EIGENVALUE ASYMPTOTICS FOR JAYNES-CUMMINGS TYPE MODELS WITHOUT MODULATIONS

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ABSTRACT. We obtain eigenvalue asymptotics for Jacobi matrices of various Jaynes-Cummings type.

1. The results

We consider a type of Jacobi matrices with unbounded entries related to some problems of quantum optics. See [1–4].

Let $\mathbb{N}^* = \{1, 2, \dots\}$ be the set of positive integers and let l^2 denote the Hilbert space of square summable complex sequences $x = (x_n)_{n\in\mathbb{N}^*}$. Let c_{00} be the subspace of sequences for which $\{n \in \mathbb{N}^* \mid x_n \neq 0\}$ is finite. We fix a real valued sequence $(\beta_n)_{n\in\mathbb{N}^*}$ and consider a linear operator J acting on $(x_n)_{n\in\mathbb{N}^*} \in c_{00}$ according to the formula

$$(Jx)_n = \begin{cases} nx_n + \beta_n x_{n+1} + \beta_{n-1}x_{n-1} & n \geq 2, \\ x_1 + \beta_1 x_2 & n = 1. \end{cases} \tag{1.1}$$

Then it is easy to establish the following elementary fact.

Proposition 1. *Assume that there exists $\rho > 0$ such that*

$$\beta_n = \mathrm{O}(n^{1-\rho}). \tag{1.2}$$

Then the closure of the operator defined by (1.1) is a self-adjoint operator J, its spectrum is discrete and bounded from below. Let $(\lambda_n(J))_{n\in\mathbb{N}^*}$ denote the sequence of eigenvalues of J repeated according to their multiplicities and ordered so that $\lambda_n(J) \leq \lambda_{n+1}(J)$ for all $n \in \mathbb{N}^*$. Then the following estimate

$$\lambda_n(J) = n + \mathrm{O}(n^{1-\rho}) \tag{1.3}$$

holds as $n \to \infty$.

The aim of this paper is to obtain sharper estimates of the asymptotic behaviour of $(\lambda_n(J))_{n\in\mathbb{N}^*}$ which can be deduced from additional assumptions made on the sequence $(\beta_n)_{n\in\mathbb{N}^*}$. Our first result is

Theorem 1. *Assume that (1.2) holds with a certain $\rho > 0$ and*

$$\beta_{n+1} - \beta_n = \mathrm{O}(n^{-\rho'}) \tag{1.4}$$

holds with a certain $\rho' > 0$. Then one has the estimate

$$\lambda_n(J) = n + \mathrm{O}(n^{1-\rho-\rho'}) \tag{1.5}$$

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Our second theorem depends on the behaviour of the sequence

$$\gamma_n = \begin{cases} \beta_{n-1}^2 - \beta_n^2 & n \geq 2, \\ -\beta_1^2 & n = 1. \end{cases} \tag{1.6}$$

Theorem 2. *Assume that* $(\beta_n)_{n\in\mathbb{N}^*}$ *satisfies the hypotheses of Theorem 1. Let* $(\gamma_n)_{n\in\mathbb{N}^*}$ *be the sequence given by* (1.6)*. If*

$$\gamma_{n+1} - \gamma_n = \mathrm{O}(n^{-\rho_1}) \tag{1.7}$$

holds with a certain $\rho_1 > 0$*, then one has*

$$\lambda_n(J) = n + \gamma_n + \mathrm{O}(n^{1-\rho-\rho_1}). \tag{1.8}$$

Remark. If it is possible to evaluate $\beta_n = b(n)$ by means of a function $b \in C^\infty((0,+\infty))$ satisfying the estimates

$$\begin{cases} b(\lambda) = \mathrm{O}(\lambda^{1-\rho}), \\ b'(\lambda) = \mathrm{O}(\lambda^{-\rho}), \end{cases}$$

then

$$\beta_{n+1} - \beta_n = \int_0^1 b'(n+s)\mathrm{d}s = \mathrm{O}(n^{-\rho}),$$

i.e., (1.4) holds with $\rho = \rho'$, and (1.5) takes the form

$$\lambda_n(J) = n + \mathrm{O}(n^{1-2\rho}).$$

If moreover

$$b''(\lambda) = \mathrm{O}(\lambda^{-1-\rho}),$$

then

$$\begin{aligned} \gamma_{n+1} - \gamma_n &= -\int_0^1 \mathrm{d}s \int_0^1 b^{2\prime\prime}(n+s-s')\mathrm{d}s' \\ &\quad - 2\int_0^1 \mathrm{d}s \int_0^1 (bb'' + b'^2)(n+s-s')\mathrm{d}s', \\ &= \mathrm{O}(n^{-2\rho}), \end{aligned}$$

i.e., (1.7) holds with $\rho_1 = 2\rho$, and (1.8) takes the form

$$\lambda_n(J) = n + \gamma_n + \mathrm{O}(n^{1-3\rho}).$$

2. Proof of Proposition 1

Let $\mathcal{B}(l^2)$ denote the algebra of bounded operators in l^2. Let $(\mathrm{e}_k)_{k\in\mathbb{N}^*}$ be the canonical basis of l^2, i.e. $\mathrm{e}_k = (\delta_{k,n})_{n\in\mathbb{N}^*}$ where

$$\delta_{k,n} = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

We denote by Λ the self-adjoint operator on l^2 satisfying

$$\Lambda\mathrm{e}_n = n\mathrm{e}_n \text{ for } n \in \mathbb{N}^*. \tag{2.1}$$

Proof of Proposition 1. The estimate (1.2) allows us to find a constant $C > 0$ such that

$$-C\Lambda^{1-\rho} \le J - \Lambda \le C\Lambda^{1-\rho} \tag{2.2}$$

holds in the sense of quadratic forms and it follows straightforwardly that Λ and J are both bounded from below and essentially self-adjoint on c_{00}.

Next we choose $\lambda > 0$ large enough and we observe that the operator

$$Q_\lambda = (J+\lambda)^{-1} - (\Lambda+\lambda)^{-1} = -(J+\lambda)^{-1}(J-\Lambda)(\Lambda+\lambda)^{-1}$$

satisfies $Q_\lambda \Lambda^\rho \in \mathcal{B}(l^2)$. However $\Lambda^{-\rho}$ is compact on l^2, hence Q_λ is compact as well and the essential spectrum $\sigma_{\mathrm{ess}}(J) = \sigma_{\mathrm{ess}}(\Lambda) = \varnothing$. Moreover (2.2) gives

$$\Lambda - C\Lambda^{1-\rho} \le J \le \Lambda + C\Lambda^{1-\rho} \tag{2.3}$$

and the min-max principle ensures

$$\lambda_n(\Lambda - C\Lambda^{1-\rho}) \le \lambda_n(J) \le \lambda_n(\Lambda + C\Lambda^{1-\rho}), \tag{2.4}$$

where

$$\lambda_n(\Lambda \pm C\Lambda^{1-\rho}) = n \pm Cn^{1-\rho} \tag{2.5}$$

is the n-th eigenvalue of $\Lambda \pm C\Lambda^{1-\rho}$. This completes the proof of (1.3). □

3. Notations and conventions

Below we describe further notations and conventions.

3.1. For any application $q\colon \mathbb{N}^* \to \mathbb{R}$ we denote by $q(\Lambda)$ the self-adjoint operator satisfying

$$q(\Lambda)\mathrm{e}_n = q(n)\mathrm{e}_n \text{ for } n \in \mathbb{N}^*, \tag{3.1}$$

i.e. the domain of $q(\Lambda)$ is $D(q(\Lambda)) = \{(x_n)_{n\in\mathbb{N}^*} \mid (q(n)x_n)_{n\in\mathbb{N}^*} \in l^2\}$.

3.2. Let B_1 and B_2 be operators acting on $\cap_{k\in\mathbb{N}} D(\Lambda^k)$, i.e., the subspace of sequences satisfying $x_n = \mathrm{O}(n^{-s})$ for every $s \in \mathbb{R}$. Then for $m \in \mathbb{R}$ we write

$$B_1 = B_2 + \mathrm{O}(\Lambda^m) \tag{3.2}$$

if and only if $\Lambda^{s-m}(B_1 - B_2)\Lambda^{-s} \in \mathcal{B}(l^2)$ holds for any $s \in \mathbb{R}$.

3.3. We also observe property

$$\left.\begin{aligned} B &= \mathrm{O}(\Lambda^m) \\ B' &= \mathrm{O}(\Lambda^{m'}) \end{aligned}\right\} \implies BB' = \mathrm{O}(\Lambda^{m+m'}), \tag{3.3}$$

which follows immediately from the inequality

$$\|\Lambda^{s-m-m'} BB' \Lambda^{-s}\| \le \|\Lambda^{(s-m')-m} B \Lambda^{-(s-m')}\| \cdot \|\Lambda^{s-m'} B' \Lambda^{-s}\|,$$

where $\|\cdot\|$ denotes the norm of $\mathcal{B}(l^2)$.

3.4. Further on all operators are acting on $\cap_{k\in\mathbb{N}} D(\Lambda^k)$ and are assumed to be closable on $\cap_{k\in\mathbb{N}} D(\Lambda^k)$. Moreover we often write $A + \mathrm{hc}$ instead of $A + A^*$.

3.5. Let $S \in \mathcal{B}(l^2)$ denote the shift operator satisfying

$$S\mathrm{e}_n = \mathrm{e}_{n+1} \text{ for } n \in \mathbb{N}^*, \tag{3.4}$$

let $b\colon \mathbb{N}^* \to \mathbb{R}$ be given by the formula

$$b(n) = \beta_n \text{ for } n \in \mathbb{N}^* \tag{3.5}$$

and set

$$J_1 = Sb(\Lambda) + b(\Lambda)S^* = Sb(\Lambda) + \mathrm{hc}\,. \tag{3.6}$$

Thus the operator J can be expressed

$$J = \Lambda + J_1. \tag{3.7}$$

3.6. We introduce the closed operator A defined on $\cap_{k\in\mathbb{N}} D(\Lambda^k)$ by

$$A = Sb(\Lambda) - b(\Lambda)S^* = Sb(\Lambda) - \mathrm{hc}\,. \tag{3.8}$$

4. Proof of theorem 1

We deduce Theorem 1 from

Proposition 2. *The domain of A is the domain of the self-adjoint operator* iA *and*

$$\tilde{J} = \mathrm{e}^{-A} J \mathrm{e}^{A} = \Lambda + \mathrm{O}(\Lambda^{1-\rho-\rho'}) \tag{4.1}$$

holds under the hypotheses of Theorem 1.

Proof of Proposition 2. See Section 6. □

Proof of Theorem 1. It is easy to see that Theorem 1 follows from Proposition 2. Indeed, (4.1) implies

$$\Lambda - C\Lambda^{1-\rho-\rho'} \le \tilde{J} \le \Lambda + C\Lambda^{1-\rho-\rho'} \tag{4.2}$$

for a certain constant $C > 0$ and the min-max principle gives

$$\lambda_n(\Lambda - C\Lambda^{1-\rho-\rho'}) \le \lambda_n(\tilde{J}) \le \lambda_n(\Lambda + C\Lambda^{1-\rho-\rho'}) \tag{4.3}$$

with $\lambda_n(\Lambda \pm C\Lambda^{1-\rho-\rho'}) = n \pm Cn^{1-\rho-\rho'}$. Hence (1.5) follows from the fact that J and $\tilde{J}$ are unitary equivalent, which ensures $\lambda_n(J) = \lambda_n(\tilde{J})$ for all $\mathbb{N}^*$. □

5. Proof of theorem 2

Similarly we can deduce the assertion of Theorem 2 from

Proposition 3. *Under the hypotheses of Theorem 2 one has*

$$\mathrm{e}^{-A} J \mathrm{e}^{A} = \Lambda + g(\Lambda) + \mathrm{O}(\Lambda^{1-\rho-\rho_1}), \tag{5.1}$$

where $g\colon \mathbb{N}^* \to \mathbb{R}$ satisfies $g(n) = \gamma_n$ for $n \in \mathbb{N}^*$.

Proof of Proposition 3. See Section 7. □

Proof of Theorem 2. Theorem 2 follows from Proposition 3. Indeed, (5.1) ensures existence of a constant $C > 0$ such that

$$\lambda_n(\Lambda + g(\Lambda) - C\Lambda^{1-\rho-\rho_1}) \leq \lambda_n(\tilde{J}) \leq \lambda_n(\Lambda + g(\Lambda) + C\Lambda^{1-\rho-\rho_1}),$$

where

$$\lambda_n(\Lambda + g(\Lambda) \pm C\Lambda^{1-\rho-\rho_1}) = n + g(n) \pm Cn^{1-\rho-\rho_1}$$

is the n-th eigenvalue of $\Lambda + g(\Lambda) \pm C\Lambda^{1-\rho-\rho_1}$, and (1.8) follows from $\lambda_n(J) = \lambda_n(\tilde{J})$. □

6. Proof of Proposition 2

We begin by a few simple lemmas.

Lemma 1. *If $q\colon \mathbb{N}^* \to \mathbb{R}$ then the commutator of $q(\Lambda)$ with the shift operator S has the form*

$$[q(\Lambda), S] = \big(q(\Lambda + I) - q(\Lambda)\big)S. \tag{6.1}$$

Proof. Indeed, the direct computation gives

$$\begin{aligned} Sq(\Lambda)\mathrm{e}_n &= Sq(n)\mathrm{e}_n = q(n)\mathrm{e}_{n+1} = q(\Lambda)S\mathrm{e}_n \\ q(\Lambda)S\mathrm{e}_n &= q(\Lambda)\mathrm{e}_{n+1} = q(n+1)\mathrm{e}_{n+1} = q(\Lambda + I)S\mathrm{e}_n \end{aligned}$$

for every $n \in \mathbb{N}^*$. □

Lemma 2. *Let A and J_1 be as in Section 3. Then*

$$[\Lambda, A] = \Lambda A - A\Lambda = J_1. \tag{6.2}$$

Proof. Using Lemma 1 with $q(n) = n$ we find

$$[\Lambda, S] = S, \tag{6.3}$$

hence

$$[\Lambda, Sb(\Lambda)] = [\Lambda, S]b(\Lambda) = Sb(\Lambda)$$

and

$$[\Lambda, A] = [\Lambda, Sb(\Lambda)] + \mathrm{hc} = Sb(\Lambda) + (Sb(\Lambda))^* = J_1.$$ □

Lemma 3. *Let g be as in Proposition 3. Then*

$$[J_1, A] = -2g(\Lambda). \tag{6.4}$$

Proof. To begin we observe that

$$g(\Lambda) = Sb(\Lambda)^2 S^* - b(\Lambda)^2 \tag{6.5}$$

follows from $Sb(\Lambda)S^*\mathrm{e}_n = Sb(\Lambda)\mathrm{e}_{n-1} = b(n-1)\mathrm{e}_n$ if $n \geq 2$ and $S^*\mathrm{e}_1 = 0$. Then

$$\begin{aligned}[J_1, A] &= [Sb(\Lambda) + b(\Lambda)S^*, Sb(\Lambda)] + \mathrm{hc} \\ &= [b(\Lambda)S^*, Sb(\Lambda)] + \mathrm{hc}\end{aligned}$$

and we complete the proof writing

$$[b(\Lambda)S^*,\ Sb(\Lambda)] = b(\Lambda)^2 - Sb(\Lambda)^2 S^* = -g(\Lambda),$$

where we used $S^*S = I$ and (6.5). □

Proof of Proposition 2. The standard expansion formula gives

$$\mathrm{e}^{A}\Lambda\mathrm{e}^{-A} = \Lambda + [\Lambda, A] + \int_0^1 (1-s)\mathrm{e}^{sA}\big[[\Lambda, A], A\big]\mathrm{e}^{-sA}\mathrm{d}s \tag{6.6}$$

and (6.2) allows us to rewrite (6.6) in the form

$$\mathrm{e}^{-A}J\mathrm{e}^{A} = \Lambda - \int_0^1 (1-s)\mathrm{e}^{(s-1)A}\big[[\Lambda, A], A\big]\mathrm{e}^{(1-s)A}\mathrm{d}s. \tag{6.7}$$

However (6.2), (6.4) and (1.4) imply

$$\big[[\Lambda, A], A\big] = [J_1, A] = -2g(\Lambda) = \mathrm{O}(\Lambda^{1-\rho-\rho'}),$$

which completes the proof due to □

Lemma 4. *For every $m \in \mathbb{R}$ one has*

$$\sup_{-1\le s\le 1} \|\Lambda^m \mathrm{e}^{sA}\Lambda^{-m}\| < \infty. \tag{6.8}$$

Proof. (a) To begin, we check that the estimate

$$[\Lambda^\varepsilon, A] = \mathrm{O}(\Lambda^{\varepsilon-\rho}) \tag{6.9}$$

holds for every $\varepsilon > 0$. Indeed, using Lemma 1 with $q(n) = n^\varepsilon$ we find

$$\begin{aligned}[\Lambda^\varepsilon, A] &= [\Lambda^\varepsilon, Sb(\Lambda)] + \mathrm{hc} \\ &= [\Lambda^\varepsilon, S]b(\Lambda) + \mathrm{hc} \\ &= \big((\Lambda+I)^\varepsilon - \Lambda^\varepsilon\big)Sb(\Lambda) + \mathrm{hc}\,.\end{aligned}$$

Hence using property (3.3) and

$$\begin{aligned}&Sb(\Lambda) = \mathrm{O}(\Lambda^{1-\rho}), \\ &(\Lambda+I)^\varepsilon - \Lambda^\varepsilon = \mathrm{O}(\Lambda^{\varepsilon-1})\end{aligned}$$

we obtain (6.9).

(b) Further on we assume $0 < \varepsilon \le \rho$ and we show that

$$\mathcal{M}_{k\varepsilon} = \sup_{-1\le s\le 1} ||\Lambda^{k\varepsilon}\mathrm{e}^{sA}\Lambda^{-k\varepsilon}|| < \infty \tag{6.10}$$

holds for every $k \in \mathbb{N}$. We introduce

$$R_{k\varepsilon}(s) = \Lambda^{(k+1)\varepsilon}\mathrm{e}^{sA}\Lambda^{-(k+1)\varepsilon} - \Lambda^{k\varepsilon}\mathrm{e}^{sA}\Lambda^{-k\varepsilon}$$

and observe that

$$R_\varepsilon(s) = \big[\mathrm{e}^{s(1-t)A}\Lambda^\varepsilon\mathrm{e}^{stA}\Lambda^{-\varepsilon}\big]_{t=0}^{t=1} = \int_0^1 \mathrm{e}^{s(1-t)A}[A, \Lambda^\varepsilon]\mathrm{e}^{stA}\Lambda^{-\varepsilon}\mathrm{d}t$$

allows us to estimate (6.9) allows us to estimate

$$\begin{aligned}\|R_{k\varepsilon}(s)\| &= \|\Lambda^{k\varepsilon}R_\varepsilon(s)\Lambda^{-k\varepsilon}\| \\ &\le \mathcal{M}_{k\varepsilon}^2\|\Lambda^{k\varepsilon}[\Lambda^\varepsilon, A]\Lambda^{-k\varepsilon}\| < \infty\end{aligned}$$

if $\mathcal{M}_{k\varepsilon} < \infty$. □

7. Proof of Proposition 3

(a) To begin, we observe that

$$[g(\Lambda), A] = \mathrm{O}(\Lambda^{1-\rho-\rho_1}) \tag{7.1}$$

follows from assumption (1.7). Indeed,

$$\begin{aligned}[g(\Lambda), A] &= [g(\Lambda), Sb(\Lambda)] + \mathrm{hc} \\ &= [g(\Lambda), S]b(\Lambda) + \mathrm{hc} \\ &= \big(g(\Lambda + I) - g(\Lambda)\big)Sb(\Lambda) + \mathrm{hc},\end{aligned}$$

hence using property (3.2) and

$$\begin{aligned}&Sb(\Lambda) = \mathrm{O}(\Lambda^{1-\rho}), \\ &g(\Lambda + I) - g(\Lambda) = \mathrm{O}(\Lambda^{-\rho_1})\end{aligned}$$

we obtain (7.1).

(b) Then the standard expansion formula gives

$$\mathrm{e}^{A}\Lambda\mathrm{e}^{-A} = \Lambda + [\Lambda, A] + \frac{1}{2}[[\Lambda, A], A] + \int_0^1 (1-s)^2\mathrm{e}^{sA}R\mathrm{e}^{-sA}\mathrm{d}s \tag{7.2}$$

with

$$R = \frac{1}{2}\big[[[\Lambda, A], A], A\big] = -[g(\Lambda), A].$$

(c) However we have $R = \mathrm{O}(\Lambda^{1-\rho-\rho_1})$ due to (7.1) and Lemma 4 allows us to deduce

$$\mathrm{e}^{A}\Lambda\mathrm{e}^{-A} = J - g(\Lambda) + \mathrm{O}(\Lambda^{1-\rho-\rho_1}) \tag{7.3}$$

from (7.2). Applying Lemma 4 once more we obtain

$$\mathrm{e}^{-A}J\mathrm{e}^{A} = \Lambda + \mathrm{e}^{-A}g(\Lambda)\mathrm{e}^{A} + \mathrm{O}(\Lambda^{1-\rho-\rho_1}). \tag{7.4}$$

(d) To complete the proof of Proposition 3 it remains to show

$$\mathrm{e}^{-A}g(\Lambda)\mathrm{e}^{A} = g(\Lambda) + \mathrm{O}(\Lambda^{1-\rho-\rho_1}). \tag{7.5}$$

However

$$\mathrm{e}^{-A}g(\Lambda)\mathrm{e}^{A} - g(\Lambda) = \int_0^1 \mathrm{e}^{-sA}[g(\Lambda), A]\mathrm{e}^{sA}\mathrm{d}s$$

and it is clear that (7.5) follows from (7.1) and Lemma 4.

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