Lower estimates of densities of solutions of elliptic equations for measures

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The purpose of this paper is to estimate from below the decreasing rate at infinity of the density of a solution to the elliptic equation

$$\mathcal{L}^*\mu = 0 \tag{1}$$

with respect to a Borel probability measure μ on $\mathbb{R}^d$, where

$$\mathcal{L}\varphi(x) = \partial_{x_i}\bigl(a^{ij}(x)\partial_{x_j}\varphi(x)\bigr) + b^i(x)\partial_{x_i}\varphi(x),$$

and the summation is taken over repeated indices, and the equation is understood in the following sense: a measure μ satisfies (1) if the coefficients a^{ij} and b^i are locally μ-integrable and for every function $\varphi \in C_0^\infty(\mathbb{R}^d)$ the equality

$$\int_{\mathbb{R}^d} \mathcal{L}\varphi\, d\mu = 0$$

is fulfilled. Throughout we assume that the matrix $A(x) = (a^{ij}(x))_{1\le i,j\le d}$ is symmetric and satisfies the following condition:

(C1) for some $p > d$ the functions a^{ij} belong to the class $W^{p,1}_{loc}(\mathbb{R}^d)$ and there exist numbers $m, M > 0$ such that for all $x, y \in \mathbb{R}^d$ we have

$$m|y|^2 \le \sum_{1\le i,j\le d} a^{ij}(x)y_iy_j \le M|y|^2.$$

If in addition to Condition (C1) we have $b^i \in L^p_{loc}(\mu)$ (or $b^i \in L^p_{loc}(\mathbb{R}^d)$), then μ is given by a continuous density $\varrho \in W^{1,p}_{loc}(\mathbb{R}^d)$ (see [1]), which we shall deal with. Equation (1) can then be rewritten as

$$\partial_{x_i}\bigl(a^{ij}\partial_{x_j}\varrho\bigr) - \partial_{x_i}(b^i\varrho) = 0,$$

understood in the weak sense. In the case where the coefficient b is locally bounded, in [2] the following estimate from below for the density ϱ as $|x| \to +\infty$ was obtained (earlier in [3] more special estimates of exponential type were obtained). Let W be a continuous increasing function on $[0,\infty)$ and $W(0) > 0$. Suppose that $|b(x)| \le W(|x|/\theta)$, where $\theta > 1$. Then there exists a positive number $C = C(d, m, M, \theta)$ such that the continuous version of the function ϱ satisfies the inequality

$$\varrho(x) \ge \varrho(0)\exp\bigl\{-C\bigl(1 + W(|x|)|x|\bigr)\bigr\}.$$

The main idea of obtaining this estimate is to apply Harnack's inequality

$$\sup_{x\in K}\varrho(x) \le C(K)\inf_{x\in K}\varrho(x)$$

for compact sets K. For obtaining lower bounds, dependence of C on the coefficients of the equation and on K is investigated in [2]. However, this approach is impossible in the case of locally unbounded b. It turns out that without any restrictions on the growth of b one can obtain estimates of the form

$$\varrho(x) \ge e^{-f(c_1|x| + c_2)}, \tag{2}$$

where c_1, c_2 are some positive numbers and the function $f \in C^2\bigl([0,\infty)\bigr)$ satisfies the conditions

(H1) $f(z) > 0$, $f'(z) > 0$, $f''(z) > 0$ if $z > 0$;

(H2) the function $e^{-f(z)}$ is convex (that is, $(e^{-f})'' \ge 0$) on the set $z > z_0$ for some $z_0 \ge 0$ and it decreases to 0 as $z \to +\infty$.

Namely, for obtaining estimate (2) it suffices, in addition to (C1), to require the following conditions:

(C2) $|b|\exp(\psi(|b|)) \in L^p(\mu)$, where $p > \min\{2, d\}$ and ψ is a nonnegative strictly increasing continuous function mapping $[0, \infty)$ onto $[0, \infty)$ such that for some $N > 0$ and all $z > 0$ one has the inequality

(H3) $\psi^{-1}(z) \le N f'(f^{-1}(z))$.

Let us give several typical examples of functions f and ψ. Let $\delta > 0$ be a given number. If $f(z) = e^z$, then one can take $\psi(z) = \delta \cdot z$ for ψ. In this case we obtain the estimate

$$\varrho(x) \ge \exp(-\widetilde{c_2}\exp(\widetilde{c_1}|x|)).$$

If $f(z) = z^{r/(r-1)}$ with $r > 1$, then $\psi(z) = \delta \cdot z^r$ is suitable. Then

$$\varrho(x) \ge \widetilde{c_2}\exp(-\widetilde{c_1}|x|^{r/(r-1)}).$$

In the case where $d = 1$, $A = I$ and $b = \varrho'/\varrho$, such estimates were obtained in [4]. Deriving (2) we show on the way that the solution density is strictly positive under a condition weaker than the exponential integrability of b (sufficiency of the latter condition was proved in [5]). For example, if we set $f(z) = e^{e^z}$ and $\psi(z) = \delta \cdot \dfrac{z}{|\ln z|^\kappa}$ for $z > 2$ and $0 < \kappa < 1$, then we obtain a condition that is sufficient for the strict positivity but is weaker than the exponential integrability of b. If $d = 1$, $A = 1$ and $b = \varrho'/\varrho$, then this new sufficient condition for positivity is close to the one obtained in [6], and the latter cannot be improved in a sense.

For a domain $\Omega \subset \mathbb{R}^d$ let $W^{q,1}(\Omega)$ denote the Sobolev space of functions belonging to $L^q(\Omega)$ along with their first order generalized partial derivatives. Let $W_0^{q,1}(\Omega)$ be the closure with respect to the standard Sobolev norm in $W^{q,1}(\Omega)$ of the class of smooth functions with compact support in Ω. Let $W_{loc}^{q,1}$ and L_{loc}^q denote the spaces of functions whose restrictions to every ball $B \subset \mathbb{R}^d$ belong to $W^{q,1}(B)$ and $L^q(B)$, respectively. Let $B(x, R)$ be the ball of radius R centered at x. If a function f is injective, then the inverse function is denoted by f^{-1}.

Since ψ is increasing, for all $\alpha \ge e^{\psi(0)}$, $\beta \ge 0$ we have

$$\alpha\beta \le \alpha\psi^{-1}(\ln\alpha) + \beta e^{\psi(\beta)}. \tag{3}$$

Set $V = e^f/f'$.

Since $(e^{-f})'' = [(f')^2 - f'']e^{-f} \ge 0$ on $[z_0, +\infty)$, we have $V' = [(f')^2 - f'']e^{-f}(f')^{-2} \ge 0$ on $[z_0, +\infty)$. In addition, V increases to $+\infty$ since the function $1/V = f'e^{-f}$ cannot be separated from zero on $[0, +\infty)$. It follows from conditions (H1) and (H3) that $f'(y) \to +\infty$ as $y \to +\infty$. Therefore, there exists $y_0 > \max\{z_0, 1\}$ such that $f'(y) \ge 1$ and $V(y) \ge e^{\psi(0)}$ whenever $y > y_0$. Let $\tau_0 := \exp\{-f(\ln y_0)\}$. Then $0 < \tau_0 < 1$. For $\tau \in (0, \tau_0)$ and $q \ge 0$ we put

$$h_q(\tau) := -\int_\tau^{\tau_0} V^2(f^{-1}(|\ln s|))\exp\{2qf^{-1}(|\ln s|)\}\,ds.$$

Lemma 1. *If conditions* (H1), (H2), (H3) *are fulfilled and* $\tau \in (0, \tau_0)$, *then*

(i) *the inequality* $V(y)\psi^{-1}(\ln V(y)) \le Ne^{f(y)}$ *is fulfilled for* $y > y_0$;

(ii) *there exists a number* $N_1 > 0$ *such that*

$$\frac{1}{V(y)}\int_{y_0}^{y} V(s)\,ds \le N_1, \quad y > y_0,$$

and, in addition,

$$h_q^2(\tau)/h_q'(\tau) \le N_1^2\exp\bigl(2qf^{-1}(|\ln\tau|)\bigr).$$

According to (C2), we have $|b| \in L^p(\mu)$ for some $p > d$. As we have already noted, in this case μ is given by a continuous density $\varrho \in W_{loc}^{1,p}(\mathbb{R}^d)$. In addition, $\|\varrho\|_{L^\infty(\mathbb{R}^d)} < \infty$ (see [3], [7]). For any $k \in \mathbb{N}$ we put

$$\Lambda := \min\{\tau_0(2\|\varrho\|_{L^\infty(\mathbb{R}^d)})^{-1}, 1\}, \quad \varrho_k = \Lambda\varrho + 1/k, \quad \xi_k := f^{-1}(|\ln\varrho_k|).$$

Then $\Lambda\varrho < 1/2$ and $\varrho_k < \tau_0$ for all natural numbers $k > 1/(2\tau_0)$. Hence substituting ϱ_k in place of τ in h_q and letting $y = \xi_k$ we have all assertions of Lemma 1.

Lemma 2. *Let $\mu = \varrho\,dx$ be a solution of equation* (1)*, let the coefficient a^{ij}, b^i satisfy conditions* (C1), (C2), *and let conditions* (H1), (H2), *and* (H3) *be fulfilled. Let $s > 1$, $s' = s/(s-1)$, where $2s' \le p$. Suppose we are given a function $\eta \in C_0^2(Q)$, where Q is a cube of the edge length 2, and let $|\eta| \le 1$. Then the following estimate holds:*

$$\int_Q |\nabla\varrho|^2 h_q'(\varrho_k)\eta^2\,dx \le N_2\Big[\int_Q |\nabla\eta|^2 \exp\{2q\xi_k\}\,dx + \Big(\int_Q \exp\{2sq\xi_k\}\eta^2\,dx\Big)^{1/s}\Big],$$

where N_2 is a number depending only on the following quantities:

$$s,\ N,\ N_1,\ \tau_0,\ m,\ M,\ d,\ \|\varrho\|_{L^\infty(\mathbb{R}^d)},\ \int_{\mathbb{R}^d} |b|^{2s'}\exp\{2s'\psi(|b|)\}\varrho\,dx.$$

Proof. For every function $\varphi \in W_0^{1,2}(Q)$ we have

$$\int_Q (A\nabla\varrho, \nabla\varphi)\,dx = \int_Q (b, \nabla\varphi)\varrho\,dx.$$

Substituting $\varphi = h_q(\varrho_k)\eta^2$, we obtain

$$\int_Q (A\nabla\varrho, \nabla\varrho)h_q'(\varrho_k)\eta^2\,dx = I + J + L,$$

$$I = -\int_Q 2(A\nabla\varrho, \nabla\eta)h_q(\varrho_k)\eta\,dx, \quad J = \int_Q 2(b, \nabla\eta)\varrho h_q(\varrho_k)\eta\,dx,$$

$$L = \int_Q (b, \nabla\varrho)h_q'(\varrho_k)\eta^2\varrho\,dx.$$

Let us estimate every summand separately. Let $\varepsilon > 0$. Then

$$I \le \varepsilon\int_Q |\nabla\varrho|^2 h_q'(\varrho_k)\eta^2\,dx + \varepsilon^{-1}M^2\int_Q |\nabla\eta|^2\frac{h_q^2(\varrho_k)}{h_q'(\varrho_k)}\,dx.$$

By estimate (ii) in Lemma 1 we have

$$I \le \varepsilon\int_Q |\nabla\varrho|^2 h_q'(\varrho_k)\eta^2\,dx + \varepsilon^{-1}(MN_1)^2\int_Q |\nabla\eta|^2 e^{2qf^{-1}(\varrho_k)}\,dx.$$

We estimate J as follows:

$$J \le \int_Q |\nabla\eta|^2\frac{h_q^2(\varrho_k)}{h_q'(\varrho_k)}\,dx + \int_Q |b|^2\varrho^2 h_q'(\varrho_k)\eta^2\,dx.$$

The first term is estimated in the same way as above. Let us consider the second term. By Hölder's inequality with exponents s' and s we have

$$\int_Q |b|^2\varrho^2 h_q'(\varrho_k)\eta^2\,dx \le \Big(\int_Q |b|^{2s'}\varrho^{2s'}V(\xi_k)^{2s'}\eta^2\,dx\Big)^{1/s'}\Big(\int_Q \exp\{2qs\xi_k\}\eta^2\,dx\Big)^{1/s}.$$

Let us estimate the first factor. Inequality (3) and estimate (i) in Lemma 1 yield

$$|b|\varrho V(\xi_k) \le \varrho\big[|b|e^{\psi(|b|)} + V(\xi_k)\psi^{-1}(\ln V(\xi_k))\big] \le |b|e^{\psi(|b|)}\varrho + N/\Lambda.$$

By using the inequalities $(x+y)^{2s'} \le 2^{2s'}(x^{2s'} + y^{2s'})$ and $\eta^2 \le 1$, we obtain

$$\int_Q |b|^{2s'}\varrho^{2s'}V(\xi_k)^{2s'}\eta^2\,dx \le 4^{s'}\|\varrho\|_{L^\infty(\mathbb{R}^d)}^{2s'-1}\int_{\mathbb{R}^d} |b|^{2s'}e^{2s'\psi(|b|)}\varrho\,dx + (2N/\Lambda)^{2s'}|Q|.$$

Therefore, there exists a number $C_1 > 0$ depending only on the quantities indicated in the lemma such that

$$J \le C_1\Big[\int_Q |\nabla\eta|^2 e^{2q\xi_k}\,dx + \Big(\int_Q e^{2sq\xi_k}\eta^2\,dx\Big)^{1/s}\Big].$$

It remains to estimate the term L. We have

$$L\le \varepsilon\int_Q |\nabla\varrho|^2 h_q'(\varrho_k)\eta^2\,dx+4\varepsilon^{-1}\int_Q |b|^2\varrho^2 h_q'(\varrho_k)\eta^2\,dx.$$

Estimating here the second term in the same way as above, we obtain

$$L\le \varepsilon\int_Q |\nabla\varrho|^2 h_q'(\varrho_k)\eta^2\,dx+4\varepsilon^{-1}C_1\Bigl(\int_Q e^{2sq\xi_k}\eta^2\,dx\Bigr)^{1/s}.$$

We observe that

$$m\int_Q |\nabla\varrho|^2 h_q'(\varrho_k)\eta^2\,dx\le\int_Q (A\nabla\varrho,\nabla\varrho)h_q'(\varrho_k)\eta^2\,dx.$$

Collecting the obtained estimates and letting $\varepsilon=m/3$ we find

$$\int_Q |\nabla\varrho|^2 h_q'(\varrho_k)\eta^2\,dx\le N_2\Bigl[\int_Q |\nabla\eta|^2 e^{2q\xi_k}\,dx+\Bigl(\int_Q e^{2sq\xi_k}\eta^2\,dx\Bigr)^{1/s}\Bigr].$$

The lemma is proven. □

For obtaining estimates of type (2) we shall employ Moser's iteration techniques (see [8], [9]). The proof of the following result of Moser can be found in [9, Lemma 7.21]. Let Ω be a domain in $\mathbb{R}^d$. For any integrable function u we put

$$u_\Omega=|\Omega|^{-1}\int_\Omega u\,dx,$$

where $|\Omega|$ is the volume of Ω.

Lemma 3. *Let Ω be a convex domain and let $v\in W^{1,1}(\Omega)$ be such that there exists $K>0$ such that, for every ball $B(x_0,R)$, one has the inequality*

$$\int_{\Omega\cap B(x_0,R)} |\nabla v|\,dx\le KR^{d-1}.$$

Then there exist positive numbers σ_0 and C depending only on d such that

$$\int_\Omega \exp\Bigl(\frac{\sigma}{K}|v-v_\Omega|\Bigr)\,dx\le C(\operatorname{diam}\Omega)^d,$$

where $\sigma=\sigma_0|\Omega|(\operatorname{diam}\Omega)^{-d}$, $\operatorname{diam}\Omega=\sup_{x,y\in\Omega}|x-y|$.

Let us fix a cub Q of unit edge.

Theorem 1. *Let $\mu=\varrho\,dx$ be a solution of equation* (1), *where the coefficients a^{ij},b^i satisfy conditions* (C1), (C2) *and let conditions* (H1), (H2), *and* (H3) *be fulfilled. Then there exist numbers $C>0$ and $\alpha>0$ such that for every measurable subset $E\subset Q$ one has*

$$\sup_{x\in Q}\exp(f^{-1}(|\ln(\Lambda\varrho)|))\le C\Bigl(\int_E \exp(-\alpha f(|\ln\Lambda\varrho|))\Bigr)^{-1/\alpha},\tag{4}$$

where Λ is defined before Lemma 1, and the numbers C and α depend only on the following quantities:

$$p,\ N,\ N_1,\ \tau_0,\ m,\ M,\ d,\ \|\varrho\|_{L^\infty(\mathbb{R}^d)},\ \int_{\mathbb{R}^d}|b|^p e^{p\psi(|b|)}\varrho\,dx.$$

Proof. Let $d>2$. Without loss of generality we may assume that

$$Q=\prod_{i=1}^d\bigl[x_i^0-\frac12,x_i^0+\frac12\bigr],\,Q_n=\prod_{i=1}^d\bigl[x_i^0-\frac12-\frac{1}{2^{n+1}},x_i^0+\frac12+\frac{1}{2^{n+1}}\bigr].$$

1. We observe that

$$\int_{Q_0}|\nabla\xi_k|^2\eta^2\,dx=\int_{Q_0}|\nabla\varrho|^2V^2(\xi_k)\eta^2\,dx=\int_{Q_0}|\nabla\varrho|^2h_0'(\varrho_k)\eta^2\,dx.$$

Let $s = p/(p-2)$. Then $s < d/(d-2)$ and $2s' = p$. Lemma 2 with $q = 0$ gives

$$\int_{Q_0} |\nabla\xi_k|^2\eta^2\,dx \le N_2\Big[\int_{Q_0} |\nabla\eta|^2\,dx + \Big(\int_{Q_0}\eta^2\,dx\Big)^{(p-2)/p}\Big]$$

Let us take two balls $B(y,r) \subset B(y,2r) \subset Q_0$. Let $\eta(x) = 1$ if $x \in B(y,r)$ and $\eta(x) = 0$ if $x \notin B(y,2r)$. Suppose also that $|\eta| \le 1$ and $|\nabla\eta| \le c_1 r^{-1}$ with some constant c_1. Substituting η in the above estimate we find

$$\int_{B(y,r)} |\nabla\xi_k|^2\,dx \le C_0 r^{d-2}.$$

Here the number C_0 depends only on the parameters indicated in the theorem but does not depend on y, r, k. Therefore, for every ball $B(y,r)$, by the Cauchy–Buniakowskii inequality we obtain the estimate

$$\int_{B(y,r)} |\nabla\xi_k|\,dx \le (C_0\omega_d)^{1/2} r^{d-1},$$

where ω_d is the volume of the d-dimensional unit ball. Applying Lemma 3 we obtain that there exist constants $\alpha > 0$ and $L > 0$ such that

$$\int_{Q_1} \exp\Big(\alpha|\xi_k - (\xi_k)_{Q_1}|\Big)\,dx \le L.$$

Then

$$\int_{Q_1} e^{\alpha\xi_k}\,dx \int_{Q_1} e^{-\alpha\xi_k}\,dx \le \Big(\int_{Q_1} \exp\{\alpha|\xi_k - (\xi_k)_{Q_1}|\}\,dx\Big)^2 \le L^2. \tag{5}$$

2. We observe that for all $\eta \in C_0^2(Q_1)$ one has the equality

$$\int_{Q_1} |\nabla e^{q\xi_k}|^2\eta^2\,dx = q^2\int_{Q_1} |\nabla\varrho|^2 V^2(\xi_k)e^{2q\xi_k}\eta^2\,dx = q^2\int_{Q_1} |\nabla\varrho|^2 h_q'(\varrho_k)\eta^2\,dx.$$

Applying Lemma 2 with $q > 0$ we obtain

$$\int_{Q_1} |\nabla e^{q\xi_k}|^2\eta^2\,dx \le q^2 N_2\Big[\int_{Q_1} |\nabla\eta|^2 e^{2q\xi_k}\,dx + \Big(\int_{Q_1} e^{2sq\xi_k}\eta^2\,dx\Big)^{1/s}\Big].$$

According to the Leibnitz formula $\nabla(e^{q\xi_k}\eta) = \eta\nabla e^{q\xi_k} + e^{q\xi_k}\nabla\eta$. Then

$$\int_{Q_1} |\nabla(e^{q\xi_k}\eta)|^2\,dx \le q^2 N_2\Big[\int_{Q_1} |\nabla\eta|^2 e^{2q\xi_k}\,dx + \Big(\int_{Q_1} e^{2sq\xi_k}\eta^2\,dx\Big)^{1/s}\Big].$$

Suppose that a smooth function $\eta = \eta_n$ vanishes outside Q_n and equals 1 on the cube Q_{n+1}. Let $|\eta_n| \le 1$ and $|\nabla\eta_n| \le c_2 2^{n+1}$ for some constant c_2 independent of n. Applying Hölder's inequality with exponents s and s' we find

$$\int_{Q_1} |\nabla(e^{q\xi_k}\eta)|^2\,dx \le (q^2+1)C_1^n\Big(\int_{Q_n} e^{2sq\xi_k}\,dx\Big)^{1/s}.$$

By the Sobolev embedding theorem we obtain

$$\Big(\int_{Q_{n+1}} |e^{q\xi_k}|^{2d/(d-2)}\,dx\Big)^{(d-2)/d} \le (q^2+1)C_2^n\Big(\int_{Q_n} e^{2sq\xi_k}\,dx\Big)^{1/s}.$$

For any measurable set E and $t \ne 0$ we put

$$F(t,E) := \Big(\int_E e^{t\xi_k}\,dx\Big)^{1/t}, \quad F(+\infty,E) = \sup_{x\in E} e^{\xi_k}.$$

Therefore,

$$F(\frac{2qd}{d-2}, Q_{n+1}) \le ((q^2+1)C_2)^{n/q} F(2qs, Q_n).$$

Set $p_n = 2qs$ and $p_{n+1} = ds^{-1}(d-2)^{-1}p_n, p_1 = \alpha$. For $s = p/(p-2)$ we obtain $s < d/(d-2)$, $\lambda = ds^{-1}(d-2)^{-1} > 1$, $p_n = \alpha\lambda^n$, $p_n \to +\infty$,

$$F(p_{n+1}, Q_{n+1}) \le C_3^{n\lambda^{-n}} F(p_n, Q_n).$$

Since $0 < \lambda < 1$, one has $\sum_{n=1}^{\infty} n\lambda^{-n} < \infty$. Hence there exists $C_4 > 0$ such that

$$F(p_{n+1}, Q_{n+1}) \le C_3^{\theta} F(\alpha, Q_1) \le C_4 F(\alpha, Q_1), \quad \theta = \sum_{n=1}^{\infty} n\lambda^{-n}.$$

It is known that $F(+\infty, Q) = \lim\limits_{t\to\infty} F(t, Q)$. Therefore, as $n \to +\infty$ we obtain

$$F(+\infty, Q) \le C_4 F(\alpha, Q_1).$$

According to (5) the inequality $F(\alpha, Q_1) \le L^2 F(-\alpha, Q_1)$ is valid. Letting $k \to \infty$ we obtain

$$\sup_{x\in Q} \exp(f^{-1}(|\ln(\Lambda\varrho)|)) \le C_4 L^2 \Bigl(\int_{Q_1} \exp(-\alpha f(|\ln \Lambda\varrho|))\Bigr)^{-1/\alpha}.$$

It remains to observe that replacing Q_1 by E increases the right-hand side. Thus, (4) is proven if $d > 2$. The cases $d = 1$ and $d = 2$ are even simpler because in the Sobolev inequality in place of the exponent $2d(d-2)^{-1}$ one can take any $r > 1$. □

Theorem 2. *Let $\mu = \varrho\, dx$ be a solution of equation* (1)*, where the coefficients a^{ij}, b^i satisfy conditions* (C1)*,* (C2) *and let conditions* (H1)*,* (H2)*, and* (H3) *be fulfilled. Then there exist numbers $c_1 > 0$ and $c_2 > 0$ such that*

$$\varrho(x) \ge e^{-f(c_1|x| + c_2)}, \quad x \in \mathbb{R}^d.$$

Proof. Let $u = \exp\bigl(\alpha f^{-1}(|\ln(\Lambda\varrho)|)\bigr)$, where α and Λ are numbers from (4), Q is an arbitrary cube of unit edge length. By Theorem 1 we obtain

$$\sup_{x\in Q} u(x) \le C|\Omega|^{-1} \sup_{\Omega} u(x)$$

for every measurable set $\Omega \subset Q$. Let us fix $x \in \mathbb{R}^d$. Let $N = [|x|] + 1$ and $x_i = ix/N$. Then $x_0 = 0$, $x_N = x$ and $|x_i - x_{i-1}| \le 1$. Let Q_i denote the cube with center at the point x_i and unit edge parallel to the vector x. For every i we have $x_{i-1} \in Q_i$, $|Q_i \cap Q_{i-1}| = 1/2$ and, therefore,

$$\sup_{Q_i} u(x) \le C|Q_i \cap Q_{i-1}|^{-1} \sup_{Q_i\cap Q_{i-1}} u(x) \le 2C \sup_{Q_{i-1}} u(x).$$

We obtain the inequality

$$\sup_{Q_i} u(x) \le 2C \sup_{Q_{i-1}} u(x).$$

Applying this inequality for all i starting with $i = N$, we find

$$u(x) = u(x_N) \le (2C)^N \sup_{Q_0} u(x) \le (2C)^N \sup_{|x|\le 2} u(x).$$

Since $N = [|x|] + 1 \le |x| + 1$, for some $\lambda_1 > 0$ and $\lambda_2 > 0$ we have

$$u(x) \le \exp(\lambda_1|x| + \lambda_2), \ x \in \mathbb{R}^d.$$

Taking into account that $\varrho = \Lambda^{-1}e^{-f(\alpha^{-1}\ln u)}$ due to the estimate $\Lambda\varrho < 1/2$ and recalling that $\Lambda^{-1} \ge 1$ and the function f is increasing, we obtain the desired estimate. □

We observe that this result gives lower bounds for the density of the stationary measure of the diffusion process with diffusion coefficient $\sqrt{2A}$ and drift b. A similar method along with techniques from [10] can be applied in the parabolic case, which will be considered in a separate work.

Example 1. Let conditions (C1) and (C2) be fulfilled and let a number $r > 1$ be given.

(i) In order to obtain the estimate

$$\varrho(x) \geq \widetilde{c_2} \exp(-\widetilde{c_1}|x|^{r/(r-1)}), \tag{6}$$

it suffices to have $\exp(\delta|b|^r) \in L^1(\mu)$ with some $\delta > 0$.

Indeed, the function $\psi(z) = \delta z^r/(2p)$ satisfies condition (H3) with $f(z) = z^{r/(r-1)}$. There exists $C(\delta) > 0$ such that $|z| \leq C(\delta)\exp(\delta|z|^r/2)$. Then $\bigl(|b|\exp(\delta|b|^r/(2p))\bigr)^p \leq C(\delta)^p \exp(\delta|b|^r)$ and so $|b|\exp(\delta|b|^r/(2p)) \in L^p(\mu)$, that is, condition (C2) is fulfilled.

(ii) In order to obtain the estimate

$$\varrho(x) \geq \exp(-\widetilde{c_2}\exp(\widetilde{c_1}|x|)), \tag{7}$$

it suffices to have $\exp(\delta|b|) \in L^1(\mu)$ with some $\delta > 0$.

Indeed, whenever $0 < \delta_1 < \delta$, the functions $\psi(z) = \delta_1 \cdot z$ and $f(z) = e^z$ satisfy (H3) with $N = 1/\delta_1$ and (C2) is fulfilled as well.

Example 2. Let $\mu = \varrho\, dx$ be a probability measure, $\varrho \in W^{1,1}_{loc}(\mathbb{R}^d)$. Then μ obviously satisfies equation (1) with $A = I$ and $b = \nabla\varrho/\varrho$, where $b(x) := 0$ if $\varrho(x) = 0$. Therefore, for obtaining estimate (6) it suffices to have $\exp(\delta|\nabla\varrho/\varrho|^r) \in L^1(\mu)$ with some $\delta > 0$, and estimate (7) follows from the inclusion $\exp(\delta|\nabla\varrho/\varrho|) \in L^1(\mu)$ with some $\delta > 0$.

For $d = 1$ the assertion in the last example was obtained in [4] (where in the case $r = 1$ the formulation contains a minor inaccuracy: $\widetilde{c_1}$ is replaced by 1; the function $\varrho(x) = \exp\bigl(-\exp(2|x|)\bigr)$ shows that one cannot get rid of $\widetilde{c_1}$). For $d > 1$ and $r = 1$ the assertion of the last example is given in Exercise 6.8.4 in book [11]; when our work was completed we learnt of the forthcoming paper [12], where in the situation of the same Example 2 the case $r > 1$ is considered. However, the methods of [4] and [12] employ in a very essential way the fact that b is of the special form $\nabla\varrho/\varrho$.

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