ESTIMATES OF SOBOLEV NORMS OF TRIANGLE TRANSFORMATIONS

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Abstract

We study increasing triangular transformations T of the n-dimensional cube $\Omega = [0,1]^n$ which transform a measure μ into a measure ν, where μ and ν are absolutely continuous Borel probability measures with densities ρ_μ and ρ_ν. It is shown that if there exist positive numbers ε and M such that $\varepsilon < \rho_\mu < M$, $\varepsilon < \rho_\nu < M$ and numbers α, $\beta > 1$ that such $p = \alpha\beta(n-1)^{-1}(\alpha+\beta)^{-1} > 1$ and $\varrho_\mu \in W^{\alpha,1}(\Omega)$, $\varrho_\nu \in W^{\beta,1}(\Omega)$, where $W^{\alpha,1}$ denotes the Sobolev class, then the transformation T belongs to the class $W^{p,1}(\Omega)$.

The so called increasing triangular transformations have been investigated in work [1]. These are transformations of the form $T = (T_1, \ldots, T_n)\colon \mathbb{R}^n \to \mathbb{R}^n$, where T_1 is a function of x_1, T_2 is a function of (x_1, x_2) and so on, T_i is a function of $(x_1, \ldots, x_i)$, and T_i is increasing in x_i. The canonical version of T is described in work [1]. Since our statements do not depend on a Lebesgue version of T (and do not depend on a μ-version as well because μ is equivalent to Lebesgue measure), we may assume that the transformation T is canonical. The main result of the paper is the following theorem.

Theorem. *Let μ and ν be absolutely continuous Borel probability measures with densities ρ_μ and ρ_ν on $\Omega = [0,1]^n$. Let T be an increasing triangular transformation that transforms the measure μ into ν. Let us assume that there exist*

(1) *positive numbers ε and M such that $\varepsilon < \rho_\mu < M$, $\varepsilon < \rho_\nu < M$;*

(2) *positive numbers α, $\beta > 1$ such that $p_n = \alpha\beta(n-1)^{-1}(\alpha+\beta)^{-1} > 1$, $\varrho_\mu \in W^{\alpha,1}(\Omega)$, and $\varrho_\nu \in W^{\beta,1}(\Omega)$.*

Then the transformation T belongs to the class $W^{p_n,1}(\Omega)$.

Proof. We shall prove the statement in the case $n = 2$. We recall that the Sobolev class $W^{p,r}(\Omega)$ (another notion is $H^{p,r}(\Omega)$) is defined as the set of functions $f \in L^p(\Omega)$ whose derivatives up to order r are elements of $L^p(\Omega)$ (regarding Sobolev classes the reader is referred to [2]). In order to show that the transformation T belongs to the Sobolev class we shall express its derivatives as functions of the densities of the measures μ and ν. It is shown in work [1] that

$$T_1(x) = F_{\nu_1}^{-1}\left(F_{\mu_1}(x)\right), \qquad T_2(x,y) = F_{\nu_{T_1(x)}}^{-1}\left(F_{\mu_x}(y)\right),$$

where μ_x and ν_x are conditional measures on $\{x\} \times [0,1]$ (on conditional measures see [3]). In our case the conditional measures determined by the densities

$$\rho_{\mu_x}(y) = \frac{\rho_\mu(x,y)}{\int_0^1 \rho_\mu(x,t)\,dt}, \qquad \varrho_{\nu_x}(y) = \frac{\rho_\nu(x,y)}{\int_0^1 \rho_\nu(x,t)\,dt}$$

with respect to Lebesgue measure. These densities are referred to as conditional densities. We shall denote the projections of the measures μ and ν to the interval $\{(x,0),\, x\in[0,1]\}$ as μ_1 and ν_1. We denote by F_ξ the distribution function of an absolutely coninuous measure ξ with a positive density ϱ_ξ defined on the interval $[0,1]$, i.e.,

$$F_\xi(x)=\int_0^x \rho_\xi(t)\,dt,\; x\in[0,1].$$

The function F_ξ has an inverse function because it is strictly increasing. Thus

$$\partial_y T_1(x)=0, \qquad \partial_x T_1(x)=\frac{F'_{\mu_1}(x)}{F'_{\nu_1}(T_1(x))}=\frac{\rho_{\mu_1}(x)}{\rho_{\nu_1}(T_1(x))},$$
$$\partial_y T_2(x,y)=\frac{F'_{\mu_x}(y)}{F'_{\nu_{T_1(x)}}(T_2(x,y))}=\frac{\rho_{\mu_x}(y)}{\rho_{\nu_{T_1(x)}}(T_2(x,y))}. \tag{1}$$

All the three functions are bounded because the densities ρ_μ and ρ_ν are bounded and are separated from zero and thus their conditional densities and the densities of their projections are separated from zero too. Therefore, they are integrable on Ω in any power. It only remains to prove that the function $\partial_x T_2$ belongs to $L^{p_2}(\Omega)$. Let the density ρ_ν be a smooth function. Then one has the equalities

$$\partial_x F_{\mu_x}(y)=\partial_x\left(\frac{\int_0^y \rho_\mu(x,t)\,dt}{\int_0^1 \rho_\mu(x,t)\,dt}\right)$$
$$=\frac{\int_0^y \partial_x\rho_\mu(x,t)\,dt\int_0^1 \rho_\mu(x,t)\,dt-\int_0^1 \partial_x\rho_\mu(x,t)\,dt\int_0^y \rho_\mu(x,t)\,dt}{\left(\int_0^1 \rho_\mu(x,t)\,dt\right)^2},$$

$$\partial_x F_{\nu_{T_1(x)}}(y)=\partial_x\left(\frac{\int_0^y \rho_\nu(T_1(x),t)\,dt}{\int_0^1 \rho_\nu(T_1(x),t)\,dt}\right)$$
$$=\frac{\int_0^y[\partial_x\rho_\nu](T_1(x),t)T_1'(x)\,dt\int_0^1 \rho_\nu(T_1(x),t)\,dt}{\left(\int_0^1 \rho_\nu(T_1(x),t)\,dt\right)^2}$$
$$-\frac{\int_0^1[\partial_x\rho_\nu](T_1(x),t)T_1'(x)\,dt\int_0^y \rho_\nu(T_1(x),t)\,dt}{\left(\int_0^1 \rho_\nu(T_1(x),t)\,dt\right)^2},$$

$$F'_{\nu_{T_1(x)}}(y)=\partial_y\left(\frac{\int_0^y \rho_\nu(T_1(x),t)\,dt}{\int_0^1 \rho_\nu(T_1(x),t)\,dt}\right)=\frac{\rho_\nu(T_1(x),y)}{\int_0^1 \rho_\nu(T_1(x),t)\,dt}.$$

The following equality holds true as well:

$$\left[\partial_x F^{-1}_{\nu_{T_1(x)}}\right](y)=-\frac{\left[\partial_x F_{\nu_{T_1(x)}}\right](F^{-1}_{\nu_{T_1(x)}}(y))}{F'_{\nu_{T_1(x)}}(F^{-1}_{\nu_{T_1(x)}}(y))}. \tag{2}$$

Indeed, suppose that $f(x,y) = F_{\nu_{T_1(x)}}(y)$ and $\varphi(x,y) = F^{-1}_{\nu_{T_1(x)}}(y)$. For any x and y we have the equality $f(x,\varphi(x,y)) = y$. Differentiation in x leads to the equality

$$\partial_x f(x,\varphi(x,y)) + \partial_y f(x,\varphi(x,y))\partial_x\varphi(x,y) = 0.$$

We obtain the equality

$$\partial_x\varphi(x,y) = -\frac{\partial_x f(x,\varphi(x,y))}{\partial_y f(x,\varphi(x,y))}. \tag{3}$$

This leads to equality (2) in the case of smooth densities. Then we obtain the following chain of equalities:

$$\partial_x T_2(x,y) = \left[\partial_x F^{-1}_{\nu_{T_1(x)}}\right](F_{\mu_x}(y))\partial_x F_{\mu_x}(y) = -\frac{\left[\partial_x F_{\nu_{T_1(x)}}\right](T_2(x,y))}{\left[F'_{\nu_{T_1(x)}}\right](T_2(x,y))}\partial_x F_{\mu_x}(y). \tag{4}$$

We get

$$\left|\left[\partial_x F_{\nu_{T_1(x)}}\right](T_2(x,y))\right| \le 2\frac{\int_0^1 |[\partial_x\rho_\nu](T_1(x),t)|\, T_1'(x)\, dt}{\int_0^1 \rho_\nu(T_1(x),t)\, dt}.$$

In addition, one has

$$|\partial_x F_{\mu_x}(y)| \le 2\frac{\int_0^1 |\partial_x\rho_\mu(x,t)|\, dt}{\int_0^1 \rho_\mu(x,t)\, dt}.$$

Due to the inequalities $\rho_\nu \ge \varepsilon$ and $\int_0^1 \rho_\mu(x,t)dt \ge \varepsilon$ for conditional density we have

$$|\partial_x T_2(x,y)| \le 4\frac{\int_0^1 |[\partial_x\rho_\nu](T_1(x),t)|\, T_1'(x)\, dt \int_0^1 |\partial_x\rho_\mu(x,t)|\, dt}{\rho_\nu(T_1(x),T_2(x,y))\int_0^1 \rho_\mu(x,t)\, dt}$$
$$\le \frac{4}{\varepsilon^2}\int_0^1 |[\partial_x\rho_\nu](T_1(x),t)|\, T_1'(x)\, dt \int_0^1 |\partial_x\rho_\mu(x,t)|\, dt. \tag{5}$$

It is easy to see that, for any function $f \in L^p(\Omega)$, where $p > 1$, the function $\int_0^1 f(x,t)\, dt$ belongs to $L^p(\Omega)$ by Fubini's theorem. It follows by Hölder's inequality that if $f \in L^\alpha(\Omega)$, $g \in L^\beta(\Omega)$, then $fg \in L^p(\Omega)$ where $p = \alpha\beta(\alpha+\beta)^{-1}$. Thus to prove our statement in the case of smooth a density ρ_ν it is enough to show that $[\partial_x\rho_\nu](T_1(x),y)T_1'(x) \in L^\beta(\Omega)$, $\partial_x\rho_\mu(x,y) \in L^\alpha(\Omega)$. The hypotheses of the theorem imply that $\partial_x\rho_\mu(x,y) \in L^\alpha(\Omega)$. By the change of variables formula and the fact that $T_1'(x) \le M/\varepsilon$ we deduce that

$$\int_\Omega |[\partial_x\rho_\nu](T_1(x),y)T_1'(x)|^\beta\, dx\, dy = \int_\Omega |\partial_x\rho_\nu(x,y)|^\beta \left(T_1'(T_1^{-1}(x))\right)^{\beta-1}\, dx\, dy$$
$$\le \frac{M^{\beta-1}}{\varepsilon^{\beta-1}}\int_\Omega |\partial_x\rho_\nu(x,y)|^\beta\, dxdy,$$

where the existence of the right hand side of the equality implies the existence of the left hand side. Thus $\partial_x T_2$ belongs to $L^{p_2}(\Omega)$ and we obtain the following estimate:

$$\| \partial_x T_2 \|_{L^{p_2}(\Omega)} \leq C \|\partial_x \rho_\nu\|_{L^\beta(\Omega)} \|\partial_x \rho_\mu\|_{L^\alpha(\Omega)}, \tag{6}$$

where C is a constant which depends only on ε and M.

From now on we do not assume that the density ρ_ν is smooth, but we suppose that the hypotheses of the theorem are fulfilled. There exists a sequence of smooth densities $\rho_{\nu^{(m)}}$ convergent to ρ_ν in the norm of $W^{\beta,1}$. In addition, we can choose it so that for $\rho_{\nu^{(m)}}$ the hypotheses of the theorem are fulfilled with the same ε, M and β for any m. Inequality (6) applied to the densities $\rho_{\nu^{(m)}}$ and the corresponding triangular transformations $T^{(m)}$ implies the boundedness of the sequence of functions $\partial_x T_2^{(m)}$ in the class $L^{p_2}(\Omega)$. Now to prove the theorem it is enough to show that the sequence of functions $T_2^{(m)}$ converges to T_2 in $L^{p_2}(\Omega)$. Notice that because the absolute values of $T_2^{(m)}$ and T_2 do not exceed 1, it is enough to establish convergence in measure. It is proved in work [1] that if a sequence of absolutely continuous probability measures ν_j defined on $\mathbb{R}^n$ converges in variation to measure ν, then sequence of canonical triangular transformations T_{μ,ν_j} converge in measure to $T_{\mu,\nu}$ (in work [4], a generalization is obtained in the case where measure μ also vary). Because convergence of densities in $W^{\beta,1}(\Omega)$ implies convergence of measures in the variation norm, the sequence $T_2^{(m)}$ converges to T_2 in $L^{p_2}(\Omega)$. The statement in the case $n = 2$ proved.

Now we apply induction on n and assume that the statement is proved if $k < n$. According to the construction of the canonical transformation T (see [1]), the first $n - 1$ coordinates of the transformation T form the canonical transformation of the projections of the measures on the $(n-1)$-dimensional cube in the hyperplane $x_n = 0$. We shall denote it by S, and the vector $(x_1, \dots, x_{n-1})$ is denoted by x. Obviously, the hypotheses of the theorem are fulfilled for the projections of our measures. Indeed, the densities of the projections are positive, bounded and separated from zero, their derivatives $\int_0^1 \partial_{x_i} \rho_\nu(x, x_n)\, dx_n$, $\int_0^1 \partial_{x_i} \rho_\mu(x, x_n)\, dx_n$ are integrable in necessary powers. Therefore, the components T_i, $i = 1, \dots, n-1$, belong to the Sobolev class $W^{p_{n-1},1}(\Omega)$, $p_{n-1} > p_n$. Thus it remains to prove the membership of $\partial_{x_i} T_n(x, x_n)$ in $L^{p_n}(\Omega)$.

We shall use the following relation for $T_n(x, x_n)$:

$$T_n(x, x_n) = F^{-1}_{\nu_{S(x)}}(F_{\mu_x}(x_n)),$$

where μ_x and ν_x are conditional measures defined on the segments $\{x\} \times [0, 1]$. The derivative of $T_n(x, x_n)$ in x_n has the same form as in (1), i.e.,

$$\partial_{x_n} T_n(x, x_n) = \frac{\rho_{\mu_x}(x_n)}{\rho_{\nu_{S(x)}}(T_n(x, x_n))}. \tag{7}$$

Hence it is integrable in any power. Suppose that the density ρ_ν is a smooth function. Then the derivative in x_i, $i<n$, has the same form as in (4), i.e.,

$$\partial_{x_i} T_n(x,x_n) = -\frac{\left[\partial_{x_i} F_{\nu_{S(x)}}\right](T_n(x,x_n))}{F'_{\nu_{S(x)}}(T_n(x,x_n))}\partial_{x_i} F_{\mu_x}(x_n).$$

Let us write out multipliers separately:

$$\left(F'_{\nu_{S(x)}}(T_n(x,x_n))\right)^{-1} = \frac{\int_0^1 \rho_\nu(S(x),t)\,dt}{\rho_\nu(S(x),T_n(x,x_n))};$$

$$\left|\left[\partial_{x_i} F_{\nu_{S(x)}}\right](T_n(x,x_n))\right| = \left|\partial_{x_i}\left(\frac{\int_0^y \rho_\nu(S(x),t)\,dt}{\int_0^1 \rho_\nu(S(x),t)\,dt}\right)\Bigg|_{y=T_n(x,x_n)}\right|$$
$$\leq \frac{2}{\int_0^1 \rho_\nu(S(x),t)\,dt}\int_0^1 \sum_{j=i}^{n-1} |[\partial_{x_j}\rho_\nu](S(x),t)\partial_{x_i} T_j(x_1,\dots,x_j)|\,dt,$$

$$|\partial_{x_i} F_{\mu_x}(x_n)| = \left|\partial_{x_i}\left(\frac{\int_0^{x_n} \rho_\mu(x,t)\,dt}{\int_0^1 \rho_\mu(x,t)\,dt}\right)\right| \leq 2\frac{\int_0^1 |\partial_{x_i}\rho_\mu(x,t)|\,dt}{\int_0^1 \rho_\mu(x,t)\,dt}.$$

Similarly to inequality (5) we obtain the estimate

$$\partial_{x_i} T_n(x,x_n) \leq \frac{4}{\varepsilon^2}\cdot \int_0^1 \sum_{j=i}^{n-1} |[\partial_{x_j}\rho_\nu](S(x),t)\partial_{x_i} T_j(x_1,\dots,x_j)|\,dt \int_0^1 |\partial_{x_i}\rho_\mu(x,t)|\,dt. \quad (8)$$

By the inductive assumption for $j=1,\dots,n-1$ the function $\partial_{x_i} T_j(x_1,\dots,x_j)$ belongs to $L^{p_j}(\Omega)$. In particular, for any j this expression belongs to $L^{p_{n-1}}(\Omega)$. The function $\int_0^1 |\partial_{x_i}\rho_\mu(x,t)|\,dt$ belongs to $L^\alpha(\Omega)$ and one has $\partial_{x_j}\rho_\nu(x,t) \in L^\beta(\Omega)$ for any j. Then by the change of variable formula (see [1, p. 7]) we obtain

$$\int_\Omega \left|[\partial_{x_j}\rho_\nu](S(x),t)\right|^\beta\,dx\,dt = \int_\Omega \left|\partial_{x_j}\rho_\nu(x,t)\right|^\beta \prod_{k=1}^{n-1}\partial_{x_k} T_k(x_1,\dots,x_k)\,dx\,dt,$$

where $\varepsilon/M \leq \partial_{x_k} T_k(x_1,\dots,x_k) \leq M/\varepsilon$ according to (7). Thus $[\partial_{x_j}\rho_\nu](S(x),t)$ belongs to $L^\beta(\Omega)$. By using Hölder's inequality we obtain that the right hand side of inequality (8) and therefore the left hand side belongs to $L^q(\Omega)$ where $1/q = 1/\alpha + 1/\beta + 1/p_{n-1}$, i.e., $q=p_n$. In addition, the following chain of equalities holds true:

$$\|\partial_{x_i} T_n\|_{L^{p_n}(\Omega)} \leq C \max_{i\leq j\leq n-1}\|\partial_{x_i} T_j\|_{L^{p_{n-1}}(\Omega)} \max_{i\leq j\leq n-1}\|\partial_{x_j}\rho_\nu\|_{L^\beta(\Omega)}\|\partial_{x_i}\rho_\mu\|_{L^\alpha(\Omega)}$$
$$\leq C \max_{i\leq j\leq n-1}\|\partial_{x_i} T_j\|_{L^{p_j}(\Omega)}\|\rho_\nu\|_{W^{\beta,1}(\Omega)}\|\rho_\mu\|_{W^{\alpha,1}(\Omega)},$$

where C is a constant depending only on ε and M. Then by induction we can obtain the estimate

$$\|\partial_{x_i} T_n\|_{L^{p_n}(\Omega)} \le C_1 \|\rho_\nu\|^{n-1}_{W^{\beta,1}(\Omega)} \|\rho_\mu\|^{n-1}_{W^{\alpha,1}(\Omega)}, \tag{9}$$

where C_1 is a constant number depending only on ε and M.

From now on we do not assume that the density ρ_ν is smooth. As in the case $n = 2$ let us find a sequence of smooth densities $\rho_{\nu^{(m)}}$ for which the hypotheses of the theorem are fulfilled with the same ε, M and β for any m, and the sequence $\rho_{\nu^{(m)}}$ converges to ρ_ν in $W^{\beta,1}(\Omega)$. By inequality (9) applied to the densities $\rho_{\nu^{(m)}}$ and the corresponding triangular transformations $T^{(m)}$, it is easy to show the boundedness of the sequence of functions $\partial_{x_i} T_n^{(m)}$ in the class $L^{p_n}(\Omega)$. The functions $T_n^{(m)}$ converge to T_n in $L^{p_n}(\Omega)$. Hence T_n is a limit of the sequence of functions $T_n^{(m)}$ in $W^{p_n,1}(\Omega)$. Theorem is completely proved.

This work has been partially supported by the RFBR Grant 04-01-00748 and the DFG Grant 436 RUS 113/343/0(R).

References

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