The uniqueness problem for subordinate resolvents with potential theoretical methods

Nicu Boboc[1)] and Gheorghe Bucur[1),2)]

[1)] *Faculty of Mathematics and Informatics, University of Bucharest, str. Academiei 14, RO-010014 Bucharest, Romania*
[2)] *Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, RO-014700 Bucharest, Romania*

Abstract. We present an analytic version of the following uniqueness problem for Markov processes: if two subprocesses of a given transient, Borel right process have the same excessive functions then they coincide. Our treatment is given in terms of subordinate sub-Markovian resolvents of kernels in the sense of P.A. Mayer and uses essentially the subordination operators introduced by G. Mokobodzki.

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Introduction

In this paper $(E, \mathcal{B})$ is a Lusin measurable space and $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is a proper sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ such that the set $\mathcal{E}(\mathcal{U})$ of all $\mathcal{U}$-excessive $\mathcal{B}$-measurable functions is min-stable, contains the positive constant functions and generates $\mathcal{B}$. We assume also that E is semisaturated with respect to $\mathcal{U}$, i.e. any $\mathcal{U}$-excessive measure dominated by a potential measure is also a potential measure. We recall that a potential measure is a σ-finite measure of the form $\mu \circ U$ where μ is a positive measure on $(E, \mathcal{B})$ and U is the initial kernel of $\mathcal{U}$. In the sequel any sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ which possesses the above properties will be called *natural*.

Let $\mathcal{U}' = (U'_\alpha)_{\alpha>0}$ be a second natural sub-Markovian resolvent on $(E, \mathcal{B})$. The resolvent $\mathcal{U}'$ is called *exact subordinate* to $\mathcal{U}$ if we have (see [3])

a) $U'_\alpha \leq U_\alpha$ for all $\alpha > 0$

b) $Uf - U'f \in \mathcal{E}(\mathcal{U})$ if $f \in p\mathcal{B}$, $Uf < \infty$, where U' is the initial kernel of $\mathcal{U}'$.

A kernel P on $(E, \mathcal{B})$ is called *exact subordination operator* with respect to $\mathcal{U}$ (see [3]) if the following properties hold:

1) $P(\mathcal{E}(\mathcal{U})) \subset \mathcal{E}(\mathcal{U})$;

2) $Ps \leq s$ for all $s \in \mathcal{E}(\mathcal{U})$;

3) $\inf(s, Ps + t - Pt + Pf) \in \mathcal{E}(\mathcal{U})$ for all $s, t \in \mathcal{E}(\mathcal{U})$ with $s < \infty$, $t < \infty$, $f \in p\mathcal{B}$;

4) For all $x \in E$ there exists $s \in \mathcal{E}(\mathcal{U})$ with $Ps(x) < s(x)$.

This notion was introduced by G. Mokobodzki (cf. [7]). We notice that if $\mathcal{V} = (V_\alpha)_{\alpha>0}$ is a second natural sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ such that $\mathcal{E}(\mathcal{U}) = \mathcal{E}(\mathcal{V})$ then a kernel P on $(E, \mathcal{B})$ will be an exact subordination operator with respect to $\mathcal{U}$ if and only if it is an exact subordination operator with respect to $\mathcal{V}$.

It is known (see [3]) that if P is an exact subordination operator with respect to $\mathcal{U}$ then there exists a natural sub-Markovian resolvent $\mathcal{U}^P = (U_\alpha^P)_{\alpha>0}$ on $(E, \mathcal{B})$ which is exact subordinate to $\mathcal{U}$, such that

$$Uf = U^P f + PUf$$

for all $f \in p\mathcal{B}$ where U^P is the initial kernel of $\mathcal{U}^P$. Conversely for any natural sub-Markovian resolvent of kernels $\mathcal{U}' = (U'_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$ which is exact subordinate to $\mathcal{U}$, there exists an exact subordination operator P with respect to $\mathcal{U}$ such that

$$Uf = U'f + PUf$$

for all $f \in p\mathcal{B}$, where U' is the initial kernel of $\mathcal{U}'$.

A function $h \in p\mathcal{B}$ is called *exact* with respect to $\mathcal{U}$ (see [1], [6]) if there exists a kernel U_h on $(E, \mathcal{B})$ such that for all $f \in p\mathcal{B}$ we have:

$$Uf = U_h f + U_h(hUf) \quad \text{and} \quad U_h(hUf) = U(hU_h f).$$

We notice that if h is exact with respect to $\mathcal{U}$ then $h < \infty$ $\mathcal{U}$-a.e. (i.e., $U(1_{[h=+\infty]}) = 0$) and the kernel U_h with the above properties is unique. Moreover it is known (see [1], [6]) that if $h \in p\mathcal{B}$ is exact with respect to $\mathcal{U}$ then for any $\alpha > 0$ the function $h + \alpha$ is also exact with respect to $\mathcal{U}$ and the family of kernels $\mathcal{U}^h = (U_{h+\alpha})_\alpha$ is a natural sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ having U_h as initial kernel which is exact subordinate to $\mathcal{U}$. In addition the kernel P^h defined by

$$P^h f = U_h(hf)$$

is an exact subordination operator with respect to $\mathcal{U}$ and we have

$$\mathcal{U}^h = \mathcal{U}^{P^h}.$$

Let $\mathcal{W} = (W_\alpha)_{\alpha>0}$ be a proper sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ such that its initial kernel W is a regular $\mathcal{U}$- excessive kernel (see [3]) and moreover there exists $f_0 \in p\mathcal{B}$, $0 < f_0 \leq 1$ with Uf_0 bounded and

$$\inf_\alpha \alpha W_\alpha(Uf_0) = 0.$$

It is known (see [3]) that the kernel W_1 is an exact subordination operator with respect to $\mathcal{U}$. We notice that if there exists $h \in p\mathcal{B}$ with $Wf = U(h \cdot f)$ for all $f \in p\mathcal{B}$ then it is known (see [6]) that h is exact with respect to $\mathcal{U}$ and we have

$$W_1 f = P^h(f)$$

for all $f \in p\mathcal{B}$ and so $\mathcal{U}^{W_1} = \mathcal{U}^h$. The problem of uniqueness for exact subordination operators is the following: Is an exact subordination operator P with respect to $\mathcal{U}$ uniquely determined by $\mathcal{E}(\mathcal{U}^P)$?

In this paper we obtain essentially two results. The first one is the following: Let P, Q be two exact subordination operators with respect to $\mathcal{U}$ such that $\mathcal{E}(\mathcal{U}^P) = \mathcal{E}(\mathcal{U}^Q)$ and such that P (resp. Q) is a regular $\mathcal{U}^P$-excessive (resp $\mathcal{U}^Q$-excessive) kernel. Then if there is no $\mathcal{U}$-absorbent point in E we have $P = Q$. This result extend a similar one obtained in [2] in the particular case when $\mathcal{U}$, $\mathcal{U}^P$, $\mathcal{U}^Q$ are such that $\mathcal{E}(\mathcal{U})$, $\mathcal{E}(\mathcal{U}^P)$ satisfy the sheaf property on a Lusin topological space E.

The second result is the following. Assume that there is no $\mathcal{U}$-finely open singleton in E. Then for any $h \in p\mathcal{B}$ which is exact with respect to $\mathcal{U}$ the kernel P^h is the unique exact subordination operator P with respect to $\mathcal{U}$ such that $\mathcal{E}(\mathcal{U}^P) = \mathcal{E}(\mathcal{U}^h)$. Particularly let $\mathcal{W} = (W_\alpha)_{\alpha>0}$ be a proper sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ such that its initial kernel W is a regular $\mathcal{U}$-excessive kernel and

$$\inf_\alpha \alpha W_\alpha(Uf_0) = 0$$

for a suitable $f_0 \in p\mathcal{B}$, $0 < f_0 \leq 1$ with Uf_0 bounded. Then the kernel W_1 is the unique exact subordination operator P with respect to $\mathcal{U}$ such that

$$\mathcal{E}(\mathcal{U}^P) = \mathcal{E}(\mathcal{U}^{W_1}).$$

This last consequence was sugested to us by L. Beznea.

For a probabilistic approach concerning the above problem of uniqueness one can see [5].

Uniqueness problem for exact subordination operators

Let $\mathcal{U}' = (U'_\alpha)_{\alpha>0}$ be a natural sub-Markovian resolvent on $(E, \mathcal{B})$ such that

$$U'_\alpha \leq U_\alpha \quad \forall \alpha > 0$$
$$Uf - U'f \in \mathcal{E}(\mathcal{U}) \quad \forall\, f \in p\mathcal{B}, Uf < \infty.$$

and let P be an exact subordination operator with respect to $\mathcal{U}$ such that $\mathcal{U}^P = \mathcal{U}'$. It is known (cf. [3]) that if $A \in \mathcal{B}$ then we have

$$R^A f - {}'R^A f = P(R^A f) - {}'R^A P(R^A f).$$

for all $f \in p\mathcal{B}$ when R^A (resp. ${}'R^A$) is the reduite kernel on $(E, \mathcal{B})$ associated with A and $\mathcal{U}$ (resp. $\mathcal{U}'$). Also the set A will be a basic set with respect to $\mathcal{U}$ if and only if it is a basic set with respect to $\mathcal{U}'$. We notice that for any $A \in \mathcal{B}$ we have

$${}'R^A f \leq R^A f \quad \forall\, f \in p\mathcal{B}$$

and that the fine topology with respect to $\mathcal{U}$ and $\mathcal{U}'$ are the same. From [4] we deduce that there exists an exact subordination operator Q with respect to $\mathcal{U}$ such that $\mathcal{E}(\mathcal{U}^Q) = \mathcal{E}(\mathcal{U}')$ and such that

$$Qs \preccurlyeq_{\mathcal{E}(\mathcal{U}')} Ps \quad \forall\, s \in \mathcal{E}(\mathcal{U})$$

for all exact subordination operator P with respect to $\mathcal{U}$ with $\mathcal{E}(\mathcal{U}^P) = \mathcal{E}(\mathcal{U}')$, where $\preccurlyeq_{\mathcal{E}(\mathcal{U}')}$ means the specific order with respect to $\mathcal{E}(\mathcal{U}')$.

Theorem 1. *Let P, Q be two exact subordination operators with respect to $\mathcal{U}$ such that $\mathcal{E}(\mathcal{U}^P) = \mathcal{E}(\mathcal{U}^Q) = \mathcal{E}(\mathcal{U}')$ and P, Q are regular $\mathcal{U}'$-excessive kernels. If there is no $\mathcal{U}$-absorbent point in E then $P = Q$.*

Proof. We consider a Ray topology $\mathcal{T}$ on E associate with $\mathcal{U}$ such that $\mathcal{B}(\mathcal{T}) = \mathcal{B}$, $U_\alpha f$ is lower semicontinuous for all positive bounded lower semicontinuous function f and $\alpha > 0$ and such that Uf_0 is bounded and continuous for a suitable $f_0 \in p\mathcal{B}$, $0 < f_0 \leq 1$. Let further A be a Borel basic set, $A^c := E \backslash A$ and $s := Uf_0$. Since

$$P(R^A s) - {}'R^A(PR^A s) = R^A s - {}'R^A s = Q(R^A s) - {}'R^A Q(R^A s),$$

$$P(1_A R^A s) = {}'R^A P(1_A R^A s), \quad Q(1_A R^A s) = {}'R^A Q(1_A R^A s)$$

we deduce the relation

$$(*) \qquad P(1_{A^c}R^A s) - {}'R^A P(1_{A^c}R^A s) = Q(1_{A^c}R^A s) - {}'R^A Q(1_{A^c}R^A s).$$

We show that the function

$$t := P(1_{A^c}R^A s) \curlywedge_{\mathcal{E}(\mathcal{U}')} {}'R^A P(1_{A^c}R^A s)$$

is zero. Indeed, let $(F_n)_n$ be an increasing sequence of Borel finely closed subsets of E such that

$$\bigcup_n F_n = A^c.$$

If we put

$$t_n := P(1_{F_n}R^A s) \curlywedge_{\mathcal{E}(\mathcal{U}')} {}'R^A P(1_{F_n}R^A s)$$

we remark that the fine carrier of t_n is included in F_n and also in A and so $t_n = 0$. Hence $t = \curlyvee_{\mathcal{E}(\mathcal{U}')} t_n = 0$. Analogously, we deduce the relation

$$Q(1_{A^c}R^A s) \curlywedge_{\mathcal{E}(\mathcal{U}')} {}'R^A Q(1_{A^c}R^A s) = 0.$$

From the above considerations, using the relation $(*)$ we get

$$(**) \qquad P(1_{A^c}R^A s) = Q(1_{A^c}R^A s)$$

Let now $x_0 \in E$ and let $(G_n)_n$ be an increasing sequence of open subsets of $(E, \mathcal{T})$ such that

$$\bar{G}_n \subset G_{n+1} \; \forall\, n \in N, \; \cup_n G_n = E \backslash \{x_0\}.$$

Since the set $\{x_0\}$ is not $\mathcal{U}$-absorbent we deduce that there exists $n_0 \in \mathbb{N}$ such that

$$R^{G_n} s(x_0) > 0 \quad \forall\, n \geq n_0.$$

On the other hand for all $n \in \mathbb{N}$ we have $R^{G_n} s = {}^{\mathcal{S}}R^{G_n} s$, where ${}^{\mathcal{S}}R^{G_n}$ is the reduite on G_n with respect to the cone $\mathcal{S}$ of all Borel supermedian functions with respect to $\mathcal{U}$. It is known that if $\mathcal{S}_k$ denote the set off all Borel supermedian functions with respect to kU_k we have

$${}^{\mathcal{S}_k}R^{G_n} s = {}^{\mathcal{S}_k}R(1_{G_n} s) = \inf\{t \in \mathcal{S}_k \mid t \geq 1_{G_{N_g}} s\}$$

and the sequence ${}^{\mathcal{S}_k}R^{G_n} s$ increases to $R^{G_n} s$ when $k \nearrow \infty$. Using now Mokobodzki's formula in computing ${}^{\mathcal{S}_k}R(1_{G_n} s)$ (see e.g. [3]) and the fact that 1_{G_n} is lower semicontinuous, we deduce that ${}^{\mathcal{S}_k}R^{G_n} s$ is also lower semicontinuous. From the above considerations it follows that the function $R^{G_n}s$ is lower semicontinuous. Since $R^{G_n}s(x_0) > 0$ for all $n \geq n_0$ it follows that there exists $\rho_0 > 0$ and an open neighbourhood D_0 of x_0 such that $R^{G_n}s(x) > \rho_0$ for all $x \in D_0$ and $n \geq n_0$. If we denote by A_n the fine closure of G_n then we get

$$G_n \subset A_n \subset \bar{G}_n,\ D_0 \cap (E \backslash \bar{G}_n) \subset A_n^c.$$

and

$$R^{G_n}s = R^{A_n}s > \rho_0 \ \text{ on } \ D_0 \cap (E \setminus \bar{G}_n).$$

Using the relation $(**)$ it follows that for all $f \in p\mathcal{B}$ we have

$$P(1_{A_n^c} f R^{A_n}s) = Q(1_{A_n^c} f R^{A_n}s)$$

and so $Pf = Qf$ for all $f \in p\mathcal{B}$ with $f = 0$ on $E \backslash \bar{G}_n$. The point x_0 being arbitrary in E and $\mathcal{T}$ having a countable basis we deduce that $P = Q$. □

Theorem 2. *Assume that $h \in p\mathcal{B}$ is an exact function with respect to $\mathcal{U}$ and that there is no $\mathcal{U}$-absorbent point in E. Then for any exact subordination operator P with respect to $\mathcal{U}$ such that $\mathcal{E}(\mathcal{U}^P) = \mathcal{E}(\mathcal{U}^h)$ and*

$$P^h U f \preccurlyeq_{\mathcal{E}(\mathcal{U}^h)} PUf \quad \forall f \in p\mathcal{B}$$

we have

$$P^h = P.$$

Proof. Let us denote $W := U^P$. Since $U_h f = Uf - P^h U f \succcurlyeq_{\mathcal{E}(\mathcal{U}^h)} Uf - PUf = U^P f$ for all $f \in p\mathcal{B}$ with $Uf < \infty$ it follows that there exists $g \in p\mathcal{B}$, $0 < g \leq 1$ such that for all $f \in p\mathcal{B}$ we have

$$Wf = U_h(gf), \quad PUf = U_h((1-g)f + hUf).$$

Let now $f_1, f_2 \in p\mathcal{B}$ with $Uf_2 < \infty$ and $Uf_1 \leq Uf_2$. Since

$$PUf_1 \preccurlyeq_{\mathcal{E}(\mathcal{U}^h)} PUf_2$$

we deduce that

$$U_h((1-g)f_1 + hUf_1) \preccurlyeq_{\mathcal{E}(\mathcal{U}^h)} U_h((1-g)f_2 + hUf_2)$$

and therefore

$$(1-g)f_1 + hUf_1 \leq (1-g)f_2 + hUf_2 \qquad \mathcal{U} - \text{a.e.}$$

Let now $f_0 \in p\mathcal{B}$, $0 < f_0 \leq 1$, be such that $s := Uf_0$ is bounded. For every strictly positive real number r we consider the sets

$$A_r := \left\{ x \in E | 1 - g(x) \geq \frac{r}{U_h f_0(x)} \right\} \text{ and } A_{r,0} := \{x \in A_r \mid \liminf_{\alpha\to\infty} \alpha U_\alpha(1_{A_r})(x) = 1\}.$$

Using ([3], Theorem 1.3.8) it follows that the set $A_r \setminus A_{r,0}$ is $\mathcal{U}$-negligible and for any finely open set $D \in \mathcal{B}$ the set $A_{r,0} \cap D$ is subbasic with respect to $\mathcal{U}$ and there exists a sequence $(f_n)_n$ in $bp\mathcal{B}$ with $Uf_n < \infty$, $f_n = 0$ on $E \setminus (A_{r,0} \cap D)$ such that

$$Uf_n \nearrow R^{A_{r,0}\cap D} s.$$

If A_r is $\mathcal{U}$-negligible for any $r > 0$ we deduce that $g = 1$ $\mathcal{U}$-a.e. on E and so $W = U_h$, $P = P^h$. Let us suppose that there exists $r > 0$ such that A_r is not $\mathcal{U}$-negligible. In this case $A_{r,0}$ is also not $\mathcal{U}$-negligible. If we consider $a \in A_{r,0}$ and a decreasing sequence $(D_n)_n$ of $\mathcal{U}$-finely open set in $\mathcal{B}$ such that $\cap_n D_n = \{a\}$ then using the fact that $U(1_D)(x) \neq 0 \; \forall \, x \in D$ where $D \in \mathcal{B}$ is $\mathcal{U}$-finely open set we deduce that

$$\sup_n U(f_0 \cdot 1_{E\setminus D_n}) = U(f_0 \cdot 1_{E\setminus\{a\}}) > 0$$

on $1_{E\setminus\{a\}}$. Since $\{a\}$ is not $\mathcal{U}$-absorbent it follows that $U(f_0 \cdot 1_{E\setminus\{a\}})(a) > 0$ and therefore there exists $n_0 \in \mathbb{N}$ such that

$$U(f_0 \cdot 1_{E\setminus D_{n_0}})(a) > 0.$$

Let us put $D := D_{n_0}$,

$$t := U(f_0 \cdot 1_{E\setminus D})$$

and let $(f_n)_n$ be a sequence in $bp\mathcal{B}$ with $f_n = 0$ on $E\setminus(D \cap A_{r,0})$ and $Uf_n \nearrow R^{D\cap A_{r,0}} t$. From

$$Uf_n \leq R^{D\cap A_{r,0}} t \leq t \leq U(f_0 \cdot 1_{E\setminus D})$$

we deduce that

$$(1-g)f_n + hUf_n \leq (1-g)f_0 \cdot 1_{E\setminus D} + h \cdot U(f_0 1_{E\setminus D}) \, \mathcal{U} - \text{a.e.}$$

Since $f_n = 0$ on $E\setminus(D \cap A_{r,0})$,

$$1 - g \geq \frac{r}{U_h f_0} \; on D \cap A_{r,0}$$

and the sequence $(hUf_n)_n$ increases to ht on $D \cap A_{r,0}$ we deduce that $\lim_n f_n = 0$ $\mathcal{U}$-a.e. on E. On the other hand we have $\mathcal{U}$-a.e.

$$(1-g)f_n \leq hU(f_0 1_{E\setminus D}) \leq hUf_0, \quad \frac{r}{U_h f_0} \cdot f_n \leq (1-g)f_n \leq hUf_0.$$

Using the fact that Uf_0 is bounded we get

$$f_n \leq \frac{h}{r}\|Uf_0\|_\infty U_h f_0 \quad \mathcal{U} - \text{a.e.}, \quad U(\frac{h}{r}\|Uf_0\|_\infty U_h f_0) \leq \frac{\|Uf_0\|_\infty}{r} Uf_0 < \infty$$

and so $\lim_n Uf_n = 0$ which contradicts the fact that

$$\lim_n Uf_n(a) = R^{D\cap A_{r,0}} t(a) > 0.$$

□

Theorem 3. *Assume that $h \in p\mathcal{B}$ is exact with respect to $\mathcal{U}$ and that there is no $\mathcal{U}$-finely open singleton in E.*

Then the kernel P^h is the unique exact subordination operator P with respect to $\mathcal{U}$ such that $\mathcal{E}(\mathcal{U}^P) = \mathcal{E}(\mathcal{U}^h)$.

Proof. From [4] there exists an exact subordination operator Q with respect $\mathcal{U}$ such that $\mathcal{E}(\mathcal{U}^Q) = \mathcal{E}(\mathcal{U}^h)$ and moreover

$$Qs \preccurlyeq_{\mathcal{E}(\mathcal{U}^h)} Ps \quad \forall s \in \mathcal{E}(\mathcal{U})$$

for all exact subordination operator P with respect to $\mathcal{U}$ with $\mathcal{E}(\mathcal{U}^P) = \mathcal{E}(\mathcal{U}^h)$. Particularly we have

$$Qs \preccurlyeq_{\mathcal{E}(\mathcal{U}^h)} P^h s \quad \forall s \in \mathcal{E}(\mathcal{U}).$$

The assertion of the theorem will be a consequence of Theorem 2 if we show that $Q = P^h$.

As in Theorem 2 we deduce that if we denote $W := U^Q$ then we have

$$U_h f \preccurlyeq_{\mathcal{E}(\mathcal{U}^h)} Wf \quad \forall f \in p\mathcal{B}, Uf < \infty.$$

Hence there exists $g \in bp\mathcal{B}$, $0 < g \leq 1$ such that

$$U_h f = W(gf) \quad \forall f \in p\mathcal{B}$$

or equivalently

$$Wf = U_h(g'f) \quad \forall f \in p\mathcal{B}$$

where $g' \in p\mathcal{B}$, $g' \geq 1$.

Let $f \in p\mathcal{B}$ with $Uf < \infty$. We have

$$QUf = Uf - Wf = U_h(f + hUf - g'f).$$

If $f_1, f_2 \in p\mathcal{B}$ are such that $Uf_1 \leq Uf_2 < \infty$ we get

$$U_h(f_1 + hUf_1 - g'f_1) = QUf_1 \preccurlyeq_{\mathcal{E}(\mathcal{U}^h)} QUf_2 = U_h(f_2 + hUf_2 - g'f_2)$$

and therefore

$$(g' - 1)(f_2 - f_1) \leq h(Uf_2 - Uf_1) \quad \mathcal{U} - \text{a.e.}$$

Particularly if $f \in p\mathcal{B}$ and $Uf < \infty$ we get

$$(g' - 1)f \leq hUf \quad \mathcal{U} - \text{a.e.}$$

Since $\mathcal{B}$ is countable generated, there exists a countable subset $\mathcal{A}$ of $p\mathcal{B}$ such that the monotone class generated by $\mathcal{A}$ is equal $p\mathcal{B}$ and any element f of $\mathcal{A}$ is bounded and $Uf < \infty$. In this case there exists a subset E_0 of E, $E_0 \in \mathcal{B}$ with $U(1_{E \setminus E_0}) = 0$ such that on E_0 we have

$$(g' - 1)f \leq hUf$$

for all $f \in \mathcal{A}$ and therefore the above inequality holds for every $f \in p\mathcal{B}$. The proof will be finished if we show that $g' = 1$ on E_0. Assume that there exists $a \in E_0$ with

$$\rho := \frac{g'(a) - 1}{h(a)} > 0.$$

Since

$$\rho f(a) \leq Uf(a) \; \forall \, f \in p\mathcal{B}$$

it follows that the set $\{a\}$ is not $\mathcal{U}$-negligible. We take now $f = 1_{\{a\}}$ and we deduce

$$\frac{g'(a) - 1}{h(a)} \leq U1_{\{a\}}(a).$$

Since $\{a\}$ is not $\mathcal{U}$-finely open then we get

$$R^{E \setminus \{a\}} U(1_{\{a\}}) = U(1_{\{a\}})$$

and so there exists a sequence $(f_n)_n$ in $bp\mathcal{B}$ with $f_n(a) = 0$ for all $n \in \mathbb{N}$ and

$$Uf_n \nearrow R^{E \setminus \{a\}} U_{1_{\{a\}}}.$$

We get

$$(g' - 1)(1_{\{a\}} - f_n) \leq h(U(1_{\{a\}}) - Uf_n) \qquad \mathcal{U} - \text{a.e.}$$

and so $g'(a) - 1 \leq 0$, which contradicts the relation $\rho > 0$. □

Theorem 4. *Let $\mathcal{W} = (W_\alpha)_{\alpha > 0}$ be a proper sub-Markovian resolvent on $(E, \mathcal{B})$ such that its initial kernel W is a regular $\mathcal{U}$-excessive kernel and*

$$\inf_\alpha \alpha W_\alpha(Uf_0) = 0$$

for some $f_0 \in p\mathcal{B}$, $0 < f_0 \leq 1$, Uf_0 bounded. If there is no $\mathcal{U}$-finely open singleton in E then W_1 is the unique exact subordination operator P with respect to $\mathcal{U}$ such that

$$\mathcal{E}(\mathcal{U}^P) = \mathcal{E}(\mathcal{U}^{W_1}).$$

Proof. Let U' be the regular $\mathcal{U}$-excessive kernel $U' = U + W$. Using ([3] Theorem 6.3.2 and Theorem 6.3.4), it follows that there exists a natural sub-Markovian resolvent $\mathcal{U}' = (U'_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$ having as initial kernel U'. Since W is a regular $\mathcal{U}$-excessive kernel there exists $h \in p\mathcal{B}$, $h \leq 1$ such that $Wf = U'(hf)$ for all $f \in p\mathcal{B}$. From ([6]) it follows that h is exact with respect to $\mathcal{U}'$ and so there exists a kernel U'_h on $(E, \mathcal{B})$ such that $W_1 f = U'_h(hf)$,

$$U'f = U'_h f + U'_h(hU'f) \text{ and } U'_h(hU'f) = U'(hU'_h f)$$

for all $f \in p\mathcal{B}$. Hence W_1 becomes an exact subordination operator with respect to $\mathcal{U}'$ and $\mathcal{U}'^{W_1} = \mathcal{U}'^h$. Obviously $\mathcal{E}(\mathcal{U}'^{W_1}) = \mathcal{E}(\mathcal{U}^{W_1})$. Using Theorem 3 it follows that for any exact subordination operator P with respect to $\mathcal{U}'$ such that

$$\mathcal{E}(\mathcal{U}'^P) = \mathcal{E}(\mathcal{U}'^h)$$

we have $P = W_1$. If P is an exact subordination operator with respect to $\mathcal{U}$ with $\mathcal{E}(\mathcal{U}^P) = \mathcal{E}(\mathcal{U}^{W_1})$ then P will be an exact subordination operator with respect to $\mathcal{U}'$ with

$$\mathcal{E}(\mathcal{U}'^P) = \mathcal{E}(\mathcal{U}^P) = \mathcal{E}(\mathcal{U}^{W_1}) = \mathcal{E}(\mathcal{U}'^{W_1})$$

and so $P = W_1$. □

Theorem 5. *Let $\mathcal{V} = (V_\alpha)_{\alpha>0}$ a second natural sub-Markovian resolvent on $(E, \mathcal{B})$ such that there is no finely open singleton in E with respect to $\mathcal{U}$ and $\mathcal{V}$. Then the following assertions are equivalent:*

1) $\mathcal{U} = \mathcal{V}$

2) $\mathcal{E}(\mathcal{U}^\alpha) = \mathcal{E}(\mathcal{V}^\alpha)$ *for any* $\alpha \geq 0$

3) *There exist $\alpha, \beta \geq 0$, $\alpha < \beta$, such that $\mathcal{E}(\mathcal{U}^\alpha) = \mathcal{E}(\mathcal{V}^\alpha)$ and $\mathcal{E}(\mathcal{U}^\beta) = \mathcal{E}(\mathcal{V}^\beta)$.*

Proof. The implications 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3) are obvious.

3) $\Longrightarrow$ 1). By hypothesis and using Theorem 3 we deduce that $(\beta-\alpha)V_\beta$ is the only exact subordination operator with respect to the resolvent $\mathcal{V}^\alpha = (V_{\alpha+\gamma})_{\gamma>0}$ such that

$$\mathcal{E}((\mathcal{V}^\alpha)^{(\beta-\alpha)V_\beta}) = \mathcal{E}(\mathcal{V}^\beta) = \mathcal{E}(\mathcal{U}^\beta).$$

On the other hand $(\beta - \alpha)U_\beta$ is an exact subordination operator with respect to $\mathcal{U}^\alpha = (U_{\alpha+\gamma})_{\gamma>0}$ and therefore, using the fact that $\mathcal{E}(\mathcal{U}^\alpha) = \mathcal{E}(\mathcal{V}^\alpha)$ it is an exact subordination operator with respect to $\mathcal{V}^\alpha$ and we have

$$\mathcal{E}((\mathcal{U}^\alpha)^{(\beta-\alpha)U_\beta}) = \mathcal{E}((\mathcal{V}^\alpha)^{(\beta-\alpha)U_\beta}).$$

Since, from hypothesis

$$\mathcal{E}(\mathcal{V}^\beta) = \mathcal{E}(\mathcal{U}^\beta)$$

we deduce using the above considerations

$$\mathcal{E}((\mathcal{V}^\alpha)^{(\beta-\alpha)V_\beta}) = \mathcal{E}(\mathcal{V}^\beta) = \mathcal{E}(\mathcal{U}^\beta) = \mathcal{E}((\mathcal{U}^\alpha)^{(\beta-\alpha)U_\beta}) = \mathcal{E}((\mathcal{V}^\alpha)^{(\beta-\alpha)U_\beta}).$$

Hence $(\beta-\alpha)U_\beta$ is an exact subordination operator with respect to $\mathcal{V}^\alpha$ such that

$$\mathcal{E}((\mathcal{V}^\alpha)^{(\beta-\alpha)U_\beta}) = \mathcal{E}((\mathcal{V}^\alpha)^{(\beta-\alpha)V_\beta})$$

and therefore, from Theorem 3 we have

$$(\beta-\alpha)U_\beta = (\beta-\alpha)V_\beta, \quad U_\beta = V_\beta.$$

Since the resolvents $\mathcal{U}^\beta$, $\mathcal{V}^P$ are bounded and sub-Markovian they coincide having the same initial kernel $U_\beta = V_\beta$ i.e.

$$U_\lambda = V_\lambda \quad \forall\, \lambda \geq \beta.$$

Because for any $\lambda < \beta$ we have

$$U_\lambda = U_\beta + \sum_{i\geq 1}(\beta-\lambda)^i U_\beta^{i+1}, \quad V_\lambda = V_\beta + \sum_{i\geq 1}(\beta-\lambda)^i V_\beta^{i+1}$$

we deduce that $U_\lambda = V_\lambda$ for all $\lambda \geq 0$. $\square$

Remark. 1. If in Theorems 1 and 2 the condition "there is no $\mathcal{U}$-absorbent point in E" is not satisfied then these results do not hold. Indeed, if $\mathcal{U}$ is the trivial sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ i.e. $\mathcal{U} = (U_\alpha)_{\alpha>0}$ where $U_\alpha f = \frac{1}{1+\alpha} f$ for all $f \in p\mathcal{B}$ then for any $\alpha \geq 0$ we have $\mathcal{E}(\mathcal{U}^\alpha) = \mathcal{E}(\mathcal{U}) = p\mathcal{B}$. In this case any point $a \in E$ is $\mathcal{U}$-absorbent.
2. In fact the condition "there is no $\mathcal{U}$-absorbent point in E" is a necessary condition such that Theorems 1 and 2 hold. Indeed, if a is a $\mathcal{U}$-absorbent point in E then the kernel $P = \dfrac{1}{2}B^{\{a\}}$ (where $B^{\{a\}}$ is the balayage kernel on $\{a\}$ with respect to $\mathcal{U}$) is an exact subordination operator with respect to $\mathcal{U}$ and we have $\mathcal{E}(\mathcal{U}) = \mathcal{E}(\mathcal{U}^P)$. Obviously P is a regular $\mathcal{U}$-excesive kernel with respect to $\mathcal{U}$.
3. If in Theorem 5 we assume that $\mathcal{E}(\mathcal{U}^\alpha) = \mathcal{E}(\mathcal{V}^\alpha)$ for only one $\alpha \in \mathbb{R}_+$ then Theorem 5 does not hold. Indeed, if we consider $\mathcal{V} = (V_\alpha)_{\alpha>0}$ such that $V_0 = 2U_0$ then we have $\mathcal{E}(\mathcal{U}) = \mathcal{E}(\mathcal{V})$ but $\mathcal{U} \neq \mathcal{V}$.

Proposition 6. *Assume that there is a point $x_0 \in E$ which is finely open. Then there exist two exact subordination operators P, Q with respect to $\mathcal{U}$ such that $\mathcal{E}(\mathcal{U}^P) = \mathcal{E}(\mathcal{U}^Q)$ and $P \neq Q$. Moreover P and Q are $\mathcal{U}$-excessive kernels and P is a regular $\mathcal{U}^P$-excessive kernel.*

Proof. We denote by P the kernel on (E, B) given by

$$Ps = \frac{1}{2} R^{\{x_0\}} s \quad \forall s \in \mathcal{E}(\mathcal{U}).$$

where $R^{\{x\}}s$ is the reduite of s on the set $\{x\}$ with respect to $\mathcal{U}$. If we denote $u = R^{\{x_0\}}1$ we have $Pf = \frac{1}{2}f(x_0)u$. Since the family of kernels $\mathcal{V} = (V_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$ given by $V_\alpha f = \dfrac{1}{1+\alpha} f(x_0)u$ is a sub-Markovian resolvent on $(E, \mathcal{B})$ such that $\inf_\alpha \alpha V_\alpha U f_0 = 0$ for $f_0 \in bp\mathcal{B}$, $0 < f_0$ with Uf_0 bounded and $P = V_1$ it follows (cf. [3]) that P is an exact subordination operator with respect to $\mathcal{U}$. We remark that P is a $\mathcal{U}$-excessive kernel and it is a regular $\mathcal{U}^P$-excessive kernel. Moreover the set

$$\{s - \frac{1}{2} s(x_0)u / s \in \mathcal{E}(\mathcal{U}), s < \infty\}$$

is solid and increasingly dense in $\mathcal{E}(\mathcal{U}^P)$.

Let now Q the following kernel on $(E, \mathcal{B})$ defined by

$$Qs = P(R^{E \setminus \{x_0\}} s), \quad s \in \mathcal{E}(\mathcal{U}).$$

Since $\{x_0\}$ is finely open it follows that there exists $s \in \mathcal{E}(\mathcal{U})$, $s < \infty$ such that

$$R^{E \setminus \{x_0\}} s(x_0) < s(x_0).$$

Obviously we have

$$Qs = \frac{1}{2} R^{E \setminus \{x_0\}} s(x_0) \cdot u \quad \forall s \in \mathcal{E}(\mathcal{U})$$

and so

$$Qf = \frac{1}{2} R^{E \setminus \{x_0\}} f(x_0) \cdot u.$$

Consequently Q is a $\mathcal{U}$-excessive kernel on $(E, \mathcal{B})$, $Qs \leq Ps$ for all $s \in \mathcal{E}(\mathcal{U})$ and $P \neq Q$.

We show now that Q is an exact subordination operator with respect to $\mathcal{U}$. By the preceding considerations it remains to show that

$$w := \inf(s, Qs + t - Qt + Qf) \in \mathcal{E}(\mathcal{U})$$

for all $s, t \in \mathcal{E}(\mathcal{U})$, $f \in p\mathcal{B}$, $s < \infty$, $t < \infty$. If $R^{E \setminus \{x_0\}} s(x_0) \geq R^{E \setminus \{x_0\}} t(x_0)$ then

$$Qs + t - Qt + Qf = \frac{1}{2}(R^{E \setminus \{x_0\}} s(x_0) - R^{E \setminus \{x_0\}} t(x_0) + R^{E \setminus \{x_0\}} f(x_0))u + t \in \mathcal{E}(\mathcal{U})$$

and therefore $w \in \mathcal{E}(\mathcal{U})$.

Assume that

$$R^{E\setminus\{x_0\}}s(x_0) < R^{E\setminus\{x_0\}}t(x_0).$$

We have

$$w = \inf(s, R^{\{x_0\}}R^{E\setminus\{x_0\}}s + (t - R^{\{x_0\}}t) + \frac{1}{2}R^{\{x_0\}}(2t - R^{E\setminus\{x_0\}}t - R^{E\setminus\{x_0\}}s + R^{E\setminus\{x_0\}}f).$$

But

$$R^{E\setminus\{x_0\}}t(x_0) + R^{E\setminus\{x_0\}}s(x_0) < 2R^{E\setminus\{x_0\}}t(x_0) \leq 2t(x_0)$$

or equivalently

$$2t(x_0) - R^{E\setminus\{x_0\}}t(x_0) - R^{E\setminus\{x_0\}}s(x_0) > 0.$$

We put

$$\beta := 2t(x_0) - R^{E\setminus\{x_0\}}t(x_0) - R^{E\setminus\{x_0\}}s(x_0) + R^{E\setminus\{x_0\}}s(x_0) + R^{E\setminus\{x_0\}}f(x_0).$$

and we have $\beta > 0$ and

$$R^{\{x_0\}}(2t - R^{E\setminus\{x_0\}}t - R^{E\setminus\{x_0\}}s + R^{E\setminus\{x_0\}}f)) = \beta u.$$

Hence

$$\begin{aligned} w &= \inf(s, R^{\{x_0\}}R^{E\setminus\{x_0\}}s + t - R^{\{x_0\}}t + \tfrac{\beta u}{2}) = \\ &= \inf(s, \inf(s, R^{\{x_0\}}R^{E\setminus\{x_0\}}s + t - R^{\{x_0\}}t) + \inf(s, \frac{\beta u}{2})) \end{aligned}$$

and so it will be sufficient to show that

$$\inf(s, R^{\{x_0\}}R^{E\setminus\{x_0\}}s + t - R^{\{x_0\}}t) \in \mathcal{E}(\mathcal{U}).$$

We have

$$R^{\{x_0\}}R^{E\setminus\{x_0\}}s(x_0) + (t - R^{\{x_0\}}t)(x_0) = R^{E\setminus\{x_0\}}s(x_0)$$

i.e.

$$\inf(s, R^{\{x_0\}}R^{E\setminus\{x_0\}}s + t - R^{\{x_0\}}t)(x_0) = R^{E\setminus\{x_0\}}s(x_0)$$

and so

$$\inf(s, R^{\{x_0\}}R^{E\setminus\{x_0\}}s + t - R^{\{x_0\}}t) = \inf(R^{E\setminus\{x_0\}}s, R^{\{x_0\}}R^{E\setminus\{x_0\}}s + t - R^{\{x_0\}}t).$$

Since

$$\inf(s', R^{\{x_0\}}s' + t - R^{\{x_0\}}t) \in \mathcal{E}(\mathcal{U})$$

for all $s', t \in \mathcal{E}(\mathcal{U})$, $s' < \infty$, $t < \infty$ we get that

$$\inf(s, R^{\{x_0\}}R^{E\setminus\{x_0\}}s + t - R^{\{x_0\}}t) \in \mathcal{E}(\mathcal{U}).$$

To finish the proof we show that $\mathcal{E}(\mathcal{U}^P) = \mathcal{E}(\mathcal{U}^Q)$. If $s \in \mathcal{E}(\mathcal{U})$ then we have

$$s - Qs = s - \frac{1}{2}R^{E\setminus\{x_0\}}s(x_0) \cdot u = s - \frac{1}{2}s(x_0) \cdot u + \frac{1}{2}(s(x_0) - R^{E\setminus\{x_0\}}s(x_0))u =$$

$$s - Ps + (s(x_0) - R^{E\setminus\{x_0\}}s(x_0))u \in \mathcal{E}(\mathcal{U}^P).$$

Conversely, let $f \in p\mathcal{B}$ with $Uf < \infty$. We have

$$(U - PU)(f) = (U - PU)(f \cdot 1_{E\setminus\{x_0\}}) + (U - PU)(f \cdot 1_{\{x_0\}}) =$$
$$= \frac{1}{2}U(f \cdot 1_{\{x_0\}}) + U(f \cdot 1_{E\setminus\{x_0\}}) - PR^{E\setminus\{x_0\}}U(f \cdot 1_{E\setminus\{x_0\}}) =$$
$$= \frac{1}{2}U(f \cdot 1_{\{x_0\}}) + U(f \cdot 1_{E\setminus\{x_0\}}) - QU(f \cdot 1_{E\setminus\{x_0\}}) \in \mathcal{E}(\mathcal{U}^Q).$$

From the above considerations we conclude that $\mathcal{E}(\mathcal{U}^P) = \mathcal{E}(\mathcal{U}^Q)$. □

Remark. 1. Proposition 6 shows that the condition "there is no $\mathcal{U}$-finely open singleton" is necessary such that Theorem 4 holds.

2. The kernel Q considered in Proposition 6 is not regular $\mathcal{U}$-exessive if x_0 is not $\mathcal{U}$-absorbent. Indeed, if $G = E\setminus\{x_0\}$ and $u = R^{\{x_0\}}1$ then we have

$$R^G u = u \text{ on } G, \quad R^G u(x_0) < u(x_0).$$

If $\alpha := \frac{1}{2}R^G 1_G(x_0)$ then we get $\alpha > 0$ and

$$Q1_G = \alpha u = \alpha R^G u \quad \text{on } G$$

but $Q1_G(x_0) = \alpha u(x_0) > \alpha R^G u(x_0)$.

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