The goal of this work is to prove the existence of a solution to the following transport equation:

$$\partial_t \mu_t + \text{div}_x (b(\mu, \cdot, \cdot) \mu_t) = 0, \quad \mu_0 = \nu, \quad (1)$$

where

$$b = (b^i)_{i=1}^d : \mathcal{M}_0(\mathbb{R}^d \times [0, 1]) \times \mathbb{R}^d \times [0, 1] \to \mathbb{R}^d$$

is a Borel mapping and $\mathcal{M}_0(\mathbb{R}^d \times [0, 1])$ is the space of finite Borel measures on $\mathbb{R}^d \times [0, 1]$ equipped with the weak topology.

We shall say that a family $\mu := (\mu_t)_{t \in [0,1]}$ of finite Borel measures (regarded also as the measure $\mu_t(dx) dt$ on $\mathbb{R}^d \times [0, 1]$) satisfies equation (1) if $b(\mu, \cdot, \cdot) \in L^1(S \times [0, 1], \mu_t(dx) dt)$ for every compact set $S \subset \mathbb{R}^d$, that is, the function $|b(\mu, \cdot, \cdot)|$ is integrable with respect to $|\mu|$ on every compact set in $\mathbb{R}^d \times [0, 1]$, and for all $t \in [0, 1]$ the following identity holds:

$$\int_{\mathbb{R}^d} \varphi(x) \mu_t(dx) - \int_{\mathbb{R}^d} \varphi(x) \nu(dx) = \int_0^t \int_{\mathbb{R}^d} (b(\mu, x, s), \nabla \varphi(x)) \mu_s(dx) ds \quad \forall \varphi \in C_0^\infty(\mathbb{R}^d).$$

This equation has been an object of intensive studies over the past decade. For recent surveys, see [1], [2], [3], and [4]; in particular, nonlinear equations are considered in [4]. A typical condition on $b$ in the linear case is the inclusion $b \in L^1([0,1], W^{1,\infty}(\mathbb{R}^d))$ or the requirement that $b$ is a BV function with respect to $x$ (see [1], [2]). Our aim is to prove the existence assuming only some conditions on the growth of $b$. In this paper we only consider probability measures. The spaces of probability measures on $\mathbb{R}^d \times [0, 1]$ and $\mathbb{R}^d$ equipped with the weak topology will be denoted by $\mathcal{P}(\mathbb{R}^d \times [0, 1])$ and $\mathcal{P}(\mathbb{R}^d)$, respectively. Our main result is the following theorem.

**Theorem 1.** Let $\nu$ be a probability measure on $\mathbb{R}^d$. Suppose that

(i) for every fixed measure $\mu \in \mathcal{P}(\mathbb{R}^d \times [0, 1])$, the mapping $x \mapsto b(\mu, x, t)$ is continuous for almost every $t$ and one has uniform convergence $b(\mu_j, x, t) \to b(\mu, x, t)$ on compact sets whenever $\mu_j \to \mu$ weakly;

(ii) there exist numbers $c \in (0, \infty)$ and $k \in \mathbb{N}$ such that for all $(x, t) \in \mathbb{R}^d \times [0, 1]$ and all $\mu \in \mathcal{P}(\mathbb{R}^d \times [0, 1])$ one has

$$b(\mu, x, t) \leq c(1 + |x|^2),$$

$$|b(\mu, x, t)| \leq c(1 + |x|^{2k}), \quad \int_{\mathbb{R}^d} |x|^{2k} \nu(dx) < \infty.$$

Then there exists a family $\mu = (\mu_t)_{t \in [0,1]}$ of probability measures satisfying (1).

Under condition (ii), condition (i) can be reformulated as follows: for every fixed measure $\mu$, the mapping $x \mapsto b(\mu, x, t)$ is continuous for a.e. $t$ and for each compact set $S \subset \mathbb{R}^d$, the mapping $b$ generates a continuous mapping

$$F : \mathcal{P}(\mathbb{R}^d \times [0, 1]) \to L^\infty([0, 1], C(S))$$

defined by $F(\mu)(t)(x) := b(\mu, x, t)$.

Our approach is based on the well-known method of “vanishing viscosity” (see, e.g., [5, Theorem 4]) combined with the Schauder theorem. We replace equation (1) by the parabolic equation

$$\partial_t \mu_t - \varepsilon \Delta \mu_t + \text{div}_x (b(\mu_t, \cdot, \cdot) \mu_t) = 0, \quad \mu_0 = \nu,$$

(3)
understood as the following integral identity for all \( t \in [0, 1] \):

\[
\int_{\mathbb{R}^d} \varphi \, d\mu_t - \int_{\mathbb{R}^d} \varphi \, d\nu = \int_0^t \int_{\mathbb{R}^d} [\varepsilon \Delta \varphi + (b, \nabla \varphi)] \, d\mu_s \, ds \quad \forall \varphi \in C_0^\infty(\mathbb{R}^d). \tag{4}
\]

Under our assumptions on \( b \) in the case where \( b \) is independent of \( \mu \), the next proposition follows from the results established in [6] and [7] (see [6, Corollary 3.3, Lemma 2.1, Lemma 2.2] and [7, Theorem 3.3]).

**Proposition 1.** Suppose that a probability measure \( \nu \) on \( \mathbb{R}^d \) has a density \( \theta_0 \in C_0^\infty(\mathbb{R}^d) \), the coefficient \( b \) does not depend on \( \mu \), and there exists numbers \( c \in (0, \infty) \) and \( k \geq 1 \) such that for all \( (x, t) \in \mathbb{R}^d \times [0, 1] \) one has

\[
(b(x, t), x) \leq c(1 + |x|^2),
\]

\[
|b(x, t)| \leq c(1 + |x|^{2k}), \quad \int_{\mathbb{R}^d} |x|^{2k} \, \nu(dx) < \infty.
\]

Then there exists a unique family \( \mu = (\mu_t)_{t \in [0, 1]} \) of probability measures on \( \mathbb{R}^d \) solving equation (3). Moreover, there exists a number \( N \) depending only on \( c \) and \( k \) and \( \int_{\mathbb{R}^d} |x|^{2k} \, d\nu \) such that

\[
\sup_{t \in [0, 1]} \int_{\mathbb{R}^d} |x|^{2k} \, d\mu_t < N.
\]

Certainly, the same is true for any interval \( [0, T] \) in place of \([0, 1] \).

First we prove our main result in the case of \( b \) independent of \( \mu \) by letting \( \varepsilon \to 0 \). This case is simpler than the general one, but the proof can be extended to the infinite-dimensional case. So we assume that \( b \) does not depend on \( \mu \), the mapping \( x \mapsto b(x, t) \) is continuous for almost every \( t \), and condition (ii) of Theorem 1 is fulfilled.

Let us fix a Borel probability measure \( \nu \) on \( \mathbb{R}^d \) and a sequence of probability measures \( \nu^n = g^n(x) \, dx \), where \( g^n \in C_0^\infty(\mathbb{R}^d) \), such that \( \{\nu^n\} \) converges weakly to \( \nu \) and one has

\[
\sup_n \int_{\mathbb{R}^d} |x|^{2k} \, d\nu^n < \infty.
\]

According to Proposition 1, for each \( n \), there exists a unique family \( \mu^n = (\mu^n_t)_{t \in [0, 1]} \) of probability measures on \( \mathbb{R}^d \) satisfying the equation

\[
\partial_t \mu^n_t - n^{-1} \Delta \mu^n_t + \text{div}_x (b \mu^n_t) = 0, \quad \mu^n_0 = \nu^n
\]

in the sense that for all \( \varphi \in C_0^\infty(\mathbb{R}^d) \) and \( t \in [0, 1] \) one has

\[
\int_{\mathbb{R}^d} \varphi \, d\mu^n_t - \int_{\mathbb{R}^d} \varphi \, d\nu^n = \int_0^t \int_{\mathbb{R}^d} [n^{-1} \Delta \varphi + (b, \nabla \varphi)] \, d\mu^n_s \, ds. \tag{5}
\]

Moreover, there exists a number \( C \) independent of \( n \) such that

\[
\sup_{t \in [0, 1]} \int_{\mathbb{R}^d} |x|^{2k} \, d\mu^n_t < C,
\]

where \( k \) is the number from condition (ii) in Theorem 1. Therefore, the set of measures \( \{\mu^n_t: t \in [0, 1], n \in \mathbb{N}\} \) is uniformly tight on \( \mathbb{R}^d \). Now we fix a countable dense set \( \mathcal{F} \subset C_0^\infty(\mathbb{R}^d) \) (the latter space is equipped with the topology of uniform convergence of all derivatives on compact sets) and take a countable dense set \( \mathcal{T} \subset [0, 1] \). We can find a subsequence \( \{\mu^n_{t_k}\} \) which converges weakly to a probability measure \( \mu_t \) for each \( t \in \mathcal{T} \). Let us prove that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} \varphi(x) \, d\mu^n_{t_k} = \int_{\mathbb{R}^d} \varphi(x) \, d\mu_t \tag{6}
\]
for all $t \in [0,1]$ and all $\varphi \in \mathcal{F}$. Let us fix a function $\varphi \in \mathcal{F}$. Then there is a number $B$ such that $|b(x,t)| \leq B$ for all $x \in \text{supp } \varphi$ and $t \in [0,1]$, so we have

$$\left| \int_{\mathbb{R}^d} \varphi \, d\mu^t_k - \int_{\mathbb{R}^d} \varphi \, d\mu^s_k \right| \leq (\|\Delta \varphi\|_{\infty} + B \|\nabla \varphi\|_{\infty})|t - s|.$$ 

Let us fix a point $t \in [0,1]$. For any $\varepsilon > 0$ there are a number $r \in T$ and a natural number $N$ such that for all numbers $k > N$ one has

$$\left| \int_{\mathbb{R}^d} \varphi \, d\mu^k_n - \int_{\mathbb{R}^d} \varphi \, d\mu^t_n \right| \leq \varepsilon/3, \quad \left| \int_{\mathbb{R}^d} \varphi \, d\mu^k_r - \int_{\mathbb{R}^d} \varphi \, d\mu^s_r \right| \leq \varepsilon/6.$$ 

Then, for all $k, l > N$, we have

$$\left| \int_{\mathbb{R}^d} \varphi \, d\mu^k_n - \int_{\mathbb{R}^d} \varphi \, d\mu^l_n \right| \leq \varepsilon.$$ 

Hence the sequence $\mu^k_n$ on $\mathbb{R}^d$ is weakly fundamental and uniformly tight. Therefore, we obtain (6) for all $t$ and all $\varphi \in \mathcal{F}$. Then equality (6) holds for all continuous functions $\varphi$ with compact support. Letting $k \to \infty$ in (5) we obtain the equality

$$\int_{\mathbb{R}^d} \varphi \, d\mu_t - \int_{\mathbb{R}^d} \varphi \, d\nu = \int_0^t \int_{\mathbb{R}^d} (b, \nabla \varphi) \, d\mu_s \, ds$$ 

because, for almost every fixed $s$, by the continuity of $x \mapsto b(x,s)$, we have

$$\int_{\mathbb{R}^d} (b(x,s), \nabla \varphi(x)) \, d\mu^k_n \to \int_{\mathbb{R}^d} (b(x,s), \nabla \varphi(x)) \, d\mu_s$$ 

and the left-hand side is uniformly bounded, which enables us to integrate in $s$ and obtain the aforementioned equality. This gives Theorem 1 in the linear case.

Let us proceed to the general case where $b$ may depend on $\mu$. We construct a solution to (1) as a weak limit of solutions to approximating nondegenerate parabolic equations with the extra terms $-\varepsilon \Delta \mu$, where the coefficient $b$ satisfies condition (ii) of Theorem 1, but in place of condition (i) we impose the following much weaker condition:

(i) the mapping $b$ is defined on the space $\mathcal{P}_{\text{abs}} \times \mathbb{R}^d \times [0,1]$, where the space $\mathcal{P}_{\text{abs}}$ consists of all absolutely continuous probability measures on $\mathbb{R}^d \times [0,1]$ and is equipped with variation distance, and for Lebesgue a.e. $(x,t)$, the mapping $\mu \mapsto b(\mu,x,t)$ is continuous in the variation distance.

In fact, we need even less: it suffices that $b$ be defined only for $\mu$ from the subset in $\mathcal{P}_{\text{abs}}$ consisting of measures of the form $\rho(x,t) \, dx \, dt$ such that $x \mapsto \rho(x,t)$ is a probability density for each $t \in [0,1]$.

Let us consider the following nonlinear parabolic equation:

$$\partial_t \mu_t - \varepsilon \Delta \mu_t + \text{div}_x (b(\mu, \cdot, \cdot) \mu_t) = 0, \quad \mu_0 = \nu, \quad (7)$$

where $\nu = \varrho_0(x) \, dx$ and $\varrho_0 \in C^\infty_0(\mathbb{R}^d)$.

First we prove the existence of a solution to equation (7). Suppose that we are given real numbers $\alpha > 0$ and $c_1 > 0$ and that, for each closed interval $J_r = [r, 1 - r]$ and each closed ball $B_R = \{ x \in \mathbb{R}^d : |x| \leq R \}$, where $r, R > 0$ we are given a number $c_2(r,R) > 0$. Let $k$ be the number from condition (ii) and let $C^\alpha(E)$ denote the Banach space of $\alpha$-Hölder functions on $E$ with its natural norm.

Let us consider the set $K \subset L^4(\mathbb{R}^d \times [0,1])$ of all functions $\varrho$ satisfying the following conditions:

$$\varrho \geq 0, \quad \int_{\mathbb{R}} \varrho(x,t) \, dx = 1, \quad \int_{\mathbb{R}^d} |x|^{2k} \varrho(x,t) \, dx \leq c_1 \quad \forall t \in [0,1],$$
\[ \| \varrho \|_{C^\alpha([J, R])} \leq c_2(r, R) \quad \forall r, R > 0, \]

\( \varrho(x, 0) = \varrho_0(x) \text{ a.e.} \) and for each \( \varphi \in C_0^\infty(\mathbb{R}^d) \) the function

\[ t \mapsto \int_{\mathbb{R}^d} \varphi(x) \varrho(x, t) \, dx \]

is Lipschitzian with some constant \( C(\varphi) \).

**Lemma 1.** The set \( \mathcal{K} \) is convex and compact in the Banach space \( L^1([0, 1] \times \mathbb{R}^d) \).

**Proof.** Indeed, given a sequence in \( \mathcal{K} \), by a diagonal argument we choose a subsequence \( \{ \varrho_n \} \) that converges uniformly on compact sets in \( \mathbb{R}^d \times (0, 1) \) (here we use the bounds on the H"older norms). Since

\[ \int_{\mathbb{R}^d} |x|^{2k} \varrho(x, t) \, dx \leq c_1 \quad \forall t \in [0, 1], \]

the set of probability measures \( \varrho(x, t) \, dx \) on \( \mathbb{R}^d \), where \( \varrho \in \mathcal{K} \) and \( t \in [0, 1] \), is uniformly tight. Hence, for each fixed \( t \in [0, 1] \), the measures \( \varrho_n(x, t) \, dx \) on \( \mathbb{R}^d \) converge weakly to a probability measure \( \mu_t \) on \( \mathbb{R}^d \), where \( \mu_0 = \nu \). Locally uniform convergence of densities shows that \( \mu = \mu_t \, dt \) has a density \( \varrho \), which is locally Hölder continuous on \( \mathbb{R}^d \times (0, 1) \) and satisfies the equality \( \mu_t = \varrho(x, t) \, dx \). For each fixed \( \varphi \in C_0^\infty(\mathbb{R}^d) \) we have

\[ \int_{\mathbb{R}^d} \varphi(x) \varrho(x, t) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^d} \varphi(x) \varrho_n(x, t) \, dx, \quad t \in [0, 1], \]

hence the left-hand side is Lipschitzian with constant \( C(\varphi) \). Therefore, \( \varrho \in \mathcal{K} \). Hence \( \varrho_n \to \varrho \) in the norm of \( L^1(\mathbb{R}^d \times [0, 1]) \); we recall that pointwise convergence of probability densities to a probability density yields convergence in mean, see [8, Theorem 2.8.9]. Obviously, \( \mathcal{K} \) is convex.

It should be noted that we obtain a convex compact set even if we omit the last condition in the definition of \( \mathcal{K} \) regulating the behavior in \( t = 0 \). However, for subsequent applications to parabolic equations the introduced class turns out to be more convenient. The probability measure with density \( \varrho \in \mathcal{K} \) on \( \mathbb{R}^d \times [0, 1] \) will be denoted by the same symbol \( \varrho \) and \( \varrho_t \) will denote both the probability density \( x \mapsto \varrho(x, t) \) on \( \mathbb{R}^d \) and the measure with this density.

Now we define a mapping \( T \): \( \mathcal{K} \to \mathcal{K} \) as follows:

\[ \chi = T(\varrho) \iff \partial_t \chi_t - \varepsilon \Delta \chi_t + \text{div}_x(b(\varrho, \cdot, \cdot) \chi_t) = 0, \quad \chi_0 = \nu. \]

According to Proposition 1 and [9, Corollary 3.9], the mapping \( T \) is well-defined. Note that the Lipschitzness of the integral of \( \varphi \in C_0^\infty(\mathbb{R}^d) \) with respect to \( \chi_t(dx) \) follows from (4) due to the uniform boundedness of \( b(\varrho, x, t) \) on \( \text{supp} \, \varphi \times [0, 1] \).

**Lemma 2.** The mapping \( T \) is continuous.

**Proof.** Let \( \varrho^n, \varrho \in \mathcal{K} \), \( \| \varrho^n - \varrho \|_{L^1} \to 0 \) and \( \chi^n = T(\varrho^n) \). Since \( \mathcal{K} \) is compact, we can find a convergent subsequence \( \{ \chi^{n_k} \} \). We prove that \( \chi := \lim_{k \to \infty} \chi^{n_k} \) satisfies equation (7) with \( b = b(\varrho, \cdot, \cdot) \). For every \( \varphi \in C_0^\infty(\mathbb{R}^d) \) we have the identity

\[ \int_{\mathbb{R}^d} \varphi(x) \chi^{n_k}(x, t) \, dx - \int_{\mathbb{R}^d} \varphi(x) \nu(dx) = \varepsilon \int_0^t \int_{\mathbb{R}^d} \Delta \varphi(x) \chi^{n_k}(x, s) \, dx \, ds + \]

\[ + \int_0^t \int_{\mathbb{R}^d} (b(\varrho^{n_k}, x, s), \nabla \varphi(x)) \chi^{n_k}(x, s) \, dx \, ds. \quad (8) \]

Set \( S := \text{supp} \, \varphi \). Since \( |b(\varrho^{n_k}, x, t)| \) is uniformly bounded on \( S \times [0, 1] \) and

\[ \| \chi^{n_k} - \chi \|_{L^1} \to 0, \quad \| \varrho^{n_k} - \varrho \|_{L^1} \to 0, \quad |b(\varrho^{n_k}, x, t) - b(\varrho, x, t)| \to 0 \quad \text{a.e.}, \]
we can let $k \to \infty$ in (8) and obtain for all $t \in [0, 1]$

$$\int_{\mathbb{R}^d} \varphi(x) \chi(x, t) \, dx - \int_{\mathbb{R}^d} \varphi(x) \nu(dx) = \varepsilon \int_0^t \int_{\mathbb{R}^d} \Delta \varphi(x) \chi(x, s) \, dx \, ds +$$

$$+ \int_0^t \int_{\mathbb{R}^d} (b(\rho, x, s), \nabla \varphi(x)) \chi(x, s) \, dx \, ds.$$

This shows that $\chi = T(\rho)$. According to the uniqueness assertion in Proposition 1 we conclude that each subsequence in $\{\chi^n\}$ contains a subsequence convergent to $\chi$. This yields that $\chi^n \to \chi$, hence $T$ is continuous. \hfill $\Box$

Applying Schauder’s fixed point theorem we conclude that there exists $\rho \in K$ such that $\rho = T(\rho)$ and the family of measures $\mu_t = \rho(x, t) \, dx$ satisfies equation (7). Hence we arrive at the following assertion.

**Proposition 2.** Suppose that a probability measure $\nu$ has a density from $C^{\infty}_0(\mathbb{R}^d)$ and $b$ and $\nu$ satisfy condition (ii) of Theorem 1 and condition (i)' above. Then there exists a family $\mu = (\mu_t)_{t \in [0, 1]}$ of probability measures on $\mathbb{R}^d$ satisfying (7). Moreover, there exists a number $N$ depending only on $c, k$ and $\int_{\mathbb{R}^d} |x|^{2k} \, d\nu$ such that

$$\sup_{t \in [0, 1]} \int_{\mathbb{R}^d} |x|^{2k} \, d\mu_t < N.$$

Now we prove Theorem 1. Let us fix a probability measure $\nu$. We can find probability measures $\nu^n = \varrho_n(x) \, dx$ weakly convergent to the measure $\nu$ such that $\varrho_n \in L^\infty(\mathbb{R}^d)$ and

$$\sup_n \int_{\mathbb{R}^d} |x|^{2k} \, d\nu^n < \infty.$$

For each $\varepsilon = n^{-1}$ we take the solution $(\mu^n_t)_{t \in [0, 1]}$ to equation (7) with $\mu^n_0 = \nu^n$. Repeating our reasoning from the linear case we find a sequence $\{n_k\}$ such that $\{\mu^n_{n_k}\}$ converges weakly to $\mu_t$ for all $t \in [0, 1]$. Denote $b(\mu, \cdot, \cdot)$ and $b(\mu^n, \cdot, \cdot)$ by $b$ and $b_k$ respectively. We have the identity

$$\int_{\mathbb{R}^d} \varphi \, d\mu_{n_k} - \int_{\mathbb{R}^d} \varphi \, d\nu_k = \int_0^t \int_{\mathbb{R}^d} [n_k^{-1} \Delta \varphi + (b_k, \nabla \varphi)] \, d\mu_k \, ds.$$

Let $S := \text{supp} \, \varphi$. Note that

$$\left| \int_0^t \int_{\mathbb{R}^d} (b_k, \nabla \varphi) \, d\mu_k \, ds \right| - \left| \int_0^t \int_{\mathbb{R}^d} (b, \nabla \varphi) \, d\mu_k \, ds \right| \leq$$

$$\leq \left\| b_k - b \right\|_{L^\infty(S \times [0, 1])} \left\| \nabla \varphi \right\| \infty +$$

$$+ \left| \int_0^t \int_{\mathbb{R}^d} (b, \nabla \varphi) \, d\mu_k \, ds \right| - \left| \int_0^t \int_{\mathbb{R}^d} (b, \nabla \varphi) \, d\mu_k \, ds \right|.$$

Letting $k \to \infty$ we obtain

$$\int_{\mathbb{R}^d} \varphi \, d\mu_t - \int_{\mathbb{R}^d} \varphi \, d\nu = \int_0^t \int_{\mathbb{R}^d} (b(\mu, \cdot, \cdot), \nabla \varphi) \, d\mu_k \, ds$$

since by our assumption the mapping $x \mapsto b(\mu, x, s)$ is continuous for a.e. $s \in [0, 1]$ and the function $(x, s) \mapsto |(b(\mu, x, s), \nabla \varphi(x))|$ is uniformly bounded. This completes the proof of Theorem 1.

It is worth noting that, according to (2), since the function $(b(\mu, x, t), \nabla \varphi(x))$ has bounded support and is uniformly bounded for each $\varphi \in C^{\infty}_0(\mathbb{R}^d)$, the function

$$t \mapsto \int_{\mathbb{R}^d} \varphi(x) \, \mu_t(dx)$$
is Lipschitzian. This function is continuously differentiable if \( b \) is continuous in \( t \).

Finally, we observe that positivity of measures is essential for our a priori estimates employed in the proof. The same techniques apply to much more general second order elliptic operators in place of the Laplacian: we only need the assumptions from [6], [7], and [9]. Extensions to the infinite-dimensional case in the spirit of [10] will be considered in a forthcoming paper.

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