A new approach to Kolmogorov equations in infinite dimensions and applications to the stochastic 2D Navier Stokes equation

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Abstract

In this note we present a new approach to solve Kolmogorov equations in infinitely many variables in weighted spaces of weakly continuous functions, including the case of non-constant possibly degenerate diffusion coefficients.

Résumé. Dans cette note nous présentons une nouvelle approche pour résoudre des équations de Kolmogorov à une infinité de variables dans des espaces à poids de fonctions faiblement continus. Le cas de coéfficients de diffusion non-constants et éventuellement dégénérés est inclus.

1 Introduction and Main result

The purpose of this note is to present a new general approach to Kolmogorov equations in infinite dimensions based on the methods first developed in [2]. We illustrate this approach through its application to the stochastic 2D Navier-Stokes equations (NSE, see [1] and the references therein) with state dependent ("multiplicative") noise, which on an open set $\Omega \subseteq \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$ is given by

$$\frac{\partial}{\partial t}u+u\cdot\nabla u = \nu\triangle u-\nabla p+f, \quad \operatorname{div} u = 0, \quad u\restriction_{\partial\Omega}=0, \quad u(x,0) = u_0(x). \tag{1.1}$$

Here $u(t,x) \in \mathbb{R}^2$ is the velocity of a fluid in $x \in \Omega$ at time $t \geq 0$, $p(t,x)$ the pressure, $f(t,x)$ an external stochastic force and ν the viscosity constant. We consider the Laplacian with Dirichlet and periodic boundary conditions.

As usual we project (1.1) onto the sub-space $H \subset L^2(\Omega \to \mathbb{R}^2)$ of divergence free vector fields by the Leray-Helmholtz projection P. Then the SPDE (1.1) becomes an SDE in H.

To describe the stochastic force f precisely, let $\{\ell_k\}_{k=1}^\infty$ be the eigenbasis of the part of Δ on H and let $\{w_t^k\}_{k=1}^\infty$ be a sequence of iid Brownian motions with $\mathcal{F}_t := \sigma\{w_t^k | 0 \leq s \leq t, k = 1,2,3,\ldots\}$ its associated filtration. If σ is an $(\mathcal{F}_t)$-adapted locally bounded separable process taking values in the space $L_2(H)$ of Hilbert-Schmidt operators on H, the series $\sum_k \int_0^t \sigma\ell_k dw_t^k$ converges in H almost surely. We denote the differential of the latter process by σdw_t and set $f = \frac{\sigma(u)dw_t}{dt}$, with a continuous map $\sigma : H \to L_2(H)$, i.e. we allow σ to depend on the solution. Thus, (1.1) turns into the following SDE in H:

$$du_t = [\nu\Delta u_t - P\,(u_t\cdot\nabla u_t)]\,dt + \sigma(u_t)dw_t \tag{1.2}$$

The usual way to obtain the Kolmogorov equations corresponding to SDE (1.2) is to reformulate the latter as a martingale problem, which is a standard approach to construct weak solutions to an SDE of type

$$du_t = \mu(u_t)dt + \sigma(u_t)dw_t \tag{1.3}$$

(cf. Stroock and Varadhan in [5] if $H = \mathbb{R}^d$): Let $\mathcal{D}$ be the set of all cylindrical functions of type

$$\Phi(u) = \phi\left(\langle \ell_1, u\rangle, \langle \ell_2, u\rangle, \ldots, \langle \ell_n, u\rangle\right), \ n \in \mathbb{N}, \phi \in C_b^2(\mathbb{R}^n). \tag{1.4}$$

Itô's formula applied to $\Phi(u_t)$, with u_t solving (1.3), yields that

$$m_\Phi(t) := \Phi(u_t) - \Phi(u_0) - \int_0^t (L\Phi)(u_s)ds, \tag{1.5}$$

is an $(\mathcal{F}_t)$-martingale, with the *Kolmogorov operator* L defined as follows:

$$L\Phi(u) = \frac{1}{2}\sum_{km}\langle \sigma(u)\ell_k, \sigma(u)\ell_m\rangle \frac{\partial^2\Phi(u)}{\partial \ell_k \partial \ell_m} + \sum_k \mu_k(u)\frac{\partial \Phi(u)}{\partial \ell_k}, \ \Phi \in \mathcal{D}, \tag{1.6}$$

where in the special case of (1.2)

$$\mu_k(u) := \langle \ell_k, \mu(u)\rangle = \langle \nu\Delta\ell_k, u\rangle + \langle u\cdot\nabla\ell_k, u\rangle, \quad k \in \mathbb{N}.$$

Then a solution to the *martingale problem* $(L, \mathcal{D})$ is a family of measures $(\mathbb{P}_u)_{u\in H}$ on $C([0,\infty), H)$, i.e. the space of continuous trajectories in H such that, for $u \in H$, first, $\mathbb{P}_u\{u_0 = u\} = 1$, and second, for $\Phi \in \mathcal{D}$, the process m_Φ is a $\mathbb{P}_u$-martingale with respect to the standard filtration on $C([0,\infty), H)$.

We confine ourselves to Markov solutions, i.e. $(\mathbb{P}_u)_{u\in H}$ form a Markov process. Then it suffices to construct the transition probability semigroup (TPS), i.e. a semi-group of Markov kernels $p_t(u, dv)$ on H such that

$$p_t\Phi(u) - \Phi(u) = \int_0^t p_s(L\Phi)(u)ds, \ t > 0, \Phi \in \mathcal{D}, \tag{1.7}$$

which is obtained from (1.5) by taking expectation. (1.7) as equations in the unknown measures $p_s(u, dv)$ are called *Kolmogorov equations* and by construction can be considered as a linearization of (1.3).

A purely analytic method of solving (1.7) was introduced in [2] and then developed in [3] (see also [4]). Its main point is the construction of the TPS p_t as a semi-group P_t of Markov operators on

$$C_{\mathbb{V}} := \big\{f : \{\mathbb{V} < \infty\} \to \mathbb{R} \ \big| \ f\restriction_{\{\mathbb{V}\le R\}} \text{ is weakly continuous } \forall\, R > 0 \text{ and } \lim_{R\to\infty}\sup_{\{\mathbb{V}\ge R\}} \mathbb{V}^{-1}|f| = 0\big\}, \tag{1.8}$$

$\mathbb{V} : H \to [0,\infty]$ being a Lyapunov function for L, i.e. $\mathbb{V}$ is of compact level sets, such that $(\lambda - L)\mathbb{V} > 0$.

To state our result precisely, let us consider the SDE (1.3) on an abstract separable Hilbert space H. Let $H_n \subset H_{n+1} \subset H$, be an increasing sequence of finite dimensional subspaces of H, $H_\infty := \cup H_n$ be dense in H, $P_n : H \to H_n$ be the corresponding orthogonal projections.

Hypothesis 1.1. The noise $\sigma : H \to L_2(H)$ is Lipschitz continuous and has block diagonal structure, that is, there exists a sequence $N_n \to \infty$ such that $P_{N_n}\sigma(u) = P_{N_n}\sigma(P_{N_n}u)$ for all $u \in H$.

Hypothesis 1.2. Let $N_n \to \infty$ be as in Hypothesis 1.1, $\sigma_n(u) := P_{N_n}\sigma(u) = P_{N_n}\sigma(P_{N_n}u)$. For all $n \in \mathbb{N}$, there exist $\mu_n \in C(H \to H_{N_n})$, and $\mathbb{V}_n \in C^2(H)$, $\mu_n(u) = \mu_n(P_{N_n}u)$, $\mathbb{V}_n(u) = \mathbb{V}_n(P_{N_n}u)$ for all $u \in H$, such that

(a) $\mathbb{V}_n > 0$;

(b) $\displaystyle \sup_{u,w \in H_{N_n}, u \neq w, |u|,|w| \leq R} \frac{\langle \mu_n(u) - \mu_n(w), u - w\rangle}{|u-w|^2} < \infty$;

(c) There exists $\lambda \in \mathbb{R}$ independent of n such that, for a.a. $u \in H_{N_n}$,

$$\limsup_{H_{N_n} \ni w \to u} \frac{\langle \mu_n(u) - \mu_n(w), u - w\rangle}{|u-w|^2} + \sup_{\xi \in H_{N_n}, |\xi|=1} |D_\xi \sigma_n|^2_{L_2}(u) \\ + \sup_{\xi \in H_{N_n}, |\xi|=1} \left\langle D_\xi \sigma_n^*(x)\xi, \sigma_n^* \frac{D\mathbb{V}_n}{\mathbb{V}_n} \right\rangle (u) + \frac{L_n \mathbb{V}_n}{\mathbb{V}_n}(u) \leq \lambda, \quad (1.9)$$

where L_n on $C^2(H_{N_n})$ is given by (1.6) with μ_n, σ_n replacing σ and μ, respectively.

Hypothesis 1.3. Let $N_n \to \infty$ be as in Hypothesis 1.1, and $\mu_n, \mathbb{V}_n, L_n$ be as in Hypothesis 1.2. There are positive functions $\mathbb{V}, \mathbb{W}$ of compact level sets, finite on H_∞, such that

(a) $\mathbb{V}_n, \mathbb{V} \in C_{\mathbb{W}}$ (the latter is defined as in (1.8)) and $\mathbb{V}_n \to \mathbb{V}$ in $C_{\mathbb{W}}$ as $n \to \infty$;

(b) For all $u \in \{\mathbb{W} < \infty\}$, $\mu(u)$ is defined, $|\mu_n - P_{N_n}\mu|(u) \leq c\frac{\mathbb{W}}{\mathbb{V}}(u)$ and $|\mu_n - P_{N_n}\mu|(u) \to 0$ as $n \to \infty$;

(c) $\displaystyle \limsup_{n\to\infty} \inf_{u \in H_{N_n}} \frac{(\lambda_* - L_n)\mathbb{V}_n}{\mathbb{W}}(u) \geq 1$ for some $\lambda_* \in \mathbb{R}$.

The following theorem is our main result in [3]. To the best of our knowledge it is the first result on solving the Kolmogorov equations (1.7) purely analytically for all points u in an explicitly specified subspace of H and with a non-constant possibly degenerate diffusion matrix in the second order part of L.

Theorem 1.4. *Let Hypotheses 1.1, 1.2, 1.3 hold. Then there exists a unique solution to* (1.7) *on* $\{\mathbb{V} < \infty\}$ *and the TPS constitutes a* C_0*-semi-group of quasi-contractions on* $C_{\mathbb{V}}$*. Furthermore, there exists a unique Markov solution* $(\mathbb{P}_u)_{u \in \{\mathbb{V}<\infty\}}$ *of* (1.5).

We now apply Theorem 1.4 to the 2D NSE (1.2). Let H be the sub-space of $L^2(\Omega \to \mathbb{R}^2)$ consisting of all divergence free vector fields, let $H_0^1 := H_0^1(\Omega \to \mathbb{R}^2)$ (note that $H_0^1 = H^1$ if $\Omega = \mathbb{T}^2$), $H^2 := H^2(\Omega \to \mathbb{R}^2)$ and let $\mu(u) := \nu\Delta u - P(u \cdot \nabla u)$ for $u \in H_0^1 \cap H^2$.

Theorem 1.5. *Let* $\sigma : H \to L_2(H, H_0^1)$ *be bounded, satisfying Hypothesis 1.1. Moreover, let* $\mathbb{V}(u) = \mathbb{V}_\varkappa(u) = e^{\varkappa|\nabla u|^2}$ *for* $\varkappa < \dfrac{\nu}{\sup_u |\sigma(u)|^2_{H\to H}}$.

Then (1.7) *for L with μ and σ as above has a unique solution on $H_0^1 \cap H$ and the respective TPS constitutes a C_0-semi-group of quasi-contractions on $C_{\mathbb{V}}$. Furthermore, there exists a unique Markov solution $(\mathbb{P}_u)_{u \in H_0^1 \cap H}$ of the corresponding martingale problem.*

Proof. Let $\mathbb{W}(u) := c\mathbb{V}(u)|\Delta u|^2$ if $u \in H_0^1 \cap H^2$, and $\mathbb{W} \equiv +\infty$ else. Let H_n be the linear hull of the first n eigenvectors of Δ, $\mathbb{V}_n(u) := \mathbb{V}(P_n u)$ and $\mu_n(u) := P_n \mu(P_n u)$, $n \in \mathbb{N}$. Then $|P_n \mu(u) - \mu_n(u)| \leq 2|u||\nabla u| \leq c|\Delta u|^2$. So Hypothesis 1.2(a)-(b) and Hypothesis 1.3(a)-(b) readily follow.

Note that for $u, \xi, \eta \in H \cap H_0^1 \cap H^2$

$$\frac{D_\xi \mathbb{V}}{\mathbb{V}}(u) = -2\varkappa\langle \Delta u, \xi\rangle, \quad \frac{D^2_{\xi\eta}\mathbb{V}}{\mathbb{V}}(u) = 4\varkappa^2 \langle \Delta u, \xi\rangle\langle \Delta u, \eta\rangle - 2\varkappa\langle \Delta\xi, \eta\rangle,$$

$$\langle \Delta u, P(u \cdot \nabla u)\rangle = \int_\Omega (\operatorname{curl} u) \operatorname{curl} P(u \cdot \nabla u) ds = \int_\Omega (\operatorname{curl} u)(u \cdot \nabla \operatorname{curl} u) ds = 0.$$

$$\begin{aligned} \text{So } \frac{L_n \mathbb{V}_n}{\mathbb{V}_n}(u) &= -2\varkappa\nu|\Delta u|^2 + 2\varkappa^2 |\sigma^*(u)\Delta u|^2 + \varkappa \left|\sigma^*(u)(-\Delta)^{\frac{1}{2}}\right|^2_{L^2(H)} \\ &\leq -2\varkappa\left(\nu - \varkappa \sup_u |\sigma(u)|^2_{H\to H}\right)|\Delta u|^2 + C \end{aligned} \tag{1.10}$$

So Hypothesis 1.3(c) follows. Furthermore, for $u, w \in H_0^1 \cap H^2 \cap H$,

$$\begin{aligned} \langle u - w, P(u \cdot \nabla u) - P(w \cdot \nabla w)\rangle &= \int_\Omega (u - w) \cdot (u \cdot \nabla u - w \cdot \nabla w)\, ds \\ &= \int_\Omega (u - w) \cdot ((u - w) \cdot \nabla u)\, ds, \end{aligned}$$

since $\int_\Omega (u - w) \cdot (w \cdot \nabla(u - w))\, ds = \frac{1}{2}\int_\Omega w \cdot \nabla |u - w|^2 ds = 0$.

So, $|\langle u - w, P(u \cdot \nabla u) - P(w \cdot \nabla w)\rangle| \leq |\Delta u| \left|(-\Delta)^{-\frac{1}{2}}|u - w|^2\right| \leq c|\Delta u||u - w|^2$.

Hence, for any $\varkappa, \varepsilon > 0$,

$$\limsup_{H_{N_n} \ni w \to u} \frac{\langle \mu_n(u) - \mu_n(w), u - w\rangle}{|u - w|^2} \leq 2\varkappa\varepsilon|\Delta u|^2 + \frac{c}{\varkappa\varepsilon}.$$

Now, using (1.10) it is easy to verify (1.9) and thus Hypothesis 1.2(c) holds. □

References

[1] F. Flandoli, M. Romito, **Markov selections and their regularity for the three-dimensional stochastic Navier-Stokes equations,** *C. R. Math. Acad. Sci. Paris* **343** (2006), 47–50.

[2] M.Röckner and Z.Sobol, **Kolmogorov equations in infinite dimensions: well-posedness and regularity of solutions, with applications to stochastic generalized Burgers equations,** *Ann. Probab.* **34** (2006), 663–727.

[3] M.Röckner and Z.Sobol, **Markov solutions for martingale problem: method of Lyapunov function,** in preparation.

[4] R.Stasi, **m-dissipativity for 2D Navier-Stokes operators with periodic boundary conditions,** in preparation.

[5] D.W.Stroock and S.R.S.Varadhan, **Multidimensional diffusion processes,** Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 233, *Springer-Verlag, Berlin-New York, 1979.*