Non-Monotone Stochastic Generalized Porous Media Equations*

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Abstract

By using the Nash inequality and a monotonicity approximation argument, existence and uniqueness of strong solutions are proved for a class of non-monotone stochastic generalized porous media equations. Moreover, we prove for a large class of stochastic PDE that the solutions stay in the smaller L^2-space provided the initial value does, so that some recent results in the literature are considerably strengthened.

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1 Introduction

Based on the classical Galerkin method of finite-dimensional approximations, a large class of nonlinear partial differential equations can be solved on a separable real Hilbert space H under certain monotonicity conditions, see e.g. [16] and the references therein for deterministic equations, and [11, 13, 5, 10, 15] and the references therein for stochastic versions. More precisely, consider for instance

$$\mathrm{d}X_t = A(t, X_t)\mathrm{d}t + B(t, X_t)\mathrm{d}W_t,$$

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where W_t is a G-valued cylindrical Brownian motion on a complete filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ for some real separable Hilbert space G, $A : V \to V^*$ is a measurable map for some reflexive Banach space V and dual V^* with embeddings $V \subset H \subset V^*$ dense and continuous, and B is a progressively measurable process in the space of Hilbert-Schmidt operators from G to H. Among other conditions for existence and uniqueness of solutions for this equation, the monotonicity is expressed as

$$ {}_{V^*}\langle A(u) - A(v), u - v\rangle_V \le c\|u - v\|_H^2, \quad u, v \in V \tag{1.1} $$

for some constant $c > 0$.

On the other hand, however, the following stochastic porous medium equation studied in [10] is not monotone on $L^2(\mathbb{R}^d; \mathrm{d}x)$:

$$ \mathrm{d}X_t = \Delta\big\{X_t|X_t|^{r-1}\big\}\mathrm{d}t + B(t, X_t)\mathrm{d}W_t, \tag{1.2} $$

where Δ is the Laplace operator on $\mathbb{R}^d$, $r > 1$ is a fixed number, and B and W are as above for $G = H := L^2(\mathbb{R}^d; \mathrm{d}x)$. Indeed, for any $c > 0$, the condition

$$ \langle \Delta(f|f|^{r-1} - g|g|^{r-1}), f - g\rangle \le c\|f - g\|_2^2, \quad f, g \in C_0^\infty(\mathbb{R}^d) $$

does not hold, where $\langle \cdot, \cdot\rangle$ and $\|\cdot\|_2$ are the inner product and norm in $L^2(\mathbb{R}^d; \mathrm{d}x)$ respectively. By combining the Sobolev inequality with Galerkin approximations, Kim [10] was able to solve this equation on $L^2(\mathbb{R}^d; \mathrm{d}x)$ for $X_0 \in L^2(\mathbb{R}^d \times \Omega; \mathrm{d}x \times \mathbb{P})$, and the unique solution is an adapted process on $L^2(\mathbb{R}^d; \mathrm{d}x)$ satisfying

$$ \mathbb{E}\int_0^T \mathrm{d}t \int_{\mathbb{R}^d} |\nabla(X_t|X_t|^{r-1})|^2(x)\mathrm{d}x < \infty. $$

The right-continuity of the solution, however, is not proved in [10].

In this paper, we show that the existence and uniqueness result for monotone equations can be extended to a class of non-monotone situations as soon as the Nash inequality holds. Indeed, our results are proved for a rather general framework in which we can also allow B to depend on the solution X. Even under the framework of Kim [10] where B is independent of X ("additive noise"), we allow B to be Hilbert-Schmidt from $L^2(\mathbb{R}^d; \mathrm{d}x)$ to H^{-1}, where H^{-1} is the dual of $H^1(\mathbb{R}^d) :=$ classical Sobolev space of order 1 in $L^2(\mathbb{R}^d; \mathrm{d}x)$, and allow X_0 to be any H^{-1}-valued $\mathscr{F}_0$-measurable random variable. Since H^{-1} is much larger than $L^2(\mathbb{R}^d; \mathrm{d}x)$ and the norm in H^{-1} is much smaller than that in $L^2(\mathbb{R}^d; \mathrm{d}x)$, our assumptions are considerably weaker than Kim's in [10]. If furthermore B_t is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^d; \mathrm{d}x)$, then our results also generalize Kim's, namely, the solution with $\mathbb{E}\|X_0\|_2^2 < \infty$ satisfies

$$ \mathbb{E}\sup_{t\in[0,T]} \|X_t\|_2^2 < \infty \quad \text{and} \quad |X|^{r-1}X \in L^2([0, T] \times \Omega \to \mathscr{F}_e; \mathrm{d}t \times \mathbb{P}), \quad T > 0, $$

where $\mathscr{F}_e$ is the completion of $C_0^\infty(\mathbb{R}^d; \mathrm{d}x)$ under the inner product $\langle f, f\rangle_{\mathscr{F}_e} := \int_{\mathbb{R}^d}\langle \nabla f, \nabla g\rangle \mathrm{d}x$. Some other properties are also derived (cf. Theorem 1.2 below). Our result, in fact, hold for a large class of (not necessarily differential) operators L replacing the Laplacian. The appropriate class are operators which are associated to Dirichlet forms satisfying a Nash-type inregularity. The reader unfamiliar with Dirichlet forms should think e.g. of L being a globally elliptic differential operator of order 2 on $\mathbb{R}^d$, $d \geq 3$.

Let us introduce our framework in detail. Let $(E, \mathscr{B}, \mathbf{m})$ be a σ-finite separable measure space and $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ a symmetric Dirichlet form on $L^2(\mathbf{m})$ (cf. [9]). Assume that the following Nash inequality

$$\|f\|_2^2 \leq C\mathscr{E}(f,f)^{d/(d+2)}, \quad f \in \mathscr{D}(\mathscr{E}), \mathbf{m}(|f|) = 1, \tag{1.3}$$

holds for some constant $C > 0$, where $\|\cdot\|_p$ is the norm in $L^p(\mathbf{m})$ for $p \geq 1$. This inequality is equivalent to the classical Sobolev inequality with dimension d if $d > 2$ (cf. [6, Theorems 2.4.2 and 2.4.6]) i.e. there exists $C_d \in (0, \infty)$ such that

$$\|f\|_{\frac{2d}{d-2}} \leq C_d \mathscr{E}(f,f)^{1/2}, \ f \in \mathscr{D}(\mathscr{E}). \tag{1.4}$$

In particular, it holds for the classical Dirichlet form generated by the Laplacian on $\mathbb{R}^d$, $d \geq 3$. We adopt the above formulation (1.3) here to include also examples with dimension ≤ 2. In particular, this inequality holds for the Dirichlet Laplace operator on bounded domains in a Riemannian manifold and on the whole Riemannian manifold provided the injectivity radius is infinite (see [3]). Moreover, (1.3) also holds for Dirichlet forms associated with stable-like processes, since according to Theorem 1.3 in [2] the Nash inequality holds for fractional Dirichlet forms with parameter $d > 0$. Let $(L, \mathscr{D}(L))$ be the associated Dirichlet operator, which is thus a negative definite self-adjoint operator on $L^2(\mathbf{m})$. We shall use $\langle \cdot, \cdot \rangle$ for the inner product in $L^2(\mathbf{m})$ and $\|\cdot\|_2$ for its norm. More generally, we set $\langle f, g \rangle := \mathbf{m}(fg) := \int fg \, \mathrm{d}\mathbf{m}$ for any two measurable functions f, g such that $fg \in L^1(\mathbf{m})$. Let $\mathscr{D}(\mathscr{E})$ be equipped with the inner product $\mathscr{E}_1 := \mathscr{E} + \langle \cdot, \cdot \rangle$ and H its dual space. H is then a separable Hilbert space equipped with the induced inner product $\langle \cdot, \cdot \rangle_H$ and norm $\|\cdot\|_H := \langle \cdot, \cdot \rangle_H^{1/2}$. For $a > 0$ we shall also consider the inner products $\mathscr{E}_a := a\mathscr{E} + \langle \cdot, \cdot \rangle$ on $\mathscr{D}(\mathscr{E})$ and their dual inner products $\langle \cdot, \cdot \rangle_{H_a}$ on H with corresponding norms $\|\cdot\|_{H_a}$ (see Section 2 below for details). If H is equipped with $\langle \cdot, \cdot \rangle_{H_a}$ (and $\|\cdot\|_{H_a}$) we denote it by H_a, hence $H_1 = H$. By continuity $1 - L$ (and hence L) extends from $\mathscr{D}(L)$ to an operator from $\mathscr{D}(\mathscr{E})$ to H, denoted by the same symbol. Finally, let $\mathscr{F}_e$ be the completion of $\mathscr{D}(\mathscr{E})$ under the inner product $\langle f, g \rangle_{\mathscr{F}_e} := \mathscr{E}(f, g)$, which is called the extended domain of the Dirichlet form (see [9]). If $d > 2$, (1.4) (hence (1.3)) immediately) implies that $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is transient in the sense of [9], that is, there exists $g \in L^1(\mathbf{m}) \cap L^\infty(\mathbf{m})$, such that $\mathscr{F}_e \subset L^1(g \cdot \mathbf{m})$ continuously. We denote the extension of $\mathscr{E}$ from $\mathscr{D}(\mathscr{E})$ to $\mathscr{F}_e$ by $\bar{\mathscr{E}}$, and denote the dual space of $\mathscr{F}_e$ by $\mathscr{F}_e^*$. Since $\mathscr{D}(\mathscr{E}) \subset \mathscr{F}_e$ densely and continuously, also $\mathscr{F}_e^* \subset H$ densely and continuously. But in general $\mathcal{F}_e^* \neq H$. We equip $\mathcal{F}_e^*$ with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{F}_e^*}$ and corresponding norm $\|\cdot\|_{\mathcal{F}_e^*}$, induced by the Riesz map $\mathscr{F}_e \ni u \mapsto \bar{\mathcal{E}}(\cdot, u) \in \mathcal{F}_e^*$. We recall that if $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is transient, then $\mathscr{F}_e \cap L^2(\mathbf{m}) = \mathscr{D}(\mathscr{E})$ (cf. [9]). If $\mathbf{m}(E) < \infty$, then (1.3) implies that $\inf \sigma(-L) > 0$ and thus that $\mathscr{D}(\mathscr{E}) = \mathscr{F}_e$, hence $H = \mathscr{F}_e^*$ and $(\mathscr{E}, (\mathscr{D}(\mathscr{E}))$ is transient in this case.

Let $r_2 > r_1 > 1$ be two constants and ν a probability measure on $[r_1, r_2]$. We consider the following stochastic partial differential equation on H:

$$\mathrm{d}X_t = \left\{ \bar{L} \int_{r_1}^{r_2} \xi(t,r)|X_t|^{r-1} X_t \nu(\mathrm{d}r) + \eta_t X_t \right\} \mathrm{d}t + B(t, X_t)\mathrm{d}W_t, \tag{1.5}$$

where W is a cylindrical Brownian motion on $L^2(\mathbf{m})$, ξ, η and B are specified in the following assumptions and $\bar{L}$ in Definition 2.3 below. For two Hilbert spaces H_1 and H_2, let $\mathscr{L}_{HS}(H_1; H_2)$ denote the Hilbert space of all Hilbert-Schmidt operators from H_1 to H_2, equipped with the usual Hilbert-Schmidt inner product. Consider the following conditions:

$(H1)$ $\xi : [0, \infty) \times [r_1, r_2] \times \Omega \to [0, \infty)$ is progressively measurable and for any $T > 0$, there exists a locally bounded function $R : [0, \infty) \to [1, \infty)$ such that $\frac{1}{R(t)} \le \xi(t, \cdot) \le R(t)$ holds on $[r_1, r_2] \times \Omega$ for all $t \in [0, T]$.

$(H2)$ η is a real-valued locally bounded progressively measurable process (i.e. $\sup_{\substack{s\in[0,T],\\ \omega\in\Omega}} |\eta_s(\omega)| < \infty$ for every $T > 0$.).

$(H3)$ For every $T > 0$ the map $B : [0, T] \times V \times \Omega \to \mathscr{L}_{HS}(L^2(\mathbf{m}); H)$ is progressively measurable such that

(i) there exists $C \in (0, \infty)$ such that for all $a \in (0, \infty)$

$$\|B(\cdot, u) - B(\cdot, v)\|_{\mathscr{L}_{HS}(L^2(\mathbf{m}), H_a)} \le C\|u - v\|_{H_a}^2 \quad \text{on } [0, T] \times \Omega \text{ for all } u, v \in V;$$

(ii)

$$\int_0^T \|B(s, 0)\|_{\mathscr{L}_{HS}(L^2(\mathbf{m});H)}^2 \mathrm{d}s \in L^{r_2}(\mathbb{P}).$$

We give examples where condition (H.3(i)) holds in Remark 2.9 at the end of Section 2 below. Obviously, when $\xi = 1, \eta = 0$ and $\nu = \delta_r$ (the Dirac measure at r), equation (1.5) reduces to (1.2). The following definition of a solution is taken from [15] (see also [11]).

First, however, we need to introduce auxiliary spaces V and V^*:
It is easy to see that $N(s) := \int_{r_1}^{r_2} |s|^{r+1} \nu(\mathrm{d}r),\ s \in \mathbb{R}$, is a Δ_2-regular Young function so that the corresponding Orlicz space $L_N(\mathbf{m})$ is a reflexive separable Banach space (see [14]). By [15, Propostion 3.1] applied to $L - 1$ instead of L the embedding $V := H \cap L_N(\mathbf{m}) \subset H$ is dense and continuous. Furthermore, V is reflexive (see [15]). Let V^* be the dual of V and N^* the dual Young function to N^* (cf. Section 2 below for details).

Definition 1.1. A continuous adapted process $\{X_t\}_{t\ge0}$ on H is called a solution to (1.5), if for any $T > 0$, $X \in L^2([0, T] \times \Omega \to H, \mathrm{d}t \times \mathbb{P})$ with

$$\int_0^T \int_{r_1}^{r_2} \|X_t\|_{r+1}^{r+1} \nu(\mathrm{d}r)\mathrm{d}t < \infty \quad \mathbb{P} - \text{a.s.} \tag{1.6}$$

such that $\mathbb{P}$-a.s.

$$
\begin{aligned}
X_t = X_0 + \bar{L}\left[\int_0^t \left(\int_{r_1}^{r_2} \xi(s,r)|X_s|^{r-1}X_s\nu(\mathrm{d}r)\right)\mathrm{d}s\right] \\
+ \int_0^t \eta_s X_s \mathrm{d}s + \int_0^t B(s,X_s)\mathrm{d}W_s, \text{ for all } t \geq 0
\end{aligned} \tag{1.7}
$$

holds in H, where the first integral in (1.7) is an L_{N^*}-valued Bochner integral which takes values in $\mathscr{D}(\bar{L})$ $\mathbb{P}$-a.s. $\forall t \geq 0$ and $\bar{L} : D(\bar{L}) \subset L_{N^*} \to V^*$ is a natural extension of $L : D(\mathscr{E}) \cap L_{N^*} \to V^*$ defined in Definition 2.3 below.

Theorem 1.2. *Assume* (1.3), $(H1), (H2)$ *and* $(H3)$.

(1) *For any $\mathscr{F}_0$-measurable H-valued random variable X_0,* (1.5) *has a unique solution in the sense of Definition 1.1. This solution is a Markov process provided ξ, η and B are constant (i.e. independent of t and ω).*

(2) *Let $\{X^{(n)}\}$ be a sequence of solutions to* (1.5)*. If $X_0^{(n)} \to X_0$ in H in probability as $n \to \infty$, then for any $t > 0$,*

$$
X_t^{(n)} \to X_t \text{ in } H \quad \text{and} \quad \int_0^t \int_{r_1}^{r_2} \|X_s^{(n)} - X_s\|_{r+1}^{r+1}\nu(\mathrm{d}r)\mathrm{d}s \to 0
$$

in probability as $n \to \infty$. Consequently, if ξ, η and B are independent of t and ω, then the transition semigroup of the solution is a Feller semigroup.

(3) *For all $p \in [2, \infty)$, $T > 0$, and some constant $c(p,T)$*

$$
\mathbb{E} \sup_{t\in[0,T]} \|X_t\|_H^p \leq c(p,T)\left[\mathbb{E}\|X_0\|_H^p + \mathbb{E}\left(\int_0^T \|B(s,0)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H)}\mathrm{d}s\right)^{\frac{p}{2}}\right]
$$

which is finite provided $p \leq 2r_2$ and $\mathbb{E}\|X_0\|_H^p < \infty$. In the latter case we have

$$
\mathbb{E}\left[\int_0^T \int_{r_1}^{r_2} \|X_t\|_{r+1}^{r+1}\nu(\mathrm{d}r)\mathrm{d}t\right]^{p/(r_2+1)} < \infty, \qquad \textit{provided } p \geq r_2 + 1.
$$

(4) *In addition, assume that $B(\cdot, 0) \in L^2\big([0,T] \times \Omega \to \mathscr{L}_{HS}(L^2(\mathbf{m}); L^2(\mathbf{m})), \mathrm{d}t \times \mathbb{P}\big)$. If $X_0 \in L^2(\mathbf{m})$ a.s. then X_t is a right-continuous process in $L^2(\mathbf{m})$ ("$L^2(\mathbf{m})$-invariance"). If moreover $\mathbb{E}\|X_0\|_2^2 < \infty$, then $\mathbb{E}\sup_{t\in[0,T]} \|X_t\|_2^2 < \infty$. If, in addition, E is a Lusin space, then $\zeta(X_t) := \int_{r_1}^{r_2} |X_t|^{(r-1)/2}X_t\nu(\mathrm{d}r) \in \mathscr{D}(\mathscr{E})$ $\mathrm{d}t \times \mathbb{P}$-a.e. with*

$$
\mathbb{E}\int_0^T \mathscr{E}(\zeta(X_t), \zeta(X_t))\mathrm{d}t < \infty. \tag{1.8}
$$

Consequently, if $\mathbb{E}(\|X_0\|_2^2 + \|X_0\|_H^{r_2+1}) < \infty$ then $\zeta(X) \in L^2([0,T] \times \Omega \to \mathscr{D}(\mathscr{E}); \mathrm{d}t \times \mathbb{P})$ for any $T > 0$.

The uniqueness and the Markov property can be proved in a standard way as in [11, 5, 15] by using the Itô formula for the square of the norm. So, the main point is to prove the existence. Since in general the map (cf. Section 3 in [15])

$$V \ni x \mapsto A(t,x) := L \int_{r_1}^{r_2} \xi(t,r)|x|^{r-1} x \nu(\mathrm{d}r) + \eta_t x \in V^*$$

is not monotone in H, known results concerning monotone stochastic SPDEs do not work directly. To make the equation monotone, in [15] we replaced H by $\mathscr{F}_e^*$, the dual space of the extended Dirichlet space $\mathscr{F}_e$, but had to assume that $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is transient. In general, the embedding $\mathscr{F}_e^* \subset H$ is dense and continuous, but $\mathscr{F}_e^*$ and $L^2(\mathbf{m})$ are incomparable except $\inf \sigma(-L) > 0$, where $\sigma(-L)$ is the spectrum of $(-L)$. Under a stronger condition than $(H3)$, namely that B is in $L^2\big([0,T] \times \Omega \to \mathscr{L}_{HS}(L^2(\mathbf{m}); \mathscr{F}_e^*), \mathrm{d}t \times \mathbb{P}\big)$, in [15] existence and uniqueness of the solution to (1.5) was proved for all $X_0 \in L^2(\Omega \to \mathscr{F}_e^*; \mathscr{F}_0, \mathbb{P})$. Since $\mathscr{F}_e^*$ and $L^2(\mathbf{m})$ are generally incomparable, the solutions constructed in [15] do not automatically provide solutions starting from points in $L^2(\mathbf{m}) \setminus \mathscr{F}_e^*$. So, in this paper we first construct solutions in H, which is larger than $L^2(\mathbf{m})$, then prove that the the solution will be in $L^2(\mathbf{m})$ for $t \geq 0$ provided the initial value is so and B is as in Theorem 1.2(4).

To construct solutions starting from all $\mathscr{F}_0$-measurable H-valued random variables, we develop an approximation argument by first considering the equation (1.5) for $L - \varepsilon$ in place of L to make the equation monotone on H, then taking the limit $\varepsilon \to 0$ we obtain a solution for the original equation. To realize this approximation procedure, the Nash inequality (1.3) will play a crucial role.

In Section 2 we first briefly recall some general results obtained in [15] concerning monotone stochastic equations, prove some technical auxiliary results and then prove a criterion for the $L^2(\mathbf{m})$-invariance of solutions. Some a priori estimates are presented in Section 3 by using the Nash inequality, which will be used in Section 4 to construct the solution to (1.5) for H-valued X_0 satisfying a moment condition. Finally, the complete proof of Theorem 1.2 is contained in Section 5.

From now on we fix $(E, \mathcal{B}, \mathbf{m})$ and $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ as above.

2 Some known results and $L^2(\mathbf{m})$-invariance

2.1 Review of known results

In this subsection we recall some results obtained recently in [15] which will be used in the sequel for constructing solutions to (1.5). In all of this subsection we assume that $\inf \sigma(-L) > 0$, hence $H = \mathscr{F}_e^*$. But at least initially we shall consider the inner product $\langle \cdot, \cdot \rangle_{\mathscr{F}_e^*}$ on H and only later $\langle \cdot, \cdot \rangle_H$.

Let $N \in C(\mathbb{R})$ be a Young function, i.e. a nonnegative, continuous, convex and even function such that $N(s) = 0$ if and only if $s = 0$, and

$$\lim_{s \to 0} \frac{N(s)}{s} = 0, \quad \lim_{s \to \infty} \frac{N(s)}{s} = \infty.$$

For any measurable function f on E with $\mathbf{m}(N(\alpha f)) < \infty$ for some $\alpha > 0$, define

$$\|f\|_N := \inf\{\lambda \geq 0 : \mathbf{m}(N(f/\lambda)) \leq 1\}.$$

Then the space

$$L_N(\mathbf{m}) := \{f : \|f\|_N < \infty\}$$

is a real separable Banach space, which is called the Orlicz space induced by the Young function N (cf. [14, Proposition 1.2.4]). There is an equivalent norm defined by using the dual function:

$$N^*(s) := \sup\{r|s| - N(r) : r \geq 0\}, \quad s \in \mathbb{R},$$

which is once again a Young function. More precisely, letting

$$\|f\|_{(N)} := \sup\{\langle f, g\rangle : \mathbf{m}(N^*(g)) \leq 1\},$$

one has (see [14, Theorem 1.2.8 (ii)])

$$\|\cdot\|_N \leq \|\cdot\|_{(N)} \leq 2\|\cdot\|_N. \tag{2.1}$$

The function N is called Δ_2-regular, if there exists a constant $c > 0$ such that

$$N(2s) \leq c\big(N(s) + 1_{\{\mathbf{m}(E)<\infty\}}\big), \quad s \in \mathbb{R}.$$

We assume that N and N^* are Δ_2-regular. By [14, Proposition 1.2.11(iii) and Theorem 1.2.13], $L_N(\mathbf{m})$ and $L_{N^*}(\mathbf{m})$ are dual spaces of each other, and hence are reflexive. By the Δ_2-regularity, $f \in L_N(\mathbf{m})$ if and only if $\mathbf{m}(N(f)) < \infty$. For simplicity, we sometimes use L_N and L_{N^*} instead of $L_N(\mathbf{m})$, $L_{N^*}(\mathbf{m})$ respectively.

Let $V := H \cap L_N(\mathbf{m})$ with $\|\cdot\|_V := \|\cdot\|_N + \|\cdot\|_H$. More precisely,

$$V = \left\{v \in L_N(\mathbf{m}) \;\middle|\; \mathscr{D}(\mathscr{E}) \cap L_{N^*}(\mathbf{m}) \ni u \mapsto \mathbf{m}(uv) \text{ is in } H\right\}.$$

Since by [15, Proposition 3.1 and its proof] $\mathscr{D}(\mathscr{E}) \cap L_{N^*}$ is dense in $\mathscr{D}(\mathscr{E})$, V is indeed embedded into H. Furthermore, V is complete, by [15, Proposition 3.1], reflexive and dense in H and L_N. Let

$$\Psi : [0,\infty) \times \mathbb{R} \times \Omega \to \mathbb{R}$$

be progressively measurable, i.e. for any $t \geq 0$, Ψ restricted to $[0,t] \times \mathbb{R} \times \Omega$ is measurable w.r.t. $\mathscr{B}([0,t]) \times \mathscr{B}(\mathbb{R}) \times \mathscr{F}_t$. We assume that for any $(t,\omega) \in [0,\infty) \times \Omega$, $\Psi(t,\cdot)(\omega)$ is continuous.

Finally, let $B : [0,\infty) \times V \times \Omega \to \mathscr{L}_{HS}(L^2(\mathbf{m}); H)$ be progressively measurable as in the last section. We shall make use of the following assumptions:

(B) For any $T > 0$, $\|B(\cdot,0)\|_{\mathscr{L}_{HS}(L^2(\mathbf{m});H)} \in L^2([0,T]\times\Omega; \mathrm{d}t\times\mathbb{P})$ and there exists a constant $c \geq 0$ such that $\|B(\cdot,u) - B(\cdot,v)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H)} \leq c\|u-v\|_H^2$ holds on $[0,T]\times\Omega$ for all $u, v \in V$.

($\mathbf{\Psi}$) For any $T>0$, there exist a nonnegative $\mathscr{F}_t$-adapted process $f \in L^1\big([0,T]\times\Omega; \mathrm{d}t\times\mathbb{P}\big)$ and a constant $c \geq 1$, such that for all $s, s_1, s_2 \in \mathbb{R}$ on $[0,T]\times\Omega$

($\mathbf{\Psi 1}$) $$(s_2 - s_1)\big(\Psi(\,\cdot\,, s_2) - \Psi(\,\cdot\,, s_1)\big) \geq 0\ .$$

($\mathbf{\Psi 2}$) $$c^{-1}N(s) - 1_{\{\mathbf{m}(E)<\infty\}} f \leq s\Psi(\,\cdot\,, s) \leq cN(s) + 1_{\{\mathbf{m}(E)<\infty\}} f.$$

($\mathbf{\Psi 3}$) $$N^*\big(\Psi(\cdot, 0)\big)\, 1_{\{\mathbf{m}(E)<\infty\}} \in L^1\big([0,T]\times\Omega; \mathrm{d}t\times\mathbb{P}\big)\ .$$

Let
$$K := L_N([0,T]\times E\times\Omega; \mathrm{d}t\times\mathbf{m}\times\mathbb{P}) \cap L^2([0,T]\times\Omega \to H; \mathrm{d}t\times\mathbb{P})$$
with norm
$$\|\cdot\|_K := \|\cdot\|_{L_N([0,T]\times E\times\Omega;\mathrm{d}t\times\mathbf{m}\times\mathbb{P})} + \|\cdot\|_{L^2([0,T]\times\Omega\to H,\mathrm{d}t\times\mathbb{P})}.$$
Then, $K \subset L^1\big([0,T]\times\Omega \to V\,;\, \mathrm{d}t\times\mathbb{P}\big)$ continuously and densely (cf. [15, Lemmas 3.7 and 3.5]). Let K^* be the dual of K. Then by [15, Lemma 2.5] K^* is the completion of $L^\infty([0,T]\times\Omega \to V^*; \mathrm{d}t\times\mathbb{P})$ w.r.t.
$$\|z^*\|_{K^*} := \sup_{\|z\|_K\leq 1} \mathbb{E}\int_0^T {}_{V^*}\langle z_t^*, z_t\rangle_V \mathrm{d}t.$$
Furthermore, $K^* \subset L^1([0,T]\times\Omega \to V^*; \mathrm{d}t\times\mathbb{P})$ and we recall that by (Ψ) and [15, Lemma 3.6(i)] for all $u \in L_N$
$$\Psi(\cdot, u) \in L^1([0,T]\times\Omega \to L_{N^*}; \mathrm{d}t\times\mathbb{P}).$$

We want to apply the existence and uniqueness result [15, Theorem 3.9] in this case. We recall that in [15], $H = \mathscr{F}_e^*$ was identified with its dual $H^* = \mathscr{D}(\mathscr{E}) = \mathscr{F}_e$ using the Riesz map comming from the inner product $\langle\cdot,\cdot\rangle_{\mathscr{F}_e^*}$ defined in the introduction. The reason is that only in this inner product we have monotonicity for our drift coefficient. Since below we want to consider other inner products on H (generating, however, equivalent norms) and to avoid confusion we are going to recall the main existence and uniqueness result from [15] in a version not based on this specific identification of H and H^*. First, we fix some notation and conventions: for a Banach space B we denote its dual by B^* and use ${}_{B^*}\langle\cdot,\cdot\rangle_B$ for their dualization. We always consider B^* with the standard dual norm $\|l\|_{B^*} := \sup_{\|v\|_B=1} l(v)$, $l \in B^*$. If B is reflexive, then $B^{**} = B$ canonically and by convention we use this below without further mentioning it. By [15, Lemma 3.4(i)] and since $\inf\sigma(-L) > 0$, the map

$$\mathscr{D}(\mathscr{E}) \ni v \mapsto -\mathscr{E}(v,\cdot) \in H \tag{2.2}$$

(i.e. the Riesz isomorphism on $(\mathscr{D}(\mathscr{E}), \mathscr{E})$ multiplied by (-1)) is the unique continuous linear extension of the map
$$\mathscr{D}(L) \ni v \mapsto \langle Lv, \cdot\rangle \in H.$$
Here, as above, $\mathscr{D}(\mathscr{E})$ is equipped with the norm $\mathscr{E}^{1/2}(u) := \mathscr{E}(u,u)^{1/2}$, $u \in D(\mathscr{E})$, which is equivalent to the norm $\mathscr{E}_1^{1/2}(u) := (\mathscr{E}(u,u) + \langle u,u\rangle)^{1/2}$, $u \in \mathscr{D}(\mathscr{E})$, since $\inf\sigma(-L) > 0$. Let us denote the map in (2.2) again by L. Let $i : h \mapsto \langle\cdot, h\rangle_{\mathscr{F}_e^*}$ be the Riesz map on

$(H, \langle\cdot,\cdot\rangle_{\mathscr{F}_e^*})$. Then clearly, $i = (-L)^{-1} : H \to H^* = \mathscr{D}(\mathscr{E})$ and by [15, Lemma 3.4(iii)] (and since $\inf \sigma(-L) > 0$)

$$-1 = i \circ L : \mathscr{D}(\mathscr{E}) \cap L_{N^*} \to H^* \subset V^*$$

uniquely extends to a continuous linear map

$$\overline{i \circ L} : L_{N^*} \to V^*. \tag{2.3}$$

The map $\overline{i \circ L}$ is of course nothing but (-1) times the natural embedding $L_{N^*} \subset V^*$ induced by the continuous and dense embedding $V \subset L_N$. So, below we always replace $\overline{i \circ L}(u)$ by $-u$ for $u \in L_{N^*}$. Now we can formulate the existence and uniqueness result [15, Theorem 3.9] in our situation:

Theorem 2.1. *Let the Young function N and its dual function N^* be Δ_2-regular, and let $\inf \sigma(-L) > 0$. Assume (H2), ($\mathbf{B}$) and ($\mathbf{\Psi}$). Then for any $X_0 \in L^2(\Omega \to H; \mathscr{F}_0; \mathbb{P})$, the equation*

$$\mathrm{d}X_t = (L\Psi(t, X_t) + \eta_t X_t)\mathrm{d}t + B(t, X_t)\mathrm{d}W_t$$

has a unique solution in the sense that X_t is a continuous adapted process in H such that $X \in K$, $-\Psi(\cdot, X) + \eta\, i(X)$ is a progressively measurable process in K^* for any $T > 0$, and $\mathbb{P}$-a.s.

$$i(X_t) = i(X_0) + \int_0^t \big\{ - \Psi(s, X_s) + \eta_s\, i(X_s)\big\}\mathrm{d}s + i\left(\int_0^t B(s, X_s)\mathrm{d}W_s\right), \quad t \geq 0, \tag{2.4}$$

holds in $i(H) = H^* = \mathscr{D}(\mathscr{E})$ (where the first integral in (2.4) *is an $L_{N^*}(\subset V^*)$-valued Bochner integral, which a posteriori is in $\mathscr{D}(\mathscr{E})$ $\mathbb{P}$-a.e. $\forall t \geq 0$) or equivalently,*

$$X_t = X_0 + L\left(\int_0^t \Psi(s, X_s)\mathrm{d}s\right) + \int_0^t \eta_s X_s \mathrm{d}s + \int_0^t B(s, X_s)\mathrm{d}W_s, \quad t \geq 0, \tag{2.5}$$

holds in H. Furthermore, $\mathbb{E}\sup_{t\in[0,T]} \|X_t\|^2_{\mathscr{F}_e^*} < \infty$ for $T > 0$ and $\mathbb{P}$-a.s.

$$\|X_t\|^2_{\mathscr{F}_e^*} = \|X_0\|^2_{\mathscr{F}_e^*} + \int_0^t \left[2\ {}_{V^*}\langle -\Psi(s, X_s) + \eta_s i(X_s), X_s\rangle_V + \|B(s, X_s)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});\mathscr{F}_e^*)}\right] ds$$
$$+ 2\int_0^t \langle X_s, B(s, X_s)\mathrm{d}W_s\rangle_{\mathscr{F}_e^*}, \quad t \geq 0.$$

We note that since by (2.4) we have that $\int_0^t \Psi(s, X_s)ds \in \mathscr{D}(\mathscr{E}) \cap L_{N^*}$, we can replace L by $\bar{L}$ in (2.5). So, (2.5) means that X is indeed a solution in the sense of Definition 1.1. We also emphasize that the existence result in [15] is considerably more general. In particular, we do not need that $\inf \sigma(-L) > 0$, but only that $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is transient. Below, however, we shall only use the weaker version formulated in Theorem 2.1 above.

The above Itô formula for the square of the norm was proved in the Appendix of [15], generalizing the version proved in the fundamental work [11] for a special case where $K := L^p([0, T] \times \Omega \to V; \mathrm{d}t \times \mathbb{P}) \cap L^2([0, T] \times \Omega \to H; \mathrm{d}t \times P)$ for some $p > 1$. Below, however, we shall apply this formula to other, but equivalent norms $\|\cdot\|_{H_a}$ on H which for $a \searrow 0$ increase to $\|\cdot\|_2$ and come from inner products $\langle\cdot,\cdot\rangle_{H_a}$ on H_a which are defined in the next subsection in which we drop the assumption that $\inf \sigma(-L) > 0$.

2.2 Some technical lemmas and change of norms

In this subsection we do neither assume $\inf \sigma(-L) > 0$ nor (1.3), unless explicitly stated. Let $a > 0$ and define the following inner product on $\mathscr{D}(\mathscr{E})$ by

$$\mathscr{E}_a(u,v) := a\mathscr{E}(u,v) + \langle u,v\rangle; \quad u,v \in \mathscr{D}(\mathscr{E}).$$

Let $\langle\cdot,\cdot\rangle_{H_a}$ be its dual inner product on H_a, i.e. the inner product induced on H by the Riesz map on $(\mathscr{D}(\mathscr{E}),\mathscr{E}_a)$ which is given by

$$\mathscr{D}(\mathscr{E}) \ni u \mapsto a\mathscr{E}(u,\cdot) + \langle u,\cdot\rangle \in H \tag{2.6}$$

and which is the unique continuous linear extension of

$$(1-aL) : \mathscr{D}(L) \subset \mathscr{D}(\mathscr{E}) \to H,$$

hence we denote it by the same symbol $1-aL$. Then $i_a := (1-aL)^{-1}$ is just the Riesz map on $(H,\langle\cdot,\cdot\rangle_{H_a})$. In particular, we have

$${}_H\big\langle i_a^{-1}u, v\big\rangle_{\mathscr{D}(\mathscr{E})} = \mathscr{E}_a(u,v)\ , \quad u,v \in \mathscr{D}(\mathscr{E}). \tag{2.7}$$

As usual we set

$$\mathscr{E}_a^{1/2}(u) := (a\mathscr{E}(u,u) + \langle u,u\rangle)^{1/2},\ u \in D(\mathscr{E}).$$

If $a \le a'$, then $\mathscr{E}_a^{1/2} \le \mathscr{E}_{a'}^{1/2} \le \sqrt{\frac{a'}{a}}\mathscr{E}_a^{1/2}$, so $\|\cdot\|_{H_a} \ge \|\cdot\|_{H_{a'}} \ge \sqrt{\frac{a}{a'}}\|\cdot\|_{H_a}$, where $\|\cdot\|_{H_a} := \langle\cdot,\cdot\rangle_{H_a}^{1/2}$.

We emphasize that for different inner products $\langle\ ,\ \rangle_{H_a}$, $a > 0$, on H the corresponding Riesz isomorphisms $i_a : H \to H^*$, $h \mapsto \langle\,\cdot\,,h\rangle_{H_a}$ depend on $a > 0$. To avoid confusion, we shall therefore always distinguish between a Hilbert space and its dual, except for $L^2(\mathbf{m})$, which we canonically identify with its dual. So, we have

$$V \subset H \xrightarrow{i_a} H^* \subset V^* \tag{2.8}$$

and

$$\mathscr{D}(\mathscr{E}) \subset L^2(\mathbf{m}) \equiv L^2(\mathbf{m})^* \subset H\ .$$

In order to apply the Itô formula from [15] to $\|X_t\|_{H_a}^2$, $t \ge 0$, we have to find the stochastic equation satisfied by $i_a(X_t)$, $t \ge 0$. To this end we first have to define and calculate the unique continuous extension

$$\overline{i_a \circ L} : L_{N^*} \to V^*$$

of

$$i_a \circ L : \mathscr{D}(\mathscr{E}) \cap L_{N^*} \to H \xrightarrow{i_a} H^* \subset V^*.$$

Lemma 2.2. *Let $a > 0$. Then the map*

$$i_a \circ L : \mathscr{D}(\mathscr{E}) \cap L_{N^*} \to V^*$$

extends continuously to L_{N^*}, and for its extension $\overline{i_a \circ L} : L_{N^*} \to V^*$ we have

$$\overline{i_a \circ L} u = \frac{1}{a}\left(\overline{(1-aL)^{-1}}^{L_{N^*}} - 1\right) u \in L_{N^*}$$

for all $u \in L_{N^*}$, $v \in V$, where as usual 1 denotes the identity map and

$$\overline{(1-aL)^{-1}}^{L_{N^*}} : L_{N^*} \to L_{N^*}$$

denotes the continuous extension of $(1-aL)^{-1} : \mathscr{D}(\mathscr{E}) \cap L_{N^*} \to L_{N^*}$ to all of L_{N^*} (which exists by a simple application of Jensen's inequality). In particular, $\overline{i_a \circ L}(L_{N^*}) \subset L_{N^*}$ and $\overline{i_a \circ L} : L_{N^*} \to L_{N^*}$ is continuous.

Altogether, we have the following diagram:

$$\begin{tikzcd}
H \cap L_N =: V & \subset & H \arrow[r, "i_a"] & H^* = \mathscr{D}(\mathscr{E}) & \subset & V^* \\
 & & \mathscr{D}(\mathscr{E}) \arrow[u, "L"] & & & \\
 & & \cup & & & \\
 & & \mathscr{D}(\mathscr{E}) \cap L_{N^*} & & & \\
 & & \cap & & & \\
 & & L_{N^*} \arrow[uuuuurrr, "\overline{i_a \circ L}"] & & &
\end{tikzcd}$$

where by [15, Proposition 3.1] (applied to the operator $-(1-\alpha L)$ instead of L) all inclusions are dense and continuous.

Proof. Let $\varepsilon > 0$. Then for all $u \in \mathscr{D}(\mathscr{E}) \cap L_{N^*}$, $v \in V$,

$$\begin{aligned}
{}_{V^*}\langle (i_a \circ L)u, v\rangle_V &= {}_{\mathscr{D}(\mathscr{E})}\langle (i_a \circ L)u, v\rangle_H = {}_{\mathscr{D}(\mathscr{E})}\big\langle (1-aL)^{-1}Lu, v\big\rangle_H \\
&= \frac{1}{a}\big(-{}_{\mathscr{D}(\mathscr{E})}\langle u, v\rangle_H + {}_{\mathscr{D}(\mathscr{E})}\big\langle (1-aL)^{-1}u, v\big\rangle_H\big) \\
&= \frac{1}{a}\Big(-\mathbf{m}(uv) + \mathbf{m}\big(\big[(1-aL)^{-1}u\big]v\big)\Big) \\
&= \frac{1}{a} \cdot \mathbf{m}\Big(\big[\big((1-aL)^{-1} - 1\big)u\big] \cdot v\Big),
\end{aligned}$$

where we used the identification of $L^2(\mathbf{m})$ with its dual (so $\mathscr{D}(\mathscr{E}) \subset L^2(\mathbf{m}) \subset H$). Using the fact that by Jensen's inequality $(1-aL)^{-1}$ with initial domain $\mathscr{D}(\mathscr{E}) \cap L_{N^*}$ is a bounded linear operator on L_{N^*}, and since by [15, Proposition 3.1] (applied to $\mathscr{E}_1$ replacing $\mathscr{E}$) $\mathscr{D}(\mathscr{E}) \cap L_{N^*}$ is dense in L_{N^*}, the assertion follows. $\square$

Now let us define the operator $\bar{L} : \mathscr{D}(\bar{L}) \subset L_{N^*} \to H$ appearing in Definition 1.1.

Definition 2.3. Let

$$\begin{aligned}\mathscr{D}(\bar{L}) :=\{u \in L_{N^*} | \exists u_n \in \mathscr{D}(\mathscr{E}) \cap L_{N^*} &\text{ and a sequence } \varepsilon_n \to 0 \\ \text{such that } \lim_{n\to\infty} u_n = u \text{ in } L_{N^*} \\ \text{and } \lim_{n\to\infty} (Lu_n - \varepsilon_n u_n) \text{ exists in } H\},\end{aligned}$$

and for $u \in \mathscr{D}(\bar{L})$ let

$$\bar{L}u := \lim_{n\to\infty} (Lu_n - \varepsilon_n u_n)(\in H).$$

The following lemma implies that $(\bar{L}, \mathscr{D}(\bar{L}))$ is well-defined. Below we add prefixes $\mathscr{D}(\mathscr{E})$, V^*, L_{N^*} in front of "lim" to indicate in which spaces the respective limit is taken.

Lemma 2.4. *Let $u \in L_{N^*}$ and u_n, ε_n, $n \in \mathbb{N}$, as in the definition of $\mathscr{D}(\bar{L})$. Then for all $a > 0$*

$$i_a(H\text{-}\lim_{n\to\infty} (Lu_n - \varepsilon_n u_n)) = \overline{i_a \circ L}u.$$

In particular, $(\bar{L}, \mathscr{D}(\bar{L}))$ is a well-defined operator from L_{N^*} to H and $\overline{i_a \circ L}u \in \mathscr{D}(\mathscr{E})$ and $i_a \circ \bar{L} = \overline{i_a \circ L}$ on $\mathscr{D}(\bar{L})$.

Proof. We have

$$\begin{aligned}i_a(H\text{-}\lim_{n\to\infty} (Lu_n - \varepsilon_n u_n)) &= \mathscr{D}(\mathscr{E})\text{-}\lim_{n\to\infty} (i_a(L - \varepsilon_n)u_n) \\ &= V^*\text{-}\lim_{n\to\infty} \frac{1}{a}((1 - a\varepsilon_n)i_a u_n - u_n) \\ &= L_{N^*}\text{-}\lim_{n\to\infty} \frac{1}{a}\left(\overline{(1 - aL)}^{L_{N^*}} u - u\right) \\ &= \overline{i_a \circ L}u\end{aligned}$$

by Lemma 2.2. □

Corollary 2.5. *Let $T > 0$ and $Z \in L^1([0, T] \to L_{N^*}, \mathrm{d}t)$. Let $t \in [0, T]$ such that*

$$\int_0^t Z_s ds \in \mathscr{D}(\bar{L})$$

and let $a > 0$. Then

$$i_a \circ \bar{L}\left(\int_0^t Z_s ds\right) = \int_0^t \overline{i_a \circ L}(Z_s) ds.$$

Proof. The assertion is an immediate consequence of Lemma 2.4 and the last part of Lemma 2.2. □

Now we can state and prove the Itô formula for the norms $\|\cdot\|_{H_a}$, $a > 0$.

Theorem 2.6. *Let X be the solution from Theorem 2.1 or, assuming (1.3) and $(H1)-(H3)$, as in Definition 1.1 (where in the latter case below we set $\Psi(t,s) := \int_{r_1}^{r_2} \xi(t,r)|s|^{r-1}s\,\nu(\mathrm{d}r)$, $s \in \mathbb{R}$, $t \geq 0$), and let $a > 0$. Then $\overline{i_a \circ L}(\Psi(\cdot, X)) + \eta\, i_a(X)$ is a progressively measurable process in K^* for any $T > 0$, and $\mathbb{P}$-a.s.*

$$i_a(X_t) = i_a(X_0) + \int_0^t [\overline{i_a \circ L}(\Psi(s, X_s)) + \eta_s\, i_a(X_s)]\mathrm{d}s + i_a\left(\int_0^t B(s, X_s)\mathrm{d}W_s\right), \quad t \geq 0. \tag{2.9}$$

Furthermore, $\mathbb{P}$-a.s.

$$\begin{aligned} \|X_t\|_{H_a}^2 = \|X_0\|_{H_a}^2 + \int_0^t \big[2\ {}_{V^*}\langle \overline{i_a \circ L}(\Psi(s, X_s)) + \eta_s\, i_a(X_s), X_s\rangle_V \\ + \|B(s, X_s)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m}); H_a)}\big]\mathrm{d}s \quad + 2\int_0^t \langle X_s, B(s, X_s)\mathrm{d}W_s\rangle_{H_a}, \quad t \geq 0. \end{aligned} \tag{2.10}$$

Proof. Applying i_a to (2.5) and (1.7) respectively, (2.9) follows from Corollary 2.5. (2.10) follows immediatley from (2.9) and the Itô formula in [15, Theorem 4.2] applied to the Hilbert space $(H_a, \langle\cdot,\cdot\rangle_{H_a})$. □

Lemma 2.7. *Let $a > 0$.*

(i) Let $v \in V$. Then $(1-aL)^{-1}v \in V$ and, in particular,

$$L(1-aL)^{-1}v = -\frac{1}{a}(v - (1-aL)^{-1}v) \in V.$$

(ii) Let $u \in L_{N^*}$, $v \in V$. Then

$${}_{V^*}\big\langle \overline{i_a \circ L}u, v\big\rangle_V = {}_{V^*}\big\langle u, L(1-aL)^{-1}v\big\rangle_V.$$

(iii) $(1-aL)^{-1} : V \to V$ is continuous. Furthermore, its dual operator $((1-aL)^{-1})^* : V^* \to V^*$ is the continuous extension of both $\overline{(1-aL)^{-1}}^{L_{N^*}} : L_{N^*} \to L_{N^*}$ defined in Lemma 2.2 and of $(1-aL)^{-1}|_{\mathscr{D}(\mathscr{E})} : \mathscr{D}(\mathscr{E}) \to \mathscr{D}(\mathscr{E})$. (Here we recall that both $\mathscr{D}(\mathscr{E}) \subset V^*$ and $L_N^* \subset V^*$ continuously and densely.)

Proof. (i) We first note that since $v \in H$, $(1-aL)^{-1}v$ is a well-defined element in $\mathscr{D}(\mathscr{E})$ and since $i_a = (1-aL)^{-1}$, we have by (2.7) for $u \in D(\mathscr{E}) \cap L_{N^*}$

$$\begin{aligned} \langle u, (1-aL)^{-1}v\rangle &= {}_H\langle u, (1-aL)^{-1}v\rangle_{\mathscr{D}(\mathscr{E})} \\ &= \langle u, v\rangle_{H_a} \\ &= {}_{\mathscr{D}(\mathscr{E})}\langle (1-aL)^{-1}u, v\rangle_H \\ &= \langle (1-aL)^{-1}u, v\rangle \\ &= \langle \overline{(1-aL)^{-1}}^{L_{N^*}} u, v\rangle. \end{aligned} \tag{2.11}$$

(cf. the proof and statement of Lemma 2.2). Since $\mathscr{D}(\mathscr{E}) \cap L_{N^*}$ is dense in L_{N^*} it follows that for fixed v the right hand side uniquely determines a continuous linear functional on L_{N^*}, since $v \in L_N$. Hence so does its left hand side. Therefore,

$$(1 - aL)^{-1} v \in L_N,$$

because $L_N = (L_{N^*})^*$.

(ii) Let $u_n \in \mathscr{D}(\mathscr{E}) \cap L_{N^*}$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} u_n = u$ in L_{N^*}. Then by Lemma 2.2

$$\begin{aligned} {}_{V^*}\langle \overline{i_a \circ L} u, v\rangle_V &= \frac{1}{a} \langle [\overline{(1-aL)^{-1}}^{L_{N^*}} - 1] u, v\rangle \\ &= \lim_{n\to\infty} \frac{1}{a} \langle (1-aL)^{-1} u_n - (1-aL)(1-aL)^{-1} u_n, v\rangle \\ &= \lim_{n\to\infty} \langle L(1-aL)^{-1} u_n, v\rangle. \end{aligned}$$

Let $v_m \in \mathscr{D}(\mathscr{E}) \subset L^2(\mathbf{m}) \subset H$, $m \in \mathbb{N}$, such that $\lim_{m\to\infty} v_m = v$ in H. Then for all $n \in \mathbb{N}$, since $L(1-aL)^{-1} u_n = \frac{1}{a}[(1-aL)^{-1} u_n - u_n] \in \mathscr{D}(\mathscr{E}) \cap L_{N^*}$

$$\begin{aligned} \langle L(1-aL)^{-1} u_n, v\rangle &= {}_{\mathscr{D}(\mathscr{E})}\langle L(1-aL)^{-1} u_n, v\rangle_H \\ &= \lim_{m\to\infty} \langle L(1-aL)^{-1} u_n, v_m\rangle \\ &= -\lim_{m\to\infty} \mathscr{E}((1-aL)^{-1} u_n, v_m) \\ &= -\lim_{m\to\infty} \mathscr{E}(u_n, (1-aL)^{-1} v_m) \\ &= -\mathscr{E}(u_n, (1-aL)^{-1} v) \\ &= -\frac{1}{a} \mathscr{E}_a(u_n, i_a v) + \frac{1}{a} \langle u_n, (1-aL)^{-1} v\rangle_H \\ &= -\frac{1}{a}\, {}_{\mathscr{D}(\mathscr{E})}\langle u_n, v\rangle_H + \frac{1}{a}\, {}_{\mathscr{D}(\mathscr{E})}\big\langle u_n, (1-aL)^{-1} v\big\rangle_H \\ &= {}_{\mathscr{D}(\mathscr{E})}\langle u_n, L(1-aL)^{-1} v\rangle_H \\ &= {}_{V^*}\langle u_n, L(1-aL)^{-1} v\rangle_V \end{aligned}$$

by (i) But again by (i) and since $L_{N^*} \subset V^*$ continuously, the latter converges to ${}_{V^*}\langle u, L(1-aL)^{-1} v\rangle_V$ as $n \to \infty$.

(iii) Since by (i)

$$(1 - aL)^{-1}(V) \subset V$$

and since $(1-aL)^{-1} : H \to \mathscr{D}(\mathscr{E}) \subset L^2(\mathbf{m}) \subset H$ is continuous, the continuity of $(1-aL)^{-1}$ on V follows from the closed graph theorem, since the topology on V is stronger than that on H. Since $L_{N^*} \subset V^*$ continuously and densely, the second statement follows from (ii).

To prove the last assertion let $u \in \mathscr{D}(\mathscr{E})$, $v \in V$. Then

$$\begin{aligned}
{}_{V^*}\big\langle ((1-aL)^{-1})^* u, v\big\rangle_V &= {}_{V^*}\big\langle u, (1-aL)^{-1} v\big\rangle_V \\
&= {}_{\mathscr{D}(\mathscr{E})}\big\langle u, (1-aL)^{-1} v\big\rangle_H \\
&= \langle u, v\rangle_{H_a} \\
&= {}_{\mathscr{D}(\mathscr{E})}\big\langle (1-aL)^{-1} u, v\big\rangle_H \\
&= {}_{V^*}\big\langle (1-aL)^{-1} u, v\big\rangle_V .
\end{aligned}$$

□

2.3 $L^2(\mathbf{m})$-invariance

Theorem 2.8. *Consider the situation of Theorem 2.6. Assume that $\mathbb{E}\|X_0\|_2^2 < \infty$, that there exist a progressively measurable $b \in L^2\big([0,T]\times\Omega \to \mathbb{R}, \mathrm{d}t \times \mathbb{P}\big)$ and $c_0 \in (0,\infty)$ such that for all $n \in \mathbb{N}$, $v \in V$*

$$\|B(\cdot, v)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m}); H_{\frac{1}{n}})} \le c_0 \|v\|^2_{H_{\frac{1}{n}}} + b^2 \quad \mathrm{d}t \times \mathbb{P} - a.s. \text{ on } [0,T]\times\Omega \tag{2.12}$$

(where we note that by assumption (B) the $\mathrm{d}t\times\mathbb{P}$-zero set is independent of $v \in V$). If there exists a constant $c > 0$ such that for all $a \in (0,1)$

$$2\, {}_{V^*}\big\langle \overline{i_a \circ L}(\Psi(s, X_s)) + \eta_s\, i_a(X_s), X_s\big\rangle_V \le c\|X_s\|^2_{H_a}, \quad \mathbb{P}\text{-}a.s. \text{ for } \mathrm{d}s\text{-}a.e.\ s \in [0,T], \tag{2.13}$$

then

$$\mathbb{E} \sup_{t\in[0,T]} \|X_t\|_2^2 < \infty \tag{2.14}$$

and, in particular, $(X_t)_{t\in[0,T]}$ is weakly continuous in $L^2(\mathbf{m})$. Furthermore, $(X_t)_{t\in[0,T]}$ is right-continuous in $L^2(\mathbf{m})$.

Proof. By (2.13), the condition on B and Theorem 2.6, we have for $0 \le r < t \le T$ and $n \in \mathbb{N}$

$$e^{-ct}\|X_t\|^2_{H_{1/n}} \le e^{-cr}\|X_r\|^2_{H_{1/n}} + \int_r^t \|B(s,X_s)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H_{1/n})} e^{-cs}\mathrm{d}s + 2\int_r^t e^{-cs}\mathrm{d}M_s^{(n)}\,, \tag{2.15}$$

where $M_t^{(n)} := \int_0^t \langle X_s, B(s,X_s)\mathrm{d}W_s\rangle_{H_{1/n}}$, $t \in [0,T]$, is a local real martingale. Therefore, setting $r = 0$ in (2.15), it follows for every stopping time $\tau \le T$

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,\tau]}\Big(\|X_t\|^2_{H_{1/n}} e^{-ct}\Big) \\
\le\, &\mathbb{E}\|X_0\|_2^2 + \mathbb{E}\int_0^\tau (c_0\|X_s\|^2_{H_{\frac{1}{n}}} + b_s^2) e^{-cs}\mathrm{d}s + 2\mathbb{E}\sup_{t\in[0,\tau]}\Big|\int_0^t e^{-cs}\mathrm{d}M_s^{(n)}\Big|\,.
\end{aligned} \tag{2.16}$$

But by the Burkholder-Davis-Gundy inequality (for $p=1$)

$$
\begin{aligned}
\mathbb{E}\sup_{t\in[0,\tau]}\Big|\int_0^t e^{-cs}\mathrm{d}M_s\Big| &\le 3\mathbb{E}\bigg(\int_0^\tau \|B^*(s,X_s)X_s\|^2_{L^2(\mathbf{m})}e^{-2cs}\mathrm{d}s\bigg)^{1/2}\\
&\le 3\mathbb{E}\bigg(\int_0^\tau \|X_s\|^2_{H_{1/n}}\|B(s,X_s)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H_{1/n})}e^{-2cs}\mathrm{d}s\bigg)^{1/2}\\
&\le 3\big(\mathbb{E}\sup_{t\in[0,\tau]}\|X_t\|^2_{H_{1/n}}e^{-ct}\big)^{1/2}\cdot\bigg(\mathbb{E}\int_0^\tau (c_0\|X_s\|^2_{H_{\frac{1}{n}}}+b_s^2)e^{-cs}\mathrm{d}s\bigg)^{1/2}. \qquad (2.17)
\end{aligned}
$$

By Grownwall's lemma (2.16) and (2.17) imply that

$$
\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_2^2=\sup_{n\in\mathbb{N}}\mathbb{E}\sup_{t\in[0,T]}\|X_t\|^2_{H_{1/n}}<\infty\ , \qquad (2.18)
$$

since $\|\cdot\|_2=\sup_n\|\cdot\|_{H_{1/n}}=\lim_{n\to\infty}\|\cdot\|_{H_{1/n}}$, so we can apply monotone convergence. In particular, X_t is weakly continuous in $L^2(\mathbf{m})$, since it is continuous in H.

Next, letting $n\to\infty$ in (2.12) by (2.18) and the Burkholder-Davis-Gundy inequality (for $p=1$) we obtain

$$
\begin{aligned}
&\limsup_{n\to\infty}\bigg\{\mathbb{E}\sup_{t\in[0,T]}\bigg|\int_0^t(\langle X_s,B(s,X_s)\mathrm{d}W_s\rangle_{H_{1/n}}-\langle X_s,B(s,X_s)\mathrm{d}W_s\rangle)\bigg|\bigg\}\\
&\le 3\limsup_{n\to\infty}\mathbb{E}\bigg(\int_0^T\|(1-n^{-1}L)^{-1}X_s-X_s\|_2^2\|B(s,X_s)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});L^2(\mathbf{m}))}\mathrm{d}s\bigg)^{1/2}\\
&\le 3\lim_{n\to\infty}\mathbb{E}\bigg(\int_0^T\|(1-n^{-1}L)^{-1}X_s-X_s\|_2(c_0\|X_s\|_2^2+b_s^2)ds\bigg)^{1/2}=0,\quad T>0.
\end{aligned}
$$

Thus, up to a subsequence, $\mathbb{P}$-a.s.

$$
\lim_{n\to\infty}\int_0^t\langle X_s,B(s,X_s)\mathrm{d}W_s\rangle_{H_{1/n}}=\int_0^t\langle X_s,B(s,X_s)\mathrm{d}W_s\rangle,\quad t\ge 0,
$$

which is a real valued continuous martingale. Hence in (2.15) we can let first $n\to\infty$ and then $t\downarrow r$, to obtain

$$
\limsup_{t\downarrow r}\|X_t\|_2\le\|X_r\|_2.
$$

On the other hand, by the $L^2(\mathbf{m})$-weak continuity of X_t we have $\liminf_{t\to r}\|X_t\|_2\ge\|X_r\|_2$. So $\|X_t\|_2$ is right-continuous and hence, X_t is right-continuous in $L^2(\mathbf{m})$ again due to the $L^2(\mathbf{m})$-weak continuity. □

Remark 2.9. (i) We emphasize that Theorem 2.8 applies to solutions as in Theorem 2.1 without the assumption $\inf\sigma(-L)>0$. We just need an Itô formula as in (2.10).

(ii) Obviously, ($H3$ (i)) implies (2.12) provided

$$\int_0^T \|B(s,0)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});L^2(\mathbf{m}))}\mathrm{d}s < \infty.$$

(iii) Now we want to describe examples in which ($H3$ (i)) holds with B non-constant in $v \in V$. The easiest is to take $B_0 : [0,T] \times \Omega \to \mathscr{L}_{HS}(L^2(\mathbf{m}), H)$ progressively measurable, $u_0 \in L^2(\mathbf{m})$ and $f : [0,T] \times \Omega \to \mathbb{R}$ progressively measurable and bounded. Then

$$B(t,v) := f(t)\langle \cdot, u_0\rangle u + B_0$$

is easily checked to satisfy ($H3$ (i)). Further examples one obtains as follows:

(M) Let $N \in \mathbb{N} \cup \{+\infty\}$ and $e_k \in L^2(\mathbf{m}) \cap L^\infty(\mathbf{m})$, $1 \le k \le N$, be an orthonormal system in $L^2(\mathbf{m})$ such that for every $1 \le k \le N$ there exists $\xi_k \in (0,\infty)$ such that for all $a \in (0,\infty)$

$$|\,_H\langle x, e_k u\rangle_{\mathscr{D}(\mathscr{E})}\,| \le \xi_k \|x\|_{H_a} \mathscr{E}_a(u,u)^{1/2} \quad \text{for all } u \in \mathscr{D}(\mathscr{E}).$$

(M) just means that each e_k is a multiplier on H_a with norm independent of $a > 0$. Choose $\mu_k \in (0,\infty)$ such that

$$\sum_{k=1}^\infty \xi_k^2 \mu_k^2 < \infty, \tag{2.19}$$

and define for $x \in H$, $B(x) \in \mathscr{L}_{HS}(L^2(\mathbf{m}); H)$ by

$$B(x)h := \sum_{k=1}^\infty \mu_k \langle e_k, h\rangle x \cdot e_k,\ h \in L^2(\mathbf{m}).$$

Indeed, (extending $\{e_k | k \in \mathbb{N}\}$ to an orthonormal basis of $L^2(\mathbf{m})$) by (M) we have for $x \in H$, $a \in (0,\infty)$

$$\begin{aligned}\|B(x)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H_a)} &= \sum_{k=1}^\infty \|B(x)e_k\|^2_{H_a}\\ &= \sum_{k=1}^\infty \mu_k^2 \|xe_k\|^2_{H_a}\\ &\le \sum_{k=1}^\infty \mu_k^2 \xi_k^2 \|x\|^2_{H_a}\end{aligned}$$

and since $x \mapsto B(x)$ is linear and $V \subset H$, condition ($H3(i)$) follows.

Now let us describe a large class of Dirichlet forms $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ for which (M) holds. Let us assume that (1.3) holds, and define the square field operator of L by

$$\Gamma(u,v) := \frac{1}{2}(L(uv) - uLv - vLu), \quad u,v \in \mathcal{A},$$

where $\{e_k | k \in \mathbb{N}\} \subset \mathcal{A} \subset \mathscr{D}(L)$ and $\mathcal{A}$ is an algebra of bounded functions which is dense in $\mathscr{D}(\mathscr{E})$ with respect to $\mathscr{E}_1$. Γ is symmetric in u, v. Suppose that there exist $\chi_n \in \mathscr{D}(L)$, $\chi_n \geq 0$, $\chi_n \to 1$ in $L^2(\mathbf{m})$ as $n \to \infty$. Then clearly

$$\mathscr{E}(u,v) = \int \Gamma(u,v)\mathrm{d}\mathbf{m} \quad \text{for all } u, v \in \mathscr{D}(\mathscr{E}).$$

Assume further that for all $u_1, u_2, v \in \mathcal{A}$

$$\Gamma(u_1 u_2, v) = u_1 \Gamma(u_2, v) + u_2 \Gamma(u_1, v)$$

which is e.g. the case if $(L, \mathcal{A})$ is a diffusion operator in the sense of [8, Appendix B, Definition 1.5], like e.g. a partial differential operator of order 2. Assume $d > 2$ and that $\Gamma(e_k, e_k) \in L^{d/2}(\mathbf{m})$. Then by (1.4) we obtain for $u \in \mathcal{A}$ and $1 \leq k \leq N$

$$\begin{aligned}
\mathscr{E}_a(e_k u, e_k u) &\leq 2a \int (u^2 \Gamma(e_k, e_k) + e_k^2 \Gamma(u,u))\mathrm{d}\mathbf{m} + \int e_k^2 u^2 \mathrm{d}\mathbf{m} \\
&\leq 2a \left(\|\Gamma(e_k, e_k)\|_{\frac{d}{2}} \|u\|^2_{\frac{2d}{d-2}} + \|e_k\|_\infty^2 \mathscr{E}(u,u) \right) + \|e_k\|_\infty^2 \|u\|_2^2 \\
&\leq 2a \left(C_d^2 \|\Gamma(e_k, e_k)\|_{\frac{d}{2}} + \|e_k\|_\infty^2 \right) \mathscr{E}(u,u) + \|e_k\|_\infty^2 \|u\|_2^2.
\end{aligned}$$

Hence (M) holds in this case with

$$\xi_k := \sqrt{2(C_d^2 \|\Gamma(e_k, e_k)\|_{\frac{d}{2}} + \|e_k\|_\infty^2)}. \tag{2.20}$$

If one wants to choose μ_k in (2.19) in a somewhat optimal way, one needs bounds on ξ_k. To this end let us assume that e_k, $1 \leq k \leq N := \infty$, is an eigenbasis of L, with corresponding eigenvalues $-\lambda_k$, $k \in \mathbb{N}$. Then one can get estimates on ξ_k in terms of merely e_k (not $\Gamma(e_k, e_k)$) and λ_k or even λ_k alone, for which the asymptotics is precisely known in a large number of cases. Note first that (1.3) then implies that $\lambda_k > 0$, $k \in \mathbb{N}$. In what follows we do not need that $d > 2$. In the present situation it is then easy to check that for all $u \in \mathcal{A}$, $k \in \mathbb{N}$,

$$\mathscr{E}(e_k u, e_k u) = \int \Gamma(e_k u, e_k u)\mathrm{d}\mathbf{m} = \int (\lambda_k u^2 + \Gamma(u,u)) e_k^2 \mathrm{d}\mathbf{m}. \tag{2.21}$$

We consider two cases:
Case 1: $d > 2$.

Then by (1.4), (2.21) and Hölder's inequality for all $u \in \mathcal{A}$, $k \in \mathbb{N}$,

$$\mathscr{E}_a(e_k u, e_k u) \leq \|e_k\|_\infty^2 \|u\|_2^2 + a(C_d^2 \lambda_k \|e_k\|_d^2 + \|e_k\|_\infty^2)\mathscr{E}(u,u) \leq \xi_k^2 \mathscr{E}_a(u,u),$$

with

$$\xi_k := \sqrt{C_d^2 \lambda_k \|e_k\|_d^2 + \|e_k\|_\infty^2}.$$

It is worth noting that if $d \le 4$, hence $d \le \frac{2d}{d-2}$, and if $\mathbf{m}(E) < \infty$, applying Hölder's inequality and (2.21) with $u := e_k$ we obtain that up to a constant $\|e_k\|_d^2$ is bounded by $\mathscr{E}(e_k, e_k) = \langle -Le_k, e_k \rangle = \lambda_k$, hence

$$\xi_k \le \text{const} \cdot (\max(\lambda_k, \|e_k\|_\infty) + 1) \tag{2.22}$$

in this case.

Case 2: $d = 1, 2$, $E \subset \mathbb{R}^d$, E open, bounded, and $L = \Delta$ with Dirichlet boundary conditions on ∂E, $\mathbf{m} = dx =$ Lebesgue measure.

In this case it is well known that for $p = \infty$, if $d = 1$, and $p \in [1, \infty)$, if $d = 2$, there exists $C_p \in (0, \infty)$ such that for all $u \in \mathscr{D}(\mathscr{E})$

$$\|u\|_p \le C_p \mathscr{E}(u, u)^{1/2},$$

hence $\|e_k\|_p \le C_p \lambda_k^{1/2}$, $k \in \mathbb{N}$, and by Sobolev's embedding for all $k \in \mathbb{N}$

$$\|e_k\|_\infty \le \text{const} \cdot \lambda_k. \tag{2.23}$$

Hence by (2.21) for all $a \in (0, \infty)$, $u \in \mathcal{A}$

$$\begin{aligned}
\mathscr{E}_a(e_k u, e_k u) &\le C\lambda_k^2 \|u\|_2^2 + a\left(\lambda_k \|u\|_4^2 \|e_k\|_4^2 + \lambda_k^2\right) \mathscr{E}(u, u) \\
&\le \tilde{C}\lambda_k^2 \|u\|_2^2 + a(\lambda_k^3 C_4^4 + \lambda_k^2) \mathscr{E}(u, u) \\
&\le \xi_k^2 \cdot \mathscr{E}_a(u, u)
\end{aligned}$$

with

$$\xi_k := \tilde{C} \cdot \left(\lambda_k^{3/2} + 1\right),$$

and the constant $\tilde{C}$ is independent of a, k, u.

We also note that if we consider Case 2 for $d = 3$, then (2.23) still holds (see e.g. [1]). In fact for nice domains E even $\sup_{k \in \mathbb{N}} \|e_k\|_\infty < \infty$ for all $d \in \mathbb{N}$. Hence by (2.22) we get

$$\xi_k \le \text{const} \cdot (\lambda_k + 1),\ k \in \mathbb{N}.$$

3 Some estimates

Let $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be as in the introduction satisfying (1.3). In this section we first present some estimates on the operator $(\varepsilon - L)^{-1/2}$ which will be used in the next section for constructing solutions of (1.5), where $(L, \mathscr{D}(L))$ is the Dirichlet operator associated with $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ (see Section 1).

Lemma 3.1. *Assume* (1.3). *For any $p \in (2, 2d/(d-2)^+)$, there exists $\alpha_p \in (0, 1/2)$ and $c_p \ge 1$, both continuous in p, such that*

$$\|(\varepsilon - L)^{-1/2}\|_{2 \to p} \le c_p \varepsilon^{-\alpha_p}, \quad \varepsilon \in (0, 1).$$

Proof. Let $P_t := \mathrm{e}^{tL}$ and $\{E_\lambda : \lambda \geq 0\}$ the spectral family of $-L$. By the spectral representation theorem we have

$$\begin{aligned}
\int_0^\infty \frac{\mathrm{e}^{-\varepsilon t}}{\sqrt{t}} P_t \mathrm{d}t &= \int_0^\infty \mathrm{d}E_\lambda \int_0^\infty \frac{\mathrm{e}^{-(\varepsilon+\lambda)t}}{\sqrt{t}} \mathrm{d}t \\
&= 2\int_0^\infty \mathrm{d}E_\lambda \int_0^\infty \mathrm{e}^{-(\varepsilon+\lambda)t^2} \mathrm{d}t = \sqrt{\pi}\int_0^\infty \frac{1}{\sqrt{\varepsilon+\lambda}} \mathrm{d}E_\lambda = \sqrt{\pi}(\varepsilon - L)^{-1/2}
\end{aligned} \tag{3.1}$$

for all $\varepsilon > 0$. By the Nash inequality (1.3), there exists $c \geq 1$ such that (cf. [6])

$$\|P_t\|_{2\to\infty} \leq ct^{-d/4}, \quad t > 0.$$

But $\|P_t\|_{2\to 2} \leq 1$. By the Riesz-Thorin interpolation theorem, we obtain

$$\|P_t\|_{2\to p} \leq ct^{-d(p-2)/4p}, \quad t > 0. \tag{3.2}$$

Taking $\delta_p := \frac{1}{2} + \frac{d(p-2)}{4p}$, we have $\delta_p \in (1/2, 1)$ since $p \in (2, 2d/(d-2)^+)$. Let $\delta_p' := \frac{1}{2} + \frac{1}{4(1-\delta_p)}$, so that $\alpha_p := \delta_p'(1-\delta_p) \in (0, \frac{1}{2})$. Then by (3.1) and (3.2), there exists $c_1 > 0$ such that for all $\varepsilon \in (0,1)$

$$\begin{aligned}
\|(\varepsilon - L)^{-1/2}\|_{2\to p} &\leq c_1 \int_0^\infty \mathrm{e}^{-\varepsilon t} t^{-\delta_p} \mathrm{d}t \leq c_1 \int_0^{\varepsilon^{-\delta_p'}} t^{-\delta_p} \mathrm{d}t + c_1 \int_{\varepsilon^{-\delta_p'}}^\infty \mathrm{e}^{-\varepsilon t} \mathrm{d}t \\
&\leq \frac{c_1 \varepsilon^{-\alpha_p}}{1-\delta_p} + \frac{c_1}{\varepsilon} \exp[-\varepsilon^{-(\delta_p'-1)}].
\end{aligned}$$

Since $\delta_p' > 1$, the last term is bounded w.r.t. $\varepsilon \in (0,1)$, so that the desired assertion holds for some $c_p \geq 1$ continuous in $p \in (2, 2d/(d-2)^+)$ and all $\varepsilon \in (0,1)$. $\square$

Lemma 3.2. *Let* (1.3) *hold and let* ε, p, c_p *and* α_p *be as in Lemma* 3.1. *Then for any* $r > p-1$ *and any* $x \in L^2(\mathbf{m}) \cap L^{r+1}(\mathbf{m})$,

$$\|(\varepsilon - L)^{-1/2} x\|_{r+1} \leq c_p \varepsilon^{-(\frac{1}{2} - (1-2\alpha_p)(p-2)/2(r-1))} \|x\|_2^{(p-2)/(r-1)} \|x\|_{r+1}^{(r+1-p)/(r-1)}.$$

Consequently, for any $\delta \in (0, 1 \wedge \frac{4}{(d-2)^+(r_2-1)})$, *there exist* $c > 0$ *and* $\alpha \in (0, 1/2)$ *such that*

$$\|(\varepsilon - L)^{-1/2} x\|_{r+1} \leq c\varepsilon^{-\alpha} \|x\|_2^\theta \, \|x\|_{r+1}^{1-\theta},$$

for $r \in [r_1, r_2], x \in L^2(\mathbf{m}) \cap L^{r_2+1}(\mathbf{m})$, $\theta \in [\delta, 1 \wedge \frac{4}{(d-2)^+(r_2-1)} - \delta]$.

Proof. Since $s := (r-1)/(r+1)$ satisfies

$$\frac{s}{\infty} + \frac{1-s}{2} = \frac{1}{r+1}, \quad \frac{s}{\infty} + \frac{1-s}{p} = \frac{1}{p(r+1)/2},$$

by the interpolation theorem

$$\|(\varepsilon - L)^{-1/2}\|_{r+1\to p(r+1)/2} \leq \|(\varepsilon - L)^{-1/2}\|_{\infty\to\infty}^s \|(\varepsilon - L)^{-1/2}\|_{2\to p}.$$

Moreover, (3.1) implies

$$\|(\varepsilon-L)^{-1/2}\|_{\infty\to\infty} \le \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{\mathrm{e}^{-\varepsilon t}}{\sqrt{t}}\mathrm{d}t \le \varepsilon^{-1/2}.$$

So, combining the above with Lemma 3.1, we obtain

$$\|(\varepsilon-L)^{-1/2}\|_{r+1\to p(r+1)/2} \le c_p\varepsilon^{-(4\alpha_p+r-1)/2(r+1)} \le c_p\varepsilon^{-1/2},\quad \varepsilon\in(0,1). \tag{3.3}$$

Let $t:=(r+1)(p-2)/(r-1)$. By Hölder inequality we obtain

$$\begin{aligned}\mathbf{m}(|(\varepsilon-L)^{-1/2}x|^{r+1}) &= \mathbf{m}(|(\varepsilon-L)^{-1/2}x|^{t}\cdot|(\varepsilon-L)^{-1/2}x|^{r+1-t})\\ &\le \mathbf{m}(|(\varepsilon-L)^{-1/2}x|^{p})^{t/p}\mathbf{m}(|(\varepsilon-L)^{-1/2}x|^{(r+1-t)p/(p-t)})^{(p-t)/p}\\ &= \|(\varepsilon-L)^{-1/2}x\|_p^{(r+1)(p-2)/(r-1)}\|(\varepsilon-L)^{-1/2}x\|_{(r+1)p/2}^{(r+1)(r+1-p)/(r-1)}.\end{aligned}$$

Combining this with (3.3) and Lemma 3.1 we prove the first assertion. Finally, for fixed $\theta\in(0,1\wedge\frac{4}{(r_2-1)(d-2)^+})$, the second assertion follows from the first by taking $p_{r,\theta}:=2+\theta(r-1)$ so that $c_{p_{r,\theta}}$ is bounded for $r\in[r_1,r_2]$ and $\theta\in[\delta,1\wedge\frac{4}{(d-2)^+(r_2-1)}-\delta]$. □

Now assume that $(H1)-(H3)$ hold. Our next aim is to apply Theorem 2.1 with $L-\varepsilon$ instead of L, i.e., we fix $\varepsilon\in(0,1)$ and consider the equation

$$\mathrm{d}X_t^\varepsilon = \big[(L-\varepsilon)\Psi(t,X_t^\varepsilon)+\eta_tX_t^\varepsilon\big]\mathrm{d}t+B_t\mathrm{d}W_t,\quad X_0^\varepsilon=X_0, \tag{3.4}$$

where

$$\Psi(t,s):=\int_{r_1}^{r_2}\xi(t,r)|s|^{r-1}s\nu(\mathrm{d}r),\quad s\in\mathbb{R},t\ge 0.$$

Define

$$N(s):=\int_{r_1}^{r_2}|s|^{r+1}\nu(\mathrm{d}r),\quad s\in\mathbb{R}.$$

It is trivial to see that both N and $N^*(s):=\inf_{r\ge 0}\{|sr|-N(r)\}$ are Δ_2-regular, which follows directly from the calculation in [15, Example 3.5] where $\nu:=\sum_{i=1}^n c_i\delta_{r_i}$ for $c_i>0$ and $r_i>1$. Then $(\boldsymbol{\Psi})$ follows from $(H1)$ and (B) from $(H3)$.

By Theorem 2.6 (applied to $L-\varepsilon$ replacing L) for any $a\in(0,\varepsilon^{-1})$ we have that $\mathbb{P}$-a.s.

$$i_a(X_t)=i_a(X_0)+\int_0^t[\overline{i_a\circ(L-\varepsilon)}(\Psi(s,X_s))+\eta_si_a(X_s)]\mathrm{d}s+i_a\left(\int_0^t B_s\mathrm{d}W_s\right),\ t\ge 0, \tag{3.5}$$

where we used that

$$i_a=(1-aL)^{-1}=\frac{1}{1-a\varepsilon}\left(1-\frac{a}{1-a\varepsilon}(L-\varepsilon)\right)^{-1}\qquad \text{for } a\in(0,\varepsilon^{-1}). \tag{3.6}$$

Furthermore, applying Lemma 2.2 with $L-\varepsilon$ replacing L and using (3.6) we obtain for all $u \in L_{N^*}$, $v \in V$, $a \in (0, \varepsilon^{-1})$

$$
{}_{V^*}\left\langle \overline{i_a \circ (L-\varepsilon)} u, v \right\rangle_V = \frac{1-a\varepsilon}{a} \langle \overline{(1-aL)^{-1}}^{L_{N^*}} u, v\rangle - \frac{1}{a}\langle u, v\rangle, \tag{3.7}
$$

which by an easy approximation argument is equal to

$$
\frac{1-a\varepsilon}{a}\langle u, \overline{(1-aL)^{-1}}^{L_N} v\rangle - \frac{1}{a}\langle u, v\rangle,
$$

where $\overline{(1-aL)^{-1}}^{L_N}$ is the unique continuous extension of $(1-aL)^{-1} : L^1(\mathbf{m}) \cap L^\infty(\mathbf{m}) \to L_N$ to all of L_N. It, however, follows immediately from (2.11) that

$$
\overline{(1-aL)^{-1}}^{L_N} v = (1-aL)^{-1} v \quad \text{for all } v \in V (= H \cap L_N), \tag{3.8}
$$

where we recall that the right hand side is by definition the Riesz map $(1-aL)^{-1} : (H, \langle \cdot, \cdot \rangle_{H_a}) \to (\mathscr{D}(\mathscr{E}), \mathscr{E}_a)$ applied to v as an element in H. Therefore, we do not distinguish $\overline{(1-aL)^{-1}}^{L_N}$ and $(1-aL)^{-1}$ below. So, altogether we obtain

$$
\begin{aligned}
&{}_{V^*}\left\langle \overline{i_a \circ (L-\varepsilon)} u, v \right\rangle_V \\
=&\frac{1-a\varepsilon}{a}\langle u, (1-aL)^{-1} v\rangle - \frac{1}{a}\langle u, v\rangle, \quad \text{for all } u \in L_{N^*},\, v \in V,\, a \in (0, \frac{1}{\varepsilon}).
\end{aligned} \tag{3.9}
$$

Therefore, by Theorem 2.1, applied to $L-\varepsilon$ in place of L, if $\mathbb{E}\|X_0\|_H^2 < \infty$ then (3.4) has a unique solution X^ε which is a continuous adapted process in H and $X^\varepsilon \in L_N([0,T] \times E \times \Omega; \mathrm{d}t \times \mathbf{m} \times \mathbb{P}) \cap L^2([0,T] \times \Omega \to H; \mathrm{d}t \times \mathbb{P})$.

Lemma 3.3. *Assume that (H1)–(H3) and* (1.3) *hold. Let $X_0 : \Omega \to H$ be $\mathscr{F}_0$-measurable such that $\mathbb{E}\|X_0\|_H^2 < \infty$. Let $T > 0$ be fixed. Then for any $q \geq 1$ there exists a constant $c(q) > 0$ such that for any $\varepsilon \in (0,1)$,*

$$
\mathbb{E} \sup_{t\in[0,T]} \|X_t^\varepsilon\|_H^{q+1} \leq c(q) \left(\mathbb{E}\|X_0\|_H^{q+1} + \mathbb{E}\left(\int_0^T \|B(s,0)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H)} \mathrm{d}s \right)^{\frac{q+1}{2}} \right) \tag{3.10}
$$

and

$$
\begin{aligned}
&\mathbb{E}\left(\int_0^T \mathrm{d}t \int_{r_1}^{r_2} \|X_t^\varepsilon\|_{r+1}^{r+1} \nu(\mathrm{d}r) \right)^q \\
\leq& c(q) \left(1 + \mathbb{E}\|X_0\|_H^{(r_2+1)q} + \mathbb{E}\left(\int_0^T \|B(s,0)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H)} \mathrm{d}s \right)^{\frac{(r_2+1)q}{2}} \right).
\end{aligned} \tag{3.11}
$$

Proof. We may assume that the right hand sides of (3.10) and (3.11) are finite. We recall that $\langle\cdot,\cdot\rangle_H = \langle\cdot,\cdot\rangle_{H_1}$, $\|\cdot\|_H = \|\cdot\|_{H_1}$.
(a) By assumptions $(H1)-(H3)$ and using the Itô formula in Theorem 2.6 and (3.9) for $a=1$, we have

$$\begin{aligned}
&\mathrm{d}\|X_t^\varepsilon\|_H^2 \\
=&2\ {}_{V^*}\langle \overline{i_1\circ(L-\varepsilon)}\Psi(t,X_t^\varepsilon)+\eta_t i_1(X_t^\varepsilon), X_t^\varepsilon\rangle_V \mathrm{d}t \\
&\quad + \|B(t,X_t^\varepsilon)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H)}\mathrm{d}t + 2\langle X_t^\varepsilon, B(t,X_t^\varepsilon)\mathrm{d}W_t\rangle_H \\
\le&(c\|X_t^\varepsilon\|_H^2 + \|B(t,0)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H)})\mathrm{d}t - 2\langle X_t^\varepsilon, \Psi(t,X_t^\varepsilon)\rangle \mathrm{d}t \\
&\quad + 2(1-\varepsilon)\langle (1-L)^{-1}X_t^\varepsilon, \Psi(t,X_t^\varepsilon)\rangle\mathrm{d}t + 2\langle X_t^\varepsilon, B(t,X_t^\varepsilon)\mathrm{d}W_t\rangle_H
\end{aligned} \tag{3.12}$$

for some constant $c>0$. Since

$$\begin{aligned}
&-2\langle X_t^\varepsilon, \Psi(t,X_t^\varepsilon)\rangle + 2(1-\varepsilon)\langle (1-L)^{-1}X_t^\varepsilon, \Psi(t,X_t^\varepsilon)\rangle \\
&\le -2\int_{r_1}^{r_2} \xi(t,r)\|X_t^\varepsilon\|_{r+1}^{r+1}\nu(\mathrm{d}r) + 2(1-\varepsilon)\int_{r_1}^{r_2}\xi(t,r)\|(1-L)^{-1}X_t^\varepsilon\|_{r+1}\|X_t^\varepsilon\|_{r+1}^r\nu(\mathrm{d}r) \\
&\le 0,
\end{aligned}$$

(3.12) implies

$$\mathrm{d}\|X_t^\varepsilon\|_H^2 \le (c\|X_t^\varepsilon\|_H^2 + \|B(t,0)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m}),H)})\mathrm{d}t + 2\langle X_t^\varepsilon, B(t,X_t^\varepsilon)\mathrm{d}W_t\rangle_H.$$

By Itô's formula, applied to the real valued semimartingale $Z_t := \|X_t^\varepsilon\|_H^2$, $t\in[0,t]$, for any $q\ge 1$ there exists $c_1(q)>0$ such that

$$\begin{aligned}
&\mathrm{d}\|X_t^\varepsilon\|_H^{q+1} \\
\le&c_1(q)(\|X_t^\varepsilon\|_H^{q+1} + \|X_t^\varepsilon\|_H^{q-1}\|B(t,0)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H)})\mathrm{d}t + (q+1)\|X_t^\varepsilon\|_H^{q-1}\langle X_t^\varepsilon, B(t,X_t^\varepsilon)\mathrm{d}W_t\rangle_H.
\end{aligned} \tag{3.13}$$

Thus, any stopping time $\tau\le T$, applying first Itô's product rule, then the Burkholder-Davis-

Gundy inequality for $p = 1$, and using $(H3)$ we obtain

$$
\begin{aligned}
&\mathbb{E} \sup_{t\in[0,\tau]} \|X_t^\varepsilon\|_H^{q+1} e^{-c_1(q)t} \\
\le&\mathbb{E}\|X_0^\varepsilon\|_H^{q+1} + c_1(q)\mathbb{E}\int_0^\tau \|X_s^\varepsilon\|_H^{q-1}\|B(s,0)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H)} e^{-c_1(q)s}\mathrm{d}s \\
&\quad + (q+1)\mathbb{E}\sup_{t\in[0,\tau]} \left| \int_0^t \|X_s^\varepsilon\|_H^{q-1} e^{-c_1(q)s}\langle X_s^\varepsilon, B(s,X_s^\varepsilon)\mathrm{d}W_s\rangle_H \right| \\
\le&\mathbb{E}\|X_0^\varepsilon\|_H^{q+1} + c_1(q)\mathbb{E}\sup_{t\in[0,\tau]}\left(\|X_t^\varepsilon\|_H^{q-1}e^{-c_1(q)t}\right)\int_0^t \|B(s,0)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H)}\mathrm{d}s \\
&\quad + 3(q+1)\mathbb{E}\left(\int_0^\tau \|X_s^\varepsilon\|_H^{2q}\|B(s,X_s^\varepsilon)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H)} e^{-2c_1(q)s}\mathrm{d}s\right)^{\frac{1}{2}} \\
\le&\mathbb{E}\|X_0^\varepsilon\|_H^{q+1} + c_1(q)\left[\mathbb{E}\sup_{t\in[0,\tau]}\left(\|X_t^\varepsilon\|_H^{q+1}e^{-c_1(q)t}\right)\right]^{\frac{q-1}{q+1}}\left[\mathbb{E}\left(\int_0^T \|B(s,0)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H)}\mathrm{d}s\right)^{\frac{q+1}{2}}\right]^{\frac{2}{q+1}} \\
&\quad + 3(q+1)c\left[\mathbb{E}\sup_{t\in[0,\tau]}\left(\|X_t^\varepsilon\|_H^{q+1}e^{-c_1(q)t}\right)\right]^{\frac{1}{2}}\left[\mathbb{E}\int_0^\tau \|X_s^\varepsilon\|_H^{q+1}e^{-c_1(q)s}ds\right]^{\frac{1}{2}} \\
&\quad + 3(q+1)\left[\mathbb{E}\sup_{s\in[0,\tau]}\left(\|X_s^\varepsilon\|_H^{q+1}e^{-c_1(q)s}\right)\right]^{\frac{q}{q+1}}\left[\mathbb{E}\left(\int_0^T \|B(s,0)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H)}\mathrm{d}s\right)^{\frac{q+1}{2}}\right]^{\frac{1}{q+1}} \\
\le&\mathbb{E}\|X_0^\varepsilon\|_H^{q+1} + \frac{1}{2}\mathbb{E}\sup_{t\in[0,\tau]}\left(\|X_t^\varepsilon\|_H^{q+1}e^{-c_1(q)t}\right) \\
&\quad + \tilde{C}(q)\left(\mathbb{E}\left[\int_0^T \|B(s,0)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H)}\mathrm{d}s\right]^{\frac{q+1}{2}} + \mathbb{E}\int_0^\tau \|X_s^\varepsilon\|_H^{q+1}e^{-c_1(q)s}\mathrm{d}s\right)
\end{aligned}
$$

for some constant $\tilde{C}(q) > 0$, where we used Young's inequality in the last step.

By Gronwall's Lemma this implies (3.10) for some $c(q) > 0$ (independent of ε).

(b) By (3.12), assumptions $(H1)$, $(H3)$, and Lemma 3.2 with $\varepsilon = 1$, there exist $\delta_1, \delta_2, \delta_3 > 0$ (independent of ε) such that

$$
\begin{aligned}
\mathrm{d}\|X_t^\varepsilon\|_H^2 \le&(c\|X_t^\varepsilon\|_H^2 + \|B(t,0)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H)})\mathrm{d}t - 2\delta_1\int_{r_1}^{r_2} \|X_t^\varepsilon\|_{r+1}^{r+1}\nu(\mathrm{d}r)\mathrm{d}t \\
&+ \delta_2\int_{r_1}^{r_2} \|X_t^\varepsilon\|_H^\theta \|X_t^\varepsilon\|_{r+1}^{r+1-\theta}\nu(\mathrm{d}r) + 2\langle X_t^\varepsilon, B(t,X_t^\varepsilon)\mathrm{d}W_t\rangle_H \\
\le&\delta_3(1 + \|X_t^\varepsilon\|_H^{r_2+1} + \|B(t,0)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H)})\mathrm{d}t \\
&- \delta_1\int_{r_1}^{r_2} \|X_t^\varepsilon\|_{r+1}^{r+1}\nu(\mathrm{d}r)\mathrm{d}t + 2\langle X_t^\varepsilon, B(t,X_t^\varepsilon)\mathrm{d}W_t\rangle_H,
\end{aligned}
$$

where the last step follows from the fact that

$$a^\theta b^{r+1-\theta} \le \frac{\delta_1}{\delta_2} b^{r+1} + c_0 a^{r+1}$$

holds for some constant $c_0 > 0$ and all $a, b \ge 0, r \in [r_1, r_2]$. This implies

$$\begin{aligned}
&\delta_1 \int_0^T \mathrm{d}t \int_{r_1}^{r_2} \|X_t^\varepsilon\|_{r+1}^{r+1} \nu(\mathrm{d}r) \\
\le &\|X_0\|_H^2 + \delta_3 \int_0^T (1 + \|X_t^\varepsilon\|_H^{r_2+1} + \|B(t,0)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H)})\mathrm{d}t + 2\int_0^T \langle X_t^\varepsilon, B(t, X_t^\varepsilon)\mathrm{d}W_t\rangle_H.
\end{aligned}$$

Therefore, (3.11) follows from (3.10) by similar arguments as above. □

4 Existence of solutions for special initial conditions

Proposition 4.1. *Consider the situation of Theorem* 1.2. *If* $\|X_0\|_H \in L^{2r_2}(\mathbb{P})$ *then* (1.5) *has a unique solution, and the solution satisfies*

$$\mathbb{E} \sup_{t\in[0,T]} \|X_t\|_H^{2r_2} + \mathbb{E}\bigg(\int_0^T \int_{r_1}^{r_2} \|X_t\|_{r+1}^{r+1} \nu(\mathrm{d}r)\mathrm{d}t\bigg)^{\frac{2r_2}{r_2+1}} < \infty, \quad \forall T > 0. \tag{4.1}$$

Proof. (a) Existence: Let $0 < \varepsilon' < \varepsilon < 1$. Then by (2.5) $\mathbb{P}$-a.s. for all $t \ge 0$

$$\begin{aligned}
X_t^\varepsilon - X_t^{\varepsilon'} &= (L-\varepsilon)\int_0^t \Psi(s, X_s^\varepsilon)\mathrm{d}s - (L-\varepsilon')\int_0^t \Psi(s, X_s^{\varepsilon'})\mathrm{d}s + \int_0^t \eta_s (X_s^\varepsilon - X_s^{\varepsilon'})\mathrm{d}s \\
&= \varepsilon\left(\frac{1}{\varepsilon}L - 1\right)\int_0^t \left(\Psi(s, X_s^\varepsilon) - \Psi(s, X_s^{\varepsilon'})\right)\mathrm{d}s + \int_0^t \eta_s (X_s^\varepsilon - X_s^{\varepsilon'})\mathrm{d}s \\
&\quad + (\varepsilon' - \varepsilon)\int_0^t \Psi(s, X_s^{\varepsilon'})\mathrm{d}s + \int_0^t (B(s, X_s^\varepsilon) - B(s, X_s^{\varepsilon'}))\mathrm{d}W_s.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&i_{\frac{1}{\varepsilon}}(X_t^\varepsilon - X_t^{\varepsilon'}) \\
=& -\varepsilon \int_0^t (\Psi(s, X_s^\varepsilon) - \Psi(s, X_s^{\varepsilon'}))\mathrm{d}s + \int_0^t \eta_s i_{\frac{1}{\varepsilon}}(X_s^\varepsilon - X_s^{\varepsilon'})\mathrm{d}s \\
&\quad + (\varepsilon' - \varepsilon)\int_0^t ((1 - \frac{1}{\varepsilon}L)^{-1})^* \Psi(s, X_s^{\varepsilon'})\mathrm{d}s + i_{\frac{1}{\varepsilon}}\left(\int_0^t (B(s, X_s^\varepsilon) - B(s, X_s^{\varepsilon'}))\mathrm{d}W_s\right).
\end{aligned}$$

where for the last term we used Lemma 2.7 (iii) and that the involved integrals are Bochner integrals in V^*.

Now we can use the Itô formula in [15, Theorem 4.2] applied to the Hilbert space $H_{\frac{1}{\varepsilon}}$ and obtain for

$$M_t^{\varepsilon,\varepsilon'} := 2\int_0^t \langle X_s^\varepsilon - X_s^{\varepsilon'}, (B(s, X_s^\varepsilon) - B(s, X_s^{\varepsilon'}))\mathrm{d}W_s\rangle_{H_{\frac{1}{\varepsilon}}}$$

by (H3(i)) that for $t \in [0,T], T>0$ fixed,

$$
\begin{aligned}
&\|X_t^\varepsilon - X_t^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2\\
&= -2\varepsilon \int_0^t \langle \Psi(s,X_s^\varepsilon) - \Psi(s,X_s^{\varepsilon'}), X_s^\varepsilon - X_s^{\varepsilon'}\rangle \mathrm{d}s\\
&\quad + 2\int_0^t \eta_s \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 \mathrm{d}s\\
&\quad + 2(\varepsilon'-\varepsilon)\int_0^t \langle \Psi(s,X_s^{\varepsilon'}), (1-\frac{1}{\varepsilon}L)^{-1}(X_s^\varepsilon - X_s^{\varepsilon'})\rangle \mathrm{d}s\\
&\quad + \int_0^t \|B(s,X_s^\varepsilon) - B(s,X_s^{\varepsilon'})\|_{\mathscr{L}_{HS}(L^2(\mathbf{m});H_{\frac{1}{\varepsilon}})}^2 ds + M_t^{\varepsilon,\varepsilon'}\\
&\le -2\varepsilon \int_0^t \int_{r_1}^{r_2} \xi(s,r)\langle X_s^\varepsilon |X_s^\varepsilon|^{r-1} - X_s^{\varepsilon'}|X_s^{\varepsilon'}|^{r-1}, X_s^\varepsilon - X_s^{\varepsilon'}\rangle \nu(dr)\mathrm{d}s \qquad (4.2)\\
&\quad + \int_0^t (2\eta_s + c^2)\|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 \mathrm{d}s\\
&\quad + 2(\varepsilon'-\varepsilon)\int_0^t \int_{r_1}^{r_2} \xi(s,r)\langle X_s^{\varepsilon'}|X_s^{\varepsilon'}|^{r-1}, (1-\frac{1}{\varepsilon}L)^{-1}(X_s^\varepsilon - X_s^{\varepsilon'})\rangle \nu(\mathrm{d}r)\mathrm{d}s + M_t^{\varepsilon,\varepsilon'}\\
&\le -\varepsilon\delta \int_0^t \int_{r_1}^{r_2} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{r+1}^{r+1}\nu(\mathrm{d}r)\mathrm{d}s\\
&\quad + c_1\int_0^t \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 \mathrm{d}s + c_1 I_t^{\varepsilon,\varepsilon'} + M_t^{\varepsilon,\varepsilon'},
\end{aligned}
$$

where we used the elementary estimate that $(x|x|^{r-1} - y|y|^{r-1})(x-y) \ge 2^{-r+1}|x-y|^{r+1}$ for all $r \in (1,\infty)$, $x,y \in \mathbb{R}$, we set $\delta := 2^{-r_2+2}\inf \xi$, $c_1 := 2\sup\eta \vee \sup\xi + c^2$ and where

$$
I_t^{\varepsilon,\varepsilon'} := \varepsilon^{\frac{3}{2}} \int_0^t \int_{r_1}^{r_2} \|(\varepsilon - L)^{-\frac{1}{2}}(1-\frac{1}{\varepsilon}L)^{-\frac{1}{2}}(X_s^\varepsilon - X_s^{\varepsilon'})\|_{r+1}\|X_s^{\varepsilon'}\|_{r+1}^r \nu(\mathrm{d}r)\mathrm{d}s.
$$

We note that $(1-\frac{1}{\varepsilon}L)^{-\frac{1}{2}}$ is a contraction on $L^{r+1}(\mathbf{m})$ and that $X_s^\varepsilon - X_s^{\varepsilon'} \in L^{r+1}(\mathbf{m})$ $\mathbb{P}\otimes ds \otimes \nu$-a.e. on $\Omega\times[0,t]\times[r_1,r_2]$. Hence by Lemma 3.2 for any given continuous function $[r_1,r_2] \ni r \mapsto \theta_r \in (0, 1\wedge \frac{4}{(d-2)^+(r_2-1)})$ there exist $c>0$ and $\alpha \in (0,\frac{1}{2})$ such that

$$
I_t^{\varepsilon,\varepsilon'} \le c\,\varepsilon^{\frac{3}{2}-\alpha} \int_0^t \int_{r_1}^{r_2} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^{\theta_r} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{r+1}^{1-\theta_r} \|X_s^{\varepsilon'}\|_{r+1}^r \nu(\mathrm{d}r)\mathrm{d}s,
$$

which by Young's inequality is dominated by

$$
\begin{aligned}
&\frac{\delta}{2}\varepsilon \int_0^t \int_{r_1}^{r_2} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{r+1}^{r+1}\nu(\mathrm{d}r)\,\mathrm{d}s \qquad (4.3)\\
&\quad + C_\delta \varepsilon^{\frac{3}{2}-\alpha}\int_0^t \int_{r_1}^{r_2} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^{\theta_r(r+1)/(r+\theta_r)} \|X_s^{\varepsilon'}\|_{r+1}^{r(r+1)/(r+\theta_r)} \nu(\mathrm{d}r)\mathrm{d}s,
\end{aligned}
$$

where $C_\delta > 0$ is a large enough constant (which is independent of $\varepsilon, \varepsilon'$ and by the continuity of $r \mapsto \theta_r$ can indeed be chosen independently of r). Now define the increasing continuous function

$$\theta_r := \frac{\theta \cdot r}{r+1-\theta},\ r \in [r_1, r_2],$$

where $\theta \in (0,1)$ is chosen so small that $(\theta_r \leq)\theta_{r_2} \in (0, 1 \wedge \frac{4}{(d-2)^+(r_2-1)})$. Then $\theta = \frac{\theta_r(r+1)}{r+\theta_r}$ for all $r \in [r_1, r_2]$ and by (4.2) and (4.3) we hence obtain

$$\begin{aligned}
\|X_t^\varepsilon - X_t^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 \leq &-\varepsilon\frac{\delta}{2}\int_0^t\int_{r_1}^{r_2}\|X_s^\varepsilon - X_s^{\varepsilon'}\|_{r+1}^{r+1}\nu(\mathrm{d}r)\mathrm{d}s \\
&+ c_1\int_0^t\|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2\mathrm{d}s \\
&+ c_1 C_\delta \varepsilon^{\frac{3}{2}-\alpha}\int_0^t\|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^\theta\int_{r_1}^{r_2}\|X_s^{\varepsilon'}\|_{r+1}^{r+1-\theta}\nu(\mathrm{d}r)\mathrm{d}s + M_t^{\varepsilon,\varepsilon'}
\end{aligned}$$

which for $\tilde{C}_\delta := c_1 C_\delta$ in turn implies for $t \leq T$

$$\begin{aligned}
&e^{-c_1 t}\|X_t^\varepsilon - X_t^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 \\
&\leq \tilde{C}_\delta \varepsilon^{\frac{3}{2}-\alpha}\sup_{s\in[0,t]}\|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^\theta\int_0^t\int_{r_1}^{r_2}\|X_s^{\varepsilon'}\|_{r+1}^{r+1-\theta}\nu(\mathrm{d}r)\mathrm{d}s \\
&\quad - \varepsilon\frac{\delta}{2}e^{-c_1 T}\int_0^t\int_{r_1}^{r_2}\|X_s^\varepsilon - X_s^{\varepsilon'}\|_{r+1}^{r+1}\nu(\mathrm{d}r)ds \\
&\quad + 2\int_0^t e^{c_1 s}\langle X_s^\varepsilon - X_s^{\varepsilon'}, (B(s, X_s^\varepsilon) - B(s, X_s^{\varepsilon'}))\mathrm{d}W_s\rangle_{H_{\frac{1}{\varepsilon}}}.
\end{aligned} \tag{4.4}$$

So, for any fixed $T > 0$ by (H3(i)) and by the Hölder and Burkholder-Davies-Gundy inequalities we have for all $t \in [0, T]$

(4.5)

$$\begin{aligned}
&E\sup_{s\in[0,t]}\|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 + \varepsilon\frac{\delta}{2}e^{-c_1 T}\int_0^t\int_{r_1}^{r_2}\|X_s^\varepsilon - X_s^{\varepsilon'}\|_{r+1}^{r+1}\nu(\mathrm{d}r)ds \\
&\leq \tilde{C}_\delta\varepsilon^{\frac{3}{2}-\alpha}e^{c_1 T}\left[\mathbb{E}\sup_{s\in[0,t]}\|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2\right]^{\theta/2}\left[\mathbb{E}\left(\int_0^T\int_{r_1}^{r_2}\|X_s^{\varepsilon'}\|_{r+1}^{r+1-\theta}\nu(\mathrm{d}r)ds\right)^{\frac{2}{2-\theta}}\right]^{\frac{2-\theta}{2}} \\
&\quad + 2c\left[E\sup_{s\in[0,t]}\|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2\right]^{\theta/2}\left[\mathbb{E}\left(\int_0^t\|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^{2(2-\theta)}ds\right)^{\frac{1}{2-\theta}}\right]^{\frac{2-\theta}{2}}.
\end{aligned}$$

Dropping the integral on the left hand side for $t \in [0,T]$ this yields

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}\|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2\\
\leq& 2^{\frac{\theta}{2-\theta}}\left(\tilde{C}_\delta e^{\frac{3}{2}-\alpha}e^{c_1T}\right)^{\frac{2}{2-\theta}}\mathbb{E}\left(\int_0^T\int_{r_1}^{r_2}\|X_s^{\varepsilon'}\|_{r+1}^{r+1-\theta}\nu(\mathrm{d}r)ds\right)^{\frac{2}{2-\theta}}\\
+ \quad & 2^{\frac{2+\theta}{2-\theta}}c^{\frac{2}{2-\theta}}\mathbb{E}\left(\int_0^t\|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^{2(2-\theta)}ds\right)^{\frac{1}{2-\theta}}.
\end{aligned}$$

But the last term is dominated by

$$\begin{aligned}
&2^{\frac{2+\theta}{2-\theta}}c^{\frac{2}{2-\theta}}\left[\mathbb{E}\sup_{s\leq t}\|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2\right]^{1/2}\left[\mathbb{E}\left(\int_0^t\|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^{2-\theta}ds\right)^{\frac{2}{2-\theta}}\right]^{1/2}\\
\leq&\frac{1}{2}\mathbb{E}\sup_{s\leq t}\|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 + C_{T,\theta}\mathbb{E}\int_0^t\|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2ds,
\end{aligned}$$

where $C_{T,\theta}$ is a constant (independent of $\varepsilon, \varepsilon'$). Hence by Gronwall's Lemma

$$\mathbb{E}\sup_{s\in[0,t]}\|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 \leq \left(\varepsilon^{\frac{3}{2}-\alpha}\tilde{C}_\delta e^{c_1T}\right)^{\frac{2}{2-\theta}}\mathbb{E}\left(\int_0^T\int_{r_1}^{r_2}\|X_s^{\varepsilon'}\|_{r+1}^{r+1-\theta}\nu(\mathrm{d}r)ds\right)^{\frac{2}{2-\theta}}. \tag{4.6}$$

Since $\|\cdot\|_{H_1}^2 \leq \frac{1}{\varepsilon}\|\cdot\|_{H_{\frac{1}{\varepsilon}}}^2$, by (3.10) applied with $q := \frac{2r_2}{r_2+1}$ and the assumption that $\|X_0\|_{H_1} \in L^{2r_2}(\mathbb{P})$, we conclude that

$$\mathbb{E}\sup_{t\in[0,T]}\|X_t^\varepsilon - X_t^{\varepsilon'}\|_{H_1}^2 \leq \varepsilon^{\frac{1+\theta-2\alpha}{2-\theta}}C \tag{4.7}$$

for some constant C (independent of $\varepsilon, \varepsilon'$). Here we applied Hölder's inequality to the right hand side of (4.6) and used that $\frac{r+1-\theta}{r+1}\frac{2}{2-\theta} \leq \frac{2r_2}{r_2+1}$ for all $\theta \in (0,1)$ and all $r \in [r_1, r_2]$. Since $\|\cdot\|_{H_{\frac{1}{\varepsilon}}}^2 \leq \|\cdot\|_{H_1}^2$, analogously one deduces from (4.5) that for some constant $C > 0$ (independent of $\varepsilon, \varepsilon'$)

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_{r_1}^{r_2}\|X_s^\varepsilon - X_s^{\varepsilon'}\|_{r+1}^{r+1}\nu(\mathrm{d}r)ds\\
&\leq C\varepsilon^{\frac{1}{2}-\alpha}\left(\mathbb{E}\sup_{t\in[0,T]}\|X_t^\varepsilon - X_t^{\varepsilon'}\|_{H_1}^2\right)^{\frac{\theta}{2}}\cdot\left(\mathbb{E}\left(\int_0^T\int_{r_1}^{r_2}\|X_s^{\varepsilon'}\|_{r+1}^{r+1}\nu(\mathrm{d}r)\mathrm{d}s\right)^{\frac{2(r+1-\theta)}{(r+1)(2-\theta)}}\right)^{\frac{2-\theta}{2}}\\
&\quad + 2cT^{1/2}\mathbb{E}\sup_{t\in[0,T]}\|X_t^\varepsilon - X_t^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2
\end{aligned} \tag{4.8}$$

So, as above by (3.11) (with q as above), (4.8) together with (4.7) imply that there exists an adapted continuous process X in $H(=H_1)$ such that for $\varepsilon_n \to 0$

$$\lim_{n\to\infty} \mathbb{E}\left(\sup_{t\in[0,T]} \|X_t^{\varepsilon_n} - X_t\|_H^2 + \int_0^T \int_{r_1}^{r_2} \|X_t^{\varepsilon_n} - X_t\|_{r+1}^{r+1} \nu(\mathrm{d}r)\mathrm{d}t \right) = 0. \tag{4.9}$$

By Fatou's lemma and Lemma 3.3 applied with $p := q+1$ in (3.10) and $q := \frac{2r_2}{r_1+1}$ in (3.11) we obtain (4.1), so in particular X satisfies (1.6). Now let us show that it also satisfies (1.7).

Claim: There exists a sequence $\varepsilon_n \to 0$ such that $\mathbb{P}$-a.s.

$$\int_0^t \Psi(s, X_s^{\varepsilon_n})ds \to \int_0^t \Psi(s, X_s)ds \quad \text{as } n\to\infty \text{ in } L_{N^*} \text{ for all } t\geq 0.$$

To prove the claim let $v \in L_N$. Then by (H1) for some $C \in (0,\infty)$

$$\begin{aligned}
&\left| \mathbf{m}\left(\int_0^t (\Psi(s, X_s^\varepsilon) - \Psi(s, X_s))ds \cdot v \right) \right| \\
\leq & C \cdot \int_0^t \int_{r_1}^{r_2} \mathbf{m}\left(\left| |X_s^\varepsilon|^{r-1} X_s^\varepsilon - |X_s|^{r-1} X_s \right| |v| \right) \nu(\mathrm{d}r) ds \\
\leq & r_2 C \int_0^t \int_{r_1}^{r_2} \mathbf{m}(|X_s^\varepsilon - X_s| (|X_s^\varepsilon| \vee |X_s|)^{r-1} |v|) \nu(\mathrm{d}r) ds
\end{aligned} \tag{4.10}$$

where we used the elementary estimate $\left| |x|^{r-1}x - |y|^{r-1}y \right| \leq r|x-y|(|x|\vee|y|)^{r-1}$; $x, y \in \mathbb{R}$. Applying Hölder's and Young's inequalities the above up to a constant can be estimated from above by

$$\begin{aligned}
& \int_0^t \int_{r_1}^{r_2} \| |X_s^\varepsilon - X_s| (|X_s^\varepsilon| \vee |X_s|)^{r-1} \|_{\frac{r+1}{r}} \|v\|_{r+1} \nu(\mathrm{d}r) ds \\
\leq & C(\delta) \int_0^t \int_{r_1}^{r_2} \| |X_s^\varepsilon - X_s| (|X_s^\varepsilon| \vee |X_s|)^{r-1} \|_{\frac{r+1}{r}}^{\frac{r+1}{r}} \nu(\mathrm{d}r) ds \\
& + \delta \int_{r_1}^{r_2} \|v\|_{r+1}^{r+1} \nu(\mathrm{d}r) \\
\leq & \tilde{C}(\delta) \int_0^t \int_{r_1}^{r_2} \|X_s^\varepsilon - X_s\|_{r+1}^{r+1} \nu(\mathrm{d}r) ds + \delta \int_0^t \int_{r_1}^{r_2} \left(\|X_s^\varepsilon\|_{r+1}^{r+1} + \|X_s\|_{r+1}^{r+1} \right) \nu(\mathrm{d}r) ds \\
& + \delta \cdot \mathbf{m}(N(v))
\end{aligned}$$

for any $\delta > 0$ and some constants $C(\delta)$, $\tilde{C}(\delta) > 0$ (only depending on δ, r_1, r_2). But by (4.9) for some sequence $\varepsilon_n \to 0$ the first term of the right hand side $\mathbb{P}$-a.s. converges to zero for all $t \geq 0$ and the second is $\mathbb{P}$-a.s. bounded by a random number $c(t)$ times δ. Hence first taking $n \to \infty$ and then $\delta \to 0$ we see that the left hand side of (4.10) converges to zero $\mathbb{P}$-a.s. for all $t \geq 0$ uniformly for all $v \in L_N$ such that $\mathbf{m}(N(v)) \leq 1$. Hence by the equivalence of the norms $\|\cdot\|_{(N^*)}$ and $\|\cdot\|_{N^*}$ on L_{N^*} (see (2.1)) the claim follows.

We have $\mathbb{P}$-a.s.

$$(4.11)\quad X_t^{\varepsilon_n} = X_0 + (L - \varepsilon_n)\int_0^t \Psi(s, X_s^{\varepsilon_n})\mathrm{d}s + \int_0^t \eta_s X_s^{\varepsilon_n}\mathrm{d}s + \int_0^t B(s, X_s^{\varepsilon_n})\mathrm{d}W_s,\ t \geq 0.$$

Obviously, by (H3(i)) and 4.7

$$\lim_{\varepsilon\to 0}\mathbb{E}\sup_{t\ \in[0,T]}\int_0^t \|B(s, X_s^\varepsilon) - B(s, X_s)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H)} ds = 0.$$

Hence by (4.9) all terms in (4.11) except for the second on the right converge in H. But hence also this term must converge in H. By Claim 1 it follows that $\mathbb{P}$-a.s.

$$\int_0^t \Psi(s, X_s)ds \in \mathscr{D}(\bar{L}) \quad \forall t \geq 0$$

and

$$(L - \varepsilon_n)\int_0^t \Psi(s, X_s^{\varepsilon_n})ds \to \bar{L}\int_0^t \Psi(s, X_s)ds \quad \text{as } n \to \infty \text{ in } H \quad \forall t \geq 0.$$

Consequently, X satisfies (1.5).

Since by Theorem 2.6 we have an Itô formula for any solution of (1.5), by exactly the same arguments as in the proof of Lemma 3.3 and choosing q as we did for our solution X constructed above, we obtain that any solution Y of (1.5) with $\|Y_0\|_H \in L^{2r_2}(\mathbb{P})$ satisfies (4.1).

It remains to prove uniqueness. So, let X, Y be two solutions of (1.5) such that $X_0 = Y_0$ and $\|X_0\|_H \in L^{2r_2}(\mathbb{P})$. Let $T > 0$ and $\varepsilon \in (0, 1)$. Then by Theorem 2.6 we have $\mathbb{P}$-a.s.

$$\begin{aligned} i_{\frac{1}{\varepsilon}}(X_t - Y_t) = &\int_0^t \left[\overline{i_{\frac{1}{\varepsilon}} \circ L}(\Psi(s, X_s) - \Psi(s, Y_s)) + \eta_s i_{\frac{1}{\varepsilon}}(X_s - Y_s)\right] ds \\ &+ i_{\frac{1}{\varepsilon}}\int_0^t (B(s, X_s) - B(s, Y_s))\mathrm{d}W_s, \quad t \geq 0. \end{aligned}$$

So, applying the Itô formula in [15, Theorem 4.2] we obtain (as in Theorem 2.6) $\mathbb{P}$-a.s.

for all $t \in [0,T]$

(4.12)

$$
\begin{aligned}
\|X_t - Y_t\|^2_{H_{\frac{1}{\varepsilon}}} &= \int_0^t 2\,{}_{V^*}\Big\langle \overline{i_{\frac{1}{\varepsilon}} \circ L}(\Psi(s,X_s) - \Psi(s,Y_s)), X_s - Y_s \Big\rangle_V ds \\
&\quad + \int_0^t \left[2\eta_s \|X_s - Y_s\|^2_{H_{\frac{1}{\varepsilon}}} + \|B(s,X_s) - B(s,Y_s)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H_{\frac{1}{\varepsilon}})} \right] ds \\
&\quad + M_t^\varepsilon \\
&\le -2\varepsilon \int_0^t \langle \Psi(s,X_s) - \Psi(s,Y_s), X_s - Y_s \rangle ds \\
&\quad + 2\varepsilon \int_0^t \langle \overline{(1 - \varepsilon^{-1}L)^{-1}}^{N^*} (\Psi(s,X_s) - \Psi_s(Y_s)), X_s - Y_s \rangle ds \\
&\quad + c_1 \int_0^t \|X_s - Y_s\|^2_{H_{\frac{1}{\varepsilon}}} ds + M_t^\varepsilon,
\end{aligned}
$$

for some constant $c_1 > 0$ and

$$
M_t^\varepsilon := 2\int_0^t \langle X_s - Y_s, (B(s,X_s) - B(s,Y_s))\mathrm{d}W_s \rangle_{H_{\frac{1}{\varepsilon}}}, \quad t \ge 0.
$$

Here we used Lemma 2.2 and the assumed Lipschitz continuity of B for the last inequality. Using the same arguments that led to (4.2) we deduce from (4.12) that

$$
\begin{aligned}
\|X_t - Y_t\|^2_{H_{\frac{1}{\varepsilon}}} \le -\varepsilon\delta \int_0^t \int_{r_1}^{r_2} \|X_s - Y_s\|^{r+1}_{r+1} \nu(\mathrm{d}r) ds \\
+ c_1 \int_0^t \|X_s - Y_s\|^2_{H_{\frac{1}{\varepsilon}}} ds + c_1 I_t^\varepsilon + M_t^\varepsilon
\end{aligned}
$$

with δ, c_1 as in (4.2) and

$$
I_t^\varepsilon := \varepsilon^{\frac{3}{2}} \int_0^t \int_{r_1}^{r_2} \|(\varepsilon - L)^{-\frac{1}{2}} (1 - \frac{1}{\varepsilon} L)^{-\frac{1}{2}} (X_s - Y_s)\|_{r+1} \|X_s - Y_s\|^r_{r+1} \nu(\mathrm{d}r) ds.
$$

Now, since $\|\cdot\|_H^2 \le \frac{1}{\varepsilon}\|\cdot\|^2_{H_{\frac{1}{\varepsilon}}}$, and proceeding in exactly the same way as in the proof of (4.6) and (4.7) we obtain that for some constant $C > 0$

$$
\mathbb{E} \sup_{s\in[0,T]} \|X_s - Y_s\|_H^2 \le C\varepsilon^{\frac{1+\theta-2\alpha}{2-\theta}} \mathbb{E} \left(\int_0^T \int_{r_1}^{r_2} \left(\|X_t\|^{r+1}_{r+1} + \|Y_t\|^{r+1}_{r+1} \right) \nu(\mathrm{d}r) \mathrm{d}t \right)^{\frac{2r_2}{r_2+1}}
$$

with α, θ as in (4.6), (4.7). Letting $\varepsilon \to 0$ shows $X_t = Y_t$ for all $t \in [0,T]$.

□

5 Proof of Theorem 1.2

Proof of Theorem 1.2(1) and (3). For any $n \geq 1$, by Proposition 4.1 we let $X^{(n)}$ be the unique solution of (1.5) with $X_0^{(n)} := X_0 1_{\{n-1 \leq \|X_0\|_H < n\}}$. Then

$$
\begin{aligned}
X_t^{(n)} &= X_0 1_{\{n-1 \leq \|X_0\|_H < n\}} + \bar{L} \int_0^t \Psi(s, X_s^{(n)}) ds \\
&\quad + \int_0^t \eta_s X_s^{(n)} \mathrm{d}s + \int_0^t B(s, X_s^{(n)}) \mathrm{d}W_s, \quad n \geq 1,
\end{aligned}
\tag{5.1}
$$

holds in H. Letting $X_t := \sum_{n=1}^{\infty} X_t^{(n)} 1_{\{n-1 \leq \|X_0\|_H < n\}}$, we obtain from (5.1) that

$$
X_t = X_0 + \bar{L} \int_0^t \Psi(s, X_s) ds + \int_0^t \eta_s X_s \mathrm{d}s + \int_0^t B(s, X_s) \mathrm{d}W_s, \quad t \geq 0,
$$

holds on $\{n - 1 \leq \|X_0\|_H < n\}$ for all $n \geq 1$. Therefore, X_t is a solution of (1.5) in the sense of Definition 1.1. Since by Theorem 2.6 we have an Itô formula for the solution X above, by exactly the same arguments as in the proof of Lemma 3.3 we obtain assertion (3).

Let Y_t be another solution with the same initial values X_0. Then both $X_t 1_{\{\|X_0\|_H \leq n\}}$ and $Y_t 1_{\{\|X_0\|_H \leq n\}}$ solve (1.5) for $B 1_{\{\|X_0\|_H \leq n\}}$ in place of B. By the uniqueness stated in Proposition 4.1 we have $X 1_{\{\|X_0\|_H \leq n\}} = Y 1_{\{\|X_0\|_H \leq n\}}$. Therefore $X = Y$ since $n \geq 1$ was arbitrary. □

Proof of Theorem 1.2(2). If $\{X_0^{(n)} : n \geq 1\}$ is uniformly bounded in H, then the desired assertion follows from Theorem 2.6 as in the proof of Proposition 4.1. In general, the proof can be completed as above by restricting the solutions first on $\{\sup_{n \geq 1} \|X_0^{(n)}\|_H \leq m\}$ then letting $m \to \infty$. For instance, if $X_t^{(n)} \to X_t$ does not hold in probability, then there exist $\varepsilon_0, \varepsilon_1 > 0$ and a subsequence $n_k \to \infty$ such that

$$
\mathbb{P}(\|X_t^{(n_k)} - X_t\|_H \geq \varepsilon_0) \geq \varepsilon_1, \quad k \geq 1. \tag{5.2}
$$

Moreover, since $X_0^{(n)} \to X_0$ in probability, we may assume further that

$$
\sum_{k=1}^{\infty} \mathbb{P}(\|X_0^{(n_k)} - X_0\|_H \geq \varepsilon_0) < \infty.
$$

Then, by the Borel-Cantelli lemma we obtain $\sup_{k \geq 1} \|X_0^{n_k}\|_H < \infty$ $\mathbb{P}$-a.s., hence

$$
\lim_{m \to \infty} \mathbb{P}(\sup_{k \geq 1} \|X_0^{n_k}\|_H > m) = 0.
$$

Therefore, letting $\Omega_m := \{\sup_{k \geq 1} \|X_0^{n_k}\|_H \leq m\}$, by the assertion with uniformly bounded initial values we obtain (recall that $1_{\Omega_m} X$ solves (1.5) with B replaced by $1_{\Omega_m} B$ for any solution X)

$$\lim_{k\to\infty}\mathbb{P}(\|X_t^{(n_k)}-X_t\|_H\geq\varepsilon_0)\leq\mathbb{P}(\Omega_m^c)+\lim_{k\to\infty}\mathbb{P}(\|X_t^{(n_k)}-X_t\|_H 1_{\Omega_m}\geq\varepsilon_0)=\mathbb{P}(\Omega_m^c)$$

which is smaller than ε_1 for large m, and hence is contradictive to (5.2). □

Proof of Theorem 1.2(4). (a) We first assume that $\mathbb{E}\|X_0\|_2^2<\infty$. Let $\varepsilon\in(0,1)$. Since $(1-\varepsilon L)^{-1}$ is contractive in $L^p(\mathbf{m})$ for $p\geq 1$ we have

$$\langle\Psi(t,v),v-(1-\varepsilon L)^{-1}v\rangle=\int_{r_1}^{r_2}\xi(t,r)\mathbf{m}(|v|^{r+1}-|v|^{r-1}v(1-\varepsilon L)^{-1}v)\nu(\mathrm{d}r)\geq 0\quad\forall v\in V.$$

This and Lemma 2.7 (i), (ii) imply that for all $v\in V$

$$\begin{aligned}{}_{V^*}\big\langle\overline{i_\varepsilon\circ L}\Psi(t,v),v\big\rangle_V&={}_{V^*}\big\langle\Psi(t,v),L(1-\varepsilon L)^{-1}v\big\rangle_V\\&=-\frac{1}{\varepsilon}\langle\Psi(t,v),v-(1-\varepsilon L)^{-1}v\rangle\\&\leq 0.\end{aligned}\tag{5.3}$$

Then Theorem 2.8 implies that X is right-continuous in $L^2(\mathbf{m})$ and $\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_2^2<\infty$. In general, letting $X^{(n)}$ be the solution with initial value $X_0 1_{\{\|X_0\|_2\leq n\}}$, we have $X=X^{(n)}$ on $\{\|X_0\|_2\leq n\}$, and hence X is right-continuous in $L^2(\mathbf{m})$ according to the results for $X_0\in L^2(\mathbf{m})$ and the arbitrariness of n.

(b) We first prove (1.8). Let $T>0$. We first note that by the left hand side of (5.3) and (H3) we have that for some constant $C>0$ independent of $\varepsilon\in(0,1)$

$$\begin{aligned}\mathbb{E}\int_0^T\frac{1}{\varepsilon}\langle\Psi(t,X_t),X_t-(1-\varepsilon L)^{-1}X_t\rangle\mathrm{d}t&\leq-\mathbb{E}\int_0^T{}_{V^*}\big\langle\overline{i_\varepsilon\circ L}\Psi(t,X_t),X_t\big\rangle_V\,\mathrm{d}t\\&\leq\mathbb{E}\|X_0\|_{H_\varepsilon}^2+C(1+\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_{H_\varepsilon}^2)\\&\leq\mathbb{E}\|X_0\|_2^2+C(1+\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_2^2)\\&<\infty\end{aligned}\tag{5.4}$$

where we used the Itô formula from Theorem 2.6 in the second step. Define

$$\zeta(s):=\int_{r_1}^{r_2}|s|^{(r-1)/2}s\nu(\mathrm{d}r),\quad s\in\mathbb{R}.$$

By $(H1)$ and the Schwartz inequality,

$$
\begin{aligned}
(\Psi(t,s)-\Psi(t,s'))(s-s') &= \int_{r_1}^{r_2} \xi(t,r)(|s|^{r-1}s-|s'|^{r-1}s')(s-s')\nu(\mathrm{d}r)\\
&= \int_{r_1}^{r_2} \xi(t,r)(s-s')\int_{s'}^{s}|u|^{r-1}\mathrm{d}u\,\nu(\mathrm{d}r)\\
&\geq \frac{(\int_{r_1}^{r_2}\xi(t,r)\int_{s'}^{s}|u|^{(r-1)/2}\mathrm{d}u\,\nu(\mathrm{d}r))^2}{\int_{r_1}^{r_2}\xi(t,r)\nu(\mathrm{d}r)}\\
&\geq c_2|\zeta(s)-\zeta(s')|^2, \quad t\in[0,T], s,s'\in\mathbb{R},
\end{aligned}
\tag{5.5}
$$

holds for some constant $c_2>0$, where we applied the fundamental theorem of calculus to ζ. In particular, since $\Psi(t,0)=0$ and $\zeta(0)=0$, it follows that

$$
\Psi(t,s)s \geq c_2\zeta(s)^2 \tag{5.6}
$$

By Lemma 5.1 below with p being the kernel corresponding to $P:=(1-\varepsilon L)^{-1}$ defined there, (5.5) and (5.6) imply

$$
\begin{aligned}
&\frac{1}{\varepsilon}\langle\Psi(t,X_t),X_t-(1-\varepsilon L)^{-1}X_t\rangle\\
=&\frac{1}{2\varepsilon}\int_E\int_E[\Psi(t,X_t(\tilde{\xi}))-\Psi(t,X_t(\xi))][X_t(\tilde{\xi})-X_t(\xi)]p(\xi,d\tilde{\xi})\mathbf{m}(d\xi)\\
&\qquad+\frac{1}{\varepsilon}\int_E(1-(1-\varepsilon L)^{-1}1)\Psi(t,X_t)X_t\,\mathrm{d}\mathbf{m}\\
\geq& c_2\frac{1}{2\varepsilon}\int_E\int_E(\zeta(X_t(\tilde{\xi}))-\zeta(X_t(\xi)))^2p(\xi,d\tilde{\xi})\mathbf{m}(d\xi)\\
&\qquad+\frac{1}{\varepsilon}\int_E(1-(1-\varepsilon L)^{-1}1)|\zeta(X_t)|^2\,\mathrm{d}\mathbf{m}\\
=&c_2\frac{1}{\varepsilon}\langle\zeta(X_t),\zeta(X_t)-(1-\varepsilon L)^{-1}\zeta(X_t)\rangle = c_2\mathscr{E}^{(\varepsilon)}(\zeta(X_t),\zeta(X_t)),
\end{aligned}
$$

where for $f\in L^2(\mathbf{m})$

$$
\mathscr{E}^{(\varepsilon)}(f,f):=\frac{1}{\varepsilon}\langle f,f-(1-\varepsilon L)^{-1}\rangle. \tag{5.7}
$$

Combining this with (5.4), we obtain

$$
\mathbb{E}\int_0^T\sup_{\varepsilon>0}\mathscr{E}^{(\varepsilon)}(\zeta(X_t),\zeta(X_t))\mathrm{d}t<\infty.
$$

Here we recall that $\mathscr{E}^{(\varepsilon)}(f,f)\nearrow\infty$ as $\varepsilon\searrow 0$ and that

$$
f\in\mathscr{D}(\mathscr{E})\Leftrightarrow\sup_{\varepsilon>0}\mathscr{E}^{(\varepsilon)}(f,f)<\infty,\ f\in L^2(\mathbf{m})
$$

in which case $\mathscr{E}(f,f)=\sup_{\varepsilon>0}\mathscr{E}^{(\varepsilon)}(f,f)$ (cf. [12, Chap. I, Theorem 2.13] or [9, Subsection 1.5]. We also note that by (1.6) and Jensen's inequality indeed $\zeta(X_t)\in L^2(\mathbf{m})$ $\mathrm{d}t\times\mathbb{P}$-a.e. Hence $\zeta(X_t)\in\mathscr{D}(\mathscr{E})$ $\mathrm{d}t\times\mathbb{P}$-a.e. and (1.8) holds.

Finally, if $\mathbb{E}\|X_0\|_H^{r_2+1}<\infty$, then Theorem 1.2(3) implies that

$$\zeta(X)=\int_{r_1}^{r_2}|X|^{r-1}X\mathrm{d}r\in L^2([0,T]\times\Omega\to L^2(\mathbf{m});\mathrm{d}t\times\mathbb{P})$$

and hence also the last part of assertion (4) is proved.

□

Lemma 5.1. *Let E be a Lusin space. Let P be a symmetric contraction on $L^2(\mathbf{m})$ which is sub-Markovian (i.e. $0\le Pf\le 1$ if $f\in L^2(\mathbf{m})$, $0\le f\le 1$).*

(i) There exists a probability kernel p on $(E,\mathcal{B})$ such that for all $\mathcal{B}$-measurable $f:E\to\mathbb{R}$ whose $\mathbf{m}$-class $\bar f$ is in $L^2(\mathbf{m})$ $P\bar f$ is the $\mathbf{m}$-class corresponding to pf where

$$Pf(\xi):=\int_E f(\tilde\xi)p(\xi,d\tilde\xi),\quad \xi\in E.$$

(ii) Let $f\in L_{N^*}$, $g\in L_N$. Then

$$E\ni\xi\mapsto p((f-f(\xi))(g-g(\xi)))(\xi)$$

is $\mathbf{m}$-integrable and

$$\mathbf{m}(f(g-Pg))=\frac{1}{2}\int\int(f(\tilde\xi)-f(\xi))(g(\tilde\xi)-g(\xi))p(\xi,d\tilde\xi)\mathbf{m}(d\xi)+\int_E(1-p1)fg\mathrm{d}\mathbf{m}.$$

Proof. (i) follows from [7, Chapter IX.11], since E is Lusin.

(ii) We first note that by Jensen's inequality and symmetry P extends to a contraction on $L^p(\mathbf{m})$ for all $p\in[1,\infty)$ and that for $\xi\in E$

(5.8)
$$p((f-f(\xi))(g-g(\xi)))(\xi)=p(fg)(\xi)-f(\xi)pg(\xi)-g(\xi)pf(\xi)+f(\xi)g(\xi)p1(\xi).$$

Since by Jensen's inequality p leaves both L_N and L_{N^*} invariant, $fg\in L^1(\mathbf{m})$ and $p1$ is bounded, it follows that the function in (5.8) is in $L^1(\mathbf{m})$. Hence integrating with respect to $\mathbf{m}$ and using the symmetry of P the assertion follows.

□

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