INDIVIDUAL BASED MODEL WITH COMPETITION IN SPATIAL ECOLOGY*

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Abstract.
We analyze an interacting particle system with a Markov evolution of birth-and-death type. We have shown that a local competition mechanism (realized via a density dependent mortality) leads to a globally regular behavior of the population in course of the stochastic evolution.

Key words. Continuous systems, spatial birth-and-death processes, correlation functions, individual based models, spatial plant ecology

AMS subject classifications. 60J80; 60K35; 82C21; 82C22

1. Introduction. Complex systems theory is a quickly growing interdisciplinary area with a very broad spectrum of motivations and applications. Having in mind biological applications, S. Levin (see [26]) characterized complex adaptive systems by such properties as diversity and individuality of components, localized interactions among components, and the outcomes of interactions used for replication or enhancement of components. In the study of these systems, proper language and techniques are delivered by the interacting particle models which form a rich and powerful direction in modern stochastic and infinite dimensional analysis. Interacting particle systems have a wide use as models in condensed matter physics, chemical kinetics, population biology, ecology (individual based models), sociology and economics (agent based models).

In this paper we consider an individual based model (IBM) in spatial ecology introduced by Bolker and Pacala [4, 5], Dieckmann and Law [6] (BDLP model). A population in this model is represented by a configuration of motionless organisms (plants) located in an infinite habit (an Euclidean space in our considerations). The habit is considered to be a continuous space as opposed to discrete spatial lattices used in the most of mathematical models of interacting particle systems. We need the infinite habit to avoid boundary effects in the population evolution and the latter moment is quite similar to the necessity to work in the thermodynamic limit for models of statistical physics. Let us also mention a recent paper [2] in which a modification of the BDLP model for the case of moving organisms (e.g., branching diffusion of the plankton) was considered.

A general IBM in the plant ecology is a stochastic Markov process in the configuration space with events comprising birth and death of the configuration points, i.e., we are dealing with a birth-and-death process in the continuum. In the particular case of the BDLP model, each plant produces seeds independently of others and then these seeds are distributed in the space accordingly to a dispersion kernel a^+. This part of the process may be considered as a kind of the spatial branching. In the same time, the model includes also a mortality mechanism. The mortality intensity consists of

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two parts. The first one corresponds to a constant intrinsic mortality value $m > 0$ s.t. any plant dies independently of others after a random time (exponentially distributed with parameter m). The second part in the mortality rate is density dependent. The latter is expressed in terms of a competition kernel a^- which describes an additional mortality rate for any given point of the configuration coming from the rest of the population, see Section 3 for the precise description of the model, in particular, (3.6). The latter formula gives the heuristic form of the Markov generator in the BDLP model.

Assuming the existence of the corresponding Markov process, we derive in Section 5 an evolution equation for correlation functions $k_t^{(n)}, n \geq 1$, of the considered model. In [4, 5], [6] this system was called the system of spatial moment equations for plant competition and, actually, this system itself was taking as a definition of the dynamics in the BDLP model. The mathematical structure of the correlation functions evolution equation is close to other well-known hierarchical systems in mathematical physics, e.g., BBGKY hierarchy for the Hamiltonian dynamics (see, e.g. [3]) or the diffusion hierarchy for the gradient stochastic dynamic in the continuum (see e.g. [21]). As in all hierarchical chains of equations, we can not expect the explicit form of the solution, and even more, the existence problem for these equations is a highly delicate question.

There is an approximative approach to produce an information about the behavior of the solutions to the hierarchical chains. This approach is called the closure procedure and consists of the following steps. The first step is to cut all correlation functions of the higher orders and the second one is to subscribe the rest correlation functions by the properly factorized correlation functions of the lower orders. As result, one obtains a finite system of non-linear equations instead of the original linear but infinite system of a hierarchical type. This closure procedure is essentially non-unique, see [7].

The aim of this paper is to study the moment equations for the BDLP model by methods of functional analysis and analysis on the configuration spaces developed in [13], [14], [15] and already applied to the non-equilibrium birth-and-death type continuous space stochastic dynamics in [16], [18]. We obtain some rigorous results concerning the existence and properties of the solution for different classes of initial conditions. One of the main question we clarify in the paper concerns the role of the competition mechanism in the regulation of the spatial structure of an evolving population. More precisely, considering the model without competition, i.e., the case $a^- \equiv 0$, we arrive in the situation of the so-called continuous contact model [9], [17], [22]. In the ecological framework, this model describes free growth of a plant population with the given constant mortality. We note that (independently on the value of the mortality $m > 0$) the considered contact model exhibits very strong clustering that is reflected in the bound (3.5) on the correlation functions at any moment of time $t > 0$. Note that this effect on the level of the computer simulation was discovered already in [2] and now it has the rigorous mathematical formulation and clarification. A direct consequence of the competition in the model is the suppression of such clustering. Namely, assuming the strong enough competition and the big intrinsic mortality m, we prove the sub-Poissonian bound for the solution to the moment equations provided such bound was true for the initial state. Moreover, we clarify specific influences of the constant and the density dependent mortality intensities separately. More precisely, the big enough intrinsic mortality m gives a uniform in time bound for each correlation function and the strong competition results ensure the regular spatial distribution of the typical configuration for any moment of time that is reflected in the sub-Poissonian bound. Joint influence of the intrinsic mortality and the competition leads to the existence of the unique invariant measure for our model which is just Dirac measure concentrated on the empty configuration. The latter means that the corresponding stochastic evolution of the population is asymptotically exhausting.

We would like to mention also the work [10] in which the BDLP model was studied in the case of the bounded habit in the stochastic analysis framework. The latter case differs essentially from the model we consider in the present paper as well as main problems studied in [10], which are related to the scaling limits for the considered processes.

2. General facts and notations. Let $\mathcal{B}(\mathbb{R}^d)$ be the family of all Borel sets in $\mathbb{R}^d$. $\mathcal{B}_b(\mathbb{R}^d)$ denotes the system of all bounded sets in $\mathcal{B}(\mathbb{R}^d)$.

The space of n-point configuration is

$$\Gamma_0^{(n)} = \Gamma_{0,\mathbb{R}^d}^{(n)} := \left\{ \eta \subset \mathbb{R}^d \middle| \, |\eta| = n \right\}, \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

where $|A|$ denotes the cardinality of the set A. The space $\Gamma_\Lambda^{(n)} := \Gamma_{0,\Lambda}^{(n)}$ for $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ is defined analogously to the space $\Gamma_0^{(n)}$. As a set, $\Gamma_0^{(n)}$ is equivalent to the symmetrization of

$$\widetilde{(\mathbb{R}^d)^n} = \left\{ (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n \middle| \, x_k \neq x_l \text{ if } k \neq l \right\},$$

i.e. $\widetilde{(\mathbb{R}^d)^n}/S_n$, where S_n is the permutation group over $\{1, \ldots, n\}$. Hence one can introduce the corresponding topology and Borel σ- algebra, which we denote by $O(\Gamma_0^{(n)})$ and $\mathcal{B}(\Gamma_0^{(n)})$, respectively. Also one can define a measure $m^{(n)}$ as an image of the product of Lebesgue measures $dm(x) = dx$ on $\left(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\right)$.

The space of finite configurations

$$\Gamma_0 := \bigsqcup_{n \in \mathbb{N}_0} \Gamma_0^{(n)}$$

is equipped with the topology which has structure of disjoint union. Therefore, one can define the corresponding Borel σ-algebra $\mathcal{B}(\Gamma_0)$.

A set $B \in \mathcal{B}(\Gamma_0)$ is called bounded if there exists $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $B \subset \bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}$. The Lebesgue—Poisson measure λ_z on Γ_0 is defined as

$$\lambda_z := \sum_{n=0}^{\infty} \frac{z^n}{n!} m^{(n)}.$$

Here $z > 0$ is the so called activity parameter. The restriction of λ_z to Γ_Λ will be also denoted by λ_z.

The configuration space

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d \;\middle|\; |\gamma \cap \Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_b(\mathbb{R}^d) \right\}$$

is equipped with the vague topology. It is a Polish space (see e.g. [15]). The corresponding Borel σ-algebra $\mathcal{B}(\Gamma)$ is defined as the smallest σ-algebra for which all mappings $N_\Lambda : \Gamma \to \mathbb{N}_0$, $N_\Lambda(\gamma) := |\gamma \cap \Lambda|$ are measurable, i.e.,

$$\mathcal{B}(\Gamma) = \sigma\left(N_\Lambda \middle| \Lambda \in \mathcal{B}_b(\mathbb{R}^d)\right).$$

One can also show that Γ is the projective limit of the spaces $\{\Gamma_\Lambda\}_{\Lambda\in\mathcal{B}_b(\mathbb{R}^d)}$ w.r.t. the projections $p_\Lambda:\Gamma\to\Gamma_\Lambda$, $p_\Lambda(\gamma):=\gamma_\Lambda$, $\Lambda\in\mathcal{B}_b(\mathbb{R}^d)$.

The Poisson measure π_z on $(\Gamma,\mathcal{B}(\Gamma))$ is given as the projective limit of the family of measures $\{\pi_z^\Lambda\}_{\Lambda\in\mathcal{B}_b(\mathbb{R}^d)}$, where π_z^Λ is the measure on Γ_Λ defined by $\pi_z^\Lambda:=e^{-zm(\Lambda)}\lambda_z$.

We will use the following classes of functions: $L^0_{\mathrm{ls}}(\Gamma_0)$ is the set of all measurable functions on Γ_0 which have a local support, i.e. $G\in L^0_{\mathrm{ls}}(\Gamma_0)$ if there exists $\Lambda\in\mathcal{B}_b(\mathbb{R}^d)$ such that $G\upharpoonright_{\Gamma_0\setminus\Gamma_\Lambda}=0$; $B_{\mathrm{bs}}(\Gamma_0)$ is the set of bounded measurable functions with bounded support, i.e. $G\upharpoonright_{\Gamma_0\setminus B}=0$ for some bounded $B\in\mathcal{B}(\Gamma_0)$.

On Γ we consider the set of cylinder functions $\mathcal{F}L^0(\Gamma)$, i.e. the set of all measurable functions G on $\big(\Gamma,\mathcal{B}(\Gamma)\big)$ which are measurable w.r.t. $\mathcal{B}_\Lambda(\Gamma)$ for some $\Lambda\in\mathcal{B}_b(\mathbb{R}^d)$. These functions are characterized by the following relation: $F(\gamma)=F\upharpoonright_{\Gamma_\Lambda}(\gamma_\Lambda)$.

The following mapping between functions on Γ_0, e.g. $L^0_{\mathrm{ls}}(\Gamma_0)$, and functions on Γ, e.g. $\mathcal{F}L^0(\Gamma)$, plays the key role in our further considerations:

$$KG(\gamma):=\sum_{\eta\Subset\gamma}G(\eta),\quad \gamma\in\Gamma, \tag{2.1}$$

where $G\in L^0_{\mathrm{ls}}(\Gamma_0)$, see e.g. [13, 24, 25]. The summation in the latter expression is taken over all finite subconfigurations of γ, which is denoted by the symbol $\eta\Subset\gamma$. The mapping K is linear, positivity preserving, and invertible, with

$$K^{-1}F(\eta):=\sum_{\xi\subset\eta}(-1)^{|\eta\setminus\xi|}F(\xi),\quad \eta\in\Gamma_0. \tag{2.2}$$

Let $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be the set of all probability measures μ on $\big(\Gamma,\mathcal{B}(\Gamma)\big)$ which have finite local moments of all orders, i.e. $\int_\Gamma|\gamma_\Lambda|^n\mu(d\gamma)<+\infty$ for all $\Lambda\in\mathcal{B}_b(\mathbb{R}^d)$ and $n\in\mathbb{N}_0$. A measure ρ on $\big(\Gamma_0,\mathcal{B}(\Gamma_0)\big)$ is called locally finite iff $\rho(A)<\infty$ for all bounded sets A from $\mathcal{B}(\Gamma_0)$. The set of such measures is denoted by $\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$.

One can define a transform $K^*:\mathcal{M}^1_{\mathrm{fm}}(\Gamma)\to\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$, which is dual to the K-transform, i.e., for every $\mu\in\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, $G\in\mathcal{B}_{\mathrm{bs}}(\Gamma_0)$ we have

$$\int_\Gamma KG(\gamma)\mu(d\gamma)=\int_{\Gamma_0}G(\eta)\,(K^*\mu)(d\eta).$$

The measure $\rho_\mu:=K^*\mu$ is called the correlation measure of μ.

As shown in [13] for $\mu\in\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ and any $G\in L^1(\Gamma_0,\rho_\mu)$ the series (2.1) is μ-a.s. absolutely convergent. Furthermore, $KG\in L^1(\Gamma,\mu)$ and

$$\int_{\Gamma_0}G(\eta)\,\rho_\mu(d\eta)=\int_\Gamma(KG)(\gamma)\,\mu(d\gamma). \tag{2.3}$$

A measure $\mu\in\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is called locally absolutely continuous w.r.t. π_z iff $\mu_\Lambda:=\mu\circ p_\Lambda^{-1}$ is absolutely continuous with respect to π_z^Λ for all $\Lambda\in\mathcal{B}_\Lambda(\mathbb{R}^d)$. In this case $\rho_\mu:=K^*\mu$ is absolutely continuous w.r.t λ_z. We denote

$$k_\mu(\eta):=\frac{d\rho_\mu}{d\lambda_z}(\eta),\quad \eta\in\Gamma_0.$$

The functions

$$k_\mu^{(n)}:(\mathbb{R}^d)^n\longrightarrow\mathbb{R}_+ \tag{2.4}$$

$$k_\mu^{(n)}(x_1,\dots,x_n) := \begin{cases} k_\mu(\{x_1,\dots,x_n\}), & \text{if } (x_1,\dots,x_n) \in \widetilde{(\mathbb{R}^d)^n} \\ 0, & \text{otherwise} \end{cases}$$

are the correlation functions well known in statistical physics, see e.g [28], [29].

We recall now the so-called Minlos lemma which plays very important role in our calculations (cf., [20]).

LEMMA 2.1. *Let $n \in \mathbb{N}$, $n \geq 2$, and $z > 0$ be given. Then*

$$\int_{\Gamma_0} \dots \int_{\Gamma_0} G(\eta_1 \cup \dots \cup \eta_n) H(\eta_1,\dots,\eta_n) d\lambda_z(\eta_1)\dots d\lambda_z(\eta_n)$$
$$= \int_{\Gamma_0} G(\eta) \sum_{(\eta_1,\dots,\eta_n)\in\mathcal{P}_n(\eta)} H(\eta_1,\dots,\eta_n) d\lambda_z(\eta)$$

for all measurable functions $G : \Gamma_0 \mapsto \mathbb{R}$ and $H : \Gamma_0 \times \dots \times \Gamma_0 \mapsto \mathbb{R}$ with respect to which both sides of the equality make sense. Here $\mathcal{P}_n(\eta)$ denotes the set of all ordered partitions of η in n parts, which may be empty.

3. Description of the model. In the present paper we study the special case of the general birth-and-death processes in continuum. The spatial birth-and-death processes describe evolution of configurations in $\mathbb{R}^d$, in which points of configurations (particles, individuals, elements) randomly appear (born) and disappear (die) in the space. Among all birth-and-death processes we will distinguish those in which new particles appear only from existing ones. These processes correspond to the models of the spatial ecology.

The simplest example of such processes is the so-called "free growth" dynamics. During this stochastic evolution the points of configuration independently create new ones distributed in the space according to a dispersion probability kernel $0 \leq a^+ \in L^1(\mathbb{R}^d)$ which is an even function. Any existing point has an infinite life time, i. e. they do not die. Heuristically, the Markov pre-generator of this birth process has the following form:

$$(L_+F)(\gamma) = \varkappa^+ \sum_{y\in\gamma} \int_{\mathbb{R}^d} a^+(x-y) D_x^+ F(\gamma) dx,$$

where

$$D_x^+ F(\gamma) = F(\gamma \cup x) - F(\gamma),$$

and $\varkappa^+ > 0$ is some positive constant.

The existence of the process associated with L_+ can be shown using the same technique as in [9], [22]. Let μ_t be the corresponding evolution of measures in time on $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$. By $k_t^{(n)}$, $n \geq 0$ we denote the dynamics of the corresponding n-th order correlation functions (provided they exist). Note, that each of such functions describes the density of the system at the moment t.

Then, using (2.3), for any continuous φ on $\mathbb{R}^d$ with bounded support, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d} \varphi(x) k_t^{(1)}(x) dx = \frac{d}{dt}\int_\Gamma \langle\varphi,\gamma\rangle d\mu_t(\gamma) = \int_\Gamma L_+\langle\varphi,\gamma\rangle d\mu_t(\gamma)$$
$$= \varkappa^+ \int_\Gamma \langle a^+ * \varphi, \gamma\rangle d\mu_t(\gamma) = \varkappa^+ \int_{\mathbb{R}^d} (a^+ * \varphi)(x) k_t^{(1)}(x) dx$$
$$= \varkappa^+ \int_{\mathbb{R}^d} \varphi(x)(a^+ * k_t^{(1)})(x) dx,$$

where $*$ denotes the classical convolution on $\mathbb{R}^d$. Hence, $k_t^{(1)}$ grows exponentially in t. In particular, for the translation invariant case one has $k_0^{(1)}(x) \equiv k_0^{(1)} > 0$ and as a result

$$k_t^{(1)} = e^{\varkappa^+ t} k_0^{(1)}. \tag{3.1}$$

One of the possibilities to prevent the density growth of the system is to include the death mechanism. The simplest one is described by the independent death rate (mortality) $m > 0$. This means that any element of a population has an independent exponentially distributed with parameter m random life time. The independent death together with the independent creation of new particles by already existing ones describe the so-called *contact model* in the continuum, see e.g. [22]. The pre-generator of such model is given by the following expression:

$$\begin{aligned}(L_{\mathrm{CM}}F)(\gamma) &= m \sum_{x\in\gamma} D_x^- F(\gamma) + (L_+F)(\gamma)\\ &= m \sum_{x\in\gamma} D_x^- F(\gamma) + \varkappa^+ \sum_{y\in\gamma} \int_{\mathbb{R}^d} a^+(x-y) D_x^+ F(\gamma) dx,\end{aligned}$$

where

$$D_x^- F(\gamma) = F(\gamma \setminus x) - F(\gamma).$$

The Markov process associated with the generator L_{CM} was constructed in [22]. This construction was generalized in [9] for more general classes of functions a^+. Let us note, that the contact model in the continuum may be used in the epidemiology to model the infection spreading process. The values of this process represent the states of the infected population. This is analog of the contact process on a lattice. Of course, such interpretation is not in the spatial ecology concept. On the other hand, contact process is a spatial branching process with a given mortality rate.

The dynamics of correlation functions in the contact model was considered in [17]. Namely, taking $m = 1$ for correctness, we have for any $n \geq 1$, $t > 0$ the correlation function of n-th order has the following form

$$\begin{aligned}k_t^{(n)}(x_1, \ldots, x_n) = &\, e^{n(\varkappa^+-1)t} \left[\bigotimes_{i=1}^n e^{tL^i_{a^+}}\right] k_0^{(n)}(x_1,\ldots,x_n)\\ &+ \varkappa^+ \int_0^t e^{n(\varkappa^+-1)(t-s)} \left[\bigotimes_{i=1}^n e^{(t-s)L^i_{a^+}}\right]\\ &\times \sum_{i=1}^n k_s^{(n-1)}(x_1,\ldots,\check{x}_i,\ldots,x_n) \sum_{j:\, j\neq i} a^+(x_i - x_j) ds,\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}&L^i_{a^+} k^{(n)}(x_1,\ldots,x_n)\\ =&\varkappa^+ \int_{\mathbb{R}^d} a^+(x_i - y) \left[k^{(n)}(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n) - k^{(n)}(x_1,\ldots,x_n)\right] dy\end{aligned}$$

and the symbol $\check{x}_i$ means that the i-th coordinate is omitted. Note that $L^i_{a^+}$ is a Markov generator and the corresponding semigroup (in L^∞ space) preserves positivity.

It was also shown in [17], that if there exists a constant $C > 0$ (independent of n) such that for any $n \geq 0$ and $(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$

$$k_0^{(n)}(x_1, \ldots, x_n) \leq n!\, C^n,$$

then for any $t \geq 0$ and almost all (a.a.) $(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$ (w.r.t. Lebesgue measure) the following estimate holds for all $n \geq 0$

$$k_t^{(n)}(x_1, \ldots, x_n) \leq \varkappa^+(t)^n (1 + a_0)^n e^{n(\varkappa^+ - 1)t} (C + t)^n n! \tag{3.3}$$

Here

$$a_0 = \|a\|_{L^\infty(\mathbb{R}^d)}, \qquad \varkappa^+(t) := \max \Big[1,\, \varkappa^+,\, \varkappa^+ e^{-(\varkappa^+ - 1)t}\Big].$$

For the translation invariant case the value $\varkappa^+ = 1$ is critical. Namely, from (3.2) we deduce that

$$k_t^{(1)} = e^{(\varkappa^+ - 1)t} k_0^{(1)}. \tag{3.4}$$

Therefore, for $\varkappa^+ < 1$ the density will exponentially decrease to 0 (as $t \to \infty$), for $\varkappa^+ > 1$ the density will exponentially increase to ∞, and for $\varkappa^+ = 1$ the density will be a constant. One can easily see from the estimate (3.3) that, in the case $\varkappa^+ < 1$, the correlation functions of all orders decrease to 0 as $t \to \infty$. On the other hand, for fixed t, the estimate (3.3) implies factorial bound in n for $k_t^{(n)}$. As result, we may expect the clustering of our system. To show clustering we start from the Poisson distribution of particles and obtain an estimate from below for the time evolutions of correlations between particles in a small region.

Hence, let $\varkappa^+ < 1$, $k_0^{(n)} = C^n$. Let B is some bounded domain of $\mathbb{R}^d$ such that

$$\alpha := \inf_{x,y \in B} a^+(x - y) > 0.$$

Let $\beta = \min\{\alpha\varkappa^+, C\}$. For any $\{x_1, x_2\} \subset B$, formula (3.2) implies

$$k_t^{(2)}(x_1, x_2) \geq 2C\varkappa^+\alpha \int_0^t e^{2(\varkappa^+ - 1)(t-s)} ds \geq 2\beta^2 t e^{2(\varkappa^+ - 1)t}.$$

We consider $t \geq 1$. One can prove by induction that for any $\{x_1, \ldots, x_n\} \subset B$, $n \geq 2$

$$k_t^{(n)}(x_1, \ldots, x_n) \geq \beta^n e^{n(\varkappa^+ - 1)t} n! \tag{3.5}$$

Indeed, for $n = 2$ this statement has been proved. Suppose that (3.5) holds for $n - 1$. Then, by (3.2), one has

$$\begin{aligned} k_t^{(n)}(x_1, \ldots, x_n) &\geq \varkappa^+ \int_0^t e^{n(\varkappa^+ - 1)(t-s)} n\beta^{(n-1)} e^{(n-1)(\varkappa^+ - 1)s} (n-1)!(n-1)\alpha ds \\ &\geq \beta^n n! e^{n(\varkappa^+ - 1)t} \int_0^t e^{-(\varkappa^+ - 1)s} ds \geq \beta^n e^{n(\varkappa^+ - 1)t} n!\,. \end{aligned}$$

As it was mentioned before, the later bound shows the clustering in the contact model. All previous consideration may be extended for the case $m \neq 1$: we should only replace 1 by m in the previous calculations.

As a conclusion we have: the presence of mortality ($m > \varkappa^+$) in the free growth model prevents the growth of density, i. e. the correlation functions of all orders decay in time. But it doesn't influence on the clustering in the system. One of the possibilities to prevent such clustering is to consider the so-called density dependent death rate. Namely, let us consider the following pre-generator:

$$(LF)(\gamma) = \sum_{x\in\gamma}\left[m + \varkappa^- \sum_{y\in\gamma\setminus x} a^-(x-y)\right] D_x^- F(\gamma) \\ + \varkappa^+ \int_{\mathbb{R}^d} \sum_{y\in\gamma} a^+(x-y) D_x^+ F(\gamma)dx. \tag{3.6}$$

Here $0 \le a^- \in L^1(\mathbb{R}^d)$ is an arbitrary, even function such that

$$\int_{\mathbb{R}^d} a^-(x)dx = 1$$

(in other words, a^- is a probability density) and $\varkappa^- > 0$ is some positive constant. It is easy to see that the operator L is well-defined, for example, on $\mathcal{F}L^0(\Gamma)$.

The generator (3.6) describes the Bolker—Dieckmann—Law—Pacala (BDLP) model, which was introduced in [4, 5, 6]. During the corresponding stochastic evolution the birth of individuals occurs independently and the death is ruled not only by the global regulation (mortality) but also by the local regulation with the kernel $\varkappa^- a^-$. This regulation may be described as a competition (e.g., for resources) between individuals in the population.

The main result of this article is presented in Section 5, Theorem 5.1. It may be informally stated in the following way:

If the mortality m and the competition kernel $\varkappa^- a^-$ are large enough, then the dynamics of correlation functions associated with the pre-generator (3.6) *preserves (sub-)Poissonian bound for correlation functions for all times.*

In particular, it prevents clustering in the model.

In the next sections we explain how to prove this fact. In the last section of the present paper we discuss the necessity to consider "large enough" death.

4. Semigroup for the symbol of the generator. The problem of the construction of the corresponding process in Γ concerns the possibility to construct the semigroup associated with L. This semigroup determines the solution to the Kolmogorov equation, which formally (only in the sense of action of operator) has the following form:

$$\frac{dF_t}{dt} = LF_t, \qquad F_t \big|_{t=0} = F_0. \tag{4.1}$$

To show that L is a generator of a semigroup in some reasonable functional spaces on Γ seems to be a difficult problem. This difficulty is hidden in the complex structure of the non-linear infinite dimensional space Γ.

In various applications the evolution of the corresponding correlation functions (or measures) helps already to understand the behavior of the process and gives candidates for invariant states. The evolution of correlation functions of the process is related heuristically to the evolution of states of our IPS. The latter evolution is formally given as a solution to the dual Kolmogorov equation (Fokker—Planck equation):

$$\frac{d\mu_t}{dt} = L^*\mu_t, \qquad \mu_t \big|_{t=0} = \mu_0, \tag{4.2}$$

where L^* is the adjoint operator to L on $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, provided, of course, that it exists.

In the recent paper [16], the authors proposed the analytic approach for the construction of a non-equilibrium process on Γ, which uses deeply the harmonic analysis technique. In the present paper we follow the scheme proposed in [16] in order to construct the evolution of correlation functions. The existence problem for the evolution of states in $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ and, as a result, of the corresponding process on Γ is not realized in this paper. It seems to be a very technical question and remains open.

Following the general scheme, first we should construct the evolution of functions which corresponds to the *symbol* (K-image) $\hat{L} = K^{-1}LK$ of the operator L in L^1-space on Γ_0 w.r.t. the weighted Lebesgue—Poisson measure. This weight is crucial for the corresponding evolution of correlation functions. It determines the growth of correlation functions in time and space. Below we start the detailed realization of the discussed scheme.

Let us set for $\eta \in \Gamma_0$

$$E^{a^\#}(\eta) := \sum_{x\in\eta}\sum_{y\in\eta\setminus x} a^\#(x-y),$$

where $a^\#$ denotes either a^- or a^+.

Proposition 4.1. *The image of L under the K-transform (or symbol of the operator L) on functions $G \in B_{bs}(\Gamma_0)$ has the following form*

$$\begin{aligned}
(\widehat{L}G)(\eta) &:= (K^{-1}LKG)(\eta)\\
&= -\Big(m|\eta| + \varkappa^- E^{a^-}(\eta)\Big)G(\eta) - \varkappa^- \sum_{x\in\eta}\sum_{y\in\eta\setminus x} a^-(x-y)G(\eta\setminus y)\\
&+ \varkappa^+ \int_{\mathbb{R}^d}\sum_{y\in\eta} a^+(x-y)G((\eta\setminus y)\cup x)dx\\
&+ \varkappa^+ \int_{\mathbb{R}^d}\sum_{y\in\eta} a^+(x-y)G(\eta\cup x)dx.
\end{aligned}$$

For the proof see [8].

With the help of Proposition 4.1, we derive the evolution equation for *quasi-observables* (functions on Γ_0) corresponding to the Kolmogorov equation (4.1). It has the following form

$$\frac{dG_t}{dt} = \widehat{L}G_t, \qquad G_t \big|_{t=0} = G_0. \tag{4.3}$$

Then in the way analogous to those in which the corresponding Fokker-Planck equation (4.2) was determined for (4.1) we get the evolution equation for the correlation functions corresponding to the equation (4.3):

$$\frac{dk_t}{dt} = \widehat{L}^*k_t, \qquad k_t \big|_{t=0} = k_0. \tag{4.4}$$

The precise form of the adjoint operator $\hat{L}^*$ will be given in Section 5. It is very important to emphasize that in the papers [4, 5] the equation (4.4) was obtained from quit heuristic arguments and, moreover, it was considered as the definition for the evolution of the BDLP model.

Let λ be the Lebesgue-Poisson measure on Γ_0 with activity parameter equal to 1.

For arbitrary and fixed $C > 0$ we consider the operator $\widehat{L}$ as a pre-generator of a semigroup in the functional space

$$\mathcal{L}_C := L^1(\Gamma_0, C^{|\eta|}\lambda(d\eta)). \tag{4.5}$$

In this section, symbol $\|\cdot\|_C$ stands for the norm of the space (4.5).

For any $\omega > 0$ we introduce the set $\mathcal{H}(\omega, 0)$ of all densely defined closed operators T on $\mathcal{L}_C$, the resolvent set $\rho(T)$ of which contains the sector

$$\mathrm{Sect}\left(\frac{\pi}{2} + \omega\right) := \left\{\zeta \in \mathbb{C} \;\middle|\; |\arg \zeta| < \frac{\pi}{2} + \omega\right\}, \quad \omega > 0$$

and for any $\varepsilon > 0$

$$||(T - \zeta 1\!\!1)^{-1}|| \le \frac{M_\varepsilon}{|\zeta|}, \quad |\arg \zeta| \le \frac{\pi}{2} + \omega - \varepsilon,$$

where M_ε does not depend on ζ.

Let $\mathcal{H}(\omega, \theta)$, $\theta \in \mathbb{R}$ denotes the set of all operators of the form $T = T_0 + \theta$ with $T_0 \in \mathcal{H}(\omega, 0)$.

REMARK 4.1. *It is well-known (see e.g., [12]), that any $T \in \mathcal{H}(\omega, \theta)$ is a generator of a semigroup $U(t)$ which is holomorphic in the sector $|\arg t| < \omega$. The function $U(t)$ is not necessarily uniformly bounded, but it is quasi-bounded, i.e.*

$$||U(t)|| \le \mathrm{const}\, |e^{\theta t}|$$

in any sector of the form $|\arg t| \le \omega - \varepsilon$.

PROPOSITION 4.2. *For any $C > 0$, $m > 0$, and $\varkappa^- > 0$ the operator*

$$(L_0 G)(\eta) := -\left(m|\eta| + \varkappa^- E^{a^-}(\eta)\right) G(\eta),$$
$$D(L_0) = \left\{G \in \mathcal{L}_C \;\middle|\; \left(m|\eta| + \varkappa^- E^{a^-}(\eta)\right) G(\eta) \in \mathcal{L}_C\right\}$$

is a generator of a contraction semigroup on $\mathcal{L}_C$. Moreover, $L_0 \in \mathcal{H}(\omega, 0)$ for all $\omega \in (0, \frac{\pi}{2})$.

Proof. It is not difficult to show that L_0 is a densely defined and closed operator in $\mathcal{L}_C$.

Let $0 < \omega < \frac{\pi}{2}$ be arbitrary and fixed. Clear, that for all $\zeta \in \mathrm{Sect}\left(\frac{\pi}{2} + \omega\right)$

$$\left|m|\eta| + \varkappa^- E^{a^-}(\eta) + \zeta\right| > 0, \quad \eta \in \Gamma_0.$$

Therefore, for any $\zeta \in \mathrm{Sect}\left(\frac{\pi}{2} + \omega\right)$ the inverse operator $(L_0 - \zeta 1\!\!1)^{-1}$, the action of which is given by

$$[(L_0 - \zeta 1\!\!1)^{-1} G](\eta) = -\frac{1}{m|\eta| + \varkappa^- E^{a^-}(\eta) + \zeta} G(\eta), \tag{4.6}$$

is well defined on the whole space $\mathcal{L}_C$. Moreover, it is a bounded operator in this space and

$$||(L_0 - \zeta 1\!\!1)^{-1}|| \le \begin{cases} \frac{1}{|\zeta|}, & \text{if } \operatorname{Re}\zeta \ge 0, \\ \frac{M}{|\zeta|}, & \text{if } \operatorname{Re}\zeta < 0, \end{cases} \tag{4.7}$$

where the constant M does not depend on ζ.

The case $\operatorname{Re}\zeta \ge 0$ is a direct consequence of (4.6) and inequality

$$m|\eta| + \varkappa^- E^{a^-}(\eta) + \operatorname{Re}\zeta \ge \operatorname{Re}\,\zeta \ge 0.$$

We prove now the bound (4.7) in the case $\operatorname{Re}\zeta < 0$. Using (4.6), we have

$$||(L_0 - \zeta 1\!\!1)^{-1} G||_C = \left\| \frac{1}{\left|m|\cdot| + \varkappa^- E^{a^-}(\cdot) + \zeta\right|}\, G(\cdot) \right\|_C = $$
$$= \frac{1}{|\zeta|} \left\| \frac{|\zeta|}{\left|m|\cdot| + \varkappa^- E^{a^-}(\cdot) + \zeta\right|} G(\cdot) \right\|_C .$$

Since $\zeta \in \operatorname{Sect}\left(\frac{\pi}{2} + \omega\right)$,

$$|\operatorname{Im}\zeta| \ge |\zeta| \left|\sin\left(\frac{\pi}{2} + \omega\right)\right| = |\zeta| \cos\omega.$$

Hence,

$$\frac{|\zeta|}{\left|m|\eta| + \varkappa^- E^{a^-}(\eta) + \zeta\right|} \le \frac{|\zeta|}{|\operatorname{Im}\zeta|} \le \frac{1}{\cos\omega} =: M$$

and (4.7) is fulfilled.

The rest of the statement of the lemma follows directly from the theorem of Hille—Yosida (see e.g., [12]). □

We define now

$$(L_1 G)(\eta) := \varkappa^- \sum_{x\in\eta} \sum_{y\in\eta\setminus x} a^-(x-y) G(\eta \setminus y), \quad G \in D(L_1) := D(L_0).$$

The lemma below implies that the operator L_1 is well-defined.

Lemma 4.3. *For any $\delta > 0$ there exists $C_0 := C_0(\delta) > 0$ such that for all $C < C_0$ the following estimate holds*

$$||L_1 G||_C \le a||L_0 G||_C, \quad G \in D(L_1), \tag{4.8}$$

with $a = a(C) < \delta$.

Proof. By modulus property

$$||L_1 G||_C = \int_{\Gamma_0} |L_1 G(\eta)|\, C^{|\eta|} \lambda(d\eta) \tag{4.9}$$

can be estimated by

$$\varkappa^- \int_{\Gamma_0} \sum_{x\in\eta} \sum_{y\in\eta\setminus x} a^-(x-y) |G(\eta\setminus y)| C^{|\eta|} \lambda(d\eta). \tag{4.10}$$

By Minlos lemma, (4.10) is equal to

$$\varkappa^- \int_{\Gamma_0} \int_{\mathbb{R}^d} \sum_{x\in\eta} a^-(x-y)|G(\eta)|C^{|\eta|+1} dy\lambda(d\eta) =$$
$$= \varkappa^- \int_{\Gamma_0} C|\eta||G(\eta)|C^{|\eta|}\lambda(d\eta) \le \frac{\varkappa^-}{m} C||L_0G||_C.$$

Therefore, (4.8) holds with

$$a = \frac{\varkappa^- C}{m}.$$

Clear, that taking

$$C_0 = \frac{\delta m}{\varkappa^-}$$

we obtain that $a < \delta$ for $C < C_0$. □

We set now

$$(L_2G)(\eta) := (L_{2,\varkappa^+}G)(\eta) = \varkappa^+ \int_{\mathbb{R}^d} \sum_{y\in\eta} a^+(x-y)G((\eta\setminus y)\cup x)dx,$$
$$G \in D(L_2) := D(L_0).$$

The operator $\big(L_2, D(L_2)\big)$ is well defined due to the lemma below:

Lemma 4.4. *For any $\delta > 0$ there exists $\varkappa_0^+ := \varkappa_0^+(\delta) > 0$ such that for all $\varkappa^+ < \varkappa_0^+$ the following estimate holds*

$$||L_2G||_C \le a||L_0G||_C, \quad G \in D(L_2), \tag{4.11}$$

with $a = a(\varkappa^+) < \delta$.

Proof. Analogously to the previous lemma we estimate

$$||L_2G||_C = \int_{\Gamma_0} |L_2G(\eta)|\, C^{|\eta|}\lambda(d\eta) \tag{4.12}$$

by

$$\varkappa^+ \int_{\Gamma_0} \int_{\mathbb{R}^d} \sum_{y\in\eta} a^+(x-y)|G((\eta\setminus y)\cup x)|C^{|\eta|} dx\lambda(d\eta). \tag{4.13}$$

By Minlos lemma, (4.13) is equal to

$$\varkappa^+ \int_{\Gamma_0} \sum_{x\in\eta} \int_{\mathbb{R}^d} a^+(x-y)|G(\eta)|C^{|\eta|} dy\lambda(d\eta) \le \frac{\varkappa^+}{m}||L_0G||_C.$$

Taking $\varkappa_0^+ = \delta m$ we prove the lemma. □

The operator defined as:

$$(NG)(\eta) = |\eta|G(\eta), \qquad G \in D(L_0) \tag{4.14}$$

is called the number operator.

REMARK 4.2. *We proved, in particular, that for* $G \in D(L_0) = D(L_1) = D(L_2)$

$$||L_1 G||_C \le \varkappa^- C ||NG||_C,$$
$$||L_2 G||_C \le \varkappa^+ ||NG||_C.$$

Finally, we consider the last part of the operator $\widehat{L}$:

$$(L_3 G)(\eta) := \varkappa^+ \int_{\mathbb{R}^d} \sum_{y \in \eta} a^+(x-y) G(\eta \cup x) dx, \quad D(L_3) := D(L_0).$$

LEMMA 4.5. *For any* $\delta > 0$ *and any* $\varkappa^+ > 0$, $C > 0$ *such that*

$$\varkappa^+ E^{a^+}(\eta) < \delta C \left(\varkappa^- E^{a^-}(\eta) + m|\eta| \right) \tag{4.15}$$

the following estimate holds

$$||L_3 G||_C \le a ||L_0 G||_C, \quad G \in D(L_3), \tag{4.16}$$

with $a = a(\varkappa^+, C) < \delta$.

Proof. Using the same tricks as in the two previous lemmas we have

$$||L_3 G||_C = \int_{\Gamma_0} |L_3 G(\eta)| \, C^{|\eta|} \lambda(d\eta)$$
$$\le \varkappa^+ \int_{\Gamma_0} \int_{\mathbb{R}^d} \sum_{y \in \eta} a^+(x-y) |G(\eta \cup x)| C^{|\eta|} dx \lambda(d\eta). \tag{4.17}$$

By Minlos lemma, (4.17) is equal to

$$\frac{\varkappa^+}{C} \int_{\Gamma_0} E^{a^+}(\eta) |G(\eta)| C^{|\eta|} \lambda(d\eta).$$

The assertion of the lemma is now trivial. □

THEOREM 4.6. *Assume that the functions* a^-, a^+ *and the constants* $\varkappa^-$, $\varkappa^+ > 0$, $m > 0$ *and* $C > 0$ *satisfy*

$$C \varkappa^- a^- \ge 2 \varkappa^+ a^+, \tag{4.18}$$

$$m > 2 \left(\varkappa^- C + \varkappa^+ \right).$$

Then, the operator $\widehat{L}$ *is a generator of a holomorphic semigroup* $\hat{U}_t$, $t \ge 0$ *in* $\mathcal{L}_C$.

Proof. The statement of the theorem follows directly from Remark 4.2, Lemma 4.5 and the theorem about the perturbation of holomorphic semigroup (see, e.g. [12]). For the reader's convenience, below we give its formulation:

For any $T \in \mathcal{H}(\omega, \theta)$ *and for any* $\varepsilon > 0$ *there exists positive constants* α, δ *such that if the operator* A *satisfies*

$$||Au|| \le a||Tu|| + b||u||, \quad u \in D(T) \subset D(A),$$

with $a < \delta$, $b < \delta$, *then* $T + A \in \mathcal{H}(\omega - \varepsilon, \alpha)$.

In particular, if $\theta = 0$ *and* $b = 0$, *then* $T + A \in \mathcal{H}(\omega - \varepsilon, 0)$

Following the proof of this theorem (see, e.g. [12]) and taking into account the fact that $L_0 \in \mathcal{H}(\omega, 0)$ for any $\omega \in (0, \frac{\pi}{2})$, one can conclude in our case that δ can be chosen equal to $\frac{1}{2}$. This is exactly the reason of appearing multiplicand 2 at the l.h.s. of (4.18). □

5. Evolution of correlation functions. Let us consider the evolution equation (4.4), which corresponds to the operator $\hat{L}^*$

$$\frac{dk_t}{dt} = \hat{L}^* k_t, \qquad k_t \Big|_{t=0} = k_0.$$

Using the general scheme, proposed in [8] we find the precise form of $\hat{L}^*$:

$$\begin{aligned}\hat{L}^* k(\eta) = &- \Big(m|\eta| + \varkappa^- E^{a^-}(\eta)\Big) k(\eta) + \varkappa^+ \sum_{x\in\eta} \sum_{y\in\eta\setminus x} a^+(x-y) k(\eta\setminus x)\\ &+ \varkappa^+ \int_{\mathbb{R}^d} \sum_{y\in\eta} a^+(x-y) k((\eta\setminus y)\cup x) dx\\ &- \varkappa^- \int_{\mathbb{R}^d} \sum_{y\in\eta} a^-(x-y) k(\eta\cup x) dx.\end{aligned}$$

The main questions which we would like to study now are the existence and properties of the solution to the hierarchical system of equations (4.4). The answers to these questions are given in the following theorem

THEOREM 5.1. *Suppose that all assumptions of Theorem 4.6 are fulfilled. Then for any initial function k_0 from the class*

$$\mathcal{K}_C := \Big\{ k : \Gamma_0 \to \mathbb{R} \mid k \cdot C^{-|\eta|} \in L^\infty(\Gamma_0, \lambda) \Big\}$$

the corresponding solution k_t to (4.4) exists and will be again the function from $\mathcal{K}_C$ for any moment of time $t \geq 0$.

Proof. Following the scheme proposed in [16], we construct the corresponding evolution of the locally finite measures on Γ_0. In order to realize this construction we consider the dual space $\mathcal{K}_C$ to the Banach space $\mathcal{L}_C$. The duality is given by the following expression

$$\langle\langle G, k \rangle\rangle := \int_{\Gamma_0} G \cdot k \, d\lambda, \quad G \in \mathcal{L}_C. \tag{5.1}$$

It is clear that $\mathcal{K}_C$ is the Banach space with the norm

$$||k|| := ||C^{-|\cdot|} k(\cdot)||_{L^\infty(\Gamma_0,\lambda)}.$$

Note also, that $k \cdot C^{-|\cdot|} \in L^\infty(\Gamma_0, \lambda)$ means that the function k satisfies the bound

$$|k(\eta)| \leq \text{const}\, C^{|\eta|} \quad \text{for } \lambda\text{-a.a. } \eta \in \Gamma_0.$$

The evolution on $\mathcal{K}_C$, which corresponds to $\hat{U}_t,\ _{t\geq 0}$ constructed in Theorem 4.6, may be determined in the following way:

$$\langle\langle G, k_t \rangle\rangle := \Big\langle\Big\langle \hat{U}_t G, k_0 \Big\rangle\Big\rangle.$$

We denote

$$\hat{U}_t^* k_0 := k_t.$$

Using the same arguments as in [16], it becomes clear that $k_t = \hat{U}_t^* k_0$ is the solution to (4.4) in the Banach space $\mathcal{K}_C$. □

It is important to emphasize that in the case of $a^- \equiv 0$ and $\varkappa^+ < m$

$$k_t^{(n)} \to 0, \quad t \to 0, \quad \text{for} \quad \text{any} \quad n \geq 1,$$

see e. g. [17]. Therefore, we may expect that the correlation functions of our model satisfy this property as well.

6. Stationary equation for the system of correlation functions. Let us consider for any $\alpha \in \mathbb{R}$ the following Banach subspace of $\mathcal{K}_C$:

$$\mathcal{K}_C^\alpha := \left\{ k \in \mathcal{K}_C \;\middle|\; k^{(0)}(\emptyset) = \alpha \right\}.$$

In this section we study the existence problem for the solutions to the stationary equation

$$\hat{L}^* k = 0 \tag{6.1}$$

in $\mathcal{K}_C^1$. The main result is formulated in the following way:

THEOREM 6.1. *Suppose that*

$$\frac{C\varkappa^-}{m} + \frac{\varkappa^+}{m} + \frac{1}{C} < 1 \tag{6.2}$$

and

$$\varkappa^- a^- \geq \varkappa^+ a^+$$

then the solution $k = (k^{(n)})_{n\geq 0}$ *to (6.1) is unique in* $\mathcal{K}_C^1$ *and such that*

$$k^{(n)} = 0, \qquad n \geq 1.$$

Proof. Let

$$\left(\hat{L}^\star k\right)(\eta) = 0.$$

The latter means that

$$\left(m\,|\eta| + \varkappa^- E^{a^-}(\eta)\right) k(\eta) = -\varkappa^- \sum_{x\in\eta} \int_{\mathbb{R}^d} k(y \cup \eta)\, a^-(x-y)\, dy +$$
$$+\varkappa^+ \sum_{x\in\eta} \sum_{y\in\eta\setminus x} a^+(x-y) k(\eta\setminus x) + \varkappa^+ \int_{\mathbb{R}^d} \sum_{y\in\eta} a^+(x-y) k((\eta\setminus y)\cup x) dx.$$

The last relation holds for any $k \in \mathcal{K}_C^1$ at the point $\eta = \emptyset$. Hence, one can consider it on $\mathcal{K}_C^0$.

Let us denote for $\eta \neq \emptyset$

$$(Sk)(\eta) = -\frac{\varkappa^-}{m\,|\eta| + \varkappa^- E^{a^-}(\eta)} \sum_{x\in\eta} \int_{\mathbb{R}^d} k(y\cup\eta)\, a^-(x-y)\, dy +$$
$$+\frac{\varkappa^+}{m\,|\eta| + \varkappa^- E^{a^-}(\eta)} \sum_{x\in\eta}\sum_{y\in\eta\setminus x} a^+(x-y) k(\eta\setminus x) +$$
$$+\frac{\varkappa^+}{m\,|\eta| + \varkappa^- E^{a^-}(\eta)} \int_{\mathbb{R}^d} \sum_{y\in\eta} a^+(x-y) k((\eta\setminus y)\cup x) dx$$

and

$$(Sk)(\emptyset) = 0.$$

Let

$$\|k\|_C = \operatorname*{ess\,sup}_{\eta\in\Gamma_0} \frac{|k(\eta)|}{C^{|\eta|}},$$

then

$$\begin{aligned}
&\|Sk\|_C \\
&\leq \|k\|_C \operatorname*{ess\,sup}_{\eta\in\Gamma_0\setminus\{\emptyset\}} \frac{\varkappa^- C}{m|\eta| + \varkappa^- E^{a^-}(\eta)} \sum_{x\in\eta} \int_{\mathbb{R}^d} a^-(x-y)\,dy \\
&\quad + \frac{\|k\|_C}{C} \operatorname*{ess\,sup}_{\eta\in\Gamma_0\setminus\{\emptyset\}} \frac{\varkappa^+}{m|\eta| + \varkappa^- E^{a^-}(\eta)} \sum_{x\in\eta} \sum_{y\in\eta\setminus x} a^+(x-y) \\
&\quad + \|k\|_C \operatorname*{ess\,sup}_{\eta\in\Gamma_0\setminus\{\emptyset\}} \frac{\varkappa^+}{m|\eta| + \varkappa^- E^{a^-}(\eta)} \int_{\mathbb{R}^d} \sum_{y\in\eta} a^+(x-y)dx \\
&\leq \|k\|_C \frac{C\varkappa^-}{m} + \|k\|_C \frac{\varkappa^+}{m} + \|k\|_C \frac{1}{C} = \|k\|_C \left(\frac{C\varkappa^-}{m} + \frac{\varkappa^+}{m} + \frac{1}{C}\right),
\end{aligned}$$

if

$$\varkappa^+ E^{a^+}(\eta) \leq \varkappa^- E^{a^-}(\eta) + m|\eta|.$$

As result,

$$\|S\| \leq \frac{1}{m} C\varkappa^- + \frac{1}{m}\varkappa^+ + \frac{1}{C} < 1.$$

The assertion of the theorem is now obvious. □

REMARK 6.1. *For any $C > 1$ one may chose $\varkappa^- > 0$ and $m > 0$ such that (6.2) is satisfied. The latter means, that, asymptotically, our system exhausted to the system with the stationary state $\delta_\emptyset(d\gamma)$ (the Dirac measure concentrated on the empty configuration $\emptyset$). In other words, the population evolving due to the BDLP dynamics is asymptotically degenerated.*

7. Further developments. In Theorem 5.1 we have shown that functions k_t is bounded by C^n for all $t > 0$, provided that k_0 satisfies initially the bound of the same type. Using approximation arguments (see e.g. [16], [18]) one may prove that the corresponding time evolution of the correlation function will be also correlation function for some probability measure on Γ. We suppose to discuss this problem as well as other probabilistic aspects of the BDLP model in a forthcoming paper. The main aim of the present paper is to analyze evolution of correlation functions. Namely, we have shown that dynamics of correlation functions stays in the space $\mathcal{K}_C$. This property seems to be very strong. To show that system of correlation functions evolving in time stays in the same space is already difficult even for the contact model. Namely, (3.3) implies that the evolution of correlation functions at some moment of time t may leave the space

$$\left\{k : \Gamma_0 \to \mathbb{R} \;\middle|\; k \cdot C^{-|\eta|} \cdot |\eta|! \in L^\infty(\Gamma_0, \lambda)\right\}.$$

The reason is that C may depend on t, which is true at least for the case $\varkappa^+ \geq 1$ ($m = 1$ at the moment). Hence, we may expect that the dynamics of correlation functions for the contact process lives in some bigger space. Of course, this is possible only for $\varkappa^+ \le 1$ since for $\varkappa^+ > 1$ density tends to infinity. Hence, let us consider the case $\varkappa^+ = 1$. One candidate for such bigger space is

$$\mathcal{R}_C := \left\{ k : \Gamma_0 \to \mathbb{R} \mid k \cdot C^{-|\eta|} \cdot (|\eta|!)^2 \in L^\infty(\Gamma_0, \lambda) \right\}.$$

Note, that the invariant measure of the contact process belongs to this space (see [17, Theorem 4.2]), provided that $d \ge 3$, a^+ has finite second moment w.r.t. the Lebesgue measure and the Fourier transform of a^+ is integrable on $\mathbb{R}^d$. Below we show that the evolution of correlation functions at any moment of time t is a function from $\mathcal{R}_C$.

Indeed, let $\varkappa^+ = 1$ and suppose that there exists $C > 0$ such that for any $n \ge 1$ and for any $x_1, \ldots, x_n \in \mathbb{R}^d$

$$k_0^{(n)}(x_1, \ldots, x_n) \le \frac{1}{2} C^n (n!)^2.$$

Then, it is clear that $k_0 \in \mathcal{R}_C$. Now, suppose that $k_t^{(n-1)} \le C^{n-1}((n-1)!)^2$. We prove the corresponding inequality for $k_t^{(n)}$ using the mathematical induction. By (3.3) we have for any $x_1, \ldots, x_n \in \mathbb{R}^d$

$$\begin{aligned} &k_t^{(n)}(x_1, \ldots, x_n) \qquad (7.1)\\ &\le \frac{1}{2} C^n (n!)^2 \\ &\quad + \int_0^t \left[\bigotimes_{i=1}^n e^{(t-s)L^i_{a^+}} \right] \sum_{i=1}^n C^{n-1}((n-1)!)^2 \sum_{j:\, j \ne i} a^+(x_i - x_j) ds \\ &= \frac{1}{2} C^n (n!)^2 + C^{n-1}((n-1)!)^2 \sum_{i=1}^n \sum_{j:\, j\ne i} \int_0^t \left(e^{2(t-s)L_{a^+}} a^+ \right)(x_i - x_j) ds, \end{aligned}$$

where for $f \in L^1(\mathbb{R}^d)$

$$L_{a^+} f(x) = \int_{\mathbb{R}^d} a(x-y)[f(y) - f(x)]dx, \quad x \in \mathbb{R}^d.$$

For the bound above we have used the fact, that for any $1 \le i \ne j \le n$

$$L^i_{a^+} a^+(x_i - x_j) = L^j_{a^+} a^+(x_i - x_j) = (L_{a^+} a^+)(x_i - x_j), \quad x_i, x_j \in \mathbb{R}^d.$$

This relation can be easily checked by simple computations.

Note, that L_{a^+} is a generator of the Markov semigroup which preserves positivity in $L^1(\mathbb{R}^d)$. Hence,

$$g_t(x) := \int_0^t \left(e^{2(t-s)L_{a^+}} a^+ \right)(x) ds \ge 0, \quad x \in \mathbb{R}^d, t \ge 0,$$

and $g_t \in L^1(\mathbb{R}^d)$. Then we have

$$g_t(x) = |g_t(x)| = \frac{1}{(2\pi)^d} \left| \int_{\mathbb{R}^d} e^{ipx} \widehat{g}_t(p) dp \right| \le \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_0^t e^{2(t-s)(\widehat{a}^+(p) - 1)} |\widehat{a}^+(p)| ds dp,$$

where symbol $\widehat{f}$ denotes the Fourier transform of the function $f \in L^1(\mathbb{R}^d)$. Therefore,

$$g_t(x) \le \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{1 - e^{2t(\widehat{a}^+(p)-1)}}{2(1 - \widehat{a}^+(p))} |\widehat{a}^+(p)| dp \le \frac{1}{2(2\pi)^d} \int_{\mathbb{R}^d} \frac{|\widehat{a}^+(p)|}{1 - \widehat{a}^+(p)} dp.$$

It was shown in [17] that under the conditions posed on function a^+ for the case of invariant measure

$$D := \int_{\mathbb{R}^d} \frac{|\widehat{a}^+(p)|}{1-\widehat{a}^+(p)} dp < \infty.$$

Finally, if additionally

$$C \geq \frac{D}{(2\pi)^d},$$

then we obtain from (7.1)

$$k_t^{(n)}(x_1,\ldots,x_n) \leq \frac{1}{2} C^n (n!)^2 + \frac{1}{2} C^{n-1}((n-1)!)^2 n(n-1) \frac{D}{(2\pi)^d} \leq C^n (n!)^2.$$

As result, $k_t \in \mathcal{R}_C$ for all $t \geq 0$.

Therefore, the dynamics of correlation functions for the contact model stays in $\mathcal{R}_C$, hence, this dynamics is really very clustering for $\varkappa^+ = m = 1$. As before, we may extend our consideration on the case $m \neq 1$.

Summarizing previous results in this section we claim that the presence of the big mortality and the big competition kernel prevents clustering in the system making it sub-Poissonian distributed. But, is it really necessary to add "big" mortality and competition kernel? Below we discuss this problem.

If we want to study the quasibounded semigroup with the generator $\widehat{L}$ on $\mathcal{L}_C$ for some $C > 0$ then, naturally, this generator should be an accretive operator in $\mathcal{L}_C$. Hence, for some $b \geq 0$ the following bound should be true

$$\int_{\Gamma_0} \operatorname{sgn}(G(\eta)) \cdot \Bigl(\bigl(\hat{L} - b1\!\!1 \bigr) G \Bigr)(\eta)\, d\lambda_C(\eta) \leq 0, \quad \forall\, G \in D(\widehat{L}),$$

since

$$C^{|\eta|} d\lambda(\eta) = d\lambda_C(\eta).$$

Let us define the "diagonal" part of the operator $\widehat{L}$:

$$\bigl(\hat{L}_{diag} G\bigr)(\eta) := -m|\eta| G(\eta) - \varkappa^- E^{a^-}(\eta) G(\eta) + \varkappa^+ \int_{\mathbb{R}^d} \sum_{y\in\eta} a^+(x-y) G((\eta\setminus y)\cup x) dx$$

and consider for some $n \geq 1$

$$G = \Bigl(0, 0, G^{(n)}, 0, 0\Bigr), \quad G^{(n)} \in L^1((\mathbb{R}^d)^n).$$

Then

$$(\widehat{L}G)(\eta) = \begin{cases} \varkappa^+ \int\limits_{\mathbb{R}^d} \sum\limits_{y\in\eta} a^+(x-y) G^{(n)}(\eta\cup x) dx, & |\eta| = n-1 \\ -\varkappa^- \sum\limits_{x\in\eta} \sum\limits_{y\in\eta\setminus x} a^-(x-y) G^{(n)}(\eta\setminus y), & |\eta| = n+1 \\ \Bigl(\hat{L}_{diag} G^{(n)}\Bigr)(\eta), & |\eta| = n \\ 0, & \text{otherwise} \end{cases}.$$

Note that $\operatorname{sgn}(G(\eta)) \equiv 0$ if $|\eta| \neq n$.

Therefore, for arbitrary $n \geq 1$

$$\begin{aligned} 0 \geq I_n &:= \int_{\Gamma_0} \operatorname{sgn}\left(G\left(\eta\right)\right) \cdot \Big(\big(\hat{L} - b1\!\!1\big) G\Big)\left(\eta\right) d\lambda_C\left(\eta\right) \\ &= \int_{\Gamma_0^{(n)}} \operatorname{sgn}\left(G\left(\eta\right)\right) \cdot \Big(\big(\hat{L}_{diag} - b1\!\!1\big) G^{(n)}\Big)\left(\eta\right) d\lambda_C\left(\eta\right) \\ &= \frac{C^n}{n!} \int_{(\mathbb{R}^d)^n} \operatorname{sgn}\Big(G^{(n)}\big(x^{(n)}\big)\Big)\Big(\big(\hat{L}_{diag} - b1\!\!1\big) G^{(n)}\Big)\big(x^{(n)}\big) dx^{(n)}. \end{aligned}$$

Let us fix some $t > 0$ and $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$. Set for $n \geq 1$

$$G^{(n)}\big(x^{(n)}\big) = t^n \prod_{k=1}^n \chi_\Lambda\left(x_k\right) = t^n 1\!\!1_{\Gamma_\Lambda^{(n)}}\big(\{x^{(n)}\}\big) \in L^1((\mathbb{R}^d)^n).$$

Then, the equality

$$\operatorname{sgn}\Big(G^{(n)}\big(x^{(n)}\big)\Big) = \prod_{k=1}^n \chi_\Lambda\left(x_k\right)$$

implies

$$\begin{aligned} 0 &\geq \frac{n!}{t^n C^n} I_n \\ &= \int_{\Lambda^n} \Bigg(-mn \prod_{k=1}^n \chi_\Lambda\left(x_k\right) - \varkappa^- E^{a^-}\left(x^{(n)}\right) \prod_{k=1}^n \chi_\Lambda\left(x_k\right) \\ &\qquad + \varkappa^+ \int_{\mathbb{R}^d} \sum_{j=1}^n a^+(x - x_j) \prod_{k \neq j} \chi_\Lambda\left(x_k\right) \chi_\Lambda\left(x\right) dx \Bigg) dx^{(n)} - b \int_{\Lambda^n} \prod_{k=1}^n \chi_\Lambda\left(x_k\right) dx^{(n)} \\ &= -\varkappa^- \int_{\Lambda^n} E^{a^-}\big(x^{(n)}\big) dx^{(n)} + \varkappa^+ \sum_{j=1}^n \prod_{k \neq j} \int_{\Lambda^{n-1}} dx_k \int_\Lambda \int_\Lambda a^+(x - x_j) dx dx_j \\ &\quad - \left(b + mn\right) \left|\Lambda\right|^n \\ &= -\varkappa^- \int_{\Lambda^n} E^{a^-}\big(x^{(n)}\big) dx^{(n)} + \varkappa^+ n \left|\Lambda\right|^{n-1} \int_\Lambda \int_\Lambda a^+(x - y) dx dy - \left(b + mn\right) \left|\Lambda\right|^n. \end{aligned}$$

We suppose, in fact, that for any $n \geq 1$

$$I_n \leq 0.$$

Since $E^{a^-}(\eta) = 0$ for $|\eta| \leq 1$ we get

$$\begin{aligned} 0 \geq \sum_{n=1}^\infty I_n &= -m \sum_{n=1}^\infty n \frac{t^n C^n}{n!} \left|\Lambda\right|^n - \varkappa^- \sum_{n=1}^\infty \frac{t^n C^n}{n!} \int_{\Lambda^n} E^{a^-}\big(x^{(n)}\big) dx^{(n)} \\ &\quad + \varkappa^+ \sum_{n=1}^\infty \frac{t^n C^n}{n!} n \left|\Lambda\right|^{n-1} \int_\Lambda \int_\Lambda a^+(x - y) dx dy - b \sum_{n=1}^\infty \frac{t^n C^n}{n!} \left|\Lambda\right|^n \\ &= -mtC \left|\Lambda\right| e^{Ct|\Lambda|} - \varkappa^- \int_{\Gamma_\Lambda} E^{a^-}\left(\eta\right) d\lambda_{Ct}\left(\eta\right) + \varkappa^+ Ct e^{Ct|\Lambda|} \int_\Lambda \int_\Lambda a^+(x - y) dx dy \end{aligned}$$

$$
\begin{aligned}
&- b\left(e^{Ct|\Lambda|}-1\right)\\
&= -mtC\left|\Lambda\right| e^{Ct|\Lambda|} - \varkappa^{-}C^{2}t^{2}\int_{\Gamma_\Lambda}\int_\Lambda\int_\Lambda a^{-}\left(x-y\right)dxdyd\lambda_{Ct}\left(\eta\right)\\
&+ \varkappa^{+}Cte^{Ct|\Lambda|}\int_\Lambda\int_\Lambda a^{+}(x-y)dxdy - b\left(e^{Ct|\Lambda|}-1\right)\\
&= e^{Ct|\Lambda|}\Bigg[Ct\left(\varkappa^{+}\int_\Lambda\int_\Lambda a^{+}(x-y)dxdy - \varkappa^{-}Ct\int_\Lambda\int_\Lambda a^{-}\left(x-y\right)dxdy - m\left|\Lambda\right|\right)\\
&-b\left(1-e^{-Ct|\Lambda|}\right)\Bigg].
\end{aligned}
$$

Therefore, for any $t>0$ and any $\Lambda\in\mathcal{B}_b(\mathbb{R}^d)$

$$
\begin{aligned}
0 \geq \varkappa^{+}&\int_\Lambda\int_\Lambda a^{+}(x-y)dxdy - \varkappa^{-}Ct\int_\Lambda\int_\Lambda a^{-}\left(x-y\right)dxdy - m\left|\Lambda\right|\\
&- b\frac{\left(1-e^{-Ct|\Lambda|}\right)}{Ct} =: B.
\end{aligned}
$$

Suppose that there exists $z>0$ such that

$$
a^{+}\left(x\right) \geq za^{-}\left(x\right), \quad x\in\mathbb{R}^d,
$$

then taking for some $\varepsilon>0$

$$
t = \varepsilon\frac{z\varkappa^{+}}{\varkappa^{-}C} > 0
$$

we obtain

$$
\begin{aligned}
B &\geq (1-\varepsilon)\varkappa^{+}z\int_\Lambda\int_\Lambda a^{-}(x-y)dxdy - m\left|\Lambda\right|\\
&- \frac{b\varkappa^{-}}{\varepsilon z\varkappa^{+}}\left(1-\exp\left(-\frac{z\varkappa^{+}}{2\varkappa^{-}}\left|\Lambda\right|\right)\right) \sim \Big((1-\varepsilon)\varkappa^{+}z - m\Big)|\Lambda|, \quad \Lambda\uparrow\mathbb{R}^d,
\end{aligned}
$$

which contradicts to $B\leq 0$. As result, m can not be arbitrary small.

Acknowledgments. The financial support of DFG through the SFB 701 (Bielefeld University) and German-Ukrainian Project 436 UKR 113/80 and 436 UKR 113/94 is gratefully acknowledged. This work was partially supported by FCT, POCI2010, FEDER. Yu.K. is very thankful to R. Law for fruitful and stimulating discussions.

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