Pathwise uniqueness of the squared Bessel process, and CIR process, with skew reflection on a deterministic time dependent curve [1]

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Summary: Let $\sigma, \delta > 0, b \geq 0$. Let $\lambda^2 : \mathbb{R}^+ \to \mathbb{R}^+$, be continuous, not necessarily absolutely continuous, and locally of bounded variation. We develop a general analytic criterion for pathwise uniqueness of

$$R_t = R_0 + \int_0^t \sigma\sqrt{|R_s|}dW_s + \int_0^t \frac{\sigma^2}{4}(\delta - bR_s)ds + (2p-1)\ell_t^0(R - \lambda^2),$$

where $p \in (0,1)$, and where $\ell_t^0(R - \lambda^2)$ is the symmetric semimartingale local time of $R - \lambda^2$ (see Theorem 2.13, and 3.2). The criterion is related to the existence of certain sub-/superharmonic functions for the associated parabolic generator, which is a complexer object than its time homogeneous counterpart. As an application, we show in Corollary 2.14 that pathwise uniqueness holds, if

$$\overline{p}d\lambda^2(s) \leq \overline{p}\frac{\sigma^2}{4}\left\{\delta - \left(\frac{1-\overline{p}}{2}\right)b\lambda^2(s)\right\}ds.$$

where $\overline{p} := sgn(2p-1)$, and sgn is the point-symmetric sign function.The inequalities are to be understood in the sense of signed measures on $\mathbb{R}^+$. For instance, if $2p-1 > 0$, $\sigma = 2$, this means that the increasing part of λ^2 is Lipschitz continuous with Bessel dimension δ as Lipschitz constant, and that the decreasing part is arbitrary.
Weak existence of R has been established in various cases (see [9]). In particular, there is no solution if $|p| > 1$ (see Remark 2.3(ii)).

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1 Introduction and motivation

For parameters $\sigma, \delta > 0, b \geq 0$, consider the Cox-Ingersoll-Ross process, i.e. the unique solution (in any probabilistic sense) of the 1-dimensional SDE

$$dR_t = \sigma\sqrt{|R_t|}dW_t + \frac{\sigma^2}{4}(\delta - bR_t)dt,$$

and denote by $(\ell_t^{a+})_{(t,a)\in\mathbb{R}^+\times\mathbb{R}}$ its associated right-continuous family of local times (upper local times), i.e. $(t,a) \mapsto \ell_t^{a+}$ is a.s. continuous in t and càdlàg in a. Applying the occupation time formula (see (8) below) we obtain

$$\int_{\mathbb{R}} \frac{\mathbb{I}_{\{a\neq 0\}}}{|a|}\ell_t^{a+}(R)da = \int_0^t \frac{\mathbb{I}_{\{R_s\neq 0\}}}{|R_s|}d\langle R, R\rangle_s \leq \sigma^2 \int_0^t \mathbb{I}_{\{R_s\neq 0\}}ds \leq \sigma^2 t,$$

so that by non-integrability of $a \mapsto \frac{1}{a}$ in any neighborhood of zero, the upper (resp. lower) local time at zero must vanish, i.e. $\ell^{0^+}(R) \equiv 0$ (resp. $\ell^{0^-}(R) \equiv 0$). Accordingly, the the symmetric local time

$$\ell^0(R) = \frac{\ell^{0^+}(R) + \ell^{0^-}(R)}{2}, \tag{1}$$

vanishes. In short, the lower (resp. upper, symmetric) local time corresponds to the right-continuous (resp. left-continuous, point-symmetric) derivative of $r \mapsto |r|$ in Tanaka's formula for $|R|$.

Given a continuous, positive function $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$, which is locally of bounded variation, and such that $\lambda \not\equiv 0$. The symmetric local time at zero $\ell^0(R-\lambda^2)$ (here $\lambda^2(t) = \lambda(t)\cdot\lambda(t)$) of the continuous semimatingale $R - \lambda^2$, where now R is a solution to (2) below, doesn't vanish. In fact, at least for $\lambda \in H^{1,2}_{loc}(\mathbb{R}^+)$, $\delta \geq 1$, the associated measure to $\ell^0(\sqrt{R} - \lambda)$, namely

$$\frac{\sigma^2}{8}|x|^{\delta-1}e^{-\frac{bx^2}{2}}\delta_{\lambda(t)}(dx)dt$$

is smooth and not identically zero (cf. [9]). From (3) below we then see that $\ell^0(R - \lambda^2)$ also doesn't vanish. On the other hand, if $\lambda \equiv 0$, as in the classical case, the measure only doesn't vanish if $\delta = 1$, so that $\ell^0(\sqrt{R})$ exists, but $\ell^0(R)$ vanishes again by (3). This is the analytical explanation for what happens in the classical semimartingale situation, and in particular in the case of (squared) Bessel processes.

For $\delta \geq 1, R_0 = r \geq 0$ a pair of continuous positive semimartingales $(R, \sqrt{R})$ has weakly been constructed in [9] with the following properties: R solves

$$R_t = R_0 + \int_0^t \sigma\sqrt{|R_s|}dW_s + \int_0^t \frac{\sigma^2}{4}(\delta - bR_s)ds + (2p-1)\ell_t^0(R - \lambda^2), \tag{2}$$

where $p \in (0,1)$ and

$$\ell_t^0(R-\lambda^2) = \int_0^t \mathbb{I}_{\{\lambda^2(s)>0\}} d\ell_s^0(R-\lambda^2).$$

In particular

$$\ell_t^0(R-\lambda^2) = \lim_{\varepsilon \downarrow 0} \frac{\sigma^2}{2\varepsilon} \int_0^t \mathbb{I}_{(-\varepsilon,\varepsilon)}(R_s - \lambda^2(s))|R_s| ds \quad a.s.$$

A solution to (2) always stays positive when started with positive initial condition (see [9], or Lemma 2.1(ii)). One can hence in that case discard the absolute value under the square root in (2).

Assuming that a solution to (2) is unique we call it p-skew squared Bessel process on λ^2 if $b=0$, $\sigma=2$, and p-skew CIR process on λ^2 if $b>0$.

The positive squareroot $\sqrt{R}$ solves

$$\begin{aligned}\sqrt{R_t} &= \sqrt{r} + \frac{\sigma}{2} W_t + \int_0^t \frac{\sigma^2}{8}\left(\frac{\delta-1}{\sqrt{R_s}} - b\sqrt{R_s}\right) ds + (2p-1)\int_0^t \mathbb{I}_{\{\lambda(s)>0\}} d\ell_s^0(\sqrt{R}-\lambda) \\ &\quad + \frac{\mathbb{I}_{\{\delta=1\}}}{2} \ell_t^{0+}(\sqrt{R}),\end{aligned}$$

and the relation

$$\ell_t^0(R-\lambda^2) = 2\int_0^t \sqrt{R_s} d\ell_s^0(\sqrt{R}-\lambda) \tag{3}$$

holds. It has been established in [9] using analytic additive functional calculus. Probabilistically, relation (3) is derived using a product formula for local times (see [12], and also [4]), noting that $R-\lambda^2 = (\sqrt{R}+\lambda)(\sqrt{R}-\lambda)$, and that $\sqrt{R} \geq 0$. Furthermore R, $\sqrt{R}$, are typical examples of diffusions with discontinuous local times (see Lemma 2.2, [10]). A construction of R in the case $0<\delta<1$ was also pointed out in various cases (see [9]).

Finally, we think that it is worth to make two remarks concerning the construction of the pair $(R, \sqrt{R})$ when $\delta \geq 1$: For the construction of R with the most general time dependent λ we had to make a detour via $\sqrt{R}$ in [9]. Our technique was to first decompose $\lambda = \beta - (-\gamma)$ as a difference of two decreasing functions, and to consider a diffusion X with the appropriate coefficients on the monotonely moving domain $E = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R} | x \geq -\gamma(t)\}$ with skew reflection on β ($\rightsquigarrow \ell_t^0(X-\beta)$), and then to lift the moving domain through a Girsanov transformation by $+\gamma(t)$ ($\rightsquigarrow \ell_t^0(X+\gamma-(\beta+\gamma)) = \ell_t^0(\sqrt{R}-\lambda)$), putting $\sqrt{R} = X+\gamma$. Since the martingale part of $\sqrt{R}$ is much more suitable for the technique of Girsanov transformation we choosed $\sqrt{R}$ as starting point. The second remark is, that we astonishingly only managed to construct (2) for increasing λ^2, so nonetheless for constants, if $2p-1<0$. For $2p-1>0$ the construction could be carried out for any $\lambda \in H^{1,2}_{loc}(\mathbb{R}^+)$.

In this this note we shall derive an analytic criterion for pathwise uniqueness of (2) with arbitrary initial condition. This is first done in Theorem 2.13 for general possibly non absolutely continuous λ^2. Uniqueness is reduced to the resolution of a certain parabolic differential equation corresponding to the generator of R. The general criterion of Theorem 2.13 is directly applied in Corollary 2.14 in order to show that, if λ^2 is locally of bounded variation, $\sigma, \delta > 0, b \geq 0$, $p \in (0,1)$, then pathwise uniqueness holds for (2) whenever

$$\overline{p} d\lambda^2(s) \leq \overline{p}\frac{\sigma^2}{4}\left\{\delta - \left(\frac{1-\overline{p}}{2}\right) b\lambda^2(s)\right\} ds,$$

where $\overline{p} := sgn(2p-1)$, and sgn is the point-symmetric sign function. The inequalities are to be understood in the sense of signed measures on $\mathbb{R}^+$. For instance, if $2p-1>0$, $\sigma = 2$, this means that the increasing part of λ^2 is Lipschitz continuous with Bessel dimension δ as Lipschitz constant, and that the decreasing part is arbitrary. Or, if $2p-1<0$, and e.g. $\lambda^2 \equiv const = c$, this means that $c \geq \frac{\delta}{b}$.
To obtain the result we made use of Kummer functions of the first kind (see Corollary 2.14). Even after "localizing "the main argument, we were not able to get any uniqueness result by using Kummer functions of the second kind (see however the proof of Corollary 2.11(ii)).

For some reasons the proof of pathwise uniqueness is not quite standard, although it ends up with the application of a generalized Gronwall inequality. Looking at the difference of $|R_t^{(1)} - R_t^{(2)}|$, where $R^{(1)}, R^{(2)}$, are two solutions, we can not use directly Le Gall's trick (see [2]), since although $\ell_t^0(R^{(1)}-R^{(2)}) \equiv 0$, there always remains a term involving the local time on λ^2. The coefficients, as well as the parabolic situation, makes simple transformations through harmonic functions as used in [3] impossible, and sup/superharmonic functions w.r.t. the time homogeneous generator may lose of their advantageous properties under parabolic boundary conditions.
Our line of arguments, is to first show that together with $R^{(1)}, R^{(2)}$, the supremum $S = R^{(1)} \vee R^{(2)}$, and the infimum $I = R^{(1)} \wedge R^{(2)}$, is also a solution. Then we have to find a good function $H(t,x)$, increasing in x, and to apply a generalized Gronwall inequality to the expectation of $H(t,S_t) - H(t,I_t)$ in order to conclude (see Corollary 2.14, and Theorem 2.13). In order to find that $S = R^{(1)} \vee R^{(2)}$, $I = R^{(1)} \wedge R^{(2)}$, is also a solution we profited from [11] (see also [5]). In order to make disappear the local time on λ^2 with the help of a good function H, we made use of simple Itô-Tanaka formulas (see Lemma 2.9), which are proved using Lemma 2.8. Lemma 2.9 is also used to show that there may be no solution to (2), if $|2p-1| > 1$ (see Corollary 2.11).
In the third section we add another pathwise uniqueness criterion in Theorem 3.2 which is developed with the help of a recent generalization of Ito's formula from [6]. In fact, it is analogous to the criterion of Theorem 2.14, but uses "true"time dependent functions. Unfortunately, we have to assume $\lambda^2 \in C^1(\mathbb{R}^+)$, but we think nonetheless that Theorem 3.2 may be useful, in particular for specialists in PDEs who hopefully might be able to better resolve the given equation. In order to keep the exposition as clear as possible, the statement and main argument for the proof of Theorems 2.13, 3.2, is only presented in global form. It can easily be localized but then loses of its clearness.

2 Pathwise uniqueness in the non absolutely continuous case

Throughout this article $\mathbb{I}_A$ will denote the indicator function of a set A. We let $\mathbb{R}^+ := \{x \in \mathbb{R} | \, x \geq 0\}$. An element of $\mathbb{R}^+ \times \mathbb{R}$ is typically represented as (t,x), i.e. the first entry is always for time, the second always for space. The time derivative is denoted by ∂_t, the space derivative by ∂_x, and the second space derivative by ∂_{xx}. Functions depending on space and time are denoted with capital letters, functions depending only on one variable are denoted with small case letters. If a function f only depends on one variable we write f', resp. f'', for its derivative, resp. second derivative.

Let $\sigma, \delta > 0$, $b \geq 0$, and $\lambda^2 : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous and locally of bounded variation. On an arbitrary complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, consider an adapted continuous process with the following properties: R solves the integral equation

$$R_t = R_0 + \int_0^t \sigma \sqrt{|R_s|} dW_s + \int_0^t \frac{\sigma^2}{4}(\delta - bR_s)ds + (2p-1)\ell_t^0(R - \lambda^2), \;\; P\text{-a.s.} \tag{4}$$

where

(1) $(W_t)_{t\geq 0}$ is a $\mathcal{F}_t$-Brownian motion starting from zero,

(2) $P[\int_0^t \{\sigma^2|R_s| + |\frac{\sigma^2}{4}(\delta - bR_s)|\}ds < \infty] = 1$,

(3) $\ell^0(R-\lambda^2)$ is the symmetric semimartingale local time of $R - \lambda^2$, i.e.

$$\begin{aligned} \frac{1}{2}\ell_t^0(R-\lambda^2) &= (R_t - \lambda^2(t))^+ - (R_0 - \lambda^2(0))^+ \\ &\quad - \int_0^t \frac{\mathbb{I}_{\{R_s - \lambda^2(s) > 0\}} + \mathbb{I}_{\{R_s - \lambda^2(s) \geq 0\}}}{2} d\{R_s - \lambda^2(s)\}, \quad t \geq 0. \end{aligned} \tag{5}$$

A process R with the given properties is called a *weak solution* to (4). In particular, one can show exactly as in [7, VI. (1.3) Proposition] that

$$\int_0^t H(s, R_s) d\ell_s^0(R-\lambda^2) = \int_0^t H(s, \lambda^2(s)) d\ell_s^0(R - \lambda^2), \tag{6}$$

for any positive Borel function H on $\mathbb{R}^+ \times \mathbb{R}$.

We say that *pathwise uniqueness* holds for (4), if, any two solutions $R^{(1)}, R^{(2)}$, on the same filtered probability space $(\Omega, \mathcal{F}, P)$, with $R_0^{(1)} = R_0^{(2)}$ P-a.s., and with same Brownian motion, are P-indistinguishable, i.e. $P[R_t^{(1)} = R_t^{(2)}] = 1$ for all $t \geq 0$.

For later purposes we introduce the upper (or right) local time of $R - \lambda^2$

$$\ell_t^{0+}(R-\lambda^2) = (R_t - \lambda^2(t))^+ - (R_0 - \lambda^2(0))^+ - \int_0^t \mathbb{I}_{\{R_s - \lambda^2(s) > 0\}} d\{R_s - \lambda^2(s)\}, \tag{7}$$

and the lower (or left) local time $\ell^{0-}(R-\lambda^2)$, which is now given through (1). Accordingly, $\ell^0(X), \ell^{0+}(X), \ell^{0-}(X)$, are defined for any continuous semimartingale X.

Another useful formula, is the *occupation times formula*: If X is a continuous semimartingale, then

$$\int_0^t H(s, X_s) d\langle X, X\rangle_s = \int_{\mathbb{R}} \int_0^t H(s,a) d\ell_s^{a+}(X) da \tag{8}$$

holds a.s. for every positive Borel function H on $\mathbb{R}^+ \times \mathbb{R}$. See e.g. [8, IV. (45.4)], [7, VI. (1.15) Exercise]. Since ℓ^{a+} has only countably many jumps in a, the formula holds for ℓ^a, and ℓ^{a-}, as well.

Let us formulate a first lemma. The statements were already proved in [9], and are simple direct consequences of well-known formulas, but we include the proof in order to keep this exposition self contained.

Lemma 2.1 *Let R be a weak solution to (4). Then:*
(i) $\int_0^t \mathbb{I}_{\{\lambda^2(s)=0\}} d\ell_s^0(R-\lambda^2) = 0$ P-a.s. for any $t \geq 0$. In particular

$$supp\{d\ell_s^0(R-\lambda^2)\} \subset \overline{\{\lambda^2(s) > 0\}}.$$

(ii) If $R_0 \geq 0$ P-a.s., then $R_t \geq 0$ P-a.s. for any $t \geq 0$.
(iii) The time of R spent at zero has Lebesgue measure zero, i.e.

$$\int_0^t \mathbb{I}_{\{R_s=0\}} ds = 0 \quad P\text{-a.s.} \quad \forall t \geq 0.$$

(iv) The time of R spent on λ^2 has Lebesgue measure zero, i.e.

$$\int_0^t \mathbb{I}_{\{R_s=\lambda^2(s)\}} ds = 0 \quad P\text{-a.s.} \quad \forall t \geq 0.$$

Proof (i) By (8) we have

$$\int_{\mathbb{R}} \frac{\mathbb{I}_{\{a\neq 0\}}}{|a|} \int_0^t \mathbb{I}_{\{\lambda^2(s)=0\}} d\ell_s^{a+}(R-\lambda^2) da \quad = \quad \int_0^t \frac{\mathbb{I}_{\{R_s-\lambda^2(s)\neq 0\}}}{|R_s-\lambda^2(s)|} \mathbb{I}_{\{\lambda^2(s)=0\}} \sigma^2 |R_s| ds \leq \sigma^2 t.$$

Since $\frac{1}{|a|}$ is not integrable in any neighborhood of zero, we obtain that

$$\int_0^t \mathbb{I}_{\{\lambda^2(s)=0\}} d\ell_s^{0+}(R-\lambda^2) = \int_0^t \mathbb{I}_{\{\lambda^2(s)=0\}} d\ell_s^{0-}(R-\lambda^2) = 0.$$

The statement thus holds for $\ell^{0+}(R-\lambda^2)$, and $\ell^{0-}(R-\lambda^2)$, and therefore also for $\ell^0(R-\lambda^2)$.
(ii) As a direct consequence of the occupation time formula $\ell_t^{0+}(R) \equiv 0$ (replace λ^2 by zero in the proof of (i)). Then, applying Tanaka's formula (cf. e.g. [7, VI. (1.2) Theorem]),

using (i) and (6), taking expectations, and cutting with $\tau_n := \inf\{t \geq 0 | |R_t| \geq n\}$, we obtain

$$\begin{aligned} E[R^-_{t\wedge\tau_n}] &= E[R^-_0] - E[\int_0^{t\wedge\tau_n} \mathbb{I}_{\{R_s\leq 0\}} \frac{\sigma^2}{4}(\delta - bR_s)ds] \\ &\quad -(2p-1)E[\int_0^{t\wedge\tau_n} \mathbb{I}_{\{R_s\leq 0\}} \mathbb{I}_{\{\lambda^2(s)>0\}} d\ell^0_s(R-\lambda^2)] \\ &\leq -E[\int_0^{t\wedge\tau_n} \mathbb{I}_{\{R_s\leq 0\}} \frac{\sigma^2}{4}(\delta - bR_s)ds] \leq 0. \end{aligned}$$

It follows that $R_{t\wedge\tau_n}$ is P-a.s. equal to its positive part $R^+_{t\wedge\tau_n}$. Letting $n \to \infty$ concludes the proof.
(iii) Due to the presence of the squareroot in the diffusion part, we have $\ell^{0+}_t(R), \ell^{0-}_t(R) \equiv 0$ (replace λ^2 by zero in the proof of (i)). Using [7, VI. (1.7) Theorem], (i) and (6), it follows P-a.s.

$$\begin{aligned} 0 = \ell^{0+}_t(R) - \ell^{0-}_t(R) &= \int_0^t \mathbb{I}_{\{R_s=0\}} \left\{ \frac{\sigma^2}{4}(\delta - bR_s)ds + (2p-1)d\ell_s(R-\lambda^2)) \right\} \\ &= \frac{\sigma^2\delta}{4} \int_0^t \mathbb{I}_{\{R_s=0\}} ds. \end{aligned}$$

(iv) As a simple consequence of the occupation time formula, we have

$$\begin{aligned} \int_0^t \mathbb{I}_{\{R_s=\lambda^2(s)\}} \mathbb{I}_{\{R_s\neq 0\}} \sigma^2 |R_s| ds &= \int_0^t \mathbb{I}_{\{R_s-\lambda^2(s)=0\}} d\langle R-\lambda^2\rangle_s \\ &= \int_{\mathbb{R}} \mathbb{I}_{\{0\}}(a) \ell^a_t(R-\lambda^2) da = 0. \end{aligned}$$

But P-a.s. $\sigma^2 |R_s| \mathbb{I}_{\{R_s\neq 0\}} > 0$ ds-a.e. by (iii) and the assertion follows.

□

From the next lemma one observes at least when λ^2 is absolutely continuous the discontinuity of the local times in the space variable at zero.

Lemma 2.2 *Let R be a weak solution to (4). We have P-a.s.:*

$$\ell^{0+}_t(R-\lambda^2) = 2p\ell^0_t(R-\lambda^2) - \int_0^t \mathbb{I}_{\{R_s=\lambda^2(s)\}} d\lambda^2(s),$$

and

$$\ell^{0-}_t(R-\lambda^2) = 2(1-p)\ell^0_t(R-\lambda^2) + \int_0^t \mathbb{I}_{\{R_s=\lambda^2(s)\}} d\lambda^2(s).$$

If λ^2 is absolutely continuous, i.e. $\lambda^2 \in H^{1,1}_{loc}(R^+)$, with $d\lambda^2(s) = (\lambda^2)'(s)ds$, then P-a.s.

$$\ell^0_t(R-\lambda^2) = \frac{1}{2p}\ell^{0+}_t(R-\lambda^2) = \frac{1}{2(1-p)}\ell^{0-}_t(R-\lambda^2).$$

Proof Since $R-\lambda^2$ is a continuous semimartingale w.r.t. P. Thus, by Tanaka's formula (7) it follows P-a.s.

$$(R_t-\lambda^2(t))^+ \quad = \quad (R_0-\lambda^2(0))^+ + \int_0^t \mathbb{I}_{\{R_s-\lambda^2(s)>0\}} d(R_s-\lambda^2(s)) + \frac{1}{2}\ell_t^{0+}(R-\lambda^2).$$

On the other hand, the symmetrized Tanaka formula (5) together with Lemma 2.1(iii) gives

$$\begin{aligned}(R_t-\lambda^2(t))^+ &= (R_0-\lambda^2(0))^+ + \int_0^t \frac{\mathbb{I}_{\{R_s-\lambda^2(s)>0\}}+\mathbb{I}_{\{R_s-\lambda^2(s)\geq 0\}}}{2} d(R_s-\lambda^2(s)) \\ &\quad +\frac{1}{2}\ell_t^0(R-\lambda^2) \\ &= (R_0-\lambda^2(0))^+ + \int_0^t \mathbb{I}_{\{R_s-\lambda^2(s)>0\}} d(R_s-\lambda^2(s)) - \frac{1}{2}\int_0^t \mathbb{I}_{\{R_s=\lambda^2(s)\}} d\lambda^2(s) \\ &\quad +p\ell_t^0(R-\lambda^2).\end{aligned}$$

Comparing the two formulas for $(R_t-\lambda^2(t))^+$ we obtain the first statement. The second follows from (1) by simple algebraic transformations. If λ^2 is absolutely continuous, then

$$\int_0^t \mathbb{I}_{\{R_s=\lambda^2(s)\}} d\lambda^2(s) = \int_0^t \mathbb{I}_{\{R_s=\lambda^2(s)\}} (\lambda^2)'(s) ds = 0$$

by Lemma 2.1(iv), and the last statement follows.

□

Remark 2.3 *(i) Using the previous Lemma 2.2 and (1), one can easily derive that*

$$(2p-1)\ell_t^0(R-\lambda^2) = \frac{\ell_t^{0+}(R-\lambda^2)-\ell_t^{0-}(R-\lambda^2)}{2} + \int_0^t \mathbb{I}_{\{R_s=\lambda^2(s)\}} d\lambda^2(s).$$

and

$$(1-p)\ell_t^{0+}(R-\lambda^2) = p\ell_t^{0-}(R-\lambda^2) - \int_0^t \mathbb{I}_{\{R_s=\lambda^2(s)\}} d\lambda^2(s).$$

(ii) Let λ^2 be absolutely continuous. Then $\int_0^t \mathbb{I}_{\{R_s=\lambda^2(s)\}} d\lambda^2(s) = 0$ by Lemma 2.1(iv). If now $|p|>1$, then a solution to (4) does not exist. In fact, by (i) it holds that $\ell^0(R-\lambda^2), \ell^{0+}(R-\lambda^2), \ell^{0-}(R-\lambda^2) \equiv 0$. Hence R must be the classical CIR process for which uniqueness is known to hold in any sense. In particular, the time dependent CIR process (t,R_t) is associated to the time dependent Dirichlet form

$$\begin{aligned}\mathcal{E}(F,G) := &\int_0^\infty\int_0^\infty \frac{\sigma^2}{2}|x|\partial_x F(t,x)\partial_x G(t,x)|x|^{\frac{\delta}{2}-1}e^{-\frac{bx}{2}} dx dt \\ &-\int_0^\infty\int_0^\infty \partial_t F(t,x)G(t,x)|x|^{\frac{\delta}{2}-1}e^{-\frac{bx}{2}} dx dt;\end{aligned}$$

and the local time $\ell^0(R-\lambda^2)$ is uniquely associated to the smooth measure

$$\frac{\sigma^2}{2}|x|^{\frac{\delta}{2}}e^{-\frac{bx}{2}}\delta_{\lambda^2(t)}(dx)dt$$

which doesn't vanish if λ^2 is different from zero on a set of positive Lebesgue measure. Therefore $\ell^0(R-\lambda^2)$ cannot vanish identically. This can be found in [9] (see also Corollary 2.11, for direct proofs with special λ's).

Remark 2.4 *In the sequel (cf. Lemma 2.5, Lemma 2.8) we shall derive the symmetric analogues of well-known formulas for right local times. In fact these results are as the proofs show direct and trivial consequences of the original results obtained for right local times by the corresponding authors. We do not claim any originality for these results. For instance, Lemma 2.5 can be found in [5, Corollary 2.6, and following remark].*

The next lemma is very useful and mainly due to S. Weinryb. It has been obtained previously in [5, Corollary 2.6, and following remark] using different and for themself important formulas for the computation of local times.

Lemma 2.5 *[cf. [11, Lemme, p.74], [5, Corollary 2.6, and following remark]] Let X,Y be two continuous semimartingales, with $X_0=Y_0$. Suppose that $\ell^{0+}(X-Y)\equiv 0$. Then the following representation formula holds for $\ell_s=\ell_s^{0+}$:*

$$\ell_t(X\vee Y)=\int_0^t \mathbb{I}_{\{Y_s<0\}}d\ell_s(X)+\int_0^t \mathbb{I}_{\{X_s\le 0\}}d\ell_s(Y). \tag{9}$$

Suppose that additionally $\ell^{0+}(Y-X)\equiv 0$. Then (9) holds also for $\ell_s=\ell_s^{0-}$, and $\ell_s=\ell_s^{0}$. In particular

$$\int_0^t \mathbb{I}_{\{Y_s=0\}}d\ell_s(X)=\int_0^t \mathbb{I}_{\{X_s=0\}}d\ell_s(Y) \tag{10}$$

holds for $\ell_s=\ell_s^{0+}$, $\ell_s=\ell_s^{0-}$, and $\ell_s=\ell_s^{0}$.

Proof If $\ell^{0+}(X-Y)\equiv 0$ then (9) for $\ell_s=\ell_s^{0+}$ is just [11, Lemme, p.74]. If additionally $\ell^{0+}(Y-X)\equiv 0$, then we can interchange X and Y in (9), and (10) follows for $\ell_s=\ell_s^{0+}$. Now we show (9) for $\ell_s=\ell_s^{0-}$, and $\ell_s=\ell_s^{0}$. By (1) it is enough to show it for $\ell_s=\ell_s^{0-}$. In the subsequent calculation we will use the formulas $\ell^{0+}(X\vee Y)+\ell^{0+}(X\wedge Y)=\ell^{0+}(X)+\ell^{0+}(Y)$, and $\ell^{0+}(X)=\ell^{0-}(-X)$, (see e.g. [7], [13], [4]), and that (9), (10), hold for $-X$, $-Y$, and $\ell_s=\ell_s^{0+}$. We have

$$\begin{aligned}
\ell_t^{0-}(X\vee Y) &= \ell_t^{0+}((-X)\wedge(-Y))\\
&= \ell_t^{0+}(-X)+\ell_t^{0+}(-Y)-\ell_t^{0+}((-X)\vee(-Y))\\
&= \ell_t^{0+}(-X)+\ell_t^{0+}(-Y)\\
&\quad -\int_0^t \mathbb{I}_{\{-Y_s\le 0\}}d\ell_s^{0+}(-X)-\int_0^t \mathbb{I}_{\{-X_s<0\}}d\ell_s^{0+}(-Y)\\
&= \int_0^t \mathbb{I}_{\{Y_s<0\}}d\ell_s^{0-}(X)+\int_0^t \mathbb{I}_{\{X_s\le 0\}}d\ell_s^{0-}(Y)
\end{aligned}$$

as desired. Since $\int_0^t \mathbb{I}_{\{-Y_s=0\}} d\ell_s^{0+}(-X) = \int_0^t \mathbb{I}_{\{-X_s=0\}} d\ell_s^{0+}(-Y)$, we can use the same calculation to see that (10) also holds for $\ell_s = \ell_s^{0-}$, and $\ell_s = \ell_s^0$.

□

Lemma 2.6 *Let $R^{(1)}, R^{(2)}$, be two solutions to (4) with same Brownian motion, on the same filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, and such that $R_0^{(1)} = R_0^{(2)}$ P-a.s. Then:*

(i) The following representation formula holds for $\ell_s = \ell_s^{0+}$, $\ell_s = \ell_s^{0-}$, and $\ell_s = \ell_s^0$:

$$\ell_t(R^{(1)} \vee R^{(2)} - \lambda^2) = \int_0^t \mathbb{I}_{\{R_s^{(2)} - \lambda^2(s) < 0\}} d\ell_s(R^{(1)} - \lambda^2) + \int_0^t \mathbb{I}_{\{R_s^{(1)} - \lambda^2(s) \leq 0\}} d\ell_s(R^{(2)} - \lambda^2).$$

(ii) The supremum $R^1 \vee R^2$, and the infimum $R^1 \wedge R^2$, are also solutions to (4).
(iii) For the supremum $S := R^1 \vee R^2$, and the infimum $I := R^1 \wedge R^2$, it holds P-a.s. that

$$S_t \mathbb{I}_{\Omega \setminus \{S_t > 0\} \cap \{I_t \geq 0\}} = I_t \mathbb{I}_{\Omega \setminus \{S_t > 0\} \cap \{I_t \geq 0\}} \quad \forall t \geq 0.$$

Proof (i) Since $R^{(1)}, R^{(2)}$, are continuous semimartingales w.r.t. P, the same is true for $R^{(1)} - \lambda^2, R^{(2)} - \lambda^2$. Using that $\ell_s((R^{(i)} - \lambda^2) - (R^{(j)} - \lambda^2)) = \ell_s(R^{(i)} - R^{(j)}) \equiv 0$, for $i \neq j$, $i, j \in \{1, 2\}$, (i) follows from Lemma 2.5. Note that if λ^2 is absolutely continuous, then upper, lower, and symmetric local times are constant multiples of each other (see Lemma 2.2, and Lemma 2.1(iii)), and the statement would follow immediately from S. Weinryb's result (cf. (proof of) Lemma 2.5).
(ii) Writing $R_t^{(1)} \vee R_t^{(2)} = (R_t^{(1)} - R_t^{(2)})^+ + R_t^{(2)}$ and applying Tanaka's formula (cf. e.g. [7, VI.(1.2)]), we easily obtain after some calculations

$$\begin{aligned} R_t^{(1)} \vee R_t^{(2)} &= r + \int_0^t \sigma \sqrt{|R_s^{(1)} \vee R_s^{(2)}|} dW_s + \int_0^t \frac{\sigma^2}{4}(\delta - b R_s^{(1)} \vee R_s^{(2)}) dt \\ &+ (2p-1) \left\{ \int_0^t \mathbb{I}_{\{R_s^{(2)} - \lambda^2(s) < 0\}} d\ell_s^0(R^{(1)} - \lambda^2) + \int_0^t \mathbb{I}_{\{R_s^{(1)} - \lambda^2(s) \leq 0\}} d\ell_s^0(R^{(2)} - \lambda^2) \right\}. \end{aligned}$$

Now, we just use (i) and conclude that $R^1 \vee R^2$ is another solution. Clearly, by linearity $R^1 \wedge R^2$ is also a solution. One just has to use the formula

$$\ell^0(X \vee Y) + \ell^0(X \wedge Y) = \ell^0(X) + \ell^0(Y),$$

which is a trival consequence of the formulas given in the proof of Lemma 2.5.
(iii) Define the function

$$h(x) := \begin{cases} -\int_0^{-x} y^{\frac{\delta}{2}} e^{\frac{by}{2}} dy & \text{for } x < 0; \\ 0 & \text{for } x \geq 0. \end{cases}$$

h is continuously differentialble with locally integrable second derivative. We may hence apply Itô's formula with h. Note that h is a harmonic function, and strictly increasing in $(-\infty, 0]$. After taking expectations and stopping w.r.t. $\tau_n := \inf\{t \geq 0 || S_t| \geq n\}$ we obtain

$$E[h(S_{t\wedge\tau_n}) - h(I_{t\wedge\tau_n})] = 0$$

for any $t \geq 0$. Letting $n \to \infty$ we get $h(S_t) = h(I_t)$ P-a.s. By continuity of the sample paths, this holds simultanuously for all $t \geq 0$. Decomposing Ω in disjoint sets

$$\{S_t > 0\} \cap \{I_t \geq 0\},\ \{S_t > 0\} \cap \{I_t < 0\},\ \{S_t \leq 0\},$$

we get

$$h(S_t)\mathbb{I}_{\{S_t \leq 0\}} = h(I_t)\mathbb{I}_{\{S_t \leq 0\}} + h(I_t)\mathbb{I}_{\{S_t > 0\}\cap\{I_t<0\}},$$

and then

$$\mathbb{I}_{\{S_t>0\}\cap\{I_t<0\}} = 0,$$

as well as

$$S_t\mathbb{I}_{\{S_t \leq 0\}} = I_t\mathbb{I}_{\{S_t \leq 0\}}$$

immediately follow.

□

Remark 2.7 *Suppose that we replace $(2p-1)\ell^0(R-\lambda^2)$ by $\frac{2p-1}{2p}\ell^{0+}(R-\lambda^2)$ (resp. $\frac{2p-1}{2(1-p)}\ell^{0-}(R-\lambda^2)$) in (4). Using Lemma 2.5 one can see as in the proof of Lemma 2.6, that with any two solutions to the modified equation (4) the sup and inf is again a solution. Consequently, as will be seen below, pathwise uniqueness can also be derived for the modified equation under the same assumptions. We have decided to work with symmetric local times, because the measures associated to symmetric local times appear naturally in integration by parts formulas for the corresponding Markov process generators. Recall that symmetric derivatives are used in distribution theory.*

For the formulation of the next lemma we need first to state some definitions and properties of *convex* functions. A function $f : \mathbb{R} \to \mathbb{R}$ is convex, provided

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

for all $x, y \in \mathbb{R}$, and $\alpha \in [0,1]$. f is then necessarily continuous. Furthermore, the left hand derivative f'^{-} (resp. right hand derivative f'^{+}) exists and is left continuous and increasing (resp. right continuous and increasing), $\{f'^{-} \neq f'^{+}\}$ is at most countable. The Schwarz derivative of f is defined as the symmetric derivative

$$f^{(\prime)} = \frac{f'^{+} + f'^{-}}{2}.$$

The second derivative $f^{('')}$ of f in the sense of distributions is a positive Radon measure, i.e. there exists a positive Radon measure $f^{('')}(dx)$, such that

$$\int_{\mathbb{R}} f(x)\varphi''(x)dx = \int_{\mathbb{R}} \varphi(x)f^{('')}(dx), \quad \forall \varphi \in C_0^2(\mathbb{R}).$$

In what follows we shall use the notations $f^{(')}$, $f^{('')}$, for distributional derivatives in general.

The following Lemma 2.8(i) is a trivial generalization of [7, VI. (1.23) Exercise] (see its proof, which we add for the reader's convenience). Lemma 2.8(i) has previously been indicated in [5, Remark, p.222]. (ii) is due to [5, Corollary 2.11].

Lemma 2.8 *Let X be a continuous semimartingale. Let f be a strictly increasing function on $\mathbb{R}$, which is the difference of two convex functions.*

(i) (cf. [7, VI. (1.23) Exercise], [5, Remark, p.222]) We have a.s. for any $a \in \mathbb{R}$

$$\ell_t^{f(a)\pm}(f(X)) = f'^{\pm}(a)\ell_t^{a\pm}(X); \quad t \geq 0.$$

In particular, if R is a solution to (4), then P-a.s.

$$\ell_t^0(f(R-\lambda^2) - f(0)) = \frac{f'^{+}(0)}{2}\ell_t^{0+}(R-\lambda^2) + \frac{f'^{-}(0)}{2}\ell_t^{0-}(R-\lambda^2); \quad t \geq 0.$$

(ii) (cf. [5, Corollary 2.11]) If f is additionally continuously differentiable, and R is a solution to (4), then P-a.s.

$$\ell_t^0(f(R) - f(\lambda^2)) = \int_0^t f'(\lambda^2(s))d\ell_s^0(R-\lambda^2).$$

Proof (i) In order to prove the second statement of (i) we first use (1), and then apply the first statement of (i) with $a = 0$, and $g(x) := f(x) - f(0)$.
The first statement of (i) for "+" follows from [7, VI. (1.23) Exercise]. In fact one shows with the help of (8), that for any Borel measurable positive g

$$\int_{\mathbb{R}} g(a)\ell_t^{f(a)+}(f(X))da = \int_{\mathbb{R}} g(a)f'^{+}(a)\ell_t^{a+}(X)da,$$

and the result follows by right continuity of $a \mapsto \ell_t^{f(a)+}(f(X))$, $a \mapsto f'^{+}(a)\ell_t^{a+}(X)$. In order to show that

$$\ell_t^{f(a)-}(f(X)) = f'^{-}(a)\ell_t^{a-}(X); \quad t \geq 0,$$

one analogously derives

$$\int_{\mathbb{R}} g(a)\ell_t^{f(a)-}(f(X))da = \int_{\mathbb{R}} g(a)f'^{-}(a)\ell_t^{a-}(X)da,$$

and then uses the left continuity of $a \mapsto \ell_t^{f(a)-}(f(X)), a \mapsto f'^-(a)\ell_t^{a-}(X)$.
(ii) The statement follows directly from [5, Corollary 2.11], $\ell^{0+}(X) = \ell^{0-}(-X)$, and (1).

□

For the purposes of this section we indicate two very simple Itô-Tanaka formulas in the next lemma. The derivation of these formulas takes advantage of the fact that the time dependency is put into a semimartingale structure. The formulas can not be compared to and are of very much simpler nature than Peskir's general Itô-Tanaka formula that we will use in the next section. However, Lemma 2.9 is useful, and allows λ^2 just to be of bounded variation. Applying Peskir's Itô-Tanaka formula we would have to impose $\lambda^2 \in C^1(\mathbb{R}^+)$.

For $F : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ we set

$$LF(t,x) = \frac{\sigma^2}{2}|x|\partial_{xx}F(t,x) + \frac{\sigma^2}{4}(\delta - bx)\partial_x F(t,x),$$

whenever this makes sense.

Lemma 2.9 *Let f be a strictly increasing function on $\mathbb{R}$, which is the difference of two convex functions. Assume moreover (for simplicity) that $f^{(\prime)}$ is absolutely continuous. Let*

$$\overline{g}(y) := \gamma \mathbb{I}_{\{y<0\}} + \frac{\alpha+\gamma}{2}\mathbb{I}_{\{y=0\}} + \alpha \mathbb{I}_{\{y>0\}}; \quad \alpha, \gamma, y \in \mathbb{R}.$$

(i) Let f additionally satisfy $f'^+(0) = f'^-(0)$. Put $F(t,x) = f(x - \lambda^2(t)) - f(0)$ and

$$H(t,x) := \overline{g}(x - \lambda^2(t))F(t,x).$$

Then P-a.s.

$$\begin{aligned} H(t,R_t) &= H(0,R_0) + \int_0^t \overline{g}(R_s - \lambda^2(s))f^{(\prime)}(R_s - \lambda^2(s))\sigma\sqrt{|R_s|}dW_s \\ &+ \int_0^t \overline{g}(R_s - \lambda^2(s))\left\{LF(s,R_s)ds - f^{(\prime)}(R_s - \lambda^2(s))d\lambda^2(s)\right\} \\ &+(\alpha p - \gamma(1-p))f'(0)\ell_t^0(R - \lambda^2). \end{aligned}$$

(ii) Let f additionally be continuously differentiable. Put $F(t,x) = f(x) - f(\lambda^2(t))$ and

$$H(t,x) := \overline{g}(x - \lambda^2(t))F(t,x).$$

Then P-a.s.

$$\begin{aligned} H(t,R_t) &= H(0,R_0) + \int_0^t \overline{g}(R_s - \lambda^2(s))f'(R_s - \lambda^2(s))\sigma\sqrt{|R_s|}dW_s \\ &+ \int_0^t \overline{g}(R_s - \lambda^2(s))\left\{LF(s,R_s)ds - f'(\lambda^2(s))d\lambda^2(s)\right\} \\ &+(\alpha p - \gamma(1-p))\int_0^t f'(\lambda^2(s))d\ell_s^0(R - \lambda^2). \end{aligned}$$

Proof (i) Applying the symmetric version of the Itô-Tanaka formula (cf. [7, VI. (1.5) Theorem] for the right (or upper) version), we obtain

$$\begin{aligned} H(t,R_t) &= \alpha(f(R_t-\lambda^2(t))-f(0))^+ - \gamma(f(R_t-\lambda^2(t))-f(0))^- \\ &= H(0,R_0) + \int_0^t \overline{g}(R_s-\lambda^2(s))df(R_s-\lambda^2(s)) + \frac{\alpha-\gamma}{2}\ell_t^0(f(R-\lambda^2)-f(0)). \end{aligned}$$

Applying again the symmetric Itô-Tanaka formula, (8), and Lemma 2.8(i), the right hand side equals

$$\begin{aligned} &H(0,R_0) + \int_0^t \overline{g}(R_s-\lambda^2(s))f^{(\prime)}(R_s-\lambda^2(s))d(R_s-\lambda^2(s)) \\ &+ \int_0^t \overline{g}(R_s-\lambda^2(s))\frac{\sigma^2}{2}|R_s|f^{(\prime\prime)}(R_s-\lambda^2(s))ds + \frac{\alpha-\gamma}{2}f'(0)\ell_t^0(R-\lambda^2), \end{aligned}$$

which easily leads to the desired conclusion.
(ii) Using Lemma 2.8(ii) instead of Lemma 2.8(i) the prooof of (ii) is nearly the same that the proof of (i). We therefore omit it.

□

Remark 2.10 *If α,β, are strictly positive, then*

$$H(t,x) := \overline{g}(x-\lambda^2(t))(f(x-\lambda^2(t))-f(0)),$$

or

$$H(t,x) := \overline{g}(x-\lambda^2(t))(f(x)-f(\lambda^2(t))),$$

is strictly increasing in x, whenever f is. Moreover functions of this type allow to get rid of the local time $\ell^0(R-\lambda^2)$. Below, we will apply Gronwall's inequality (see Theorem 2.12) to functions

$$g(t) = E[H(t,S_t)-H(t,I_t)],$$

using the Itô-Tanaka formula of Lemma 2.9 (resp. apply Peskir's Itô-Tanaka formula in Theorem 3.1), and derive pathwise uniqueness in Theorem 2.13 (resp. 3.2). For this purpose it is important to find the right sub-/superharmonic functions (see Theorem 2.13, 3.2).

As a simple application of the preceding Lemma 2.9, we present the next corollary. It provides for some special λ's a different proof of the fact that is derived in Remark 2.3(ii) for general time dependent λ. The idea for its proof is similar to the idea used in [3] to show that the p-skew Brownian motion doesn't exist if $|p|>1$.

Corollary 2.11 *(i) Let* $R_0 = \lambda^2(0)$*, and* $d\lambda^2(t) = \frac{\sigma^2}{4}\left\{\delta - b\lambda^2(s)\right\} ds$*, or* $d\lambda^2(t) = \frac{\sigma^2\delta}{4} ds$*. Then there is no solution to (4), if* $|2p-1| > 1$*.*
(ii) Let $0 < R_0 = c \equiv \lambda^2$*, Then there is no solution to (4), if* $|2p-1| > 1$*.*

Proof (i) Let us to the contrary assume that there is a solution. Then we can apply Lemma 2.9(i) with $f(x) = x$, and $\alpha = p - 1$, $\gamma = -p$, if $p > 1$ (resp. $\alpha = 1 - p$, $\gamma = p$, if $p < 0$). If $d\lambda^2(t) = \frac{\sigma^2}{4}\left\{\delta - b\lambda^2(s)\right\} ds$, it follows

$$0 \le H(t, R_t) \le \int_0^t \overline{g}(R_s - \lambda^2(s))\sigma\sqrt{R_s} dW_s, \quad 0 \le t < \infty,$$

which holds pathwise, hence also with t replaced by $t \wedge \tau_n$, where $\tau_n := \inf\{t \ge 0 | |R_t| \ge n\}$. Clearly $\tau_n \nearrow \infty$ P-a.s. It follows that the P-expectation of $H(t, R_t)$ is zero, hence $R \equiv \lambda^2$ P-a.s., which is impossible. In case $d\lambda^2(t) = \frac{\sigma^2\delta}{4} ds$ we first note that $R_0 = \lambda^2(0) \ge 0$, implies P-a.s. $\frac{\sigma^2}{2}|R_t| = \frac{\sigma^2}{2} R_t$ for all t, by Lemma 2.1(ii). Then we apply Lemma 2.9(i) with $f(x) = e^{\frac{bx}{2}}$ and conclude in the same manner as before with $f(x) = x$.
(ii) Let us to the contrary assume that there is a solution. Let $g : \mathbb{R} \to \mathbb{R}^+$ be such that $g(x) = 0$, if $x \le 0$, $g \in C^1(\mathbb{R})$, $g(x) = x^{\frac{\delta}{2}+2}$, for $x \in [0, \frac{c}{2}]$. Suppose further that $g'(x)$ is negative if $x \ge c$, and positive if $x \le c$. Define

$$f_g(x) := \begin{cases} -\int_0^{-x} y^{\frac{\delta}{2}} e^{\frac{by}{2}} dy & \text{for } x < 0; \\ \int_0^x g(y) y^{-\frac{\delta}{2}} e^{\frac{by}{2}} dy & \text{for } x \ge 0. \end{cases}$$

Then $f_g \in C^1(\mathbb{R})$ is strictly increasing, with locally integrable second derivative, and

$$Lf_g(x) = \frac{\sigma^2}{2} x^{1-\frac{\delta}{2}} e^{\frac{bx}{2}} g'(x) \mathbb{I}_{x \ne 0}.$$

Now, we can apply Lemma 2.9(ii) with $f(x) = f_g(x)$, and $\alpha = p - 1$, $\gamma = -p$, if $p > 1$ (resp. $\alpha = 1 - p$, $\gamma = p$, if $p < 0$). It follows

$$0 \le H(t, R_t) = \int_0^t \overline{g}(R_s - c) f_g'(R_s)\sigma\sqrt{R_s} dW_s + \frac{\sigma^2}{2}\int_0^t \overline{g}(R_s - c) R_s^{1-\frac{\delta}{2}} e^{\frac{bR_s}{2}} g'(R_s) ds.$$

By our assumptions on g, the bounded variation part is non-positive. Thus we may conclude analoguously to (i), that $f_g(R_t) = f_g(c)$, and hence $R \equiv c$, which is impossible.

□

We will make use of the following generalization of Gronwall's inequality. Its proof can be found in [1, Appendixes, 5.1. Theorem].

Theorem 2.12 *Let* μ^+ *be a Borel measure (finite on compacts!) on* $[0, \infty)$*, let* $\varepsilon \ge 0$*, and let* g *be a Borel measurable function that is bounded on bounded intervals and satisfies*

$$0 \le g(t) \le \varepsilon + \int_{[0,t)} g(s)\mu^+(ds).$$

Then

$$g(t) \le \varepsilon e^{\mu^+([0,t))}.$$

We are now prepared to formulate our main theorem.

Theorem 2.13 *Let f be a strictly increasing function on $\mathbb{R}$, which is the difference of two convex functions. Let $f^{('')}$ be absolutely continuous. Suppose either that $f'^{+}(0) = f'^{-}(0)$, and $F(t,x) = f(x-\lambda^2(t)) - f(0)$, or that f is continuously differentiable, and $F(t,x) = f(x) - f(\lambda^2(t))$. Suppose further that*

$$(\partial_t + L)F(t,x) = F(t,x)\mu(dt) + sgn(2p-1)\nu(dt) \quad \text{for (a.e.) } x \geq 0, \tag{11}$$

where $\mu(dt) = \mu^+(dt) - \mu^-(dt)$ is a signed Borel measure, with continuous positive part $\mu^+(dt)$, $\nu(dt)$ is a positive Borel measure, and (11) is in the sense of distributions. Then pathwise uniqueness holds for (4).

Proof Let $\overline{g}$ be as in Lemma 2.9, with $\alpha = 1-p$, $\gamma = p$, and

$$H(t,x) := \overline{g}(x-\lambda^2(t))F(t,x),$$

Let $R^{(1)}, R^{(2)}$, be two solutions to (4) with same Brownian motion, same initial condition, and on the same filtered probability space $(\Omega, \mathcal{F}, P)$. By Lemma 2.6 we know that $S = R^{(1)} \vee R^{(2)}$, and $I = R^{(1)} \vee R^{(2)}$, are also solutions to (2). Define the stopping time $\tau_n := \inf\{t \geq 0 : |S_t| \wedge |I_t| \geq n\}$. Then clearly $\tau_n \nearrow \infty$ P-a.s. Applying Lemma 2.9, we obtain for $Z = S$, and for $Z = I$,

$$\begin{aligned} E[H(t\wedge\tau_n, Z_{t\wedge\tau_n})] &= E_r[H(t\wedge\tau_n, Z_0)] \\ &+E\left[\int_0^{t\wedge\tau_n} \overline{g}(Z_s-\lambda^2(s))d\left\{\int_0^s LF(u,Z_u)du - \int_0^s f^{(')}(\widetilde{Z}_u)d\lambda^2(u)\right\}\right], \end{aligned}$$

where either $\widetilde{Z}_s = Z_s - \lambda^2(s)$, or $\widetilde{Z}_s = \lambda^2(s)$. By Lemma 2.6(iii) we know that P-a.s.

$$S_t \mathbb{I}_{\Omega\setminus\{S_t>0\}\cap\{I_t\geq 0\}} = I_t \mathbb{I}_{\Omega\setminus\{S_t>0\}\cap\{I_t\geq 0\}} \quad \forall t \geq 0.$$

We can therefore neglect what happens outside $\{S_t > 0\} \cap \{I_t \geq 0\}$. Thus, by assumption (11)

$$\begin{aligned} &E\left[H(t\wedge\tau_n, S_{t\wedge\tau_n}) - H(t\wedge\tau_n, I_{t\wedge\tau_n})\right] \\ &\quad = E\left[\int_0^{t\wedge\tau_n} \left(H(s,S_s) - H(s,I_s)\right)\mu(ds)\right] \\ &+sgn(2p-1)E\left[\int_0^{t\wedge\tau_n} \left(\overline{g}(S_s-\lambda^2(s)) - \overline{g}(I_s-\lambda^2(s))\right)\nu(ds)\right], \end{aligned}$$

which is further, since $sgn(2p-1)\overline{g}$ is decreasing, estimated from above by

$$E\left[\int_0^{t\wedge\tau_n} \left(H(s,S_s) - H(s,I_s)\right)\mu^+(ds)\right],$$

and then again, since $H(s,S_s) - H(s,I_s)$ is positive, by

$$E\left[\int_0^t \left(H(s\wedge\tau_n, S_{s\wedge\tau_n}) - H(s\wedge\tau_n, I_{s\wedge\tau_n})\right)\mu^+(ds)\right].$$

Applying Fubini's theorem and Theorem 2.12, we obtain that

$$E\left[H(t\wedge\tau_n, S_{t\wedge\tau_n}) - H(t\wedge\tau_n, I_{t\wedge\tau_n})\right] = 0, \quad 0\le t<\infty.$$

Since H increases in the space variable, for any fixed time, it follows that $S_{\cdot\wedge\tau_n}$ and $I_{\cdot\wedge\tau_n}$ are P-indistinguishable. Letting $n\to\infty$ we see that $S = I$, hence $R^{(1)} = R^{(2)}$, and pathwise uniqueness is shown.

□

Corollary 2.14 *Let $\overline{p} := sgn(2p-1)$. Pathwise uniqueness holds for (4), whenever*

$$\overline{p}d\lambda^2(s) \le \overline{p}\frac{\sigma^2}{4}\left\{\delta - \left(\frac{1-\overline{p}}{2}\right)b\lambda^2(s)\right\}ds.$$

Proof Let $f(x) = x$, and $F(t,x) = f(x-\lambda^2(t)) - f(0)$. Then

$$(\partial_t + L)F(t,x) = -\frac{\sigma^2 b}{4}F(t,x) + \frac{\sigma^2}{4}\left(\delta - b\lambda^2(t)\right)dt - d\lambda^2(t)$$

Putting $\mu(dt) = -\frac{\sigma^2 b}{4}dt$, and

$$\nu(dt) = sgn(2p-1)\left\{\frac{\sigma^2}{4}\left(\delta - b\lambda^2(t)\right)dt - d\lambda^2(t)\right\},$$

we conclude by Theorem 2.13 that pathwise uniqueness holds, if

$$sgn(2p-1)d\lambda^2(s) \le sgn(2p-1)\frac{\sigma^2}{4}\left\{\delta - b\lambda^2(s)\right\}ds.$$

This holds for $p\in(0,1)$, and $b\ge 0$. If $2p-1>0$, and $b>0$, we may refine our argument letting $f(x) = e^{\frac{bx}{2}}$. Then

$$(\partial_t + L)F(t,x) = \frac{b}{2}F(t,x)d\left\{\frac{\sigma^2\delta}{4}t - \lambda^2(t)\right\} + \frac{b}{2}\left\{\frac{\sigma^2\delta}{4}dt - d\lambda^2(t)\right\} \quad \text{for a.e. } x\ge 0,$$

and we apply again Theorem 2.13 with $\mu^+(dt) = \nu(dt) = \frac{b}{2}\left\{\frac{\sigma^2\delta}{4}dt - d\lambda^2(t)\right\}$, so that pathwise uniqueness holds whenever

$$d\lambda^2(t) \le \frac{\sigma^2\delta}{4}dt.$$

Putting both cases together, we obtain the statement.

□

3 Pathwise uniqueness in the C^1-case

In the previous section we derived a general criterion for pathwise uniqueness but using only special time dependent functions F built by functions f that do not depend on time (cf. Theorem 2.13). In this section we shall develop a general criterion using "true" time dependent functions. We will use Peskir's Itô-Tanaka formula (see [6, Theorem 2.1]), and will therefore have to assume that $\lambda^2 \in C^1(\mathbb{R}^+)$. We couldn't improve the results of the preceding section, but think that the results derived below may be of use for future purposes.

Let

$$\Gamma(\lambda^2) := \{(s,x) \in \mathbb{R}^+ \times \mathbb{R}^+ | x = \lambda^2(s)\}.$$

Consider the linear operator

$$\mathcal{L}F(t,x) = (\partial_t + L)F(t,x) = \frac{\sigma^2}{2}|x|\partial_{xx}F(t,x) + \frac{\sigma^2}{4}(\delta - bx)\partial_x F(t,x) + \partial_t F(t,x).$$

Let

$$\begin{aligned}\mathcal{M} \;:=\; & C(\mathbb{R}^+ \times \mathbb{R}) \cap \{H \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R} \setminus \Gamma(\lambda^2)) \,|\, \partial_x H(t,\lambda^2(t)\pm),\ \partial_t H(t,\lambda^2(t)\pm), \\ & \text{and } \partial_{xx} H(t,\lambda^2(t)\pm) \text{ exists in } \mathbb{R}\}.\end{aligned}$$

By Lemma 2.1(iii)

$$\left(\int_0^t G(s,R_s)\mathcal{L}H(s,R_s)ds\right)_{t\geq 0}$$

is well-defined for any $H \in \mathcal{M}$, G bounded and measurable.

The next Lemma is a direct consequence of [6, Theorem 2.1].

Lemma 3.1 *Let $F \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$, such that $F(t,\lambda^2(t)) = 0$ for all $t \geq 0$. Set $H(t,x) = \overline{g}(x - \lambda^2(s))F(t,x)$, where $\overline{g}$ is defined as in Lemma 2.9. Then $H \in \mathcal{M}$, and*

$$\begin{aligned}& H(t,R_t) = H(0,R_0) + \int_0^t \overline{g}(R_s - \lambda^2(s))\partial_x F(s,R_s)\sigma\sqrt{|R_s|}dW_s \\ & + \int_0^t \overline{g}(R_s - \lambda^2(s))\mathcal{L}F(s,R_s)ds + (\alpha p - \gamma(1-p))\int_0^t \partial_x F(s,\lambda^2(s))d\ell_s^0(R - \lambda^2).\end{aligned}$$

Proof The first statement is clear. In order to prove the second, first observe that by [6, Theorem 2.1], the following Itô-formula holds:

$$
\begin{aligned}
H(t,R_t) &= H(0,R_0) + \int_0^t \frac{1}{2}(\partial_t H(s,R_s+) + \partial_t H(s,R_s-))ds \\
&\quad + \int_0^t \frac{1}{2}(\partial_x H(s,R_s+) + \partial_x H(s,R_s-))dR_s \\
&\quad + \frac{1}{2}\int_0^t \partial_{xx} H(s,R_s)\mathbb{I}_{\{R_s\neq\lambda^2(s)\}}d[R]_s \\
&\quad + \frac{1}{2}\int_0^t (\partial_x H(s,R_s+) - \partial_x H(s,R_s-))d\ell_s^0(R-\lambda^2).
\end{aligned}
$$

Since $\int_0^t \mathbb{I}_{\{R_s=\lambda^2(s)\}}ds = 0$ P-a.s. by Lemma 2.1(iv), we obtain P-a.s.

$$
\begin{aligned}
H(t,R_t) &= H(0,R_0) + \int_0^t \partial_t H(s,R_s)ds \\
&\quad + \int_0^t \partial_x H(s,R_s)\sigma\sqrt{|R_s|}dW_s \\
&\quad + \int_0^t \frac{\sigma^2}{4}(\delta - bR_s)\partial_x H(s,R_s)ds \\
&\quad + \int_0^t \frac{1}{2}(\partial_x H(s,\lambda^2(s)+) + \partial_x H(s,\lambda^2(s)-))(2p-1)d\ell_s^0(R-\lambda^2) \\
&\quad + \int_0^t \frac{\sigma^2}{2}|R_s|\partial_{xx} H(s,R_s)ds \\
&\quad + \frac{1}{2}\int_0^t (\partial_x H(s,R_s+) - \partial_x H(s,R_s-))d\ell_s^0(R-\lambda^2). \qquad (12)
\end{aligned}
$$

We have

$$\partial_x H(s,\lambda^2(s)+) - \partial_x H(s,\lambda^2(s)-) = (\alpha-\gamma)\partial_x F(s,\lambda^2(s)),$$

and

$$\partial_x H(s,\lambda^2(s)+) + \partial_x H(s,\lambda^2(s)-) = (\alpha+\gamma)\partial_x F(s,\lambda^2(s)).$$

Replacing these terms in (12) we get

$$
\begin{aligned}
H(t,R_t) &= H(0,R_0) + \int_0^t \partial_x H(s,R_s)\sigma\sqrt{|R_s|}dW_s + \int_0^t \overline{g}(R_s-\lambda^2(s))\mathcal{L}F(s,R_s)ds \\
&\quad + \left((2p-1)\frac{\alpha+\gamma}{2} + \frac{\alpha-\gamma}{2}\right)\int_0^t \partial_x F(s,\lambda^2(s))d\ell_s^0(R-\lambda^2)
\end{aligned}
$$

as stated.

□

Theorem 3.2 *Let $\beta(t) \in L^1_{loc}(\mathbb{R})$. Let $F \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ be such that $F(t,x)$ is strictly increasing in x for every fixed $t \geq 0$, and*

$$F(t, \lambda^2(t)) = 0 \quad \forall t \geq 0.$$

Let $H(t,x) := \overline{g}(x - \lambda^2(s))F(t,x)$, where $\overline{g}$ is as in Lemma 2.9, with $\alpha = 1-p$, $\gamma = p$. Suppose further that

$$\mathcal{L}H(t,x) = \beta(t)H(t,x) + \overline{g}(x - \lambda^2(t))v(t), \text{ for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^+ \setminus \Gamma(\lambda^2)$$

where $v \geq 0$, if $p > \frac{1}{2}$, or $v \leq 0$, if $p < \frac{1}{2}$. Then pathwise uniqueness holds for (4).

Proof The proof is exactly the same than the proof of Theorem 2.13. We therefore omit it.

□

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