Compact excessive functions and Markov processes: a general case and applications

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Abstract. In this paper we present simple conditions for a Markovian resolvent of kernels on a general state space to be associated with a right process. We apply this to the construction of Brownian motion on abstract Wiener space and to identify new potential theoretic properties for it. In particular, we can define natural associated capacities and obtain new results for the solution of the Dirichlet problem with measurable boundary data.

Mathematics Subject Classification (2000): 60J45, 60J40, 60J35, 47D07, 31C25.
Key words: abstract Wiener space, infinite dimensional Brownian motion, capacity, Dirichlet problem, controlled convergence

1. Introduction

The purpose of this paper is twofold: First, we want to present simple conditions for a Markovian resolvent $\mathcal{U}$ of kernels on a general (Lusin measurable) state space E to be associated with a right process, a classical problem even on more regular state spaces. As a consequence we can define associated capacities. Furthermore, various other potential theoretic notions, techniques and results become available for $\mathcal{U}$. Second, we show that the said conditions are fulfilled for the resolvent of the Brownian semigroup on an abstract Wiener space. Thus, we obtain a new construction of the classical Brownian motion on abstract Wiener space first studied by L. Gross in [Gr 67]. But, in addition, we can then apply the powerful machinery of potential theory to this process, since our construction implies that this Brownian motion falls into the class of (Borel) right processes (see e.g. [BeBo 04] for the precise definition). In particular, we have naturally associated capacities as in the finite dimensional case. This positively answers an old question of R. Carmona from the seventies of last century.

We also adapt and employ a technique, earlier developed by the second named author, for essentially finite dimensional (more, precisely, locally compact) state spaces, named "controlled convergence". Thus, we obtain new results on the boundary behaviour of the Dirichlet problem on an abstract Wiener space, when the boundary data are merely measurable.

2. Framework and main results

Let $(E, \mathcal{B})$ be a Lusin measurable space (i.e. it is measurable isomorphic to a Borel subset of a metrizable compact space endowed with the Borel σ-algebra) and $\mathcal{L}$ be a vector lattice of bounded $\mathcal{B}$-measurable real-valued functions on E, $1 \in \mathcal{L}$, and $\mathcal{F}_0$ be a countable subset of $\mathcal{L}_+$ separating the points of E such that the topologies on E generated by $\mathcal{L}$ and $\mathcal{F}_0$ coincide.

Let further $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be a Markovian resolvent of kernels on $(E, \mathcal{B})$, such that

(a) $U_\alpha(\mathcal{L}) \subset \mathcal{L}$ for all $\alpha > 0$
(b) $\lim\limits_{\alpha\to\infty} \|\alpha U_\alpha f - f\|_\infty = 0$ for all $f \in \mathcal{L}$.

We shall denote by $\mathcal{E}(\mathcal{U})$ the set of all $\mathcal{B}$-measurable $\mathcal{U}$*-excessive functions:* $u \in \mathcal{E}(\mathcal{U})$ if and only if u is a positive numerical $\mathcal{B}$-measurable function, $\alpha U_\alpha u \le u$ for all $\alpha > 0$ and $\lim_{\alpha\to\infty} \alpha U_\alpha u(x) = u(x)$ for all $x \in E$. If $\beta > 0$ we denote by $\mathcal{U}_\beta$ the sub-Markovian resolvent of kernels $(U_{\beta+\alpha})_{\alpha>0}$.

Note that $\sigma(\mathcal{F}_0) = \mathcal{B}$ and using (b) one obtains that for all $\beta > 0$ we have:

$$\sigma\big(\mathcal{E}(\mathcal{U}_\beta)\big) = \mathcal{B} \quad \text{and} \quad \mathcal{E}(\mathcal{U}_\beta) \text{ is min-stable.} \tag{2.1}$$

Recall that a *Ray cone* associated with $\mathcal{U}_\beta$ is a cone $\mathcal{R}$ of bounded $\mathcal{U}_\beta$-excessive functions such that: $U_\alpha(\mathcal{R}) \subset \mathcal{R}$ for all $\alpha > 0$, $U_\beta\big((\mathcal{R} - \mathcal{R})_+\big) \subset \mathcal{R}$, $\sigma(\mathcal{R}) = \mathcal{B}$, it is min-stable, separable in the supremum norm and $1 \in \mathcal{R}$. The topology on E generated by a Ray cone is called *Ray topology.*

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If $M \in \mathcal{B}$ and $u \in \mathcal{E}(\mathcal{U}_\beta)$, then the *reduced function* (with respect to $\mathcal{U}_\beta$) *of* u *on* M is the function $R_\beta^M u$ defined by

$$R_\beta^M u := \inf\{v \in \mathcal{E}(\mathcal{U}_\beta) \mid v \geq u \text{ on } M\} .$$

Then (see e.g. [BeBo 04]) $R_\beta^M u$ is universally $\mathcal{B}$-measurable. Let

$$\widehat{R_\beta^M} u := \sup_{\alpha>0} \alpha U_{\beta+\alpha}(R_\beta^M u).$$

The set $M \in \mathcal{B}$ is called *polar* (resp. μ-*polar*; where μ is a σ-finite measure on $(E, \mathcal{B})$) if $\widehat{R_\beta^M} 1 = 0$ (resp. $\widehat{R_\beta^M} 1 = 0$ μ-a.e.). Recall that if $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is the resolvent associated with a right process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ with state space E, i.e.

$$U_\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f \circ X_t \, \mathrm{d}t$$

for all $\alpha > 0$, $x \in E$ and $f \in p\mathcal{B}$ (:= the set of all positive $\mathcal{B}$-measurable functions on E), then by a theorem of Hunt we have:

$$R_\beta^M u(x) = E^x(e^{-\alpha D_M} u \circ X_{D_M} \,;\, D_M < \infty) , \quad \widehat{R_\beta^M} u(x) = E^x(e^{-\alpha T_M} u \circ X_{T_M} \,;\, T_M < \infty) ,$$

where $D_M(\omega) := \inf\{t \geq 0 \mid X_t(\omega) \in M\}$, $T_M(\omega) := \inf\{t > 0 \mid X_t(\omega) \in M\}$, $\omega \in \Omega$.

Theorem 2.1. (a) *The topology on* E *generated by* $\mathcal{L}$ *is a Ray topology.*

(b) *Assume that:*

(∗) *there exists a* $\mathcal{U}_\beta$*-excessive compact function* v *which is finite* $\mathcal{U}$*-a.e. and has compact level sets, i.e.* $U_\beta(1_{[v=\infty]}) = 0$ *and the set* $[v \leq \alpha]$ *is compact for all* $\alpha > 0$.

Then the resolvent $\mathcal{U}$ *is associated with a Borel right (Markov) process with state space* E*. If* v *is real-valued, then the process is càdlàg.*

(c) *Assume that (∗) holds and let* $p := U_\beta f_0$*, with* $0 < f_0 \leq 1$, $f_0 \in p\mathcal{B}$*, and let* μ *be a finite measure on* $(E, \mathcal{B})$*. Then the following assertions hold:*

(i) *The functional* $M \mapsto c_\mu(M)$, $M \subset E$*, defined by*

$$c_\mu(M) := \inf\{\mu(R_\beta^G p) \mid M \subset G \text{ open}\}$$

is a Choquet capacity on E*; see e.g.* [BeBo 04]*. If the function* v *is finite* μ*-a.e., then the capacity* c_μ *is* tight*, i.e. there exists an increasing sequence* $(K_n)_n$ *of compact sets such that* $\inf_n c_\mu(E \setminus K_n) = 0$.

(ii) *Let* $M \in \mathcal{B}$*. Then*

$$c_\mu(M) = \mu(R_\beta^M p) = \sup\{\nu(p \cdot 1_M) \mid \nu \circ U_\beta \leq \mu \circ U_\beta\} .$$

The set M *will be* μ*-polar and* μ*-negligible if and only if* $c_\mu(M) = 0$.

Remark. A function v as in Theorem 2.1(b) above is also called *Lyapunov function*.

Sketch of the proof. (For a detailed proof we refer to [BeCoRö 07]). We shall outline two steps of the proof of (b).

(I) Starting with property (2.1), one can show (cf. [BeBo 04] and [BeBoRö 06]) that there exist a larger Lusin topological space E_1, $E \subset E_1$, $E \in \mathcal{B}_1$ (= the Borel σ-algebra on E_1) and a Borel right process with state space E_1 having as associated resolvent an extension $\mathcal{U}^1 = (U_\alpha^1)_{\alpha>0}$ of $\mathcal{U}$ to E_1, $U_\alpha^1(1_{E_1 \setminus E}) = 0$.

(II) By assumption (∗) it follows that there exists an increasing sequence $(K_n)_n$ of Ray compact sets such that $\inf_n R_\beta^{E_1 \setminus K_n} 1 = 0$ $\mathcal{U}^1$-a.e. Consequently, the set $E_1 \setminus E$ is polar and therefore $\mathcal{U}$ is the resolvent associated with the restriction of the process to E. □

3. Application to the construction of Brownian motion on an abstract Wiener space

Let (E, H, μ) be an *abstract Wiener space*, i.e. $(H, \langle\ ,\ \rangle)$ is a separable real Hilbert space with corresponding norm $|\cdot|$, which is continuously and densely embedded into a Banach space $(E, \|\cdot\|)$, which is hence also separable; μ is a Gaussian measure on $\mathcal{B}$ (= the Borel σ-algebra of E), that is, each $\ell \in E'$, the dual space of E, is normally distributed with mean zero and variance $|\ell|^2$. Here we use the standard continuous and dense embeddings

$$E' \subset (H' \equiv) H \subset E .$$

We recall that the embedding $H \subset E$ is automatically compact (see Ch.III, Section 2 in [Bo 98]). One can show that the norm $\|\cdot\|$ is *measurable* in the sense of L. Gross (cf. [Gr 67]). Hence also the Gaussian measures μ_t, $t>0$, exist on $\mathcal{B}$, whose variance are given by $t|\ell|^2$, $\ell \in E'$, $t>0$. So,

$$\mu_1 = \mu \, .$$

For $x \in E$, the probability measure $p_t(x,\cdot)$ is defined by

$$p_t(x,A) := \mu_t(A-x) \, , \quad \text{for all } A \in \mathcal{B}.$$

Let $(P_t)_{t>0}$ be the associated family of Markovian kernels:

$$P_t f(x) := \int_E f(y)\, p_t(x, \mathrm{d}y) = \int_E f(x+y)\, \mu_t(\mathrm{d}y) \, , \quad f \in p\mathcal{B},\ x \in E.$$

By Proposition 6 in [Gr 67] it follows that $(P_t)_{t\geq 0}$ (where $P_0 := \mathrm{id}_E$) induces a strongly continuous semigroup of contractions on the space $\mathcal{C}_u(E)$ of all bounded uniformly continuous real-valued functions on E. Let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be the associated strongly continuous resolvent of contractions. Taking $\mathcal{L} = \mathcal{C}_u(E)$ it follows that $\mathcal{U}$ satisfies (a) and (b).

Theorem 3.1. *There exists a Borel right (Markov) process with continuous paths and state space E, having $(P_t)_{t\geq 0}$ as transition function.*

Sketch of the proof. (For a detailed proof we refer to [BeCoRö 07]). The proof is based on an application of Theorem 2.1. The main point is to verify condition $(*)$ (the existence of a real-valued compact excessive function). It turns out that it is sufficient to do this on the subspace E_0 of E considered in [Ku 82] and [AlRö 88]. $\square$

Let $\mathcal{W} = (\Omega, \mathcal{F}, \mathcal{F}_t, W_t, \theta_t, P^x)$ be the path continuous Borel right process with state space E, having $(P_t)_{t\geq 0}$ as transition function, given by Theorem 3.1; $\mathcal{W}$ is called the *Brownian motion* on E.

By Remark 3.5 in [Gr 67] it follows that the process $\mathcal{W}$ is *transient*, i.e. the potential kernel

$$Uf = \int_0^\infty P_t f \; \mathrm{d}t$$

is proper (that is, there exists a bounded strictly positive $\mathcal{B}$-measurable function f such that Uf is finite). Let $M \in \mathcal{B}$ and P_{T_M} be the associated *hitting kernel*,

$$P_{T_M} f(x) = E^x(f \circ W_{T_M}\, ;\, T_M < \infty) \, , \quad x \in E,\ f \in p\mathcal{B}.$$

If $u \in \mathcal{E}(\mathcal{U})$, then $P_{T_M} u = \widehat{R^M} u$, and for each $x \in E \setminus M$ the measure $f \mapsto P_{T_M} f(x)$ is carried by the boundary ∂M of M.

Remark. By assertion *c*) of Theorem 2.1 (since condition $(*)$ is verified) we obtain a natural capacity associated with the Brownian motion on an abstract Wiener space, consequently we answer to the question formulated by R. Carmona in [Ca 80], page 41.

4. Dirichlet problem and controlled convergence

Following [Go 72], a real-valued function f defined on an open set $V \subset E$ is called *harmonic* on V, if it is locally bounded, Borel measurable, finely continuous and there exists $\rho > 0$ such that

$$f(x) = P_{T_{E\setminus B_r(x)}} f(x)$$

for all $r < \rho$ whenever $\bar{B}_r(x) \subset V$; $\bar{B}_r(x)$ denotes the closed ball or radius r centered at x, the *fine topology* is the topology on E generated by $\mathcal{E}(\mathcal{U}_\beta)$.

We shall denote by $H^V : p\mathcal{B}(\partial V) \to p\mathcal{B}(V)$ the kernel defined by

$$H^V f := P_{T_{E\setminus V}} \bar{f} \restriction_V \, , \quad f \in p\mathcal{B}(\partial V),$$

where $\bar{f}$ is a Borel measurable extension of f to E, hence

$$H^V f(x) = E^x(f \circ W_{T_{E\setminus V}}\, ;\, T_{E\setminus V} < \infty) \, , \quad x \in V.$$

$H^V f$ is called the *stochastic solution of the Dirichlet problem* for f (cf. [Go 72]).

By Corollary 1.2 and Remark 3.4 in [Gr 67] it follows that if V is *strongly regular* (i.e., for each $y \in \partial V$ there exists a cone K in E with vertex y such that $V \cap K = \emptyset$; a cone in E with vertex y is the closed convex hull of the set $\{y\} \cup \bar{B}_r(z)$ and $y \notin \bar{B}_r(z)$) and $f \in \mathcal{C}_b(\partial V)$, then $H^V f$ is harmonic on V and

$\lim\limits_{V\ni x\to y} H^V f(x) = f(y)$ for all $y \in \partial V$. Furthermore, if $f \in p\mathcal{B}(\partial V)$ is bounded, then $H^V f$ is harmonic on V. Consequently, for every $f \in p\mathcal{B}(\partial V)$, $H^V f$ is the sum of a series of positive harmonic functions on V.

Let $f : \partial V \to \bar{\mathbb{R}}$ and $h, k : V \to \mathbb{R}$ be such that $k \geq 0$. We say that h *converges to f controlled by k*, if the following conditions hold: For every set $A \subset V$ and $y \in \partial V \cap \bar{A}$ we have

(c1) If $\limsup\limits_{A\ni x\to y} k(x) < \infty$, then $f(y) \in \mathbb{R}$ and $f(y) = \lim\limits_{A\ni x\to y} h(x)$.

(c2) If $\lim\limits_{A\ni x\to y} k(x) = \infty$, then $\lim\limits_{A\ni x\to y} \frac{h(x)}{1+k(x)} = 0$.

Remark. Following [Co 95] and [Co 98], the controlled convergence intends to offer a new method for setting and solving the Dirichlet problem for general open sets and general boundary data. In the above definition the function f should be interpreted as being the boundary data of the harmonic function h. The function k is controlling the convergence of the solution h to the given boundary data f. Note that the case $k = 0$ corresponds to the classical solution: $\lim\limits_{V\ni x\to y} h(x) = f(y)$ for any boundary point y.

Theorem 4.1. *Let $V \subset E$ be a strongly regular open set, λ be a finite measure on V and $\hat{\lambda}$ be the measure on ∂V defined by $\hat{\lambda} := \lambda \circ H^V$. If $f \in \mathcal{L}^1_+(\hat{\lambda})$, then there exists $g \in p\mathcal{B}(\partial V)$ such that $k := H^V g \in \mathcal{L}^1_+(\hat{\lambda})$ and $H^V f$ converges to f controlled by k on the set $[k < \infty]$.*

Proof. See [BeCoRö 07]. □

Acknowledgement. Financial support by the Deutsche Forschungsgemeinschaft, through Project GZ:436 RUM 113/23/0-1 is gratefully acknowledged. The first named author acknowledges support also from the Romanian Ministry of Education and Research (CEEX Program, Project 2-CEx06-11-10).

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