LARGE DEVIATIONS FOR STOCHASTIC TAMED 3D NAVIER-STOKES EQUATIONS

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ABSTRACT. In this paper, using weak convergence method, we prove a large deviation principle of Freidlin-Wentzell type for the stochastic tamed 3D Navier-Stokes equations driven by multiplicative noise, which was investigated in [21].

1. INTRODUCTION

Let $\mathbb{D} = \mathbb{R}^3$ or $\mathbb{T}^3$ (=three dimensional torus). Consider the following stochastic tamed 3D Navier-Stokes equation in $\mathbb{D}$:

$$\begin{cases} \mathrm{d}\mathbf{u}(t) = \Big[\nu\Delta\mathbf{u}(t) - (\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t) + \nabla p(t) - g_N(|\mathbf{u}(t)|^2)\mathbf{u}(t)\Big]\mathrm{d}t \\ \qquad\qquad + \mathbf{f}(t,\mathbf{u}(t))\mathrm{d}t + \displaystyle\sum_{k=1}^{\infty}\Big[\nabla\tilde{p}_k(t) + \mathbf{h}_k(t,\mathbf{u}(t))\Big]\mathrm{d}W_t^k, \\ \mathrm{div}\mathbf{u}(t) = 0, \quad \mathbf{u}(0) = \mathbf{u}_0. \end{cases} \tag{1}$$

Here, $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^3$, $\mathbf{u} = (u^1, u^2, u^3)$ is the velocity field, $p(t,x)$ and $\tilde{p}_k(t,x)$ are unknown scalar functions. The function $g_N : \mathbb{R}^+ \to \mathbb{R}^+$ is smooth and satisfies

$$\begin{cases} g_N(r) = 0, & \text{if } r \leqslant N, \\ g_N(r) = (r-N)/\nu, & \text{if } r \geqslant N+1, \\ 0 \leqslant g_N'(r) \leqslant 2/(\nu\wedge 1), & r \geqslant 0. \end{cases} \tag{2}$$

$\{W_t^k; t \geqslant 0, k = 1, 2, \cdots\}$ is a sequence of independent one dimensional standard Brownian motions on some complete filtered probability space $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\geqslant 0})$. The stochastic integral is understood as Itô's integral, and the coefficients

$$\begin{aligned} \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \ni (t,x,\mathbf{u}) &\to \mathbf{f}(t,x,\mathbf{u}) \in \mathbb{R}^3, \\ \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \ni (t,x,\mathbf{u}) &\to \mathbf{h}(t,x,\mathbf{u}) \in \mathbb{R}^3 \times l^2, \end{aligned}$$

satisfy assumptions **(H1)** and **(H2)** specified below. Here l^2 denotes the Hilbert space consisting of all sequences of square summable real numbers with standard norm $\|\cdot\|_{l^2}$.

It was proven in [21] that under **(H1)** and **(H2)** below, there exists a unique strong solution for equation (1) in the sense of PDE and SDE. The ergodicity of equation (1) was also proved in [21]. By virtue of the presence of the taming function g_N, equation (1) can be considered as a modified stochastic Navier-Stokes equation. Without the term g_N, equation (1) is the stochastic 3D Navier-Stokes equation for which it is well known that there exists a martingale solution (cf. [15, 13]). But the uniqueness is open, and one only knows the existence of a locally unique strong solution (cf. [16, 27]). Moreover, without the term g_N, the existence and ergodicity of invariant measures for equation (1) have already been studied by using Kolmogorov operators (cf. [1, 8, 9]) and the existence of Markovian selections (cf. [12, 13]). We would like to emphasize that the deterministic tamed equation has the following feature: if there exists a bounded solution to the classical 3D Navier-Stokes equation, then this solution also solves the tamed equation.

The purpose of this paper is to study the small noise large deviation for the stochastic tamed 3D Navier-Stokes equation (1).

The large deviations for stochastic partial differential equations (SPDE) have been studied by many authors. For example, a large deviation principle (LDP) for stochastic reaction diffusion equations with non-Lipschitz reaction term was established by Cerrai and Röckner in [7]. An LDP for a Burgers'-type SPDE was considered by Cardon-Weber in [6]. A uniform LDP for parabolic SPDE was proved by Chenal and Millet in [5]. An LDP for stochastic reaction diffusion equations was established by Sowers in [23]. A small time large deviation principle for stochastic parabolic equations was obtained by the second named author in [24]. For the general theory of large deviations, the reader is referred to the monograph [10]. Because of the different nature of the nonlinearities for different types of equations, the large deviations for SPDEs has to be dealt with on individual bases.

To establish the large deviation principle for the above stochastic tamed Navier-Stokes equation, we shall adopt the weak convergence method developed by Dupuis-Ellis in [11, 2, 3]. This method has been proved to be very effective for various types of stochastic equations (cf. [4, 18, 19, 26, 14, 27]). The main advantage of this method is that one avoids the use of the usual complicated time discretization and the proof of exponential tightness. Our proof is inspired by Liu's method [14] and also works for 2D stochastic Navier-Stokes equations.

This paper is organized as follows: Section 2 contains preliminaries and the formulation of our main result. In Section 3, we shall prove the main result by the weak convergence method.

2. Preliminary and Main Result

Throughout this paper, we fix $T > 0$ and assume $\nu = 1$. We shall use the following convention: The letter C with or without subscripts will denote a positive constant, whose value may change from one place to another.

2.1. **An abstract criterion for the Laplace principle.** It is well known that there exists a Hilbert space $\mathbb{U}$ so that $l^2 \subset \mathbb{U}$ with Hilbert-Schmidt embedding operator J and $\{W^k(t), k \in \mathbb{N}\}$ is a Brownian motion with values in $\mathbb{U}$, whose covariance operator is given by $Q = J \circ J^*$. For example, one can take $\mathbb{U}$ as the completion of l^2 with respect to the norm generated by the scalar product

$$\langle h, h' \rangle_{\mathbb{U}} := \sum_{k=1}^{\infty} \frac{h_k h'_k}{k^2}, \quad h, h' \in l^2.$$

For a Polish space $\mathbb{B}$, we denote by $\mathcal{B}(\mathbb{B})$ the Borel σ-field, and by $\mathbb{C}_T(\mathbb{B})$ the continuous function space from $[0, T]$ to $\mathbb{B}$, which is endowed with the uniform distance so that $\mathbb{C}_T(\mathbb{B})$ is again a Polish space. Define

$$\mathscr{H} := \left\{ h = \int_0^{\cdot} \dot{h}(s) \mathrm{d}s : \ \dot{h} \in L^2(0, T; l^2) \right\} \tag{3}$$

with the norm

$$\|h\|_{\mathscr{H}} := \left(\int_0^T \|\dot{h}(s)\|_{l^2}^2 \mathrm{d}s \right)^{1/2},$$

where the dot denotes the weak derivative. Let μ be the law of the Brownian motion W in $\mathbb{C}_T(\mathbb{U})$ with covariance space l^2. Then

$$(\mathbb{C}_T(\mathbb{U}), \mathscr{H}, \mu)$$

forms an abstract Wiener space.

For $T, M > 0$, set

$$\mathcal{D}_M := \{h \in \mathscr{H} : \|h\|_{\mathscr{H}} \leqslant M\}$$

and

$$\mathcal{A}_M^T := \left\{ \begin{array}{l} h : [0,T] \to l^2 \text{ is a continuous and } (\mathcal{F}_t)\text{-adapted} \\ \text{process, and for almost all } \omega, \quad h(\cdot,\omega) \in \mathcal{D}_M \end{array} \right\}. \tag{4}$$

We equip $\mathcal{D}_M$ with the weak topology in $\mathscr{H}$. Then

$$\mathcal{D}_M \text{ is a compact Polish space.} \tag{5}$$

Let $\mathbb{S}$ be a Polish space and let $I : \mathbb{S} \to [0,\infty]$ be given.

Definition 2.1. *The function I is called a rate function if for every $a < \infty$, the set $\{f \in \mathbb{S} : I(f) \leqslant a\}$ is compact in $\mathbb{S}$.*

Let $\{Z_\epsilon : \mathbb{C}_T(\mathbb{U}) \to \mathbb{S}, \epsilon \in (0,1)\}$ be a family of measurable mappings. Assume that there is a measurable map $Z_0 : \mathscr{H} \mapsto \mathbb{S}$ such that

$(\mathbf{LD})_\mathbf{1}$ For any $M > 0$, if a family $\{h^\epsilon, \epsilon \in (0,1)\} \subset \mathcal{A}_M^T$ as random variables in $\mathcal{D}_M$ converges in distribution to $h \in \mathcal{A}_M^T$, then for some subsequence ϵ_k, $Z_{\epsilon_k}\left(\cdot + \frac{h^{\epsilon_k}(\cdot)}{\sqrt{\epsilon_k}}\right)$ converges in distribution to $Z_0(h)$ in $\mathbb{S}$.

$(\mathbf{LD})_\mathbf{2}$ For any $M > 0$, $Z_0 : \mathcal{D}_M \to \mathbb{S}$ is weakly continuous. Equivalently, if $\{h_n, n \in \mathbb{N}\} \subset \mathcal{D}_M$ weakly converges to $h \in \mathscr{H}$, then for some subsequence h_{n_k}, $Z_0(h_{n_k})$ converges to $Z_0(h)$ in $\mathbb{S}$.

For each $f \in \mathbb{S}$, define

$$I(f) := \frac{1}{2} \inf_{\{h \in \mathscr{H}:\ f = Z_0(h)\}} \|h\|^2_{\mathscr{H}}, \tag{6}$$

where $\inf \emptyset = \infty$ by convention. Then under $(\mathbf{LD})_\mathbf{2}$, $I(f)$ obviously is a rate function.

We recall the following result from [3] (see also [28, Theorem 4.4]).

Theorem 2.2. *Under $(\mathbf{LD})_\mathbf{1}$ and $(\mathbf{LD})_\mathbf{2}$, $\{Z_\epsilon, \epsilon \in (0,1)\}$ satisfies the Laplace principle with the rate function $I(f)$ given by (6). More precisely, for each real bounded continuous function g on $\mathbb{S}$:*

$$\lim_{\epsilon \to 0} \epsilon \log \mathbb{E}^\mu \left(\exp\left[-\frac{g(Z_\epsilon)}{\epsilon} \right] \right) = -\inf_{f \in \mathbb{S}} \{g(f) + I(f)\}, \tag{7}$$

where $\mathbb{E}^\mu$ denotes expectation with respect to μ. In particular, the family of $\{Z_\epsilon, \epsilon \in (0,1)\}$ satisfies the large deviation principle in $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ with rate function $I(f)$. More precisely, let ν_ϵ be the law of Z_ϵ in $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$, then for any $A \in \mathcal{B}(\mathbb{S})$

$$-\inf_{f \in A^o} I(f) \leqslant \liminf_{\epsilon \to 0} \epsilon \log \nu_\epsilon(A) \leqslant \limsup_{\epsilon \to 0} \epsilon \log \nu_\epsilon(A) \leqslant -\inf_{f \in \bar{A}} I(f),$$

where the closure and the interior are taken in $\mathbb{S}$, and $I(f)$ is defined by (6).

2.2. **Statement of the Main Result.** In order to formulate our result, we need to introduce some functional analytic framework.

To simplify the notations we simultaneously use $\mathbb{D}$ to denote the whole space $\mathbb{R}^3$ or the three dimensional torus $\mathbb{T}^3$. We assume that the reader can distinguish these two cases below.

Let $C_0^\infty(\mathbb{D};\mathbb{R}^3)$ denote the set of all smooth functions from $\mathbb{D}$ to $\mathbb{R}^3$ with compact supports. For $p \geqslant 1$, let $L^p(\mathbb{D};\mathbb{R}^3)$ be the vector valued L^p-space in which the norm is denoted by $\|\cdot\|_{L^p}$. For $\alpha \geqslant 0$, let H^α be the usual Sobolev space on $\mathbb{D}$ with values in $\mathbb{R}^3$, i.e., the closure of $C_0^\infty(\mathbb{D};\mathbb{R}^3)$ with respect to the norm:

$$\|\mathbf{u}\|_{H^\alpha} = \left(\int_{\mathbb{D}} |(I-\Delta)^{\alpha/2}\mathbf{u}|^2 \mathrm{d}x\right)^{1/2}.$$

Here as usual, $(I-\Delta)^{\alpha/2}$ is defined by Fourier transform. For two separable Hilbert spaces $\mathbb{K}$ and $\mathbb{H}$, $L_2(\mathbb{K};\mathbb{H})$ will denote the space of all Hilbert-Schmidt operators from $\mathbb{K}$ to $\mathbb{H}$ with norm $\|\cdot\|_{L_2(\mathbb{K};\mathbb{H})}$.

Set for $\alpha \geqslant 0$

$$\mathbb{H}^\alpha := \{\mathbf{u} \in H^\alpha : \mathrm{div}(\mathbf{u}) = 0\}, \tag{8}$$

where the divergence is taken in the sense of Schwartz distributions. Then $(\mathbb{H}^\alpha, \|\cdot\|_{H^\alpha})$ is a separable Hilbert space. We shall denote the norm $\|\cdot\|_{H^\alpha}$ in $\mathbb{H}^\alpha$ by $\|\cdot\|_{\mathbb{H}^\alpha}$. We remark that $\mathbb{H}^0$ is a closed linear subspace of the Hilbert space $L^2(\mathbb{R}^3;\mathbb{R}^3) = H^0$.

Let $\mathscr{P}$ be the orthogonal projection from $L^2(\mathbb{D};\mathbb{R}^3)$ to $\mathbb{H}^0$. Since we are considering the torus or the full space, it is well known that $\mathscr{P}$ commutes with the derivative operators, and that $\mathscr{P}$ can be restricted to a bounded linear operator from H^m to $\mathbb{H}^m$. For any $\mathbf{u} \in \mathbb{H}^2$, define

$$A(\mathbf{u}) := \mathscr{P}\Delta\mathbf{u} - \mathscr{P}((\mathbf{u}\cdot\nabla)\mathbf{u}) - \mathscr{P}(g_N(|\mathbf{u}|^2)\mathbf{u}) \tag{9}$$

and for $k \in \mathbb{N}$

$$B(t,\mathbf{u}) := (\mathscr{P}\mathbf{h}_k(t,\mathbf{u}))_{k\in\mathbb{N}}. \tag{10}$$

Letting the operator $\mathscr{P}$ act on both sides of equation (1), system (1) can be written as the following equivalent abstract stochastic evolution equation:

$$\begin{cases} \mathrm{d}\mathbf{u}(t) = \Big[A(\mathbf{u}(t)) + \mathscr{P}\mathbf{f}(t,\mathbf{u}(t))\Big]\mathrm{d}t + B(t,\mathbf{u}(t))\mathrm{d}W_t, \\ \mathbf{u}(0) = \mathbf{u}_0 \in \mathbb{H}^1. \end{cases} \tag{11}$$

Consider the following small perturbation of equation (11):

$$\begin{cases} \mathrm{d}\mathbf{u}_\epsilon(t) = \Big[A(\mathbf{u}_\epsilon(t)) + \mathscr{P}\mathbf{f}(t,\mathbf{u}_\epsilon(t))\Big]\mathrm{d}t + \sqrt{\epsilon}B(t,\mathbf{u}_\epsilon(t))\mathrm{d}W_t, \\ \mathbf{u}_\epsilon(0) = \mathbf{u}_0 \in \mathbb{H}^1, \end{cases} \tag{12}$$

and assume that

(H1) There exist a constant $C_{T,\mathbf{f}} > 0$ and a function $H_{\mathbf{f}}(t,x) \in L^1([0,T]\times\mathbb{D})$ such that for any $t \in [0,T], x \in \mathbb{D}, \mathbf{u} \in \mathbb{R}^3$ and $j = 1,2,3$

$$\begin{aligned} |\partial_{x^j}\mathbf{f}(t,x,\mathbf{u})|^2 + |\mathbf{f}(t,x,\mathbf{u})|^2 &\leqslant C_{T,\mathbf{f}}\cdot|\mathbf{u}|^2 + H_{\mathbf{f}}(t,x), \\ |\partial_{u^j}\mathbf{f}(t,x,\mathbf{u})| &\leqslant C_{T,\mathbf{f}}. \end{aligned}$$

(H2) There exist a constant $C_{T,\mathbf{h}} > 0$ and a function $H_{\mathbf{h}}(t,x) \in L^1([0,T]\times\mathbb{D})$ such that for any $t \in [0,T], x \in \mathbb{D}, \mathbf{u} \in \mathbb{R}^3$ and $i,j = 1,2,3$

$$\|\mathbf{h}(t,x,\mathbf{u})\|_{l^2}^2 + \|\partial_{x^j}\mathbf{h}(t,x,\mathbf{u})\|_{l^2}^2 + \|\partial_{x^j}^2\mathbf{h}(t,x,\mathbf{u})\|_{l^2}^2 \leqslant C_{T,\mathbf{h}}\cdot|\mathbf{u}|^2 + H_{\mathbf{h}}(t,x)$$

and

$$\|\partial_{u^j}\mathbf{h}(t,x,\mathbf{u})\|_{l^2} + \|\partial_{x^j}\partial_{u^i}\mathbf{h}(t,x,\mathbf{u})\|_{l^2}^2 + \|\partial_{u^ju^i}^2\mathbf{h}(t,x,\mathbf{u})\|_{l^2} \leqslant C_{T,\mathbf{h}}.$$

It was proved in [21, 20] that under **(H1)** and **(H2)**, there exists a unique functional

$$\Phi_\epsilon : \mathbb{C}_T(\mathbb{U}) \to \mathbb{C}_T(\mathbb{H}^1) \tag{13}$$

such that $\mathbf{u}_\epsilon(t,\omega) := \Phi_\epsilon(W(\cdot,\omega))(t)$ satisfies

$$\mathbf{u}_\epsilon(\cdot,\omega) \in L^2(0,T;\mathbb{H}^2) \quad \text{for } P\text{-almost all } \omega \in \Omega \tag{14}$$

and

$$\mathbf{u}_\epsilon(t) = \mathbf{u}_0 + \int_0^t \Big[A(\mathbf{u}_\epsilon(s)) + \mathscr{P}\mathbf{f}(s,\mathbf{u}_\epsilon(s))\Big]\mathrm{d}s + \sqrt{\epsilon}\int_0^t B(s,\mathbf{u}_\epsilon(s))\mathrm{d}W_s.$$

Our main result is the following:

Theorem 2.3. *Assume* **(H1)** *and* **(H2)** *hold. Then* $\{\mathbf{u}_\epsilon, \epsilon \in (0,1)\}$ *satisfies the large deviation principle in* $\mathbb{C}_T(\mathbb{H}^1) \cap L^2(0,T;\mathbb{H}^2)$ *with rate function* I *given by*

$$I(f) := \frac{1}{2}\inf_{\{h\in\mathscr{H}:\ f=\mathbf{u}_h\}} \|h\|^2_{\mathscr{H}}, \tag{15}$$

where $\mathbf{u}_h$ *solves the following equation:*

$$\mathbf{u}_h(t) = \mathbf{u}_0 + \int_0^t [A(\mathbf{u}_h) + \mathscr{P}(\mathbf{f}(s,\mathbf{u}_h))]\mathrm{d}s + \int_0^t B(s,\mathbf{u}_h(t))\dot{h}(s)\mathrm{d}s. \tag{16}$$

Remark 2.4. *The existence and uniqueness of strong solutions in the sense of PDE of equation (16) can be proved as done in* [22].

2.3. **Some Estimates.** For any $\mathbf{u},\mathbf{v} \in \mathbb{H}^2$, write

$$[\![A(\mathbf{u}),\mathbf{v}]\!] := \langle A(\mathbf{u}), (I-\Delta)\mathbf{v}\rangle_{\mathbb{H}^0}. \tag{17}$$

We first recall the following result (cf. [21, Lemma 2.3]). For the reader's convenience, a proof is provided here.

Lemma 2.5. *For any* $\mathbf{u} \in \mathbb{H}^2$, *we have*

$$\|A(\mathbf{u})\|_{\mathbb{H}^0} \leqslant C(1 + \|\mathbf{u}\|^6_{\mathbb{H}^0} + \|\mathbf{u}\|^2_{\mathbb{H}^2}) \tag{18}$$

and

$$[\![A(\mathbf{u}),\mathbf{u}]\!] \leqslant -\frac{1}{2}\|\mathbf{u}\|^2_{\mathbb{H}^2} - \frac{1}{2}\||\mathbf{u}|\cdot|\nabla\mathbf{u}|\|^2_{L^2} + N\|\nabla\mathbf{u}\|^2_{\mathbb{H}^0} + \|\mathbf{u}\|^2_{\mathbb{H}^0}. \tag{19}$$

Proof. First of all, by Gagliado-Nirenberge's inequality and Young's inequality, we have

$$\begin{aligned}
\|A(\mathbf{u})\|_{\mathbb{H}^0} &\leqslant \|\mathbf{u}\|_{\mathbb{H}^2} + C\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{\mathbb{H}^0} + C\|\mathbf{u}\|^3_{L^6}\\
&\leqslant \|\mathbf{u}\|_{\mathbb{H}^2} + C\|\mathbf{u}\|^{3/4}_{\mathbb{H}^0}\|\mathbf{u}\|^{5/4}_{\mathbb{H}^2} + C\|\mathbf{u}\|^{3/2}_{\mathbb{H}^0}\|\mathbf{u}\|^{3/2}_{\mathbb{H}^2}\\
&\leqslant C(1 + \|\mathbf{u}\|^2_{\mathbb{H}^2} + \|\mathbf{u}\|^6_{\mathbb{H}^0}).
\end{aligned}$$

which gives the first assertion.

For inequality (19), we have

$$\begin{aligned}
\langle \mathscr{P}\Delta\mathbf{u}, (I-\Delta)\mathbf{u}\rangle_{\mathbb{H}^0} &= -\|(I-\Delta)\mathbf{u}\|^2_{\mathbb{H}^0} + \langle \mathbf{u}, (I-\Delta)\mathbf{u}\rangle_{\mathbb{H}^0}\\
&= -\|\mathbf{u}\|^2_{\mathbb{H}^2} + \|\nabla\mathbf{u}\|^2_{\mathbb{H}^0} + \|\mathbf{u}\|^2_{\mathbb{H}^0},
\end{aligned}$$

and by Young's inequality

$$\begin{aligned}
\langle -\mathscr{P}((\mathbf{u}\cdot\nabla)\mathbf{u}), (I-\Delta)\mathbf{u}\rangle_{\mathbb{H}^0} &\leqslant \frac{1}{2}\|(I-\Delta)\mathbf{u}\|^2_{\mathbb{H}^0} + \frac{1}{2}\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|^2_{L^2}\\
&\leqslant \frac{1}{2}\|\mathbf{u}\|^2_{\mathbb{H}^2} + \frac{1}{2}\||\mathbf{u}|\cdot|\nabla\mathbf{u}|\|^2_{L^2},
\end{aligned}$$

where

$$|\mathbf{u}|^2 = \sum_{k=1}^{3} |u^k|^2, \quad |\nabla \mathbf{u}|^2 = \sum_{k,i=1}^{3} |\partial_i u^k|^2.$$

Noting that

$$g_N(|\mathbf{u}|^2) \geqslant |\mathbf{u}|^2 - N \quad \text{ and } \quad g_N'(|\mathbf{u}|^2) \geqslant 0, \tag{20}$$

we have

$$\begin{aligned}
& \langle -\mathscr{P}(g_N(|\mathbf{u}|^2)\mathbf{u}), (I-\Delta)\mathbf{u}\rangle_{\mathbb{H}^0} \\
= \;& -\langle \nabla(g_N(|\mathbf{u}|^2)\mathbf{u}), \nabla \mathbf{u}\rangle_{L^2} - \langle g_N(|\mathbf{u}|^2)\mathbf{u}, \mathbf{u}\rangle_{L^2} \\
= \;& -\sum_{k,i=1}^{3} \int_{\mathbb{R}^3} \partial_i u^k \cdot \partial_i (g_N(|\mathbf{u}|^2) u^k) \mathrm{d}x - \int_{\mathbb{R}^3} |\mathbf{u}|^2 \cdot g_N(|\mathbf{u}|^2) \mathrm{d}x \\
\leqslant \;& -\sum_{k,i=1}^{3} \int_{\mathbb{R}^3} \partial_i u^k \cdot \left(g_N(|\mathbf{u}|^2) \cdot \partial_i u^k - g_N'(|\mathbf{u}|^2) \partial_i |\mathbf{u}|^2 \cdot u^k \right) \mathrm{d}x \\
= \;& -\int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 \cdot g_N(|\mathbf{u}|^2) \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^3} g_N'(|\mathbf{u}|^2) |\nabla |\mathbf{u}|^2|^2 \mathrm{d}x \\
\leqslant \;& -\int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 \cdot |\mathbf{u}|^2 \mathrm{d}x + N \|\nabla \mathbf{u}\|_{\mathbb{H}^0}^2.
\end{aligned}$$

Combining the above calculations yields (19). □

We also need the following lemmas.

Lemma 2.6. *For any $\mathbf{u}, \mathbf{v} \in \mathbb{H}^2$ and $\alpha \in (3/2, 2)$, we have*

$$[\![A(\mathbf{u}) - A(\mathbf{v}), \mathbf{u} - \mathbf{v}]\!] \leqslant -\frac{1}{2} \|\mathbf{u} - \mathbf{v}\|_{\mathbb{H}^2}^2 + C \|\mathbf{u} - \mathbf{v}\|_{\mathbb{H}^\alpha}^2 \cdot (1 + \|\mathbf{u}\|_{\mathbb{H}^1}^4 + \|\mathbf{v}\|_{\mathbb{H}^1}^4). \tag{21}$$

Proof. By (17) and $ab \leqslant \frac{1}{4}a^2 + b^2$, we have

$$\begin{aligned}
[\![A(\mathbf{u}) - A(\mathbf{v}), \mathbf{u} - \mathbf{v}]\!] \; = \;& \langle \Delta(\mathbf{u} - \mathbf{v}), (I-\Delta)(\mathbf{u} - \mathbf{v})\rangle_{\mathbb{H}^0} \\
& -\langle (\mathbf{u} \cdot \nabla)\mathbf{u} - (\mathbf{v} \cdot \nabla)\mathbf{v}, (I-\Delta)(\mathbf{u} - \mathbf{v})\rangle_{L^2} \\
& -\langle g_N(|\mathbf{u}|^2)\mathbf{u} - g_N(|\mathbf{v}|^2)\mathbf{v}, (I-\Delta)(\mathbf{u} - \mathbf{v})\rangle_{L^2} \\
\leqslant \;& -\frac{1}{2} \|\mathbf{u} - \mathbf{v}\|_{\mathbb{H}^2}^2 + \|\mathbf{u} - \mathbf{v}\|_{\mathbb{H}^1}^2 \\
& + \|(\mathbf{u} \cdot \nabla)\mathbf{u} - (\mathbf{v} \cdot \nabla)\mathbf{v}\|_{L^2}^2 \\
& + \|g_N(|\mathbf{u}|^2)\mathbf{u} - g_N(|\mathbf{v}|^2)\mathbf{v}\|_{L^2}^2.
\end{aligned}$$

By Hölder's inequality and Sobolev's embedding theorem we have, for any $\alpha \in (3/2, 2)$

$$\begin{aligned}
\|(\mathbf{u} \cdot \nabla)\mathbf{u} - (\mathbf{v} \cdot \nabla)\mathbf{v}\|_{L^2}^2 \; \leqslant \;& 2\|\mathbf{u} - \mathbf{v}\|_{L^\infty}^2 \cdot \|\nabla \mathbf{u}\|_{L^2}^2 + 2\|\mathbf{v}\|_{L^6}^2 \cdot \|\nabla(\mathbf{u} - \mathbf{v})\|_{L^3}^2 \\
\leqslant \;& C\|\mathbf{u} - \mathbf{v}\|_{\mathbb{H}^\alpha}^2 \cdot \|\mathbf{u}\|_{\mathbb{H}^1}^2 + C\|\mathbf{v}\|_{\mathbb{H}^1}^2 \cdot \|\mathbf{u} - \mathbf{v}\|_{\mathbb{H}^\alpha}^2
\end{aligned}$$

and by $|g_N'(r)| \leqslant 2$ ($\nu = 1$)

$$\begin{aligned}
\|g_N(|\mathbf{u}|^2)\mathbf{u} - g_N(|\mathbf{v}|^2)\mathbf{v}\|_{L^2}^2 \; \leqslant \;& C\|\mathbf{u} - \mathbf{v}\|_{L^6}^2 \cdot \||\mathbf{u}|^2 + |\mathbf{v}|^2\|_{L^3}^2 \\
\leqslant \;& C\|\mathbf{u} - \mathbf{v}\|_{\mathbb{H}^1}^2 \cdot (\|\mathbf{u}\|_{\mathbb{H}^1}^4 + \|\mathbf{v}\|_{\mathbb{H}^1}^4).
\end{aligned}$$

Combining the above calculations, we obtain (21). □

Lemma 2.7. *There exists a constant $C>0$ such that for every $\mathbf{u}\in\mathbb{H}^2$ and $t\in[0,T]$*

$$\begin{aligned}\|\mathscr{P}(\mathbf{f}(t,\mathbf{u}))\|_{\mathbb{H}^0}^2 &\leqslant C\|\mathbf{u}\|_{\mathbb{H}^0}^2+\|H_{\mathbf{f}}(t)\|_{L^1(\mathbb{D})}, &(22)\\ \|B(t,\mathbf{u})\|_{L_2(l^2;\mathbb{H}^0)}^2 &\leqslant C\|\mathbf{u}\|_{\mathbb{H}^0}^2+\|H_{\mathbf{h}}(t)\|_{L^1(\mathbb{D})}, &(23)\\ \|B(t,\mathbf{u})\|_{L_2(l^2;\mathbb{H}^1)}^2 &\leqslant C(\|\mathbf{u}\|_{\mathbb{H}^1}^2+\|H_{\mathbf{h}}(t)\|_{L^1(\mathbb{D})}), &(24)\\ \|B(t,\mathbf{u})\|_{L_2(l^2;\mathbb{H}^2)}^2 &\leqslant C(\|\mathbf{u}\|_{\mathbb{H}^2}^2+\|\mathbf{u}\|_{\mathbb{H}^1}^4+\|H_{\mathbf{h}}(t)\|_{L^1(\mathbb{D})}). &(25)\end{aligned}$$

Proof. We only prove the last one. The others are easier. Since $\mathscr{P}$ is a bounded linear operator from H^m to $\mathbb{H}^m$, by **(H2)** we have

$$\begin{aligned}\|B(t,\mathbf{u})\|_{L_2(l^2;\mathbb{H}^2)}^2 &\leqslant C(\|\mathbf{h}(t,\mathbf{u})\|_{L_2(l^2;L^2)}^2+\|\nabla^2\mathbf{h}(t,\mathbf{u})\|_{L_2(l^2;L^2)}^2)\\ &\leqslant C(\|\mathbf{u}\|_{\mathbb{H}^0}^2+\|H_{\mathbf{h}}(t)\|_{L^1(\mathbb{D})}+\|\nabla^2\mathbf{h}(t,\mathbf{u})\|_{L_2(l^2;L^2)}^2),\end{aligned}$$

where $\nabla=(\partial_{x_1},\partial_{x_2},\partial_{x_3})$.

Noting that

$$\begin{aligned}\nabla^2\mathbf{h}(t,\mathbf{u}) &= (\nabla_x^2\mathbf{h})(t,\mathbf{u})+2(\nabla_x\partial_{u^i}\mathbf{h})(t,\mathbf{u})\nabla u^i\\ &\quad+(\partial_{u^i}\mathbf{h})(t,\mathbf{u})\nabla^2u^i+(\partial^2_{u^iu^j}\mathbf{h})(t,\mathbf{u})\nabla u^i\otimes\nabla u^j,\end{aligned}$$

by **(H2)** we have

$$\begin{aligned}\|\nabla^2\mathbf{h}(t,\mathbf{u})\|_{L_2(l^2;L^2)}^2 &\leqslant C(\|\mathbf{u}\|_{\mathbb{H}^0}^2+\|H_{\mathbf{h}}(t)\|_{L^1(\mathbb{D})}+\|\mathbf{u}\|_{\mathbb{H}^1}^2+\|\mathbf{u}\|_{\mathbb{H}^2}^2+\|\mathbf{u}\|_{\mathbb{H}^1}^4)\\ &\leqslant C(\|\mathbf{u}\|_{\mathbb{H}^2}^2+\|H_{\mathbf{h}}(t)\|_{L^1(\mathbb{D})}+\|\mathbf{u}\|_{\mathbb{H}^1}^4).\end{aligned}$$

(25) now follows by combining the above calculations. ☐

The following lemma is a direct consequence of **(H1)** and **(H2)**.

Lemma 2.8. *There exists a constant $C>0$ such that for any $\mathbf{u},\mathbf{v}\in\mathbb{H}^0$ and $t\in[0,T]$*

$$\begin{aligned}\|\mathscr{P}(\mathbf{f}(t,\mathbf{u})-\mathbf{f}(t,\mathbf{v}))\|_{\mathbb{H}^0} &\leqslant C\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}^0}, &(26)\\ \|B(t,\mathbf{u})-B(t,\mathbf{v})\|_{L_2(l^2;\mathbb{H}^0)} &\leqslant C\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}^0}. &(27)\end{aligned}$$

3. Proof of Theorem 2.3

For proving Theorem 2.3, the main task is to verify $\mathbf{(LD)_1}$ and $\mathbf{(LD)_2}$ for

$$\begin{aligned}\mathbb{S} &:= \mathbb{C}_T(\mathbb{H}^1)\cap L^2(0,T;\mathbb{H}^2),\\ Z_\epsilon &:= \Phi_\epsilon(W),\quad \epsilon\in(0,1),\\ Z_0(h) &:= \mathbf{u}_h,\quad h\in\mathscr{H}.\end{aligned}$$

Let $h_\epsilon\in\mathcal{A}_M^T$ converge as $\epsilon\to0$ in probability to $h\in\mathcal{A}_M^T$ as random variables in $\mathscr{H}$. Set

$$\mathbf{u}^\epsilon(t,\omega):=\Phi_\epsilon\Big(W(\cdot,\omega)+\frac{1}{\sqrt{\epsilon}}h_\epsilon(\cdot,\omega)\Big)(t), \tag{28}$$

where Φ_ϵ is given by (13). Thanks to $h_\epsilon\in\mathcal{A}_M^T$, i.e., $\int_0^T\|h_\epsilon(s,\omega)\|_{l^2}^2\mathrm{d}s\leqslant M$, by Girsanov's theorem, $\mathbf{u}^\epsilon$ solves the following control equation:

$$\begin{aligned}\mathbf{u}^\epsilon(t) &= \mathbf{u}_0+\int_0^t\Big[A(\mathbf{u}^\epsilon(s))+\mathscr{P}\mathbf{f}(s,\mathbf{u}^\epsilon(s))\Big]\mathrm{d}s\\ &\quad+\int_0^t B(s,\mathbf{u}^\epsilon(s))\dot{h}_\epsilon(s)\mathrm{d}s+\sqrt{\epsilon}\int_0^t B(s,\mathbf{u}^\epsilon(s))\mathrm{d}W_s. &(29)\end{aligned}$$

We first prove the following uniform estimate.

Lemma 3.1. *There exists a positive constant $C_{T,M,N,\mathbf{u}_0} > 0$ such that for any $\epsilon \in [0,1)$*

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|\mathbf{u}^\epsilon(t)\|^2_{\mathbb{H}^1}\right) + \int_0^T \mathbb{E}\|\mathbf{u}^\epsilon(s)\|^2_{\mathbb{H}^2}\mathrm{d}s \leqslant C_{T,M,N,\mathbf{u}_0}. \tag{30}$$

Here $\mathbf{u}^0(t) := \mathbf{u}_h(t)$, where $\mathbf{u}_h(t)$ solves equation (16).

Proof. We only prove (30) for $\mathbf{u}^\epsilon(t)$ with $\epsilon \in (0,1)$. The case of $\mathbf{u}^0(t) := \mathbf{u}_h(t)$ can be proved similarly. Below, the constant C will be independent of ϵ.

Consider the evolution triple

$$\mathbb{H}^2 \subset \mathbb{H}^1 \subset \mathbb{H}^0.$$

By Lemmas 2.5, 2.7 and (14) we know

$$\int_0^T \|A(\mathbf{u}^\epsilon(s))\|_{\mathbb{H}^0}\mathrm{d}s + \int_0^T \|B(s,\mathbf{u}^\epsilon(s))\|^2_{L_2(l^2;\mathbb{H}^1)}\mathrm{d}s < +\infty, \quad P-a.s..$$

Thus, by Itô's formula (cf. [17]) we have

$$\begin{aligned}
\|\mathbf{u}^\epsilon(t)\|^2_{\mathbb{H}^1} &= \|\mathbf{u}_0\|^2_{\mathbb{H}^1} + 2\int_0^t [\![A(\mathbf{u}^\epsilon(s)),\mathbf{u}^\epsilon(s)]\!]\mathrm{d}s + 2\int_0^t \langle \mathbf{f}(s,\mathbf{u}^\epsilon(s)),\mathbf{u}^\epsilon(s)\rangle_{\mathbb{H}^1}\mathrm{d}s \\
&\quad +2\int_0^t \langle B(s,\mathbf{u}^\epsilon(s))\dot{h}_\epsilon(s),\mathbf{u}^\epsilon(s)\rangle_{\mathbb{H}^1}\mathrm{d}s + \epsilon\int_0^t \|B(s,\mathbf{u}^\epsilon(s))\|^2_{L_2(l^2;\mathbb{H}^1)}\mathrm{d}s \\
&\quad +2\sqrt{\epsilon}\int_0^t \langle B(s,\mathbf{u}^\epsilon(s))\mathrm{d}W_s,\mathbf{u}^\epsilon(s)\rangle_{\mathbb{H}^1} \\
&=: \|\mathbf{u}_0\|^2_{\mathbb{H}^1} + I_1^\epsilon(t) + I_2^\epsilon(t) + I_3^\epsilon(t) + I_4^\epsilon(t) + I_5^\epsilon(t).
\end{aligned} \tag{31}$$

For I_1^ϵ, by (19) we have

$$\begin{aligned}
I_1^\epsilon(t) &\leqslant -\int_0^t \|\mathbf{u}^\epsilon(s)\|^2_{\mathbb{H}^2}\mathrm{d}s - \int_0^t \||\mathbf{u}^\epsilon(s)|\cdot|\nabla\mathbf{u}^\epsilon(s)|\|^2_{L^2}\mathrm{d}s \\
&\quad +N\int_0^t \|\nabla\mathbf{u}^\epsilon(s)\|^2_{\mathbb{H}^0}\mathrm{d}s + 2\int_0^t \|\mathbf{u}^\epsilon(s)\|^2_{\mathbb{H}^0}\mathrm{d}s.
\end{aligned}$$

For I_2^ϵ, by (22) and Young's inequality we have

$$\begin{aligned}
I_2^\epsilon(t) &\leqslant 2\int_0^t \|\mathbf{f}(s,\mathbf{u}^\epsilon(s))\|^2_{\mathbb{H}^0}\mathrm{d}s + \frac{1}{2}\int_0^t \|\mathbf{u}^\epsilon(s)\|^2_{\mathbb{H}^2}\mathrm{d}s \\
&\leqslant C\int_0^t \Big[\|\mathbf{u}^\epsilon(s)\|^2_{\mathbb{H}^0} + 1\Big]\mathrm{d}s + \frac{1}{2}\int_0^t \|\mathbf{u}^\epsilon(s)\|^2_{\mathbb{H}^2}\mathrm{d}s.
\end{aligned}$$

For I_3^ϵ, recalling $h_\epsilon \in \mathcal{D}_M$, by (24) and Young's inequality we have

$$\begin{aligned}
I_3^\epsilon(t) &\leqslant 2\int_0^t \Big[\|B(s,\mathbf{u}^\epsilon(s))\|_{L_2(l^2;\mathbb{H}^1)}\cdot\|\dot{h}_\epsilon(s)\|_{l^2}\cdot\|\mathbf{u}^\epsilon(s)\|_{\mathbb{H}^1}\Big]\mathrm{d}s \\
&\leqslant 2M\left(\int_0^t \Big[\|B(s,\mathbf{u}^\epsilon(s))\|^2_{L_2(l^2;\mathbb{H}^1)}\cdot\|\mathbf{u}^\epsilon(s)\|^2_{\mathbb{H}^1}\Big]\mathrm{d}s\right)^{1/2} \\
&\leqslant 2M\left(\sup_{s\in[0,t]}\|\mathbf{u}^\epsilon(s)\|_{\mathbb{H}^1}\right)\cdot\left(\int_0^t \|B(s,\mathbf{u}^\epsilon(s))\|^2_{L_2(l^2;\mathbb{H}^1)}\mathrm{d}s\right)^{1/2} \\
&\leqslant \frac{1}{4}\left(\sup_{s\in[0,t]}\|\mathbf{u}^\epsilon(s)\|^2_{\mathbb{H}^1}\right) + C_M\int_0^t \Big[\|\mathbf{u}^\epsilon(s)\|^2_{\mathbb{H}^1} + 1\Big]\mathrm{d}s.
\end{aligned}$$

Define for $R>0$

$$\tau_R^\epsilon := \inf\left\{t\in[0,T] : \int_0^t \|B(s,u^\epsilon(s))\|^2_{L_2(l^2;\mathbb{H}^1)}\mathrm{d}s \geqslant R\right\}.$$

For I_5^ϵ, by Burkholder's inequality and Young's inequality, we similarly have

$$\begin{aligned}
\mathbb{E}\left(\sup_{s\in[0,t\wedge\tau_R^\epsilon]}|I_5^\epsilon(s)|^2\right) &\leqslant C\mathbb{E}\left(\int_0^{t\wedge\tau_R^\epsilon}\Big[\|B(s,\mathbf{u}^\epsilon(s))\|^2_{L_2(l^2;\mathbb{H}^1)}\cdot\|\mathbf{u}^\epsilon(s)\|^2_{\mathbb{H}^1}\Big]\mathrm{d}s\right)^{1/2}\\
&\leqslant \frac{1}{4}\mathbb{E}\left(\sup_{s\in[0,t\wedge\tau_R^\epsilon]}\|\mathbf{u}^\epsilon(s)\|^2_{\mathbb{H}^1}\right)+C\mathbb{E}\int_0^{t\wedge\tau_R^\epsilon}\Big[\|\mathbf{u}^\epsilon(s)\|^2_{\mathbb{H}^1}+1\Big]\mathrm{d}s.
\end{aligned}$$

Set

$$g(t) := \mathbb{E}\left(\sup_{s\in[0,t\wedge\tau_R^\epsilon]}\|\mathbf{u}^\epsilon(s)\|^2_{\mathbb{H}^1}\right).$$

Combining the above calculations we get

$$\begin{aligned}
g(t)+\mathbb{E}\int_0^{t\wedge\tau_R^\epsilon}\|\mathbf{u}^\epsilon(s)\|^2_{\mathbb{H}^2}\mathrm{d}s &\leqslant C\|\mathbf{u}_0\|^2_{\mathbb{H}^1}+C_{M,N}\mathbb{E}\int_0^{t\wedge\tau_R^\epsilon}\Big[\|\mathbf{u}^\epsilon(s)\|^2_{\mathbb{H}^1}+1\Big]\mathrm{d}s\\
&\leqslant C\|\mathbf{u}_0\|^2_{\mathbb{H}^1}+C_{M,N}\int_0^t(g(s)+1)\mathrm{d}s,
\end{aligned}$$

which implies by Gronwall's inequality that

$$\mathbb{E}\left(\sup_{s\in[0,t\wedge\tau_R^\epsilon]}\|\mathbf{u}^\epsilon(s)\|^2_{\mathbb{H}^1}\right)+\mathbb{E}\int_0^{T\wedge\tau_R^\epsilon}\|\mathbf{u}^\epsilon(s)\|^2_{\mathbb{H}^2}\mathrm{d}s\leqslant C.$$

Since $\lim_{R\to\infty}\tau_R^\epsilon = T$, by Fatou's lemma, we obtain (30). □

Let $\mathbf{u}_h$ solve equation (16). Set

$$\mathbf{w}_\epsilon(t) := \int_0^t B(s,\mathbf{u}_h(s))(\dot{h}_\epsilon(s)-\dot{h}(s))\mathrm{d}s. \tag{32}$$

Lemma 3.2. $\mathbf{w}_\epsilon$ *converges in probability to zero in* $\mathbb{C}_T(\mathbb{H}^2)$ *as* $\epsilon\to 0$.

Proof. Let $\{\mathbf{e}_k, k\in\mathbb{N}\}$ be an orthonormal basis of $\mathbb{H}^2$ and Π_n the projection from $\mathbb{H}^2$ to $\mathbb{H}^2_n$:=span$\{\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n\}$. Define

$$\mathbf{w}^n_\epsilon(t) := \int_0^t \Pi_n B(s,\mathbf{u}_h(s))(\dot{h}_\epsilon(s)-\dot{h}(s))\mathrm{d}s.$$

Then by Hölder's inequality and since $h_\epsilon, h\in\mathcal{A}_M^T$, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[0,T]}\|\mathbf{w}^n_\epsilon(t)-\mathbf{w}_\epsilon(t)\|^2_{\mathbb{H}^2}\right)\\
&\leqslant \mathbb{E}\left(\int_0^T\|(I-\Pi_n)B(s,\mathbf{u}_h(s))(\dot{h}_\epsilon(s)-\dot{h}(s))\|_{\mathbb{H}^2}\mathrm{d}s\right)^2\\
&\leqslant \mathbb{E}\left(\int_0^T\|(I-\Pi_n)B(s,\mathbf{u}_h(s))\|_{L_2(l^2;\mathbb{H}^2)}\cdot\|\dot{h}_\epsilon(s)-\dot{h}(s)\|_{l^2}\mathrm{d}s\right)^2\\
&\leqslant 2M^2\cdot\mathbb{E}\left(\int_0^T\|(I-\Pi_n)B(s,\mathbf{u}_h(s))\|^2_{L_2(l^2;\mathbb{H}^2)}\mathrm{d}s\right).
\end{aligned}$$

By the dominated convergence theorem, (25) and (30)(with $\epsilon = 0$), we get

$$\lim_{n\to\infty} \sup_{\epsilon\in(0,1)} \mathbb{E}\left(\sup_{t\in[0,T]} \|\mathbf{w}_\epsilon^n(t) - \mathbf{w}_\epsilon(t)\|_{\mathbb{H}^2}^2\right) = 0. \tag{33}$$

Fix $n \in \mathbb{N}$. We have that

$$\mathscr{H} \ni \tilde{h} \mapsto \int_0^t \Pi_n B(s, \mathbf{u}_h(s))\tilde{h}(s)\mathrm{d}s$$

by (25) and (30)(with $\epsilon = 0$) is $P-$a.s. a continuous linear map on $\mathscr{H}$ for all $t \in [0, T]$ with values in $\mathbb{H}_n^2$. Hence, if $(\tilde{h}_k)$ is a sequence of random variables with values in $\mathscr{H}$, P-a.s. weakly converging to $0 \in \mathscr{H}$, then

$$\int_0^t \Pi_n B(s, \mathbf{u}_h(s))\tilde{h}_k(s)\mathrm{d}s \overset{k\to\infty}{\longrightarrow} 0, \quad P - a.s.. \tag{34}$$

Clearly, by (25) and (30)(with $\epsilon = 0$)

$$[0, T] \ni t \mapsto \int_0^t \Pi_n B(s, \mathbf{u}_h(s))\tilde{h}_k(s)\mathrm{d}s, \quad k \in \mathbb{N}$$

are equi-continuous, $P-$a.s.. Hence the convergence in (34) is uniform in $t \in [0, T]$, i.e., takes place in $\mathbb{C}_T(\mathbb{H}^2)$. Since a sequence of random variables in a metric space converges in probability if and only if any of its subsequences has a subsequence which $P-$a.s. converges to the same limit point, from the above we conclude that, since h_ϵ converges in probability to h as random variables in $\mathscr{H}$, $\mathbf{w}_\epsilon^n$ converges in probability to zero in $\mathbb{C}_T(\mathbb{H}^2)$ as $\epsilon \to 0$.

Note that for any $\eta > 0$

$$\begin{aligned} P\left(\sup_{t\in[0,T]} \|\mathbf{w}_\epsilon(t)\|_{\mathbb{H}^2} \geqslant 2\eta\right) &\leqslant P\left(\sup_{t\in[0,T]} \|\mathbf{w}_\epsilon^n(t) - \mathbf{w}_\epsilon(t)\|_{\mathbb{H}^2} \geqslant \eta\right) \\ &\quad + P\left(\sup_{t\in[0,T]} \|\mathbf{w}_\epsilon^n(t)\|_{\mathbb{H}^2} \geqslant \eta\right). \end{aligned}$$

The result now follows by combining this with (33). ∎

We now prove:

Lemma 3.3. $\mathbf{u}^\epsilon$ *defined in (28) converges in probability to* $\mathbf{u}^0 := \mathbf{u}_h$*, defined by equation (16), in* $\mathbb{S} = \mathbb{C}_T(\mathbb{H}^1) \cap L^2(0, T; \mathbb{H}^2)$ *as* $\epsilon \to 0$.

Proof. Set $\mathbf{v}_\epsilon := \mathbf{u}^\epsilon - \mathbf{u}_h$. Then

$$\begin{aligned} \mathbf{v}_\epsilon(t) &= \int_0^t (A(\mathbf{u}^\epsilon(s)) - A(\mathbf{u}_h(s)))\mathrm{d}s + \int_0^t \mathscr{P}(\mathbf{f}(s, \mathbf{u}^\epsilon(s)) - \mathbf{f}(s, \mathbf{u}_h(s)))\mathrm{d}s \\ &\quad + \int_0^t (B(s, \mathbf{u}^\epsilon(s))\dot{h}_\epsilon(s) - B(s, \mathbf{u}_h(s))\dot{h}(s))\mathrm{d}s + \sqrt{\epsilon}\int_0^t B(s, \mathbf{u}^\epsilon(s))\mathrm{d}W_s. \end{aligned} \tag{35}$$

As in Lemma 3.1, by Itô's formula we have

$$\begin{aligned} \|\mathbf{v}_\epsilon(t)\|_{\mathbb{H}^1}^2 &= 2\int_0^t [\![A(\mathbf{u}^\epsilon(s)) - A(\mathbf{u}_h(s)), \mathbf{v}_\epsilon(s)]\!]\mathrm{d}s \\ &\quad + 2\int_0^t \langle \mathscr{P}(\mathbf{f}(s, \mathbf{u}^\epsilon(s)) - \mathbf{f}(s, \mathbf{u}_h(s))), \mathbf{v}_\epsilon(s)\rangle_{\mathbb{H}^1}\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+2\int_0^t \langle (B(s,\mathbf{u}^\epsilon(s))-B(s,\mathbf{u}_h(s)))\dot{h}_\epsilon(s),\mathbf{v}_\epsilon(s)\rangle_{\mathbb{H}^1}\mathrm{d}s\\
&+2\int_0^t \langle B(s,\mathbf{u}_h(s))(\dot{h}_\epsilon(s)-\dot{h}(s)),\mathbf{v}_\epsilon(s)\rangle_{\mathbb{H}^1}\mathrm{d}s\\
&+2\sqrt{\epsilon}\int_0^t \langle B(s,\mathbf{u}^\epsilon(s))\mathrm{d}W_s,\mathbf{v}_\epsilon(s)\rangle_{\mathbb{H}^1}\\
&+\epsilon\int_0^t \|B(s,\mathbf{u}^\epsilon(s))\|^2_{L_2(l^2;\mathbb{H}^1)}\mathrm{d}s\\
=:\ & I_1^\epsilon(t)+I_2^\epsilon(t)+I_3^\epsilon(t)+I_4^\epsilon(t)+I_5^\epsilon(t)+I_6^\epsilon(t).
\end{aligned}
$$

Set for $R>0$ and $\epsilon\in[0,1)$

$$\theta_R^\epsilon := \inf\left\{t\geqslant 0 : \|\mathbf{u}^\epsilon(t)\|^2_{\mathbb{H}^1}+\int_0^t\|\mathbf{u}^\epsilon(s)\|^2_{\mathbb{H}^2}\mathrm{d}s>R\right\}$$

and

$$\tau_R^\epsilon := \theta_R^\epsilon\wedge\theta_R^0.$$

Then, by (30) we have

$$
\begin{aligned}
\sup_\epsilon P(\theta_R^\epsilon<T) &= \sup_\epsilon P\left(\sup_{t\in[0,T]}\left[\|\mathbf{u}^\epsilon(t)\|^2_{\mathbb{H}^1}+\int_0^t\|\mathbf{u}^\epsilon(s)\|^2_{\mathbb{H}^2}\mathrm{d}s\right]>R\right)\\
&\leqslant \sup_\epsilon \mathbb{E}\left(\sup_{t\in[0,T]}\|\mathbf{u}^\epsilon(t)\|^2_{\mathbb{H}^1}+\int_0^T\|\mathbf{u}^\epsilon(s)\|^2_{\mathbb{H}^2}\mathrm{d}s\right)/R\\
&\leqslant \frac{C_{T,M,N,\mathbf{u}_0}}{R}
\end{aligned}
$$

and

$$\sup_\epsilon P(\tau_R^\epsilon<T)\leqslant\frac{C_{T,M,N,\mathbf{u}_0}}{R^2}. \tag{36}$$

For I_1^ϵ, by (21) we have

$$
\begin{aligned}
I_1^\epsilon(t\wedge\tau_R^\epsilon) &\leqslant -\int_0^{t\wedge\tau_R^\epsilon}\|\mathbf{v}_\epsilon(s)\|^2_{\mathbb{H}^2}\mathrm{d}s+C_R\int_0^{t\wedge\tau_R^\epsilon}\|\mathbf{v}_\epsilon(s)\|^2_{\mathbb{H}^\alpha}\mathrm{d}s\\
&\leqslant -\frac{3}{4}\int_0^{t\wedge\tau_R^\epsilon}\|\mathbf{v}_\epsilon(s)\|^2_{\mathbb{H}^2}\mathrm{d}s+C_R\int_0^{t\wedge\tau_R^\epsilon}\|\mathbf{v}_\epsilon(s)\|^2_{\mathbb{H}^0}\mathrm{d}s,
\end{aligned}
$$

where we have used Young's inequality and the following interpolation inequality

$$\|\mathbf{v}\|^2_{\mathbb{H}^\alpha}\leqslant C_\alpha\|\mathbf{v}\|^\alpha_{\mathbb{H}^2}\cdot\|\mathbf{v}\|^{2-\alpha}_{\mathbb{H}^0},\quad \alpha\in(0,2).$$

For I_2^ϵ, by Young's inequality and (26) we have

$$I_2^\epsilon(t\wedge\tau_R^\epsilon)\leqslant\frac{1}{4}\int_0^{t\wedge\tau_R^\epsilon}\|\mathbf{v}_\epsilon(s)\|^2_{\mathbb{H}^2}\mathrm{d}s+C\int_0^{t\wedge\tau_R^\epsilon}\|\mathbf{v}_\epsilon(s)\|^2_{\mathbb{H}^0}\mathrm{d}s.$$

For I_3^ϵ, by (27) we have

$$
\begin{aligned}
I_3^\epsilon(t\wedge\tau_R^\epsilon) &\leqslant C\int_0^{t\wedge\tau_R^\epsilon}\|\mathbf{v}_\epsilon(s)\|_{\mathbb{H}^0}\cdot\|\dot{h}_\epsilon(s)\|_{l^2}\cdot\|\mathbf{v}_\epsilon(s)\|_{\mathbb{H}^2}\mathrm{d}s\\
&\leqslant \frac{1}{4}\int_0^{t\wedge\tau_R^\epsilon}\|\mathbf{v}_\epsilon(s)\|^2_{\mathbb{H}^2}\mathrm{d}s+C\int_0^{t\wedge\tau_R^\epsilon}\|\dot{h}_\epsilon(s)\|^2_{l^2}\|\mathbf{v}_\epsilon(s)\|^2_{\mathbb{H}^0}\mathrm{d}s.
\end{aligned}
$$

Hence

$$\begin{aligned} I_1^\epsilon(t\wedge\tau_R^\epsilon)+I_2^\epsilon(t\wedge\tau_R^\epsilon)+I_3^\epsilon(t\wedge\tau_R^\epsilon)+\frac{1}{4}\int_0^{t\wedge\tau_R^\epsilon}\|\mathbf{v}_\epsilon(s)\|_{\mathbb{H}^2}^2\mathrm{d}s \\ \leqslant C_R\int_0^{t\wedge\tau_R^\epsilon}(1+\|\dot{h}_\epsilon(s)\|_{l^2}^2)\cdot\|\mathbf{v}_\epsilon(s)\|_{\mathbb{H}^0}^2\mathrm{d}s. \end{aligned} \tag{37}$$

For I_5^ϵ, by Doob's maximal inequality and (24) we have

$$\begin{aligned} \mathbb{E}\left(\sup_{t\in[0,T]}|I_5^\epsilon(t\wedge\tau_R^\epsilon)|^2\right) &\leqslant \epsilon\mathbb{E}\left(\int_0^{T\wedge\tau_R^\epsilon}\|B(s,\mathbf{u}^\epsilon(s))\|_{L_2(l^2;\mathbb{H}^1)}^2\cdot\|\mathbf{v}_\epsilon(s)\|_{\mathbb{H}^1}^2\mathrm{d}s\right) \\ &\leqslant \epsilon\cdot C_{T,R}. \end{aligned} \tag{38}$$

Likewise, for I_6^ϵ we have

$$\mathbb{E}\left(\sup_{t\in[0,T]}|I_6^\epsilon(t\wedge\tau_R^\epsilon)|\right)\leqslant\epsilon\cdot C_{T,R}. \tag{39}$$

We now deal with the hard term I_4^ϵ. By (32), (35) and Itô's formula we have

$$\begin{aligned} \frac{1}{2}I_4^\epsilon(t) &= \langle\mathbf{v}_\epsilon(t),\mathbf{w}_\epsilon(t)\rangle_{\mathbb{H}^1}-\int_0^t {}_{\mathbb{H}^2}\langle\mathbf{w}_\epsilon(s),A(\mathbf{u}^\epsilon(s))-A(\mathbf{u}_h(s))\rangle_{\mathbb{H}^0}\mathrm{d}s \\ &\quad -\int_0^t\langle\mathbf{w}_\epsilon(s),\mathbf{f}(s,\mathbf{u}^\epsilon(s))-\mathbf{f}(s,\mathbf{u}_h(s))\rangle_{\mathbb{H}^1}\mathrm{d}s \\ &\quad -\int_0^t\langle\mathbf{w}_\epsilon(s),B(s,\mathbf{u}^\epsilon(s))\dot{h}_\epsilon(s)-B(s,\mathbf{u}_h(s))\dot{h}(s)\rangle_{\mathbb{H}^1}\mathrm{d}s \\ &\quad -\sqrt{\epsilon}\int_0^t\langle\mathbf{w}_\epsilon(s),B(s,\mathbf{u}^\epsilon(s))\mathrm{d}W_s\rangle_{\mathbb{H}^1} \\ &=: I_{41}^\epsilon(t)+I_{42}^\epsilon(t)+I_{43}^\epsilon(t)+I_{44}^\epsilon(t)+I_{45}^\epsilon(t). \end{aligned}$$

For I_{41}^ϵ, we have

$$I_{41}^\epsilon(t\wedge\tau_R^\epsilon)\leqslant\frac{1}{4}\|\mathbf{v}_\epsilon(t\wedge\tau_R^\epsilon)\|_{\mathbb{H}^1}^2+\|\mathbf{w}_\epsilon(t\wedge\tau_R^\epsilon)\|_{\mathbb{H}^1}^2. \tag{40}$$

For I_{42}^ϵ, by (18) we have

$$\begin{aligned} I_{42}^\epsilon(t\wedge\tau_R^\epsilon) &\leqslant \int_0^{t\wedge\tau_R^\epsilon}\|\mathbf{w}_\epsilon(s)\|_{\mathbb{H}^2}\cdot\|A(\mathbf{u}^\epsilon(s))-A(\mathbf{u}_h(s))\|_{\mathbb{H}^0}\mathrm{d}s \\ &\leqslant C\int_0^{t\wedge\tau_R^\epsilon}\|\mathbf{w}_\epsilon(s)\|_{\mathbb{H}^2}\cdot(\|\mathbf{u}^\epsilon(s)\|_{\mathbb{H}^2}^2+\|\mathbf{u}_h(s)\|_{\mathbb{H}^2}^2+C_R)\mathrm{d}s \\ &\leqslant C_{T,R}\sup_{s\in[0,T]}\|\mathbf{w}_\epsilon(s)\|_{\mathbb{H}^2}. \end{aligned} \tag{41}$$

Similarly, we have

$$I_{43}^\epsilon(t\wedge\tau_R^\epsilon)\leqslant C_{T,R}\sup_{s\in[0,T]}\|\mathbf{w}_\epsilon(s)\|_{\mathbb{H}^2}. \tag{42}$$

For I_{44}^ϵ, by (23) and Hölder's inequality we have

$$\begin{aligned} I_{44}^\epsilon(t\wedge\tau_R^\epsilon) \leqslant \sup_{s\in[0,T]}\|\mathbf{w}_\epsilon(s)\|_{\mathbb{H}^2}\int_0^{t\wedge\tau_R^\epsilon}\Big[&\|B(s,\mathbf{u}^\epsilon(s))\|_{L_2(l^2;\mathbb{H}^0)}\|\dot{h}_\epsilon(s)\|_{l^2} \\ &+\|B(s,\mathbf{u}_h(s))\|_{L_2(l^2;\mathbb{H}^0)}\|\dot{h}(s)\|_{l^2}\Big]\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&\leqslant C_T \cdot M \sup_{s\in[0,T]} \|\mathbf{w}_\epsilon(s)\|_{\mathbb{H}^2} \left(\int_0^{t\wedge\tau_R^\epsilon} \left[\|\mathbf{u}^\epsilon(s)\|_{\mathbb{H}^0}^2 + \|\mathbf{u}_h(s)\|_{\mathbb{H}^0}^2 + 1 \right] \mathrm{d}s \right)^{1/2} \\
&\leqslant C_{T,M,R} \cdot \sup_{s\in[0,T]} \|\mathbf{w}_\epsilon(s)\|_{\mathbb{H}^2}. \qquad (43)
\end{aligned}
$$

Similar to (38), we have

$$
\mathbb{E}\left(\sup_{t\in[0,T]} |I_{45}^\epsilon(t\wedge\tau_R^\epsilon)|^2 \right) \leqslant \epsilon \cdot C_{T,R}. \qquad (44)
$$

Combining (37)-(43), we obtain

$$
\begin{aligned}
&\frac{1}{2}\|\mathbf{v}_\epsilon(t\wedge\tau_R^\epsilon)\|_{\mathbb{H}^1}^2 + \frac{1}{4}\int_0^{t\wedge\tau_R^\epsilon} \|\mathbf{v}_\epsilon(s)\|_{\mathbb{H}^2}^2 \mathrm{d}s \\
&\quad \leqslant C_R \int_0^{t\wedge\tau_R^\epsilon} (1 + \|\dot{h}_\epsilon(s)\|_{l^2}^2) \cdot \|\mathbf{v}_\epsilon(s)\|_{\mathbb{H}^1}^2 \mathrm{d}s \\
&\qquad + C_{T,M,R} \cdot \sup_{s\in[0,T]} \|\mathbf{w}_\epsilon(s)\|_{\mathbb{H}^2} + \xi_\epsilon,
\end{aligned}
$$

where

$$
\xi_\epsilon := \sup_{t\in[0,T]} |I_{45}^\epsilon(t\wedge\tau_R^\epsilon)| + \sup_{t\in[0,T]} |I_5^\epsilon(t\wedge\tau_R^\epsilon)| + \sup_{t\in[0,T]} |I_6^\epsilon(t\wedge\tau_R^\epsilon)|.
$$

Set

$$
g_\epsilon(t) := \sup_{s\in[0,t]} \|\mathbf{v}_\epsilon(s)\|_{\mathbb{H}^1}^2 + \int_0^t \|\mathbf{v}_\epsilon(s)\|_{\mathbb{H}^2}^2 \mathrm{d}s.
$$

Then

$$
\begin{aligned}
g_\epsilon(t\wedge\tau_R^\epsilon) &\leqslant C_{T,M,R} \cdot \left(\sup_{s\in[0,T]} \|\mathbf{w}_\epsilon(s)\|_{\mathbb{H}^2} + \xi_\epsilon \right) \\
&\quad + C_R \int_0^t (1 + \|\dot{h}_\epsilon(s)\|_{l^2}^2) \cdot g_\epsilon(s\wedge\tau_R^\epsilon) \mathrm{d}s.
\end{aligned}
$$

By Gronwall's inequality we obtain

$$
\begin{aligned}
g_\epsilon(T\wedge\tau_R^\epsilon) &\leqslant C_{T,M,R} \cdot \left(\sup_{s\in[0,T]} \|\mathbf{w}_\epsilon(s)\|_{\mathbb{H}^2} + \xi_\epsilon \right) \cdot \exp\left\{ C_R \int_0^T (1 + \|\dot{h}_\epsilon(s)\|_{l^2}^2) \mathrm{d}s \right\} \\
&\leqslant C_{T,M,R} \cdot \left(\sup_{s\in[0,T]} \|\mathbf{w}_\epsilon(s)\|_{\mathbb{H}^2} + \xi_\epsilon \right) \cdot \exp(C_R(T + M^2)).
\end{aligned}
$$

Therefore, by Lemma 3.2 and (38), (39) and (44), we have

$$
g_\epsilon(T\wedge\tau_R^\epsilon) \to 0 \ \text{ in probability as } \epsilon \to 0. \qquad (45)
$$

Lastly, note that for any $\delta > 0$.

$$
P(g_\epsilon(T) \geqslant \delta) \leqslant P(g_\epsilon(T\wedge\tau_R^\epsilon) \geqslant \delta) + P(\tau_R^\epsilon < T).
$$

The desired convergence now follows from (45) and (36). □

Proof of Theorem 2.3:

Let h_ϵ be a sequence in $\mathcal{A}_N^T$ which converges in distribution to h. Since $\mathcal{D}_N$ is compact and the law of W is tight, $\{h_\epsilon, W\}$ is tight in $\mathcal{D}_N \times \mathbb{C}_T(\mathbb{U})$ by the definition of tightness. Hence for some sequence $\epsilon_k \downarrow 0$, $\{h_{\epsilon_k}, W\}$ weakly converges to some probability measure ν on $\mathcal{D}_N \times \mathbb{C}_T(\mathbb{U})$. Note that the law of h is just $\nu(\cdot, \mathbb{C}_T(\mathbb{U}))$. By Skorohod's representation theorem, there are random variables $\tilde{h}_{\epsilon_k}, \tilde{W}^{\epsilon_k}, k \in \mathbb{N}$ and $\tilde{h}, \tilde{W}$ on a probability space $(\tilde{\Omega}, \tilde{P})$ such that

(1) $(\tilde{h}_{\epsilon_k}, \tilde{W}^{\epsilon_k})$ a.s. converges to $(\tilde{h}, \tilde{W})$ in $\mathcal{D}_N \times \mathbb{C}_T(\mathbb{U})$;

(2) $(\tilde{h}_{\epsilon_k}, \tilde{W}^{\epsilon_k})$ has the same law as (h_{ϵ_k}, W) for each $k \in \mathbb{N}$;

(3) The law of $\{\tilde{h}, \tilde{W}\}$ is ν, and the law of h is the same as $\tilde{h}$.

We remark that in the proof of Lemma 3.3, one can replace (h_{ϵ_k}, W) by $(\tilde{h}_{\epsilon_k}, \tilde{W}_{\epsilon_k})$. Thus, using Lemma 3.3, we get

$$\Phi_{\epsilon_k}\left(\tilde{W}_{\epsilon_k} + \frac{\tilde{h}_{\epsilon_k}(\cdot)}{\sqrt{\epsilon_k}}\right) \to \mathbf{u}^{\tilde{h}}, \quad \text{in probability,}$$

where Φ_{ϵ_k} is the strong solution functional given by (13). From this, we get

$$\Phi_{\epsilon_k}\left(W + \frac{h_{\epsilon_k}(\cdot)}{\sqrt{\epsilon_k}}\right) \to \mathbf{u}^{h}, \quad \text{in distribution,}$$

hence $\mathbf{(LD)_1}$ holds.

$\mathbf{(LD)_2}$ can be proved as in Lemma 3.3. The theorem now follows from Theorem 2.2.

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