A remark on the generator of a right-continuous Markov process

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Summary. Given a right-continuous Markov process $(X_t)_{t\geq 0}$ on a second countable metrizable space E with transition semigroup $(p_t)_{t\geq 0}$, we prove that there exists a σ-finite Borel measure μ with full support on E, and a closed and densely defined linear operator $(\mathcal{L}_p, D(\mathcal{L}_p))$ generating $(p_t)_{t\geq 0}$ on $L^p(E;\mu)$. In particular, we solve the corresponding Cauchy problem in $L^p(E;\mu)$ for any initial condition $u \in D(\mathcal{L}_p)$. Furthermore, for any real $\beta > 0$ we show that there exists a generalized Dirichlet form which is associated to $(e^{-\beta t}p_t)_{t\geq 0}$. If the β-subprocess of $(X_t)_{t\geq 0}$ corresponding to $(e^{-\beta t}p_t)_{t\geq 0}$, $\beta > 0$, is μ-special standard then all results from generalized Dirichlet form theory become available, and Fukushima's decomposition holds for $u \in D(\mathcal{L}_2)$. If $(X_t)_{t\geq 0}$ is transient, then β can be chosen to be zero.

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1 Introduction and motivation

The notion of an infinitesimal generator of a Markov process is central and widely used in the theory of Markov processes to describe their properties by analytic means (cf. [1]). In general, however, it is difficult to prove in what sense the infinitesimal generator describes the Markov process uniquely. In the simple case when the state space is locally compact and the transition semigroup maps the set of all continuous functions vanishing at infinity into itself, this is classical (cf. e.g. [6]). If the Markov process is symmetric, this is also well-understood by working on $L^2(\mu)$ where μ is the symmetrizing measure (cf. e.g. [3], or the generalization [4]). In this note we show that for any right-continuous Markov process on a polish state space E one can completely determine its generator on $L^2(\mu)$ where μ is a suitable reference measure on E, such that the Markov process is uniquely determined by it μ-a.e. In particular it has (at least in the transient case) a corresponding generalized Dirichlet form associated to it, which, in case the Markov process belongs to the well-known class of special standard processes, is quasi-regular. In this case all the theory of quasi-regular generalized Dirichlet forms from [7], in particular its well-developed L^2-potential theory, and the theory of smooth measures and additive functionals from [8], applies. Moreover, the extended version of the well-known Fukushima decomposition to the case of generalized Dirichlet forms, which in its full generality is an extension of the classical semimartingale decomposition, holds (see the final Remark 3.3(ii), (iii) and (iv)). A main point is that the above mentioned reference measure can be constructed to have full topological support, that is, $\mu(U) > 0$, if $\emptyset \neq U \subset E$, U open. In particular, if the Markov process is Feller, i.e. its transition semigroup maps the set of all bounded continuous functions into itself, then its infinitesimal generator on $L^2(\mu)$ determines the Markov process uniquely (not only up to a μ-zero set) among all Feller Markov processes. In particular, we fully recover the classical results.

2 Construction of L^p-semigroups

Let E be a separable and metrizable topological space. Adjoining an extra point Δ (the cemetery) to the measurable space $(E, \mathcal{B}(E))$ let $E_\Delta := E \cup \{\Delta\}$ and $\mathcal{B}(E_\Delta) = \mathcal{B}(E) \cup \{B \cup \{\Delta\} | B \in \mathcal{B}(E)\}$. As usual, any function f on E is considered as a function on E_Δ with $f(\Delta) := 0$.
We shall first give here the exact definition of what we mean by a *right-continuous Markov process*. Denote by $P(G)$ the set of all probability measures on a measurable space $(G, \mathcal{G})$ and let $\mathcal{G}^*$ be the σ-algebra of universally measurable sets in G, i.e. $\mathcal{G}^* := \bigcap_{P \in P(G)} \mathcal{G}^P$ and $\mathcal{G}^P$ is the completion of $\mathcal{G}$ w.r.t. P. Furthermore $\mathcal{G}_b$, $\mathcal{G}^+$, $\mathcal{G}_b^+$, denote the bounded, positive, bounded and positive, respectively, measurable functions on G. We denote by $C(E)$, $C_b(E)$, the continuous, respectively bounded and continuous functions $f : E \to \mathbb{R}$. Since E is metrizable, we have $\sigma(C(E)) = \mathcal{B}(E)$.

Definition 2.1 $\mathbb{M} = (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ *is called a (temporally homogeneous)* **right-continuous Markov process** *with state space E, life time ζ, and corresponding filtration $(\mathcal{M}_t)_{t\geq 0}$, if*

(M.1) $X_t : \Omega \to E_\Delta$ is $\mathcal{M}_t/\mathcal{B}(E_\Delta)$- measurable for all $t \geq 0$, $X_t(\omega) = \Delta \Leftrightarrow t \geq \zeta(\omega)$ for all $\omega \in \Omega$, where $(\mathcal{M}_t)_{t\geq 0}$ is a filtration on $(\Omega, \mathcal{M})$ and $\zeta : \Omega \to [0, \infty]$ is $\mathcal{M}$-measurable.

(M.2) For all $t \geq 0$ there exists a map $\theta_t : \Omega \to \Omega$ (called the shift operator or simply the shift) such that $X_s \circ \theta_t = X_{s+t}$ for all $s \geq 0$.

(M.3) $(P_x)_{x\in E_\Delta}$ is a family of probability measures on $(\Omega, \mathcal{M})$, such that $x \mapsto P_x[B]$ is $\mathcal{B}(E_\Delta)^*$–measurable for all $B \in \mathcal{M}$ and $\mathcal{B}(E_\Delta)$–measurable for all $B \in \sigma(X_t | t \geq 0)$.

(M.4) (Markov property) For all $A \in \mathcal{B}(E_\Delta), s, t \geq 0$, and $x \in E_\Delta$

$$P_x[X_{t+s} \in A | \mathcal{M}_t] = P_{X_t}[X_s \in A] \; P_x\text{-a.s.}.$$

(M.5) (Normal property) $P_x[X_0 = x] = 1$ *for all* $x \in E_\Delta$.

(M.6) (Right continuity) $t \mapsto X_t(\omega)$ *is right continuous on* $[0, \infty)$ *for all* $\omega \in \Omega$.

From now on assume that we are given a *right-continuous Markov process*

$$\mathbb{M} = (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$$

with state space E, and life time ζ. Let E_x denote the expectation w.r.t. P_x. Since $(X_t)_{t\geq 0}$ is measurable by $(M.6)$,

$$p_t f(x) := p_t(x, f) := E_x[f(X_t)], \; x \in E, \; t \geq 0, \; f \in \mathcal{B}(E)_b,$$

defines a sub-Markovian semigroup of kernels on $(E, \mathcal{B}(E))$, generally referred to as the *transition semigroup* of the given Markov process $\mathbb{M}$. The semigroup property follows directly from the Markov property $(M.4)$.

Let $(R_\alpha)_{\alpha>0}$ denote the *family of resolvent kernels* associated to $(p_t)_{t\geq 0}$, i.e. $R_\alpha(x, B) = \int_0^\infty e^{-\alpha t} p_t 1_B(x) dt$, $\forall B \in \mathcal{B}(E)$, $x \in E$, and $\alpha \geq 0$. Here we use the notation 1_B for the characteristic function of B.

Let $\{x_n | n \geq 1\}$ be a countable dense subset in E. Fix an arbitrary $\gamma > 0$. Define a measure on E by

$$m(dy) := \sum_{n\geq 1} \frac{1}{2^n} R_\gamma(x_n, dy).$$

Under the following assumption

(T) there is $0 < \varphi \in \mathcal{B}(E)_b^*$ with $E_x[\int_0^\infty \varphi(X_t)dt] < \infty$ for all $x \in E$,

let us define

$$m_T(dy) := \sum_{n\geq 1} \frac{1}{2^n c_n} R_0(x_n, dy),$$

where $c_n = E_{x_n}[\int_0^\infty \varphi(X_t)dt]$, $n \geq 1$. Clearly, $c_n > 0$ for all n by $(M.5)$, $(M.6)$. If condition (T) holds then $\mathbb{M}$ is said to be *transient*. We extend m, m_T, to $(E_\Delta, \mathcal{B}(E_\Delta))$ by setting $m(\{\Delta\}) = 0$, $m_T(\{\Delta\}) = 0$. Obviously, m is a finite measure, whereas m_T is in general only σ-finite.

Let $\beta \geq 0$. The next lemma directly implies that m (resp. m_T) is subinvariant for

$$p_t^\beta := e^{-\beta t} p_t; \;\; t \geq 0,$$

whenever $\beta \geq \gamma$ (resp. $\beta = 0$).

Lemma 2.2 *(i) Let $\beta \geq 0$, $p \geq 1$. For any $f \in \mathcal{B}(E)_b^+$, $t \geq 0$, we have*

$$\left(\int_E \left(p_t^\beta f(y)\right)^p m(dy)\right)^{1/p} \leq e^{-(\beta-\frac{\gamma}{p})t}\left(\int_E f^p(y)m(dy)\right)^{1/p}.$$

In particular, if $\|\cdot\|_p$ denotes the norm in $L^p(E;m)$, then for any $g \in \mathcal{B}(E)_b$ we have $\|p_t^\beta g\|_p \leq e^{-(\beta-\frac{\gamma}{p})t}\|g\|_p$.
(ii) Under assumption (T) we have for $f \in \mathcal{B}(E)_b^+$, $t \geq 0$, $p \geq 1$,

$$\left(\int_E (p_t f(y))^p m_T(dy)\right)^{1/p} \leq \left(\int_E f^p(y)m_T(dy)\right)^{1/p}.$$

In particular, if $\|\cdot\|_{T,p}$ denotes the norm in $L^p(E;m_T)$, then for any $g \in \mathcal{B}(E)_b \cap L^p(E;m_T)$ we have $\|p_t g\|_{T,p} \leq \|g\|_{T,p}$.

Proof (i) Let $f \in \mathcal{B}(E)_b^+$. Observe that $(p_t f)^p(x_n) \leq p_t(f^p)(x_n)$ by Jensen's inequality. Then

$$\int_E \left(p_t^\beta f(y)\right)^p m(dy) \leq e^{-p\beta t}\sum_{n\geq 1}\frac{1}{2^n}R_\gamma p_t(f^p)(x_n).$$

Using the Markov property for $(X_t)_{t\geq 0}$, i.e. that P_{x_n}-a.s., one has $E_{x_n}[f^p(X_t)\circ\vartheta_s|\mathcal{M}_s] = E_{X_s}[f^p(X_t)]$, and that f^p is positive, one can easily see that

$$R_\gamma p_t(f^p)(x_n) \quad = \quad e^{\gamma t}E_{x_n}\left[\int_t^\infty e^{-\gamma s}f^p(X_s)ds\right] \leq e^{\gamma t}R_\gamma(f^p)(x_n),$$

so that

$$\int_E \left(p_t^\beta f(y)\right)^p m(dy) \leq e^{-(p\beta-\gamma)t}\sum_{n\geq 1}\frac{1}{2^n}R_\gamma(f^p)(x_n) = e^{-(p\beta-\gamma)t}\int_E f^p(y)m(dy).$$

(ii) Using simply Jensen's inequality, and the Markov property (M.4), we obtain

$$\begin{aligned}\int_E (p_t f)(y)^p m_T(dy) &\leq \sum_{n\geq 1}\frac{1}{2^n c_n}E_{x_n}\left[\int_0^\infty E_{X_s}[f(X_t)^p]ds\right]\\ &= \sum_{n\geq 1}\frac{1}{2^n c_n}E_{x_n}\left[\int_t^\infty f(X_s)^p ds\right]\\ &\leq \int_E f(y)^p m_T(dy).\end{aligned}$$

□

Next we show that m, m_T, has full support.

Lemma 2.3 *Let $\emptyset \neq U \subset E$, U be open. Then $m(U) > 0$, resp. $m_T(U) > 0$.*

Proof Suppose to the contrary that $m(U) = 0$, resp. $m_T(U) = 0$. Since $\{x_n | n \geq 1\}$ is dense in E, there is some $n_0 \geq 1$ with $x_{n_0} \in U$. Otherwise U would be empty. Since E is a metric space, there is a continuous $f : E \to [0,1]$ with $f(x_{n_0}) = 1$ and $f(E \setminus U) \subset \{0\}$. Since $1_U \geq f$ pointwise

$$0 = m(U) \geq \int_E f(y) m(dy) = \sum_{n \geq 1} \frac{1}{2^n} R_\gamma f(x_n),$$

resp.

$$0 = m_T(U) \geq \int_E f(y) m_T(dy) = \sum_{n \geq 1} \frac{1}{2^n c_n} R_0 f(x_n),$$

It follows that $R_\gamma f(x_n) = 0$, resp. $R_0 f(x_n) = 0$, for every n. In particular $R_{\gamma'} f(x_n) = 0$ for any $\gamma' \geq \gamma$ in both cases. Since the last is the argument we need, we can now (up to this end) treat both cases simultaneously. By right continuity of $(X_t)_{t \geq 0}$, and Lebesgue's theorem, we have $\lim_{t \downarrow 0} p_t g(y) = g(y)$ for any $g \in C_b(E)$, $y \in E$. Applying this, Fubini's theorem, and Lebesgue's theorem, we obtain

$$0 = \lim_{\gamma' \to \infty} \gamma' R_{\gamma'} f(x_n) = \lim_{\gamma' \to \infty} \int_0^\infty e^{-t} p_{\frac{t}{\gamma'}} f(x_n) dt = f(x_n)$$

for every n which contradicts $f(x_{n_0}) = 1$. Therefore $m(U) > 0$, resp. $m_T(U) > 0$.

□

Proposition 2.4 *(i) $((p_t^\beta)_{t \geq 0}, C_b(E))$, $\beta \geq 0$, uniquely extends to a submarkovian C_0-semigroup $(T_t^\beta := e^{-\beta t} T_t)_{t \geq 0}$ on $L^p(E; m)$. It is a semigroup of contractions on $L^p(E; m)$ for any $\beta \geq \frac{\gamma}{p}$.*
(ii) Suppose that assumption (T) holds, and let $\mathcal{A} := \{f \cdot R_1 \varphi | f \in C_b(E)\}$. Then $((p_t^\beta)_{t \geq 0}, \mathcal{A})$, $\beta \geq 0$, uniquely extends to a submarkovian C_0-semigroup $(T_t^\beta := e^{-\beta t} T_t)_{t \geq 0}$ of contractions on $L^p(E; m_T)$.

Proof (i) The assertion follows easily from the fact that $C_b(E) \subset L^p(E; m)$ densely, from the right-continuity of $t \mapsto f(X_t)$, $f \in C_b(E)$, and Lemma 2.2(i). In particular, the strong continuity of $(p_t^\beta)_{t \geq 0}$ on $C_b(E)$ can be extended to the whole space $L^p(E; m)$ by a 3ε-argument. The submarkovian property in $L^p(E; m)$, i.e. $0 \leq T_t^\beta f \leq 1$ for any $f \in L^p(E; m)$, $0 \leq f \leq 1$, is directly inherited from $(p_t^\beta)_{t \geq 0}$.
(ii) Obviously, $\varphi \in L^p(E; m_T)$ for all $p \geq 1$, and since m_T is p_t-supermedian by Lemma 2.2(ii), the same is true for $R_1 \varphi$. Therefore $f \cdot R_1 \varphi \in L^p(E; m_T)$, for any $f \in C_b(E)$, and all $p \geq 1$. In particular since $R_1 \varphi > 0$ pointwise, it follows that $\mathcal{A} \subset L^p(E; m_T)$ densely. Indeed, let $g \in L^p(E; m_T)'$ such that $\int_E g f R_1 \varphi \, dm_T = 0$ for all $f \in C_b(E)$. Then since $g R_1 \varphi \in L^1(E; m_T)$, it follows as above that $g R_1 \varphi = 0$, hence $g = 0$ m_T-a.e. The assertion (ii) now follows as in (i), if we can show the strong continuity of $(p_t^\beta)_{t \geq 0}$ on $\mathcal{A}$. We will carry it out for $\beta = 0$, which is enough. Applying the simple Markov property, we obtain

$$p_t(f \cdot R_1 \varphi)(x) = E_x \left[f(X_t) E_x \left[\int_0^\infty e^{-s} \varphi(X_{s+t}) ds | \mathcal{M}_t \right] \right] = e^t E_x \left[f(X_t) \int_t^\infty e^{-s} \varphi(X_s) ds \right].$$

Hence $\lim_{t\downarrow 0} p_t(f\cdot R_1\varphi) = f\cdot R_1\varphi$ pointwise, and consequently in $L^p(E;m_T)$. This concludes the proof.

□

3 Definition of the generalized Dirichlet form and the L^p-Cauchy problem

From now on let

$$\mu = m, \quad resp. \quad \mu = m_T \quad if\ we\ assume\ (T).$$

We would like to define the generalized Dirichlet form related to $(p_t^\beta)_{t\geq 0}$.
Let $\beta \geq \frac{\gamma}{p}$, resp. $\beta = 0$ if we know that assumption (T) holds. Then by Proposition 2.4, $(T_t^\beta)_{t\geq 0}$ is a *strongly continuous contraction semigroup* on $L^p(E;\mu)$, which is submarkovian. For $p \geq 1$, define the $L^p(E;\mu)$-generator $(L_p, D(L_p))$:

$$D(L_p) := \{u \in L^p(E;\mu) | \exists \lim_{t\downarrow 0} \frac{1}{t}(T_t^\beta u - u) \text{ in } L^p(E;\mu)\}$$

$$L_p u := \lim_{t\downarrow 0} \frac{1}{t}(T_t^\beta u - u) \text{ if } u \in L^p(E;\mu).$$

Let $(\hat{L}_2, D(\hat{L}_2))$ be the adjoint operator of $(L_2, D(L_2))$ in $L^2(E;\mu)$, $(\cdot,\cdot)$ the inner product in $L^2(E;\mu)$. Then by [7, I.Examples 4.9(ii)], $(T_t^\beta)_{t\geq 0}$, regarded as semigroup on $L^2(E;\mu)$, is associated to the following *generalized Dirichlet form*

$$\mathcal{E}(u,v) := \begin{cases} (-L_2 u, v) & \text{for } u \in D(L_2),\ v \in L^2(E;\mu) \\ (-\hat{L}_2 v, u) & \text{for } v \in D(\hat{L}_2),\ u \in L^2(E;\mu). \end{cases}$$

Remark 3.1 *(i) Obviously, $p_t^\beta f$ is a μ-version of $T_t^\beta f$ for any $f \in \mathcal{B}(E)_b \cap L^p(E;\mu)$, $\beta \geq 0$.*
(ii) Let $(\hat{T}_t^\beta)_{t\geq 0}$ be the adjoint semigroup of $(T_t^\beta)_{t\geq 0}$ in $L^2(E;\mu)$. Clearly $(\hat{T}_t^\beta)_{t\geq 0}$ is positivity preserving since $(T_t^\beta)_{t\geq 0}$ is positivity preserving. Furthermore $(\hat{T}_t^\beta)_{t\geq 0}$ is sub-Markovian for every $\beta \geq \gamma$ (resp. $\beta \geq 0$ if (T) holds). Indeed, let $0 \leq f \leq 1$, $f \in \mathcal{B}(E) \cap L^2(E;\mu)$, and $g \in \mathcal{B}(E)_b^+ \cap L^2(E;\mu)$. Then by Lemma 2.2 and since $(T_t^\beta)_{t\geq 0}$ is positivity preserving

$$\int_E g(1 - \hat{T}_t^\beta f)d\mu \geq \int_E T_t^\beta g(1-f)d\mu \geq 0.$$

Therefore $\hat{T}_t^\beta f \leq 1$. $0 \leq \hat{T}_t^\beta f$ follows since $(\hat{T}_t^\beta)_{t\geq 0}$ is positivity preserving.
Similarly, for any $\beta \geq \frac{\gamma}{2}$ (resp. $\beta \geq 0$ if (T) holds), $(\hat{T}_t^\beta)_{t\geq 0}$ is a $L^2(E;\mu)$-contraction since $(T_t^\beta)_{t\geq 0}$ is a $L^2(E;\mu)$-contraction.

Applying our preceding results (see however Remark 3.3), we obtain:

Theorem 3.2 *(i) Let* $\mathbb{M} = (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ *be a right-continuous normal Markov process on a separable and metrizable state space* E*. Let* $(p_t)_{t\geq 0}$ *be the corresponding transition semigroup. Given any* $\beta > 0$ *there exists a finite measure* μ *with full support on* E*, and a generalized Dirichlet form* $\mathcal{E}$ *on* $L^2(E;\mu)$ *with associated semigroup* $(T_t^\beta)_{t\geq 0}$*, such that* $p_t^\beta f := e^{-\beta t} p_t f$ *is an* μ*-version of* $T_t^\beta f$ *for any* $f \in \mathcal{B}(E)_b$*,* $t \geq 0$*.*
(ii) If (T) holds, we may choose $\beta = 0$ *in (i), simply* μ *is then in general only* σ*-finite.*

We now formulate the $L^p(E;\mu)$-Cauchy problem related to the given arbitrary Markov process. Let $(\mathcal{L}_p, D(\mathcal{L}_p))$ be its $L^p(E;\mu)$-generator, i.e. $\mathcal{L}_p u = \lim_{t\downarrow 0} \frac{1}{t}(T_t u - u)$ in $L^p(E;\mu)$, $u \in D(\mathcal{L}_p)$. Then it is well-known and easily verified that $D(L_p) = D(\mathcal{L}_p)$ and

$$L_p u = -\beta u + \mathcal{L}_p u, \quad u \in D(\mathcal{L}_p).$$

We are looking to continuously differentiable functions $u : [0,\infty) \to L^p(E;\mu)$, such that $u(t) \in D(\mathcal{L}_p)$, $\forall t \geq 0$, and which solve the following problem with initial condition $u(0) = g \in D(\mathcal{L}_p)$:

$$\begin{aligned} \partial_t u &= \mathcal{L}_p u \\ u(0) &= g. \end{aligned}$$

A solution to this problem is called a *solution to the* $L^p(E;\mu)$*-Cauchy problem related to* $(\mathcal{L}_p, D(\mathcal{L}_p))$ with initial condition $g \in D(\mathcal{L}_p)$. It is well-known (cf. e.g. [5, Theorem 1.2.5 c)]) that the unique solution is given by

$$u(t) = T_t g; \quad t \geq 0.$$

Analogously, for a given $\beta \geq 0$, we can speak about a *solution to the* $L^p(E;\mu)$*-Cauchy problem related to* $(L_p = -\beta\cdot + \mathcal{L}_p, D(L_p))$ with initial condition $g \in D(L_p)$. Clearly, $u(t)$ solves $L^p(E;\mu)$-Cauchy problem related to $(\mathcal{L}_p, D(\mathcal{L}_p))$ with initial condition $g \in D(\mathcal{L}_p)$, if and only if $e^{-\beta t}u(t)$ solves $L^p(E;\mu)$-Cauchy problem related to $(L_p = -\beta\cdot + \mathcal{L}_p, D(L_p))$ with initial condition $g \in D(L_p)$.

Remark 3.3 *(i) For the last results concerning the Cauchy problem condition (T) didn't play any role. Condition (T) was only used to ensure the contraction property of* $(T_t)_{t\geq 0}$ *on* $L^p(E;m_T)$ *which further implies that the corresponding operator* $(\mathcal{L}_p, D(\mathcal{L}_p))$ *is negative definite. We could therefore have restricted our attention in this case to* $\mu = m$*.*
(ii) Suppose that $\mathbb{M}$ *(satisfying (M.1)-(M.6)) has additionally left limits up to* ζ P_μ*-a.e. and that* E *is a metrizable co-Souslin space. Then* $\mathbb{M}$ *is automatically* μ*-tight (see [4, IV. Theorem 1.15, Remark 1.16]).*
(iii) Suppose that $\mathbb{M}$ *(satisfying (M.1)-(M.6)) additionally satisfies the strong Markov property (see e.g. [4] for the definition). Then* $\mathbb{M}$ *is a so-called right process. Suppose also*

that E is a metrizable co-Souslin space. If the generalized Dirichlet form satisfies the sector condition, then $\mathbb{M}$ is μ-special standard (see [2]). Since the existence of left limits up to ζ P_μ-a.e. is part of the definition of a μ-special standard process, $\mathbb{M}$ is then automatically μ-tight by (ii). The investigation of the general non-sectorial case will be the subject of forthcoming work.

(iv) If the β-subprocess of $\mathbb{M}$ corresponding to $(p_t^\beta)_{t\geq 0}$, $\beta \geq 0$, is actually a μ-tight μ-special standard process, then $\mathcal{E}$ in Theorem 3.2 is automatically quasi-regular by [7, IV. Theorem 3.1.], so that we may apply the full potential theory of quasi-regular generalized Dirichlet forms from [7], and the theory of smooth measures and additive functionals from [8]. In particular, by Remark 3.1(ii) the dual semigroup is sub-Markovian if $\beta \geq \gamma$ (resp. $\beta \geq 0$ if (T) holds) and thus the Fukushima decomposition for generalized Dirichlet forms, which is in general an extension of the classical semimartingale decomposition, holds for all $u \in D(L_2)$ (cf. [8, Theorem 4.5(i)]) if we choose such a β. The conclusions which can be drawn from this consideration will also be the subject of forthcoming work.

References

[1] Dynkin, E. B.: Markov processes. Vols. I, II. Die Grundlehren der Mathematischen Wissenschaften, Bände 121, 122, Academic Press Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg 1965.

[2] Fitzsimmons, P. J.: On the quasi-regularity of semi-Dirichlet forms, Potential Anal. 15 (2001), no. 3, 151–185.

[3] Fukushima, M.,Oshima, Y., Takeda, M.: Dirichlet forms and Symmetric Markov processes. Berlin-New York: Walter de Gruyter 1994.

[4] Ma, Z.M., Röckner, M.: Introduction to the Theory of (Non-Symmetric) Dirichlet Forms. Berlin: Springer 1992.

[5] Pazy, A.: Semi-groups of linear operators and applications to partial differential equations, (Lecture notes, University of Maryland (College Park, Md.), Department of Mathematics; 10), 1974.

[6] Revuz, D., Yor, M.: Continuous martingales and Brownian motion. Third edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 293. Springer-Verlag, Berlin, 1999.

[7] Stannat, W.: The theory of generalized Dirichlet forms and its applications in analysis and stochastics, Mem. Amer.Math. Soc., **142** (1999), no. 678.

[8] Trutnau, G.: Stochastic calculus of generalized Dirichlet forms and applications to stochastic differential equations in infinite dimensions, Osaka J. Math. 37 (2000), 13-41.