}$$

Now, for fixed $x, y \in \mathbb{R}^n, t > 0$, let

$$x_s = (x-y)h_s + y, \qquad s \in [0,t],$$

where $h \in C^1([0,t],\mathbb{R})$ such that $h_0 = 0$ and $h_t = 1$. By Lemma 2.2, (2.3) and noting that $\sigma^*\sigma \geq \tilde{\sigma}_0^2$, we have, since $s \mapsto P_s\log P_{t-s}f(x_s)$ is absolutely continuous by Lemma 2.2, that

$$\begin{aligned} &P_t\log f(x) - \log P_tf(y) \\ =&\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(P_s\log P_{t-s}f)(x_s)\mathrm{d}s \\ =&-\frac{1}{2}\int_0^t \left\{P_s\|\sigma\nabla\log P_{t-s}f\|^2(x_s) + h_s'\langle x-y, \nabla P_s\log P_{t-s}f\rangle(x_s)\right\}\mathrm{d}s \\ \leq&-\frac{1}{2}\int_0^t \left\{\mathrm{e}^{-Ks}\|\tilde{\sigma}_0\nabla P_s\log P_{t-s}f\|^2(x_s) + |h_s'|\cdot\|\tilde{\sigma}_0^{-1}(x-y)\|\cdot\|\tilde{\sigma}_0\nabla P_s\log P_{t-s}f\|\right\}\mathrm{d}s \\ \leq&\frac{\|\tilde{\sigma}_0^{-1}(x-y)\|^2}{2}\int_0^t \mathrm{e}^{Ks}|h_s'|^2\mathrm{d}s. \end{aligned}$$

Letting

$$h_s = \frac{1 - \mathrm{e}^{-Ks}}{1 - \mathrm{e}^{-Kt}}, \quad s \in [0, t],$$

we complete the proof. $\square$

3 Proof of Theorem 1.1

For any $n \geq 1$, let $\pi_n : \mathbb{H} \to \mathbb{H}_n := \mathrm{span}\{e_1, \cdots, e_n\}$ be the orthogonal projection. Let $W_t^n = \pi_n W_t, A_n = \pi_n A, \sigma_n = \pi_n \sigma, \sigma_{i,n} = \pi_n \tilde{\sigma}_i (i = 0, 1)$, and $F_n = \pi_n F$. By $(H1)$ and $(H2)$ we have

$$A_n x = Ax, \quad \sigma_{0,n} x = \tilde{\sigma}_0 x, \quad x \in \mathbb{H}_n. \tag{3.1}$$

Consider the following SDE on $\mathbb{H}_n$:

$$\mathrm{d}X_t^n = (A_n X_t^n + F_n(X_t^n))\mathrm{d}t + \sigma_n(X_t^n)\mathrm{d}W_t^n, \quad X_0^n = \pi_n X_0.$$

By $(H3)$ we see that both $b_n(x) := A_n x + F_n(x)$ and $\sigma_n(x)$ are Lipschitzian in $x \in \mathbb{H}_n$. So, this equation has a unique solution. Let P_t^n be the associated Markov semigroup. Moreover, by $(H4)$, since $A \leq 0$ and by (3.1) we have

$$\begin{aligned}
&2\langle \sigma_{0,n}^{-1}(b_n(x) - b_n(y)), \sigma_{0,n}^{-1}(x - y)\rangle + \|\sigma_{0,n}^{-1}(\sigma_n(x) - \sigma_n(y))\|_{HS}^2 \\
&\leq 2\langle \tilde{\sigma}_0^{-1}(F(x) - F(y)), \tilde{\sigma}_0^{-1}(x - y)\rangle + \|\tilde{\sigma}_0^{-1}(\sigma(x) - \sigma(y))\|_{HS}^2 \\
&\leq K\|\tilde{\sigma}_0^{-1}(x - y)\|^2, \quad x, y \in \mathbb{H}_n.
\end{aligned}$$

Thus, Theorem 2.1 implies that for $f \in C_b(\mathbb{H}_n)$

$$P_t^n \log f(x) \leq \log P_t^n f(y) + \frac{K\|\tilde{\sigma}_0^{-1}(x - y)\|^2}{2(1 - \mathrm{e}^{-Kt})}, \quad t > 0, x, y \in \mathbb{H}^n. \tag{3.2}$$

So, to derive the inequality for P_t, we need only to prove that

$$\lim_{n \to \infty} \mathbb{E}\|X_t^n - X_t\|^2 = 0, \quad X_0 = x \in \bigcup_{n \geq 1} \mathbb{H}_n. \tag{3.3}$$

Indeed, this implies that for any Lipschitzian function f on $\mathbb{H}$, such that $f = f \circ \pi_m$ for some $m \in \mathbb{N}$

$$\lim_{n\to\infty} |P_t f(x) - P_t^n f(\pi_n x)| \le \|f\|_{Lip} \lim_{n\to\infty} \mathbb{E}\|X_t - X_t^n\| = 0, \quad \forall x \in \bigcup_{n\ge1} \mathbb{H}_n.$$

Therefore, by letting $n \to \infty$ in (3.2) we derive the desired log-Harnack inequality for such Lipschitzian functions first for $x \in \bigcup_{n\ge1} \mathbb{H}_n$, but then since this set is dense in $\mathbb{H}$ and $P_t f$ is continuous for all such f, hence also for $\log(f+\varepsilon)$, we obtain it for all $x \in \mathbb{H}$. Finally, we extend it for all $f \in \mathscr{B}_b(\mathbb{H})$ by the monotone class theorem.

In order to prove (3.3), let

$$Y_t = \int_0^t T_{t-s}\tilde{\sigma}_0 \mathrm{d}W_s, \quad Y_t^n = \pi_n Y_t = \int_0^t \mathrm{e}^{(t-s)A_n}\sigma_{0,n}\mathrm{d}W_s^n, \quad t > 0.$$

By $(H2)$ we have

$$\sup_{s\in[0,t]} \mathbb{E}\|Y_s\|^2 < \infty,$$

so that the dominated convergence theorem implies

$$\lim_{n\to\infty} \mathbb{E}\|Y_t - Y_t^n\|^2 = 0, \quad \lim_{n\to\infty} \int_0^t \mathbb{E}\|Y_s - Y_s^n\|^2 \mathrm{d}s = 0. \tag{3.4}$$

Let

$$Z_t = X_t - Y_t, \qquad Z_t^n = X_t^n - Y_t^n.$$

By (3.4) it suffices to prove

$$\lim_{n\to\infty} \mathbb{E}\|Z_t - Z_t^n\|^2 = 0. \tag{3.5}$$

We have

$$\mathrm{d}Z_t = (AZ_t + F(Z_t + Y_t))\mathrm{d}t + \sigma_1(Z_t + Y_t)\mathrm{d}W_t, \tag{3.6}$$

$$dZ_t^n = (A_n Z_t^n + F_n(Z_t^n + Y_t^n))\mathrm{d}t + \sigma_{1,n}(Z_t^n + Y_t^n)\mathrm{d}W_t^n. \tag{3.7}$$

To be precise, (3.6) is first meant in the mild sense. But by our assumptions it also has a unique variational solution (see e.g. [12]). Since both are analytically weak solutions and these are unique (see e.g. the recent paper [4], where uniquenss of analytically weak solutions is proved for an even more general class of equations), we see that Z_t defined above solves (3.6) in the variational sense, so that Itô's formula applies to $\|Z_t - Z_t^n\|^2$. Due to (3.1) we have

$$\begin{aligned}\mathrm{d}(Z_t - Z_t^n) =& \Big(A(Z_t - Z_t^n) + F(X_t) - F_n(X_t^n)\Big)\mathrm{d}t \\ &+ (\sigma_1(X_t) - \pi_n\sigma_1(X_t^n))\mathrm{d}W_t^n + \sigma_1(X_t)\mathrm{d}(W_t - W_t^n).\end{aligned}$$

So, by Iô's formula and $(H3)$ we obtain

$$
\begin{aligned}
\mathrm{d}\|Z_t - Z_t^n\|^2 &\le C_1\Big(\|F(X_t) - F_n(X_t^n)\| \cdot \|X_t - X_t^n\| \\
&\qquad + \|\sigma_1(X_t) - \pi_n\sigma_1(X_t^n)\|_{HS}^2 + \sum_{i>n}\|\sigma_1(X_t)e_i\|^2\Big)\mathrm{d}t \\
&\le C_2\Big(\|Z_t - Z_t^n\|^2 + \|Y_t - Y_t^n\|^2 + \|(1-\pi_n)F(X_t)\|^2 + \sum_{i>n}\|\sigma_1(X_t)e_i\|^2\Big)\mathrm{d}t
\end{aligned} \tag{3.8}
$$

for some constants $C_1, C_2 > 0$. Since by $(H3)$ we have

$$\sup_{s\in[0,t]} \mathbb{E}\Big(\|F(X_s)\|^2 + \|\sigma_1(X_s)\|_{HS}^2\Big) \le C_3 \sup_{s\in[0,t]}(1 + \mathbb{E}\|X_s\|^2) < \infty,$$

by the dominated convergence theorem

$$\varepsilon_n := C_2\mathbb{E}\int_0^t \Big(\|Y_s - Y_s^n\|^2 + \|(1-\pi_n)F(X_s)\|^2 + \sum_{i>n}\|\sigma_1(X_s)e_i\|^2\Big)\mathrm{d}s \to 0$$

as $n \to \infty$. Thus, it follows from (3.8) that

$$\lim_{n\to\infty} \mathbb{E}\mathrm{e}^{-C_2 t}\|Z_t - Z_t^n\|^2 \le \lim_{n\to\infty} \varepsilon_n = 0.$$

Therefore, (3.5) holds. $\square$

4 Proof of Corollary 1.2

It is sufficient to prove (1.3) for nonnegative $f \in \mathscr{B}_b(\mathbb{H})$. Applying the log-Harnack inequality in Theorem 1.1 for $1+\varepsilon f$ in place of f, we obtain from the elementary inequality $r \le \log(1+r) + r^2,\ r \ge 0,$

$$P_t f(y) - \varepsilon\|f\|_\infty^2 \le P_t\frac{\log(1+\varepsilon f)}{\varepsilon}(y) \le \frac{1}{\varepsilon}\log(1+\varepsilon P_t f(x)) + \frac{c_t\|x-y\|_0^2}{\varepsilon}, \quad \varepsilon > 0, x, y \in \mathbb{H},$$

where $c_t := \frac{K}{2(1-\mathrm{e}^{-Kt})}$. Letting first $y \to x$ in $\|\cdot\|_0$ and then $\varepsilon \to 0$, we obtain

$$\limsup_{\|y-x\|_0\to 0} P_t f(y) \le P_t f(x).$$

Similarly, we have

$$P_t\frac{\log(1+\varepsilon f)}{\varepsilon}(x) - \frac{c_t\|x-y\|_0^2}{\varepsilon} \le \frac{1}{\varepsilon}\log(1+\varepsilon P_t f(y)) \le P_t f(y).$$

Letting first $y \to x$ in $\|\cdot\|_0$ then $\varepsilon \to 0$, we arrive at

$$P_t f(x) \le \liminf_{\|x-y\|_0 \to 0} P_t f(y).$$

Therefore, $P_t f$ is $\|\cdot\|_0$ continuous. The second part of assertion (1.1) is then an easy consequence.

Now, let $p_t(x,y)$ be the transition density of P_t with respect to μ. By Theorem 1.1, for any positive $f \in \mathscr{B}_b(\mathbb{H})$ we have

$$\mathrm{e}^{P_t \log f(x)} \le \exp\left[\frac{K\|x-y\|_0^2}{2(1-\mathrm{e}^{-Kt})}\right] P_t f(y), \quad x, y \in \mathbb{H}.$$

Thus,

$$\mathrm{e}^{P_t \log f(x)} \int_{\mathbb{H}} \exp\left[-\frac{K\|x-y\|_0^2}{2(1-\mathrm{e}^{-Kt})}\right]\mu(\mathrm{d}y) \le \int_{\mathbb{H}} P_t f(y)\mu(\mathrm{d}y) = C\mathrm{e}^{\alpha t}\mu(f).$$

For fixed $x \in \mathbb{H}$, applying this inequality to $f = n \wedge p_t(x,\cdot)$ then letting $n \to \infty$, we obtain

$$\mathrm{e}^{\int_{\mathbb{H}} p_t(x,z)\log p_t(x,z)\mu(\mathrm{d}z)} \int_{\mathbb{H}} \exp\left[-\frac{K\|x-y\|_0^2}{2(1-\mathrm{e}^{-Kt})}\right]\mu(\mathrm{d}y) \le C\mathrm{e}^{\alpha t}.$$

This implies (2).

By approximations it remains to prove (3) for bounded positive f with $\mu(f) = 1$. By Theorem 1.1 for $P_t^* f$ in place of f, we obtain

$$P_t \log P_t^* f(x) \le \log P_t P_t^* f(y) + \frac{K\|x-y\|_0^2}{2(1-\mathrm{e}^{-Kt})}, \quad x, y \in \mathbb{H}.$$

So, for any $\pi \in \mathscr{C}(f\mu, \mu)$, integrating both sides with respect to $\pi(\mathrm{d}x, \mathrm{d}y)$ we arrive at

$$\mu((P_t^* f)\log P_t^*) \le \mu(\log P_t P_t^* f) + \frac{K}{2(1-\mathrm{e}^{-Kt})}\int_{\mathbb{H}\times\mathbb{H}} \|x-y\|_0^2 \pi(\mathrm{d}x, \mathrm{d}y).$$

Since Jensen's inequality implies

$$\mu(\log P_t P_t^* f) \le \log \mu(P_t P_t^* f) = \log \mu(f) = 0,$$

this implies the desired entropy-cost inequality.

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