Finite time extinction for solutions to fast diffusion stochastic porous media equations

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Abstract

We prove that the solutions to fast diffusion stochastic porous media equations have finite time extinction with strictly positive probability. To cite this article: V. Barbu, G. Da Prato, M. Röckner, C. Acad. Sci. Paris,....

Résumé

Nous prouvons l'extinction avec une probabilité strictement positive pour les solutions des équations des milieux poreux avec diffusion rapide. Pour citer cet article: V. Barbu, G. Da Prato, M. Röckner, C. Acad. Sci. Paris,....

1 Introduction

Consider the stochastic porous media equation

$$\begin{cases} dX(t) - \rho\Delta(|X|^\alpha(t) \text{ sign } X(t))dt - \Delta(\tilde{\Psi}(X(t))dt = \sigma(X(t))dW(t), \text{ in } (0,\infty)\times\mathcal{O}, \\ X = 0 \quad \text{on } (0,\infty)\times\partial\mathcal{O}, \quad X(0,x) = x \quad \text{on } \mathcal{O}, \end{cases} \tag{1}$$

where $\rho > 0$, $\alpha \in (0,1)$, $\tilde{\Psi}$ is a continuous monotonically non decreasing function of linear growth and $\sigma(X)dW = \sum_{k=1}^\infty \mu_k X e_k d\beta_k, \quad t \ge 0$, where $\{\beta_k\}$ is a sequence of independent real Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ and $\{e_k\}$ is an orthormal basis in $L^2(\mathcal{O})$ which for convenience will be taken as the eigenfunction system for the Laplace operator with Dirichlet boundary conditions, i.e., $-\Delta e_k = \lambda_k e_k$ in $\mathcal{O}$, $e_k = 0$ on $\partial\mathcal{O}$, where $\mathcal{O}$ is an open and bounded subset of $\mathbb{R}^d$, with smooth boundary $\partial\mathcal{O}$. We shall assume that $\sum_{k=1}^\infty \mu_k^2\lambda_k^2 < \infty$. Equation (1) for $0 < \alpha < 1$ is relevant in the mathematical modelling of the dynamics of an ideal gas in a porous medium and, in particular, in a plasma fast diffusion model (for $\alpha = 1/2$) (see e.g. [4]). The existence and uniqueness of a strong solution in the sense to be defined below was studied in [1],[2],[3],[5] for more general nonlinear stochastic equations of the form (1). In [3] (see also [1]) it was also proven that for $\alpha = 0$ and $d = 1$ the solution $X = X(t,x)$ to (1) has the finite extinction property: $\mathbb{P}(\tau \le n) \ge 1 - \frac{|x|_{-1}}{\rho\gamma} \left(\int_0^n e^{-C_N s} ds\right)^{-1}$ for $|x|_{-1} < C_N^{-1}\rho\gamma$ where $\tau = \inf\{t \ge 0 : |X(t,x)|_{-1} = 0\} = \sup\{t \ge 0 : |X(t,x)|_{-1} > 0\}$ and C_N, γ are constants related to the Wiener process W and respectively to the domain $\mathcal{O} \subset \mathbb{R}^1$.

The following notations will be used in the sequel. $H = L^2(\mathcal{O})$, $p \ge 1$, with the norm denoted by $|\cdot|_2$ and scalar product $\langle\cdot,\cdot\rangle$. $H^{-1}(\mathcal{O})$ is the dual of the Sobolev space $H_0^1(\mathcal{O})$ and is endowed with the scalar product $\langle u, v\rangle_{-1} = \langle u, (-\Delta)^{-1}v\rangle$, where Δ is the Laplace operator with domain $H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$. All processes $X = X(t)$ arising here are adapted with respect to the filtration $\{\mathcal{F}_t\}$. For a Banach space E, $L^p_W(0,T;E)$ denotes the space of all adapted processes in $L^p(0,T;E)$. We shall use standard notation for Sobolev spaces and spaces of integrable functions on $\mathcal{O}$.

2 The main result

Definition 2.1 *Let $x \in H$. An H-valued continuous $(\mathcal{F}_t)$-adapted process $X = X(t,x)$ is called a solution to (1) on $[0,T]$ if $X \in L^p(\Omega\times(0,T)\times\mathcal{O})\cap L^2(0,T;L^2(\Omega,H))$, $p \geq 2$, such that $\mathbb{P}$-a.s. $\forall\, j \in \mathbb{N}$, $t \in [0,T]$,*

$$\begin{array}{rcl}\langle X(t,x),e_j\rangle &=& \langle x,e_j\rangle + \int_0^t \int_{\mathcal{O}} (\rho|X(s,x)(\xi)|^\alpha \operatorname{sign} X(s,x)(\xi) + \tilde{\Psi}(X(s,x)(\xi)))\Delta e_j(\xi)d\xi ds \\ && + \sum_{k=1}^\infty \mu_k \int_0^t \langle X(s,x)e_k, e_j\rangle d\beta_k(s),\end{array} \tag{2}$$

For $x \in L^p(\mathcal{O})$, $p \geq 4$ and $d = 1,2,3$ there is a unique solution $X \in L^\infty_W(0,T;L^p(\Omega,H))$ to (1) in the sense of Definition 2.1. Moreover, if $x \geq 0$ a.e. in $\mathcal{O}$ then $X \geq 0$ a.e. in $\Omega \times [0,T] \times \mathcal{O}$).

By the proof of [3, Theorem 2.2] and [3, Proposition 3.4] we also know that for $\lambda \to 0$,

$$\left\{\begin{array}{l} X_\lambda \to X \quad \text{strongly both in } L^2(0,T;L^2(\Omega,L^2(\mathcal{O}))) \text{ and in } L^2(\Omega;C([0,T];H)), \\ \text{weakly in } L^p(\Omega\times(0,T)\times\mathcal{O}), \text{and weak}^* \text{ in } L^\infty(0,T;L^p(\Omega;L^p(\mathcal{O}))), \end{array}\right. \tag{3}$$

where X_λ, $\lambda > 0$, is the solution to approximating equation

$$\left\{\begin{array}{l} dX_\lambda(t) - \Delta(\Psi_\lambda(X_\lambda(t)) + \lambda X_\lambda(t) + \tilde{\Psi}(X_\lambda(t)))dt = \sigma(X_\lambda(t))dW(t), \\ \Psi_\lambda(X_\lambda) + \lambda X_\lambda + \tilde{\Psi}(X_\lambda) = 0 \quad \text{on } \partial\mathcal{O}, \quad X_\lambda(0,x) = x, \\ \Psi_\lambda(x) = \frac{1}{\lambda}\,(x - (1+\lambda\Psi_0)^{-1}(x)) = \Psi_0((1+\lambda\Psi_0)^{-1}(x)), \quad \Psi_0(x) = \rho|x|^\alpha \operatorname{sign} x. \end{array}\right. \tag{4}$$

Everywhere in the sequel $X = X(t,x)$ is the solution to (1) in the sense of Definition 2.1 where $x \in L^4(\mathcal{O})$. Below γ shall denote the minimal constant arising in the Sobolev embedding $L^{\alpha+1}(\mathcal{O}) \subset H^{-1}(\mathcal{O})$ (see (7) below) and $C^* = \sum_{k=1}^\infty \mu_k^2\, |e_k|^2_{H^1_0(\mathcal{O})} = \sum_{k=1}^\infty \mu_k^2\, \lambda_k^2$. Theorem 2.2 is the main result of the paper.

Theorem 2.2 *Assume that $d = 1,2,3$ and that $0 < \alpha < 1$ if $d = 1,2$, $\frac{1}{5} \leq \alpha < 1$ if $d = 3$. Let $\tau := \inf\{t \geq 0 : \; |X(t,x)|_{-1} = 0\}$. Then we have $|X(t,x)|_{-1} = 0$, for $t \geq \tau$, $\mathbb{P}$-a.s.. Furthermore*

$$\mathbb{P}(\tau \leq t) \geq 1 - \frac{|x|_{-1}^{1-\alpha}}{(1-\alpha)\rho\gamma^{1+\alpha}} \left(\int_0^t e^{-(1-\alpha)C^* s} ds\right)^{-1}.$$

In particular, if $|x|_{-1}^{1-\alpha} < \frac{\rho\gamma^{1+\alpha}}{C^*}$, then $\mathbb{P}(\tau < \infty) > 0$, and if $C^* = 0$, then $\tau \leq \frac{|x|_{-1}^{1-\alpha}}{(1-\alpha)\rho\gamma^{1+\alpha}}$.

Remark 1 This result extends to $\mathcal{O} \subset \mathbb{R}^d$ with $d \geq 4$, if $\alpha \in [\frac{d-2}{d+2}, 1)$. However, we have to strengthen the assumption on μ_k, $k \in \mathbb{N}$, see [1, Section 4] and in particular [6, Remark 2.9(iii)] for a detailed discussion.

3 Proof of Theorem 2.2

We shall proceed as in the proof of [3, Theorem 4.2]. Consider the solution $X_\lambda \in L^2_W(0,T;L^2(\Omega;H^1_0(\mathcal{O})))$ to equation (4). Then by applying the classical Itô formula to the real valued semi-martingale $|X_\lambda(t)|_{-1}^2, t \in [0,T]$, and to the function $\varphi_\varepsilon(r) = (r+\varepsilon^2)^{(1-\alpha)/2}, \quad r \in \mathbb{R}$, we find that

$$
\begin{aligned}
&d\varphi_\varepsilon(|X_\lambda(t)|_{-1}^2) + (1-\alpha)(|X_\lambda(t)|_{-1}^2+\varepsilon^2)^{-(1+\alpha)/2}\langle X_\lambda(t), \Psi_\lambda(X_\lambda(t)) + \lambda X_\lambda(t) + \tilde{\Psi}_\lambda(X_\lambda(t))\rangle dt \\
&= \tfrac{1}{2}\ \sum_{k=1}^\infty \mu_k^2 (1-\alpha) \tfrac{|X_\lambda(t)e_k|_{-1}^2(|X_\lambda(t)|_{-1}^2+\varepsilon^2)-(1-\alpha)^2|\langle X_\lambda(t)e_k, X_\lambda(t)\rangle_{-1}|^2)}{(|X_\lambda(t)|_{-1}^2+\varepsilon^2)^{(3+\alpha)/2}}\ dt \\
&+\langle \sigma(X_\lambda(t))dW(t), \varphi_\varepsilon'(|X_\lambda(t)|_{-1}^2)X_\lambda(t)\rangle_{-1} \\
&\le \tfrac{1}{2}\ \sum_{k=1}^\infty \mu_k^2\ \tfrac{(1-\alpha)|X_\lambda(t)e_k|_{-1}^2}{(|X_\lambda(t)|_{-1}^2+\varepsilon^2)^{(1+\alpha)/2}} dt + \langle \sigma(X_\lambda(t))dW(t), \varphi_\varepsilon'(|X_\lambda(t)|_{-1}^2)X_\lambda(t)\rangle_{-1} \\
&\le C^* \frac{(1-\alpha)|X_\lambda(t)e_k|_{-1}^2}{(|X_\lambda(t)|_{-1}^2+\varepsilon^2)^{(1+\alpha)/2}} dt + \langle \sigma(X_\lambda(t))dW(t), \varphi_\varepsilon'(|X_\lambda(t)|_{-1}^2)X_\lambda(t)\rangle_{-1}
\end{aligned} \tag{5}
$$

Then letting $\lambda \to 0$, by (3) we get that

$$
\liminf_{\lambda\to 0} \int_0^T \langle \Psi_\lambda(X_\lambda(t)), X_\lambda(t)\rangle dt \ge \rho \int_0^T |X(t)|_{L^{1+\alpha}(\mathcal{O})}^{1+\alpha} dt, \quad \mathbb{P}\text{-a.s.}
$$

and hence

$$
\begin{aligned}
&\varphi_\varepsilon(|X(t)|_{-1}^2) + (1-\alpha)\rho \int_r^t \frac{|X(s)|_{L^{\alpha+1}(\mathcal{O})}^{\alpha+1}}{(|X(s)|_{-1}^2+\varepsilon^2)^{(1+\alpha)/2}} ds \le \varphi_\varepsilon(|X(r)|_{-1}^2) \\
&+C^* \int_r^t \frac{(1-\alpha)|X(s)|_{-1}^2}{(|X(s)|_{-1}^2+\varepsilon^2)^{(1+\alpha)/2}} ds \\
&+2 \int_r^t \langle \sigma(X(s))dW(s), \varphi_\varepsilon'(|X(s)|_{-1}^2)X(s)\rangle_{-1}, \quad \mathbb{P}\text{-a.s., } r<t.
\end{aligned} \tag{6}
$$

Next by the Sobolev embedding theorem we have

$$
|u|_{-1} \le \gamma |u|_{L^{\alpha+1}(\mathcal{O})},\ \forall\ u \in L^{\alpha+1}(\mathcal{O}),\ \text{if } d>2 \text{ and } \alpha \ge \tfrac{d-2}{d+2},\ \text{and } \forall\ \alpha>0,\ \text{if d=1,2.} \tag{7}
$$

Then substituting (7) into (6) we get

$$
\begin{aligned}
&\varphi_\varepsilon(|X(t)|_{-1}^2) + (1-\alpha)\rho\gamma^{1+\alpha} \int_r^t \frac{|X(s)|_{-1}^{\alpha+1}}{(|X(s)|_{-1}^2+\varepsilon^2)^{(1+\alpha)/2}} ds \le \varphi_\varepsilon(|X(r)|_{-1}^2) \\
&+C^* \int_r^t \frac{(1-\alpha)|X(s)|_{-1}^2}{(|X(s)|_{-1}^2+\varepsilon^2)^{(1+\alpha)/2}} ds \\
&+ \int_r^t \langle \sigma(X(s))dW(s), \varphi_\varepsilon'(|X(s)|_{-1}^2)X(s)\rangle_{-1}, \quad \mathbb{P}\text{-a.s., } r<t.
\end{aligned} \tag{8}
$$

Now for $\epsilon \to 0$ we have

$$
\begin{aligned}
&|X(t)|_{-1}^{1-\alpha} + (1-\alpha)\rho\gamma^{1+\alpha} \int_r^t 1_{\{|X(s)|_{-1}>0\}} ds \le |X(r)|_{-1}^{1-\alpha} + C^*(1-\alpha) \int_r^t |X(s)|_{-1}^{1-\alpha} ds \\
&+(1-\alpha) \int_r^t \langle \sigma(X(s))dW(s), |X(s)|_{-1}^{-(\alpha+1)} X(s)\rangle_{-1},\ \mathbb{P}\text{-a.s., } r<t.
\end{aligned}
$$

Hence by Itô's product rule

$$
\begin{aligned}
&e^{-(1-\alpha)C^*t}|X(t)|_{-1}^{1-\alpha} + (1-\alpha)\rho\gamma^{1+\alpha} \int_r^t e^{-(1-\alpha)C^*s} 1_{\{|X(s)|_{-1}>0\}} ds \\
&\le e^{-(1-\alpha)C^*r}|X(r)|_{-1}^{1-\alpha} \\
&\quad +(1-\alpha) \int_r^t e^{-(1-\alpha)C^*s} \langle \sigma(X(s))dW(s), |X(s)|_{-1}^{-(\alpha+1)} X(s)\rangle_{-1}, \quad \mathbb{P}\text{-a.s., } r<t.
\end{aligned} \tag{9}
$$

From this it immediately follows that $e^{-(1-\alpha)C^*t}|X(t)|_{-1}^{1-\alpha},\ t \ge 0$, is an $(\mathcal{F}_t)$-supermartingale, hence $|X(t)|_{-1} = 0$ for all $t \ge \tau$. So, (9) with $r=0$ after taking expectation implies that $\int_0^t e^{-(1-\alpha)C^* s}\mathbb{P}(\tau > s)ds \leq \frac{|x|_{-1}^{1-\alpha}}{(1-\alpha)\rho\gamma^{1+\alpha}}, \quad t \geq 0$. This implies that $\mathbb{P}(\tau > t) \leq \frac{|x|_{-1}^{1-\alpha}}{(1-\alpha)\rho\gamma^{1+\alpha}} \left(\int_0^t e^{-(1-\alpha)C^* s} ds\right)^{-1}$, $t \geq 0$, and the assertion follows. □

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